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Research article

Gorenstein projective modules over Milnor squares of rings

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Abstract: We construct a class of Gorenstein-projective modules over Milnor squares of rings. As an application, we obtain Gorenstein-projective modules over Morita context rings with two bimodule homomorphisms zero in the general setting instead of Artin algebras or Noetherian rings.

Keywords: Gorenstein-projective module; Milnor square of ring; Morita context ring; fiber product category
Mathematics Subject Classification: 16E05, 18G05, 18G25

1. Introduction

Let A_i and A be four rings with units for $i = 0, 1, 2$. Let $\pi_i : A_i \to A_0$ and $p_i : A \to A_i$ be ring homomorphisms for $i = 1, 2$. We fix a commutative diagram of rings.

$$\begin{array}{ccc} A & \xrightarrow{p_1} & A_1 \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle \pi_1} \\ A_2 & \xrightarrow{\pi_2} & A_0 \end{array} \qquad (*)$$

If the sequence $0 \longrightarrow A \xrightarrow{(p_1,p_2)} A_1 \oplus A_2 \xrightarrow{\binom{\pi_1}{-\pi_2}} A_0$ is exact, it is called a *pullback*, and A is called a *pullback* ring. For the given π_n for $n = 1, 2$, we can always define a ring $A := \{(x, y) \in A_1 \times A_2 \mid (x)\pi_1 = (y)\pi_2\}$, and define $p_n : A \to A_n$ as the canonical projections for $n = 1, 2$ such that it is a standard pullback. The ring A is unique up to isomorphism, and we shall always consider the standard case throughout the paper. If, moreover, one of π_1 and π_2 is surjective, then it is called a *Milnor square* of rings (see [13]), which gives the short exact sequence

$$0 \longrightarrow A \xrightarrow{(p_1,p_2)} A_1 \oplus A_2 \xrightarrow{\binom{\pi_1}{-\pi_2}} A_0 \longrightarrow 0 \qquad (\dagger)$$

Note that if π_1 is surjective, then so is p_2.

Milnor squares of rings appear in many aspects in mathematics, such as algebraic K-theory, representation theory, and homological algebra (for instance, see [4, 8, 12, 13]). In [13], they were studied by the author Milnor in his construction of projective A-modules from projective A_1- and A_2-modules, and then the establishment of the Mayer–Vietoris sequence of K-groups. As Milnor's philosophy of investigation of projective A-modules, Facchini and Vámos studied injective and flat A-modules from injective and flat A_1- and A_2-modules, respectively, in [4]. Herbara and Prihoda also investigated infinitely generated projective modules in [8] within the framework of Milnor squares of rings. In view of the development of Gorenstein homological algebra (see, for instance, [2, 9]), it is essential to study Gorenstein-projective modules in the context of pullback rings, and the question of how to construct Gorenstein-projective A-modules from Gorenstein-projective A_1- and A_2-modules arises naturally.

In this paper, we give a method to answer this question under certain conditions. Given two Gorenstein-projective A_1- and A_2-modules G_1 and G_2, and an isomorphism $g : A_0 \otimes_{A_1} G_1 \to A_0 \otimes_{A_2} G_2$ of A_0-modules, we can define the standard pullback module G of the triple (G_1, G_2, g) (see Section 2.3). Our main result Theorem 1.1 shows that G is a Gorenstein-projective A-module.

Theorem 1.1. *Let $A_1 \stackrel{\pi_1}{\to} A_0 \stackrel{\pi_2}{\leftarrow} A_2$ be homomorphisms of rings such that both π_1 and π_2 are surjective, and let A be the pullback ring. Denote by A_n-*GProj *the category of Gorenstein-projective A_n-modules for $n = 1, 2$. Assume that the following conditions hold.*

(i) $\mathrm{Tor}_1^{A_2}(A_0, H) = 0$ *for all* $H \in A_2$-GProj.

(ii) $\mathrm{Add}(\mathrm{Ker}(\pi_n)) \subseteq A_n$-$\mathrm{GProj}^{\perp}$ *for* $n = 1, 2$.

Let (G_1, G_2, g) be a triple such that G_n is a Gorenstein-projective A_n-module for $n = 1, 2$ and $g : A_0 \otimes_{A_1} G_1 \simeq A_0 \otimes_{A_2} G_2$ is an A_0-isomorphism. Then the pullback module of the triple (G_1, G_2, g) is a Gorenstein-projective A-module.

Moreover, when A_2 is strongly left CM-free (see Section 3), the conditions (i) and (ii) in Theorem 1.1 can be weakened as follows:

Corollary 1.2. *Assume that A_2 is strongly left CM-free. If* $\mathrm{Add}(\mathrm{Ker}(\pi_1)) \subseteq \mathrm{A_1}$-$\mathrm{GProj}^{\perp}$, *and G_n is a Gorenstein-projective A_n-module for $n = 1, 2$, then the pullback module of the triple (G_1, G_2, g) is a Gorenstein-projective A-module.*

An important class of examples of Milnor squares of rings with both π_1 and π_2 surjective can be provided by the Morita context rings with two bimodule homomorphisms zero. The current works (see [5,7]) about the construction of Gorenstein-projective modules over these kinds of rings are mainly concentrated in the settings of Artin algebras and Noetherian rings, while our results, Theorem 1.3, as an application of Theorem 1.1, extends these results to the general rings.

Theorem 1.3. *Let $\Lambda_{(0,0)} = \left(\begin{smallmatrix} \Lambda & N \\ M & B \end{smallmatrix}\right)_{(0,0)}$ be a Morita context ring with two bimodule homomorphisms zero. Assume that the following conditions hold.*

(1) $\mathrm{Tor}_1^{\Lambda}(M, H) = 0$ *for all* $H \in \Lambda$-GProj, *and* $\mathrm{Add}({}_{\mathrm{B}}\mathrm{M}) \subseteq \mathrm{B}$-$\mathrm{GProj}^{\perp}$,

(2) $\mathrm{Tor}_1^{B}(N, G) = 0$ *for all* $G \in B$-GProj, *and* $\mathrm{Add}({}_{\Lambda}\mathrm{N}) \subseteq \Lambda$-$\mathrm{GProj}^{\perp}$.

Let $U \in \Lambda$-GProj, $V \in B$-GProj. *If there exist a Λ-module X, a B-module Y, and two short exact sequences*

$$0 \longrightarrow N \otimes_B V \stackrel{g}{\longrightarrow} X \stackrel{s}{\longrightarrow} U \longrightarrow 0$$

and

$$0 \longrightarrow M \otimes_{\Lambda} U \xrightarrow{f} Y \xrightarrow{t} V \longrightarrow 0$$

in Λ-Mod *and* B-Mod*, respectively, then* $(X, Y, (1_M \otimes s)f, (1_N \otimes t)g)$ *is a Gorensteon-projective* $\Lambda_{(0,0)}$-*module.*

The rest of this paper is structured as follows: In Section 2, we fix some notations and recall basic facts for later use. In Section 3, we prove the main result Theorem 1.1 in Theorem 3.3, while Lemmas 3.1 and 3.2 are the preparations. In Section 4, we prove Theorem 1.3 in Corollary 4.1.

2. Preliminaries

In this section, we recall basic definitions and facts for later proofs.

Let A be an associative ring with unit. We denote by A-Mod the category of all left A-modules. As usual, A-Proj is the full subcategory of A-Mod consisting of all projective modules. For a module ${}_AX$, we denote by $\mathrm{Add}({}_AX)$ the full subcategory of A-Mod consisting of modules isomorphic to direct summands of direct sums of ${}_AX$. Then we have $A\text{-Proj} = \mathrm{Add}({}_AA)$.

Let $\cdots \to X_2 \to X_1 \to X_0 \to \cdots$ be a sequence in A-Mod and $Y \in A$-Mod. It is called $\mathrm{Hom}_A(-, Y)$-exact if itself is exact and applying $\mathrm{Hom}_A(-, Y)$ preserves its exactness. Let $\cdots \to P_1 \xrightarrow{d_1} P_0 \xrightarrow{\delta} X \to 0$ be a projective resolution of ${}_AX$. We denote $\mathrm{Ker}(d_i)$ by $\Omega_i X$ for $i \geq 1$ and $\mathrm{Ker}(\delta)$ by $\Omega_0 X$.

The composite of two homomorphisms $f : X \to Y$ and $g : Y \to Z$ will be denoted by fg instead of gf. Thus, the image of $x \in X$ under f is written as $(x)f$ or xf, and the image of f is denoted by $\mathrm{Im}(f)$.

The composite of two functors $F : C \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ will be denoted by $G \circ F$, which is a functor from $C \to \mathcal{E}$. Let (F, G) be an adjoint pair where $F : C \to \mathcal{D}$ and $G : \mathcal{D} \to C$. We denote by $\eta : \mathrm{Id}_C \to GF$ the unit, and by $\epsilon : FG \to \mathrm{Id}_{\mathcal{D}}$ the counit. We need the following fact.

Lemma 2.1. *Let* (F, G) *be an adjoint pair. Then*

(1) *The adjunction isomorphism* $\eta_{X,Y} : \mathrm{Hom}_{\mathcal{D}}(FX, Y) \to \mathrm{Hom}_C(X, GY)$ *is defined by* $f \mapsto \eta_X(Gf)$ *with the inverse map* $\eta^{-1}_{X,Y}$ *defined by* $g \mapsto (Fg)\epsilon_Y$*, where* η_X *and* ϵ_Y *are the corresponding unit and counit maps.*

(2) *Let*

$$\begin{array}{ccc} U & \xrightarrow{\alpha} & V \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ GU' & \xrightarrow{G\beta} & GV' \end{array}$$

be a commutative diagram in C*. Then the diagram*

$$\begin{array}{ccc} FU & \xrightarrow{F\alpha} & FV \\ {\scriptstyle (f)\eta^{-1}_{U,U'}}\downarrow & & \downarrow{\scriptstyle (g)\eta^{-1}_{V,V'}} \\ U' & \xrightarrow{\beta} & V' \end{array}$$

is commutative, and vice versa.

Next, we recall the definition of approximations. Let $\mathcal{D}$ be a full additive subcategory of an additive category C and X an object in C. A morphism $f : X \to D$ in C is called a *left* $\mathcal{D}$*-approximation* of X if $D \in \mathcal{D}$ and $\mathrm{Hom}_{C}(f, D') : \mathrm{Hom}_{C}(D, D') \to \mathrm{Hom}_{C}(X, D')$ is surjective for any object $D' \in \mathcal{D}$. Dually, a morphism $f : D \to X$ in C is called a *right $\mathcal{D}$-approximation* of X if $D \in \mathcal{D}$ and $\mathrm{Hom}_{C}(D', f) : \mathrm{Hom}_{C}(D', D) \to \mathrm{Hom}_{C}(D', X)$ is surjective for any object $D' \in \mathcal{D}$. Note that left and right approximations are termed as preenvelopes and precovers in ring theory, respectively.

Let $0 \to X \xrightarrow{\alpha} T \to Y \to 0$ be a short exact sequence with ${}_AT$ projective. Then α is a left $\mathrm{Add}({}_AA)$-approximation of X if and only if $\mathrm{Ext}^1_A(Y, P) = 0$ for all $P \in A$-Proj if and only if it is $\mathrm{Hom}_A(-, P)$-exact for every $P \in A$-Proj.

The following homological facts are often used.

Lemma 2.2. *Let A and B be unitary rings, and let ${}_AX$ be an A-module, ${}_BU$ a B-module, and ${}_BM_A$ a B-A-bimodule. If $\mathrm{Tor}^A_1(M, X) = \mathrm{Ext}^1_B(M, U) = 0$, then*

$$\mathrm{Ext}^1_B(M \otimes_A X, U) \simeq \mathrm{Ext}^1_A(X, \mathrm{Hom}_B(M, U)).$$

Proof. Consider the short exact sequences

$$0 \longrightarrow {}_AK \xrightarrow{f} {}_AP \longrightarrow {}_AX \longrightarrow 0,$$

and

$$0 \longrightarrow {}_BU \longrightarrow {}_BI \xrightarrow{g} {}_BL \longrightarrow 0,$$

where $P \in A$-Proj, $I \in B$-Inj. Since $\mathrm{Tor}^A_1(M, X) = \mathrm{Ext}^1_B(M, U) = 0$ by assumption, applying ${}_BM \otimes_A -$ and $\mathrm{Hom}_B(M, -)$ yield two short exact sequences

$$0 \longrightarrow {}_BM \otimes_A K \longrightarrow {}_BM \otimes_A P \longrightarrow {}_BM \otimes_A X \longrightarrow 0,$$

and

$$0 \longrightarrow \mathrm{Hom}_B(M, U) \longrightarrow \mathrm{Hom}_B(M, I) \xrightarrow{g*} \mathrm{Hom}_B(M, L) \longrightarrow 0,$$

respectively. Note that Hom is a functor of two variables. Then we have the following commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & {}_A(X, {}_B(M, U)) & \longrightarrow & {}_A(P, {}_B(M, U)) & \longrightarrow & {}_A(K, {}_B(M, U)) & \longrightarrow & \mathrm{Ext}^1_A(X, {}_B(M, U)) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & {}_A(X, {}_B(M, I)) & \longrightarrow & {}_A(P, {}_B(M, I)) & \longrightarrow & {}_A(K, {}_B(M, I)) & \longrightarrow & 0 \\
 & & \downarrow {\scriptstyle {}_A(X, g^*)} & & \downarrow & & \downarrow {\scriptstyle {}_A(K, g^*)} & & \\
0 & \longrightarrow & {}_A(X, {}_B(M, L)) & \longrightarrow & {}_A(P, {}_B(M, L)) & \xrightarrow{{}_A(f, (M, L))} & {}_A(K, {}_B(M, L)) & \longrightarrow & \mathrm{Ext}^1_A(X, {}_B(M, L)) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \mathrm{Ext}^1_B(M \otimes_A X, U) & & 0 & & \mathrm{Ext}^1_B(M \otimes_A K, U) & & \\
 & & \downarrow & & & & \downarrow & & \\
 & & 0 & & & & 0 & &
\end{array}$$

In fact, the exactness of the left column follows from the commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 \longrightarrow & {}_B(M\otimes_A X, U) & \longrightarrow & {}_B(M\otimes_A X, I) & \longrightarrow & {}_B(M\otimes_A X, L) & \longrightarrow & \mathrm{Ext}^1_B(M\otimes_A X, U) & \longrightarrow 0 \\
 & \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq & \\
0 \longrightarrow & {}_A(X, {}_B(M, U)) & \longrightarrow & {}_A(X, {}_B(M, I)) & \xrightarrow{{}_A(X, g^*)} & {}_A(X, {}_B(M, L)) & \longrightarrow & \mathrm{Coker}({}_A(X, g^*)) & \longrightarrow 0
\end{array}$$

Similarly, one can verify that the right column and the middle row are exact. Then by the Snake Lemma, we have $\mathrm{Ext}^1_A(X, {}_B(M, U)) \simeq \mathrm{Ext}^1_B(M \otimes_A X, U)$. □

2.1. Gorenstein-projective modules

Recall that a *complete projective resolution* is an exact sequence

$$P^\bullet = \cdots P_2 \xrightarrow{d_2^P} P_1 \xrightarrow{d_1^P} P_0 \xrightarrow{d_0^P} P^0 \xrightarrow{d_P^0} P^1 \xrightarrow{d_P^0} P^2 \to \cdots$$

such that $\mathrm{Hom}_A(P^\bullet, P)$ is exact for every projective A-module P. An A-module X is called *Gorenstein-projective* if there exits a complete projective resolution complex $P^\bullet$ such that $X \simeq \mathrm{Im}(d_0^P) = \mathrm{Ker}(d_P^0)$. One can observe that if $P^\bullet$ is a complete projective resolution, then all the images and kernels of $P^\bullet$ are Gorenstein-projective. We denote by A-GProj the category of all Gorenstein-projective A-modules, and let $A\text{-GProj}^\perp := \{{}_AX \mid \mathrm{Ext}^1_A(M, X) = 0, \forall {}_AM \in A\text{-GProj}\}$. We list the following well-known properties for later use (see also [9]).

Lemma 2.3. *Let* $0 \to G' \to G \to G'' \to 0$ *be a short exact sequence in* A-Mod.

(1) *If G and G'' are Gorenstein-projective, then G' is Gorenstein-projective.*
(2) *If G, G' are Gorenstein-projective, and* $\mathrm{Ext}^1_A(G'', P) = 0$ *for all projective A-modules P, then G'' is Gorenstein-projective.*

Lemma 2.4. *For any module ${}_AG$, the following two conditions are equivalent.*

(1) *${}_AG$ is Gorenstein-projective.*
(2) *There exists a* $\mathrm{Hom}_A(-, P)$*-exact sequence* $0 \to G \to Q^0 \to Q^1 \to \cdots$ *for every projective A-module P, and* $\mathrm{Ext}^i_A(G, P) = 0$ *for all $i \geq 1$ and all projective A-modules P.*
Moreover, if $M \in A$-Mod such that $\mathrm{projdim}({}_AM) < \infty$, *then $M \in A\text{-GProj}^\perp$.*

2.2. Modules under change of rings

Let $f : A \to B$ be a homomorphism of rings. Then every B-module U can be viewed as an A-module by defining $a \cdot u := (a) f u$ for all $a \in A$ and $u \in U$. Thus we get the so-called restriction functor ${}_A(-) : B\text{-Mod} \to A\text{-Mod}$. Moreover, there is an adjoint pair $(B \otimes_A -, {}_A(-))$ of functors whose unit is the canonical homomorphism of A-modules:

$$\eta_X : X \longrightarrow {}_AB \otimes_A X, \quad x \mapsto 1_B \otimes x \ \text{ for } \ x \in X$$

for $X \in A$-Mod, and the counit

$$\epsilon_U : B \otimes_A U \longrightarrow U, \quad b \otimes u \mapsto bu \ \text{ for } \ u \in U, b \in B,$$

for $U \in B$-Mod.

By Lemma 2.1, we have the following fact.

Lemma 2.5. *Let* $X \in A$-Mod, $U \in B$-Mod. *Then*

(1) *There is an adjunction isomorphism*

$$\eta_{X,U} : \mathrm{Hom}_B(B \otimes_A X, U) \to \mathrm{Hom}_A(X, U), \quad f \mapsto (f)\eta_{X,U} : x \mapsto (1_B \otimes x)f, \ x \in X$$

with the inverse map $\eta^{-1}_{X,U}$ *defined by* $(g)\eta^{-1}_{X,U} = (1_B \otimes g)\epsilon_U : b \otimes x \mapsto b((x)g), \ x \in X, \ b \in B.$

(2) *The cannonical isomorphism* $B \otimes_A (A)^I \to (B)^I$ *maps* $b \otimes (a_i)$ *to* $b(a_i f)$ *is the image of* f^I *under* $\eta^{-1}_{A^I,B^I}$, *where* I *denotes any cardinal number and* f^I *denotes the diagonal map.*

If f is surjective, then $B \otimes_A X \simeq X/\mathrm{Ker}(f)X$ for $X \in A$-Mod, and we have the following lemma.

Lemma 2.6. *Let* $X \in A$-Mod, $U \in B$-Mod. *Then*

(1) *The unit map* $\eta_X : X \to B \otimes_A X$ *is surjective.*
(2) *The counit map* $\epsilon_U : B \otimes_A U \longrightarrow U$ *is an isomorphism.*
(2) *Let* $g \in \mathrm{Hom}_A(X, U)$. *If* g *is surjective, then so is* $(g)\eta^{-1}_{X,U}$.

Remark that the unit map in Lemma 2.6(1) is actually the composite $X \to X/\mathrm{Ker}(f)X \simeq B \otimes_A X$.
Next we study when a ring extension preserves Gorenstein-projective modules.

Lemma 2.7. *Let* $f : A \to B$ *be a surjective homomorphism of rings. Suppose that* $\mathrm{Tor}^A_1(B, H) = 0$ *for all* $H \in A$-GProj, *and* Add(Ker(f)) $\subseteq$ A-GProj$^{\perp}$. *If* $X \in A$-GProj, *then* $B \otimes_A X \in B$-GProj.

Proof. Since $X \in A$-Gproj, there exists a complete projective resolution $P^\bullet$ such that $\mathrm{Ker}(d^0_P) = X$. Since $\mathrm{Tor}^A_1(B, H) = 0$ for all Gorenstein-projective A-modules, we have that $B \otimes_A P^\bullet$ is exact with $\mathrm{Ker}(1_B \otimes d^0_P) = B \otimes_A X$.

Next we shall show that $\mathrm{Hom}_B(B \otimes_A P^\bullet, Q)$ is exact for every $Q \in B$-Proj, while by the isomorphism $\mathrm{Hom}_B(B \otimes_A P^\bullet, Q) \cong \mathrm{Hom}_A(P^\bullet, Q)$, we only need to show that $\mathrm{Hom}_A(P^\bullet, Q)$ is exact. Then it is sufficient to show that ${}_AQ \subseteq A$-GProj$^{\perp}$. Since Add(Ker(f)) $\subseteq$ A-GProj$^{\perp}$, by the short exact $0 \longrightarrow \mathrm{Ker}(f) \longrightarrow A \longrightarrow B \longrightarrow 0$, we have $\mathrm{Add}({}_AB) \subseteq A$-GProj$^{\perp}$, and thus ${}_AQ \subseteq A$-GProj$^{\perp}$. Then it follows that $\mathrm{Hom}_A(P^\bullet, Q)$ is exact. □

2.3. Modules over Milnor squares of rings

In this subsection, we mainly study the module category of a pullback ring.

Let $(*)$ be a Milnor square of rings. From whom we obtain four left adjoint functors listed in the following diagram

$$\begin{array}{ccc} A\text{-Mod} & \xrightarrow{F} & A_1\text{-Mod} \\ {\scriptstyle L}\downarrow & & \downarrow{\scriptstyle F_1} \\ A_2\text{-Mod} & \xrightarrow{F_2} & A_0\text{-Mod} \end{array}$$

where F, L denote the left adjoint functors $A_1 \otimes_A -$ and $A_2 \otimes_A -$, respectively, and F_n denotes the left adjoint functors $A_0 \otimes_{A_n} -$ for $n = 1, 2$. We will fix the above notations in the rest of the paper. Note that there is a natural isomorphism of functors $\pi : F_1F \to A_0 \otimes_A - \to F_2L$. As for F_n, we denote the unit map by $\eta_n : X_n \longrightarrow F_nX_n$ for $X_n \in A_n$-Mod, and the counit map $\epsilon_n : F_nY \longrightarrow Y$ for $Y \in A_0$-Mod.

Following [4], we study the category of A-Mod by the so-called *fiber product category* $\mathcal{F}$ of A_1-Mod and A_2-Mod. An object in $\mathcal{F}$ is a triple (X_1, X_2, x), where $X_n \in A_n$-Mod, $n = 1, 2$, and $x : F_1X_1 \to F_2X_2$

is an A_0-isomorphism. A homomorphism from a triple (X_1, X_2, x) to another triple (Y_1, Y_2, y) in $\mathcal{F}$ is a pair (α_1, α_2) of A_n-homomorphisms $\alpha_n : X_n \to Y_n$ such that the following square commutes.

$$\begin{array}{ccc} F_1X_1 & \xrightarrow{F_1\alpha_1} & F_1Y_1 \\ \downarrow{\scriptstyle x} & & \downarrow{\scriptstyle y} \\ F_2X_2 & \xrightarrow{F_2\alpha_2} & F_2Y_2 \end{array} \tag{2.1}$$

Then for an A-module X, there is an associated triple (FX, LX, π_X) where $\pi_X : F_1FX \to F_2LX$ is the composite of the cannonical A_0-isomorphisms $F_1FX \xrightarrow{\simeq} A_0 \otimes_A X \xrightarrow{\simeq} F_2LX$.

Conversely, for a triple (X_1, X_2, x) in $\mathcal{F}$, there is an associated pullback module $\mathrm{Pb}(X_1, X_2, x) := \{(x_1, x_2) \in X_1 \times X_2 \mid (1_{A_1} \otimes x_1)x = 1_{A_2} \otimes x_2\}$. Write X for $\mathrm{Pb}(X_1, X_2, x)$ for brevity. Then X gives the following standard pullback diagram of A-modules:

$$\begin{array}{ccc} X & \xrightarrow{j_1} & X_1 \\ {\scriptstyle j_2}\downarrow & & \downarrow{\scriptstyle \eta_1 x} \\ X_2 & \xrightarrow{\eta_2} & F_2X_2 \end{array} \tag{2.2}$$

and the short exact sequence $0 \longrightarrow X \xrightarrow{(j_1, j_2)} X_1 \oplus X_2 \xrightarrow{\binom{\eta_1 x}{-\eta_2}} F_2X_2 \longrightarrow 0$, where j_n is the canonical projection for $n = 1, 2$. We denote by $\widetilde{j_1}$ and $\widetilde{j_2}$ the image of j_1 and j_2 under the isomorphism $\mathrm{Hom}_A(X, X_1) \simeq \mathrm{Hom}_{A_1}(FX, X_1)$ and $\mathrm{Hom}_A(X, X_2) \simeq \mathrm{Hom}_{A_2}(LX, X_2)$, respectively.

In this language, we summarize Milnor's classical description of projective modules over Milnor squares of rings as follows.

Lemma 2.8. [13] *An A-module P is projective if and only if there is a triple (P_1, P_2, p) in $\mathcal{F}$ such that P_n is projective as A_n-module for $n = 1, 2$, and $P \simeq \mathrm{Pb}(P_1, P_2, p)$.*

Moreover, if (P_1, P_2, p) is such a triple in $\mathcal{F}$ with P its pullback module, then $FP \simeq P_1$, and $LP \simeq P_2$.

In the following lemma, we discuss the case that for every triple (X_1, X_2, x) with X its pullback module, it holds that $FX \simeq X_1$ and $LX \simeq X_2$.

Lemma 2.9. *If π_n is surjective for $n = 1, 2$, then both $\widetilde{j_1} : FX \to X_1$ and $\widetilde{j_2} : LX \to X_2$ are isomorphisms.*

Proof. Since π_n is surjective, p_n is also surjective for $n = 1, 2$ by the pullback diagram $(*)$. Now consider the pullback diagram of A-modules with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Ker}(j_1) & \xrightarrow{i_1} & X & \xrightarrow{j_1} & X_1 \\ & & \downarrow{\scriptstyle \simeq} & & {\scriptstyle j_2}\downarrow & & \downarrow{\scriptstyle \eta_1 x} \\ 0 & \longrightarrow & K_2X_2 & \longrightarrow & X_2 & \xrightarrow{\eta_2} & F_2X_2 \end{array}$$

where K_2 denotes $\mathrm{Ker}(\pi_2)$, and $\mathrm{Ker}(j_1) = (0, K_2X_2)$. By Lemma 2.6(1), the unit map η_n is surjective for $n = 1, 2$. Then j_n is also surjective by the pullback diagram, and so is $\widetilde{j_n}$ by Lemma 2.6(3).

Next, we shall show that $\widetilde{j_n}$ is injective. We only prove the case for $n = 1$, since the proof of $n = 2$ is similar. Applying F to the upper row of the above diagram gives the exact sequence

$$F\mathrm{Ker}(j_1) \xrightarrow{Fi_1} FX \xrightarrow{\widetilde{j_1}} X_1 \longrightarrow 0$$

We claim that $\mathrm{Im}(Fi_1) = 0$. Actually, let $a_1 \otimes (0, k_2x_2) \in F\mathrm{Ker}(j_1)$ with $a_1 \in A_1, k_2 \in K_2, x_2 \in X_2$. Since $\eta_1 x$ is surjective, there exists a $x_1 \in X_1$ such that $(x_1, x_2) \in X$. Then $a_1 \otimes (0, k_2x_2) = a_1 \otimes (0, k_2)(x_1, x_2) = a_1(0, k_2) \otimes (x_1, x_2)$ in FX, but $(0, k_2)$ obviously lies in $\mathrm{Ker}(p_1)$, thus $a_1(0, k_2) = 0$, and our claim follows, which implies that $\widetilde{j_1}$ is an isomorphism. □

A module $_AX$ is called *separated* (see [12]), if there exists an A-monomorphism $X \to X_1 \oplus X_2$ for some A_n-module X_n, $n = 1, 2$. By Lemma 2.8, a projective module is separated. Thus, a Gorenstein-projective module is also separated by the definition that it can be embedded into a projective module. The following lemma, which can be obtained from [12, Corollary 3.3], shows that a separated module must be a pullback module.

Lemma 2.10. *Let X be an A-module. Then the following are equivalent.*

(1) *X is separated.*
(2) *$(p_1 \otimes 1_X, p_2 \otimes 1_X) : X \longrightarrow FX \oplus LX$ is a monomorphism.*
(3) *There is a short exact sequence*

$$0 \longrightarrow X \longrightarrow FX \oplus LX \xrightarrow{\binom{\eta_1\pi_X}{-\eta_2}} F_2LX \longrightarrow 0\,.$$

(4) *X is isomorphic to a pullback module of a triple in $\mathcal{F}$.*

Proof. (1) ⇒ (2) Since X is separated, there exists a monomorphism $(c_1, c_2) : X \to X_1 \oplus X_2$ with A_n-module X_n, Note that $p_1 \otimes 1_X$ and $p_2 \otimes 1_X$ are actually the unit maps corresponding to the left adjoint functions F and L, respectively. Then (c_1, c_2) factors through $FX \oplus LX$; that is, there is a commutative diagram.

$$\begin{array}{ccc} X & \xrightarrow{(p_1\otimes 1_X, p_2\otimes 1_X)} & FX \oplus LX \\ {\scriptstyle (c_1,c_2)}\big\downarrow & \swarrow{\scriptstyle c} & \\ X_1 \oplus X_2 & & \end{array}$$

where $c = \begin{pmatrix} (c_1)\eta^{-1}_{X,X_1} & 0 \\ 0 & (c_2)\eta^{-1}_{X,X_2} \end{pmatrix}$. This shows that $(p_1 \otimes 1_X, p_2 \otimes 1_X)$ is indeed a monomorphism.

(2) ⇒ (3) This follows from the commutative diagram

$$\begin{array}{ccccccccc} & & X & \xrightarrow{(p_1\otimes 1_X, p_2\otimes 1_X)} & FX \oplus LX & \longrightarrow & A_0 \otimes_A X & \longrightarrow & 0 \\ & & \downarrow & & \| & & \downarrow{\scriptstyle \simeq} & & \\ 0 & \longrightarrow & X' & \longrightarrow & FX \oplus LX & \xrightarrow{\binom{\eta_1\pi_X}{-\eta_2}} & F_2LX & \longrightarrow & 0 \end{array} \tag{2.3}$$

where the upper row is obtained from applying $- \otimes_A X$ to (†), and X' denotes $\mathrm{Ker}\binom{\eta_1\pi_X}{-\eta_2}$. Since $(p_1 \otimes 1_X, p_2 \otimes 1_X)$ is injective, it follows that $X \simeq X'$.

(3) ⇒ (4) (3) implies that X is isomprphic to $\mathrm{Ker}\binom{\eta_1\pi_X}{-\eta_2}$, which is the pullback module of the triple (FX, LX, π_X).

(4) ⇒ (1) is trivial. □

Finally, we have an analogue of Lemma 2.8.

Proposition 2.11. *Suppose that π_n is surjective for $n = 1, 2$. Then an A-module X is separated if and only if there is a triple (X_1, X_2, x) in $\mathcal{F}$ such that $X \simeq \mathrm{Pb}(X_1, X_2, x)$.*

Moreover, if (X_1, X_2, x) is a triple in $\mathcal{F}$ with X its pullback module, then $FX \simeq X_1$ and $LX \simeq X_2$.

Proof. This follows from Lemmas 2.10 and 2.9. □

3. Proof of the main results

In this section, we always assume that $(*)$ is a Milnor square of rings such that π_n is surjective for $n = 1, 2$.

We call a triple (G_1, G_2, g) in $\mathcal{F}$ a Gorenstein triple if $G_n \in A_n$-GProj for $n = 1, 2$. Our purpose is to construct a Gorenstein-projective A-module from a given Gorenstein triple.

We know that each Gorenstein-projective module can be embedded into a projective module such that the cokernel is also Gorenstein-projective by its definition, while the following lemma will lift this property to a whole Gorenstein triple, which will be crucial to our later proof.

Lemma 3.1. *Assume that* $\mathrm{Add}({}_{\mathrm{A_n}}\mathrm{Ker}(\pi_\mathrm{n})) \subseteq \mathrm{A_n}\text{-}\mathrm{GProj}^{\perp}$ *for $n = 1, 2$. Then for a Gorenstein triple (G_1, G_2, g), there exists a (T_1, T_2, t) and a homomorphism $(\alpha_1, \alpha_2) : (G_1, G_2, g) \to (T_1, T_2, t)$ in $\mathcal{F}$ such that ${}_{A_n}T_n \in \mathrm{Add}({}_{A_n}A_n)$ and α_n is a left $\mathrm{Add}({}_{A_n}A_n)$-approximation of G_n, for $n = 1, 2$.*

Proof. Our proof proceeds in two steps:

Step 1. Construction of (T_1, T_2, t). Since $G_1 \in A_1$-GProj, there is a monomorphism $a : G_1 \to P_1$ which is a left $\mathrm{Add}({}_{A_1}A_1)$-approximation of G_1. By choosing a projective module Q_1, we can modify a as $a_1 : G_1 \overset{(a,0)}{\to} P_1 \oplus Q_1 \simeq (A_1)^I$, where I denotes any cardinal number. Similarly, there is also a monomorphism $a_2 : G_2 \to (A_2)^J$, which is a left $\mathrm{Add}({}_{A_2}A_2)$-approximation of G_2. Now write T_1 for $(A_1)^I \oplus (A_1)^J$ and T_2 for $(A_2)^I \oplus (A_2)^J$. Considering the canonical isomorphism

$$F_1T_1 \xrightarrow{\simeq} F_1(A_1)^I \oplus F_1(A_1)^J \xrightarrow{\begin{pmatrix} t_1^I & 0 \\ 0 & t_1^J \end{pmatrix}} (A_0)^I \oplus (A_0)^J \xrightarrow{\begin{pmatrix} (t_2^I)^{-1} & 0 \\ 0 & (t_2^J)^{-1} \end{pmatrix}} F_2(A_2)^I \oplus F_2(A_2)^J \xrightarrow{\simeq} F_2T_2$$

where t_n^I denotes the image of $(\pi_n)^I$ under the isomorphism $\mathrm{Hom}_{A_n}((A_n)^I, (A_0)^I) \to \mathrm{Hom}_{A_0}(F_n(A_n)^I, (A_0)^I)$, which is an isomorphism by Lemma 2.5(2), and t_n^J is defined in the same way for $n = 1, 2$. Then we naturally get a triple (T_1, T_2, t) in $\mathcal{F}$.

Step 2. Construction of (α_1, α_2). Our aim is to construct two homomorphisms $b_1 : G_1 \to (A_1)^J$ and $b_2 : G_2 \to (A_2)^I$ such that $(\alpha_1, \alpha_2) := ((a_1, b_1), (b_2, a_2)) : (G_1, G_2, g) \to (T_1, T_2, t)$ is a homomorphism in $\mathcal{F}$. If so, it will follow from a_n being an $\mathrm{Add}({}_{A_n}A_n)$-approximation that α_n is also an $\mathrm{Add}({}_{A_n}A_n)$-approximation for $n = 1, 2$, and then we will complete the proof.

Actually, by the definition of t and by the diagram (2.1), our desired b_n should make the following diagram commute.

$$\begin{array}{ccc} F_1G_1 & \xrightarrow{(F_1a_1,F_1b_1)} & F_1(A_1)^I \oplus F_1(A_1)^J \\ & & \downarrow{\scriptsize\begin{pmatrix} t_1^I & 0 \\ 0 & t_1^J \end{pmatrix}} \\ \simeq \downarrow g & & (A_0)^I \oplus (A_0)^J \\ & & \downarrow{\scriptsize\begin{pmatrix} (t_2^I)^{-1} & 0 \\ 0 & (t_2^J)^{-1} \end{pmatrix}} \\ F_2G_2 & \xrightarrow{(F_2b_2,F_2a_2)} & F_2(A_2)^I \oplus F_2(A_2)^J \end{array} \tag{3.1}$$

that is, $b_1 : G_1 \to (A_1)^J$ and $b_2 : G_2 \to (A_2)^I$ should satisfy $(F_1b_1)t_1^J(t_2^J)^{-1} = gF_2a_2$, and $(F_1a_1)t_1^I(t_2^I)^{-1} = gF_2b_2$ respectively.

To obtain $b_1 : G_1 \to (A_1)^J$, we need to consider the composite map $g(F_2a_2)t_2^J : F_1G_1 \to (A_0)^J$ in A_0-Mod, and denote it by g_0. Let $\widetilde{g_0}$ be the image of g_0 under the isomorphism $\eta_{G_1,(A_0)^J}$: $\mathrm{Hom}_{A_0}(F_1G_1,(A_0)^J) \simeq \mathrm{Hom}_{A_1}(G_1,(A_0)^J)$. By the assumption that $\mathrm{Add}({}_{\mathrm{A}_1}\mathrm{Ker}(\pi_1)) \subseteq \mathrm{A}_1\text{-GProj}^{\perp}$, we have $\mathrm{Ext}^1_{A_1}(G_1,\mathrm{Ker}(\pi_1)^J) = 0$. Then applying $\mathrm{Hom}_{A_1}(G_1,-)$ to the short exact sequence

$$0 \to \mathrm{Ker}(\pi_1)^J \to (A_1)^J \xrightarrow{(\pi_1)^J} (A_0)^J \to 0$$

preserves its exactness. Thus there exists a $b_1 : G_1 \to (A_1)^J$ such that $b_1(\pi_1)^J = \widetilde{g_0}$. Note that by Lemma 2.5(1), we have $g_0 = (F_1\widetilde{g_0})\epsilon_1 = (F_1b_1)(F_1(\pi_1)^J)\epsilon_1$ with ϵ_1 the counit map $F_1(A_0)^J \to (A_0)^J$, but again by Lemma 2.5(1) and the definition of t_1^J, we have $(F_1(\pi_1)^J)\epsilon_1 = t_1^J$, and then it follows that $(F_1b_1)t_1^J = g_0 := g(F_2a_2)t_2^J$, as desired. Similarly, by considering the composite map $g^{-1}(F_1a_1)t_1^I$, the existence of the desired b_2 can also be proved. This completes the proof. □

The following lemma exhibits a situation that makes an A-module X lie in the left Ext-orthogonal group of $\mathrm{Add}({}_AA)$, which is a necessary condition for ${}_AX$ to be Gorenstein-projective.

Lemma 3.2. *Suppose that F_2 sends Gorenstein-projective A_2-modules to Gorenstein-projective A_0-modules. Let $0 \to X \to T \to Y \to 0$ be a short exact sequence in A-Mod where T is projective and both X and Y are pullback modules of two Gorenstein triples. If $\mathrm{Tor}_1^A(A_n,X) = 0$, and $\mathrm{Tor}_1^A(A_n,Y) = 0$ for $n = 1,2$, then we have $\mathrm{Ext}^1_A(X,P) = 0$ for all $P \in A$-Proj.*

Proof. Take $P \in A$-Proj and write P_n for $A_n \otimes_A P \in A_n$-Proj for $n = 0,1,2$. By applying $- \otimes_A P$ to (†), we have the short exact sequence

$$0 \to P \to P_1 \oplus P_2 \to P_0 \to 0.$$

Applying $\mathrm{Hom}_A(Y,-)$ to the above sequence yields the long exact sequence

$$\cdots \to \mathrm{Ext}^1_A(Y,P_0) \to \mathrm{Ext}^2_A(Y,P) \to \mathrm{Ext}^2_A(Y,P_1 \oplus P_2) \to \cdots.$$

Then we only need to show that $\mathrm{Ext}^1_A(Y,P_0) = 0$, and $\mathrm{Ext}^2_A(Y,P_1 \oplus P_2) \simeq \mathrm{Ext}^1_A(X,P_1 \oplus P_2) = 0$. If so, it will follow that $\mathrm{Ext}^2_A(Y,P) = 0$, and thus $\mathrm{Ext}^1_A(X,P) = 0$ by the dimension shifting.

We first show that $\mathrm{Ext}^1_A(X,P_n) = 0$ for $n = 1,2$. Assume that X is a pullback module of a Gorenstein triple (X_1,X_2,x). By Lemma 2.9, we have $FX \simeq X_1$, and $LX \simeq X_2$, and thus $FX \in A_1$-GProj and $LX \in A_2$-GProj. Then, by Lemma 2.2 and the assumption that $\mathrm{Tor}_1^A(A_n,X) = 0$ for $n = 1,2$, we have $\mathrm{Ext}^1_A(X,P_1) \simeq \mathrm{Ext}^1_{A_1}(FX,P_1) = 0$, and $\mathrm{Ext}^1_A(X,P_2) \simeq \mathrm{Ext}^1_{A_2}(LX,P_2) = 0$.

Next, we show that $\mathrm{Ext}_A^1(Y, P_0) = 0$. Since Y is also a pullback module of a Gorenstein triple, we similarly have $FY \in A_1$-GProj and $LY \in A_2$-GProj. Then, by the assumption on F_2, we have $A_0 \otimes_A Y \simeq F_2LY \in A_0$-GProj. Applying $- \otimes_A Y$ to (†) yields a long exact sequence

$$0 \longrightarrow \mathrm{Tor}_1^A(A_1, Y) \oplus \mathrm{Tor}_1^A(A_2, Y) \longrightarrow \mathrm{Tor}_1^A(A_0, Y) \longrightarrow Y \longrightarrow FY \oplus LY.$$

Since Y is separated, it follows from Lemma 2.10(2) that $\mathrm{Tor}_1^A(A_0, Y) \simeq \mathrm{Tor}_1^A(A_1, Y) \oplus \mathrm{Tor}_1^A(A_2, Y) = 0$. Then by Lemma 2.2, we have $\mathrm{Ext}_A^1(Y, P_0) \simeq \mathrm{Ext}_{A_0}^1(A_0 \otimes_A Y, P_0) = 0$. □

Now it is time to prove our main result.

Theorem 3.3. *Let (G_1, G_2, g) be a Gorenstein triple in $\mathcal{F}$, and $G := \mathrm{Pb}(G_1, G_2, g)$. Assume that the following conditions hold.*

(i) $\mathrm{Tor}_1^{A_2}(A_0, H) = 0$ *for all* $H \in A_2$-GProj.

(ii) $\mathrm{Add}(\mathrm{Ker}(\pi_\mathrm{n})) \subseteq \mathrm{A_n\text{-}GProj}^\perp$ *for* $n = 1, 2$.

Then we have that

(a) *There exists a short exact sequence* $0 \to G \to T \to G^1 \to 0$ *in* A-Mod *where T is projective, and G^1 is a pullback module of a Gorenstein triple in $\mathcal{F}$, and* $\mathrm{Tor}_1^A(A_n, G^1) = 0$ *for* $n = 1, 2$.

(b) $\Omega_0 G$ *is a pullback module of a Gorenstein triple in $\mathcal{F}$, and* $\mathrm{Tor}_1^A(A_n, G) = 0$ *for* $n = 1, 2$.

(c) $G \in A$-GProj.

Proof. By the conditions (i), and (ii) for $n = 2$, it follows from Lemma 2.7 that F_2 sends Gorenstein-projective A_2-modules to A_0-modules, which will be used as a fact in the following proof.

(a) By Lemma 3.1, we have a triple (T_1, T_2, t) and a homomorphism (α_1, α_2) from (G_1, G_2, g) to (T_1, T_2, t) in $\mathcal{F}$ with α_n an $\mathrm{Add}(A_n)$-approximation of G_n. Since G_n can be embedded into a projective module, the embedding map must factor through α_n, and it follows that α_n is injective, and we have the short exact sequence in A_n-mod

$$0 \longrightarrow G_n \xrightarrow{\alpha_n} T_n \xrightarrow{\beta_n} L_n \longrightarrow 0 \tag{3.2}$$

where $L_n \in A_n$-GProj by Lemma 2.3(2) for $n = 1, 2$. Then, by applying F_n to (3.2) and by the condition (i) and diagram (2.2), we obtain the commutative diagram with an exact lower row

$$\begin{array}{ccccccccc} & & F_1G_1 & \xrightarrow{F_1\alpha_1} & F_1T_1 & \longrightarrow & F_1L_1 & \longrightarrow & 0 \\ & & \simeq\downarrow g & & \simeq\downarrow t & & \downarrow l & & \\ 0 & \longrightarrow & F_2G_2 & \xrightarrow{F_2\alpha_2} & F_2T_2 & \longrightarrow & F_2L_2 & \longrightarrow & 0 \end{array} \tag{3.3}$$

where the induced map l is an isomorphism by the Five Lemma. Denote by T the pullback module of the triple (T_1, T_2, t). By the definition of pullback modules, we obtain the left two columns of the following commutative diagram with exact columns and the lower two rows

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & G & \overset{\alpha}{\dashrightarrow} & T & \dashrightarrow & G^1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & G_1 \oplus G_2 & \xrightarrow{\left(\begin{smallmatrix}\alpha_1 & 0 \\ 0 & \alpha_2\end{smallmatrix}\right)} & T_1 \oplus T_2 & \xrightarrow{\left(\begin{smallmatrix}\beta_1 & 0 \\ 0 & \beta_2\end{smallmatrix}\right)} & L_1 \oplus L_2 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \left(\begin{smallmatrix}\eta_1 g \\ -\eta_2\end{smallmatrix}\right)} & & \downarrow{\scriptstyle \left(\begin{smallmatrix}\eta_1 t \\ -\eta_2\end{smallmatrix}\right)} & & \downarrow{\scriptstyle \left(\begin{smallmatrix}\eta_1 l \\ -\eta_2\end{smallmatrix}\right)} & & \\
0 & \longrightarrow & F_2G_2 & \xrightarrow{F_2\alpha_2} & F_2T_2 & \longrightarrow & F_2L_2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{3.4}$$

where G^1 denotes $\mathrm{Ker}\left(\begin{smallmatrix}\eta_1 l \\ -\eta_2\end{smallmatrix}\right)$. In fact, the commutativity of the lower two squares follows from (3.3) and the functoriality of η_n for $n = 1, 2$. Then, by the Snake Lemma, we obtain our desired short exact sequence, the induced first row

$$0 \to G \xrightarrow{\alpha} T \to G^1 \to 0.$$

In fact, from the exact third column in (3.4), we see that G^1 is the pullback module of the Gorenstein triple (L_1, L_2, l) in $\mathcal{F}$, and from the upper two rows in diagram (3.4), we deduce the commutative two diagrams by Lemma 2.1(2)

$$\begin{array}{ccccccccc}
0 & \longrightarrow & FG & \xrightarrow{F\alpha} & LT & \longrightarrow & FG^1 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle \simeq} & & \\
0 & \longrightarrow & G_1 & \xrightarrow{\alpha_1} & T_1 & \longrightarrow & L_1 & \longrightarrow & 0
\end{array}
\qquad
\begin{array}{ccccccccc}
0 & \longrightarrow & LG & \xrightarrow{L\alpha} & LT & \longrightarrow & LG^1 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle \simeq} & & \\
0 & \longrightarrow & G_2 & \xrightarrow{\alpha_2} & T_2 & \longrightarrow & L_2 & \longrightarrow & 0
\end{array}$$

where all the vertical isomorphisms follow from Lemma 2.9, which implies that both $F\alpha$ and $L\alpha$ are injective, and thus $\mathrm{Tor}_1^A(A_n, G^1) = 0$ for $n = 1, 2$.

(b) Choose a short exact sequence $0 \to \Omega_0 G \to P \xrightarrow{\beta} G \to 0$ with ${}_A P$ projective. Since both G and P are separated, we have, by Lemma 2.10(3), the two exact columns of the following commutative diagram with exact rows

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega_0 G & \longrightarrow & P & \xrightarrow{\beta} & G & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C_1 \oplus C_2 & \longrightarrow & FP \oplus LP & \xrightarrow{\left(\begin{smallmatrix}F\beta & 0 \\ 0 & L\beta\end{smallmatrix}\right)} & FG \oplus LG & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \left(\begin{smallmatrix}\eta_1 c \\ -\eta_2\end{smallmatrix}\right)} & & \downarrow{\scriptstyle \left(\begin{smallmatrix}\eta_1 \pi_P \\ -\eta_2\end{smallmatrix}\right)} & & \downarrow{\scriptstyle \left(\begin{smallmatrix}\eta_1 \pi_G \\ -\eta_2\end{smallmatrix}\right)} & & \\
0 & \longrightarrow & F_2C_2 & \longrightarrow & F_2LP & \xrightarrow{F_2L\beta} & F_2LG & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{3.5}$$

where C_1 and C_2 are kernels of $F\beta$ and $L\beta$, respectively, both of which are Gorenstein-projective modules by Lemma 2.3(1). c is induced from the diagram

$$\begin{array}{ccccccccc} & & F_1C_1 & \longrightarrow & F_1FP & \longrightarrow & F_1FG & \longrightarrow & 0 \\ & & {\scriptstyle\simeq}\ \vdots\ {\scriptstyle c} & & {\scriptstyle\simeq}\ \downarrow\ {\scriptstyle \pi_P} & & {\scriptstyle\simeq}\ \downarrow\ {\scriptstyle \pi_G} & & \\ 0 & \longrightarrow & F_2C_2 & \longrightarrow & F_2LP & \longrightarrow & F_2LG & \longrightarrow & 0 \end{array}$$

where the commutativity of the right square follows from the functoriality of $\pi : F_1F \to F_2L$, and the exactness of the lower row follows from the condition (i). Then, by the Snake Lemma, the first column is exact. This shows that Ω_0G is isomorphic to the pullback module of the Gorenstein triple (C_1, C_2, c) in $\mathcal{F}$. Now, by a similar proof in (a), we have $\mathrm{Tor}_1^A(A_n, G) = 0$ for $n = 1, 2$.

(c) **Step 1.** We show that $\mathrm{Ext}_A^1(G, P) = 0$ for every $P \in A$-Proj.

Actually, this follows from Lemma 3.2 and the short exact sequence

$$0 \to G \overset{\alpha}{\to} T \to G^1 \to 0 \tag{3.6}$$

in (a), since G is a pullback module of a Gorenstein triple by the assumption, and $\mathrm{Tor}_1^A(A_n, G) = 0$ for $n = 1, 2$ by (b), and G^1 is a pullback module of a Gorenstein triple such that $\mathrm{Tor}_1^A(A_n, G^1) = 0$ for $n = 1, 2$ by (a).

Step 2. We show that there is a $\mathrm{Hom}_A(-, P)$-exact sequence

$$T^+ : 0 \longrightarrow G \overset{\alpha}{\longrightarrow} T \longrightarrow T^1 \longrightarrow \cdots \longrightarrow T^i \overset{d_T^i}{\longrightarrow} T^{i+1} \longrightarrow \cdots$$

for every $P \in A$-Proj.

Since G^1 is a pullback module of a Gorenstein triple, again by (a) and by induction on $i \geq 1$, we obtain a series of short exact sequences of A-modules

$$0 \longrightarrow G^i \overset{\alpha^i}{\longrightarrow} T^i \longrightarrow G^{i+1} \longrightarrow 0$$

with T^i projective and G^i the pullback module of a Gorenstein triple such that $\mathrm{Tor}_1^A(A_n, G^i) = 0$ for $n = 1, 2$. Then, by Lemma 3.2, we have $\mathrm{Ext}_A^1(G^i, P) = 0$ for all $i \geq 1$ and $P \in A$-Proj. Thus, by splicing these short exact sequences for all $i \geq 1$, together with (3.6), we obtain the desired P^+.

Step 3. We show that $\mathrm{Ext}_A^i(G, P) = 0$ for all $i \geq 2$ and $P \in A$-Proj.

This is equivalent to showing that $\mathrm{Ext}_A^1(\Omega_i G, P) = 0$ for all $i \geq 0$ and $P \in A$-Proj. But this similarly follows from the induction on $\Omega_i G$ by (b) for $i \geq 0$, and applying Lemma 3.2 to the short exact sequences $0 \to \Omega_{i+1}G \to T_i \to \Omega_i G \to 0$ and $0 \to \Omega_0 G \to T_0 \to G \to 0$ with T_0, T_i projective.

Finally, by Lemma 2.4, we get that G is a Gorenstein-projective A-module. □

A ring is said to be *strongly left CM-free* (see [1]) if each Gorenstein-projective left module over it is projective. We have the following corollary.

Corollary 3.4. *Let* (G_1, G_2, g) *be a Gorenstein triple in* $\mathcal{F}$*, and* $G := \mathrm{Pb}(G_1, G_2, g)$*. Assume that* A_2 *is strongly left CM-free, and* $\mathrm{Add}(\mathrm{Ker}(\pi_1)) \subseteq \mathrm{A}_1\text{-GProj}^{\perp}$*. Then* $G \in A$-GProj.

4. Applications to Morita context rings

Morita context rings play an important role in ring theory. They provide many important examples and frameworks of kinds of problems (for instance, see [6, 11]). In this section we construct Gorenstein-projective modules over Morita context rings with two bimodule homomorphisms zero $\Lambda_{(0,0)} = \left(\begin{smallmatrix} \Lambda & N \\ M & B \end{smallmatrix}\right)_{(0,0)}$, of which the multiplication is defined as

$$\begin{pmatrix} \lambda & n \\ m & b \end{pmatrix}\begin{pmatrix} \lambda' & n' \\ m' & b' \end{pmatrix} = \begin{pmatrix} \lambda\lambda' & \lambda n' + nb' \\ m\lambda' + bm' & bb' \end{pmatrix}.$$

A left $\Lambda_{(0,0)}$-module is identified with a quadruple $({}_\Lambda X, {}_B Y, f, g)$, where $f \in \mathrm{Hom}_B(M \otimes_\Lambda X, Y)$ and $g \in \mathrm{Hom}_\Lambda(N \otimes_B Y, X)$, and a homomorphism from a triple (X, Y, f, g) to another triple (X', Y', f', g') is a pair (α,β) with $\alpha \in \mathrm{Hom}_\Lambda(X, X')$ and $\beta \in \mathrm{Hom}_B(Y, Y')$ such that the two diagrams are commutative

$$\begin{array}{ccc} M \otimes_\Lambda X & \xrightarrow{f} & Y \\ {\scriptstyle 1_M \otimes \alpha}\downarrow & & \downarrow{\scriptstyle \beta} \\ M \otimes_\Lambda X' & \xrightarrow{f'} & Y' \end{array} \qquad \text{and} \qquad \begin{array}{ccc} N \otimes_B Y & \xrightarrow{g} & X \\ {\scriptstyle 1_N \otimes \beta}\downarrow & & \downarrow{\scriptstyle \alpha} \\ N \otimes_B Y' & \xrightarrow{g'} & X' \end{array}$$

For more details about Morita context rings, we refer the reader to [6], for instance.

Let A_1 be the upper triangular matrix ring $\left(\begin{smallmatrix} \Lambda & N \\ 0 & B \end{smallmatrix}\right)$, A_2 be the lower triangular matrix ring $\left(\begin{smallmatrix} \Lambda & 0 \\ M & B \end{smallmatrix}\right)$, and A_0 be the diagonal matrix ring $\left(\begin{smallmatrix} \Lambda & 0 \\ 0 & B \end{smallmatrix}\right)$. Then $\Lambda_{(0,0)}$ can be interpreted as a pullback ring through the following commutative diagram.

$$\begin{array}{ccc} \Lambda_{(0,0)} & \xrightarrow{p_1} & A_1 \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle \pi_1} \\ A_2 & \xrightarrow{\pi_2} & A_0 \end{array}$$

where p_n and π_n are canonical projections for $n = 1, 2$. By the characterization of Gorenstein-projective modules over triangular matrix rings in [3], we obtain the following corollary, which allows us to obtain Gorenstein-projective $\Lambda_{(0,0)}$-modules from Gorenstein-projective Λ-modules and B-modules.

Corollary 4.1. *Assume that* (1) $\mathrm{Tor}_1^\Lambda(M, H) = 0$ *for all* $H \in \Lambda$-GProj, *and* $\mathrm{Add}({}_\mathrm{B}\mathrm{M}) \subseteq \mathrm{B}\text{-}\mathrm{GProj}^\perp$,

(2) $\mathrm{Tor}_1^B(N, G) = 0$ *for all* $G \in B$-GProj, *and* $\mathrm{Add}({}_\Lambda\mathrm{N}) \subseteq \Lambda\text{-}\mathrm{GProj}^\perp$.

Let $U \in \Lambda$-GProj, $V \in B$-GProj. *If there exist a* Λ*-module* X*, a* B*-module* Y*, and two short exact sequences*

$$0 \longrightarrow N \otimes_B V \xrightarrow{g} X \xrightarrow{s} U \longrightarrow 0$$

and

$$0 \longrightarrow M \otimes_\Lambda U \xrightarrow{f} Y \xrightarrow{t} V \longrightarrow 0$$

in Λ-Mod *and* B-Mod, *respectively, then* $(X, Y, (1_M \otimes s)f, (1_N \otimes t)g) \in \Lambda_{(0,0)}$-Gproj.

Proof. Write W for $(X, Y, (1_N \otimes t)g, (1_M \otimes s)f)$, W_1 for $(X, V, 0, g)$, W_2 for $(U, Y, f, 0)$, and W_0 for $(U, V, 0, 0)$. Then $W_0 \simeq A_0 \otimes_{A_1} W_1 \simeq A_0 \otimes_{A_2} W_2$, and one can verify that the following commutative diagram is a pullback diagram,

$$\begin{array}{ccc} W & \xrightarrow{(1_X, t)} & W_1 \\ {\scriptstyle (s, 1_Y)}\downarrow & & \downarrow{\scriptstyle (s, 1_V)} \\ W_2 & \xrightarrow{(1_U, t)} & W_0 \end{array}$$

By the condition (2) and the short exact sequence

$$0 \longrightarrow N \otimes_B V \xrightarrow{g} X \xrightarrow{s} U \longrightarrow 0,$$

it follows from Theorem [3, Theorem 2.5] that $W_1 := (X, V, g) \in A_1$-GProj. Dually, by the condition (1) and the short exact sequence

$$0 \longrightarrow M \otimes_\Lambda U \xrightarrow{f} Y \xrightarrow{t} V \longrightarrow 0,$$

we have $W_2 := (U, Y, f) \in A_2$-GProj. Now we only need to show that the conditions (i) and (ii) in Theorem 3.3 are satisfied here, and then it will follow that $W \in \Lambda_{(0,0)}$-GProj.

Actually, by [5, Lemma 3.9], it follows from $\mathrm{Add}({}_\Lambda \mathrm{N}) \subseteq \Lambda\text{-GProj}^\perp$ that $\mathrm{Add}({}_\Lambda \mathrm{N}, 0, 0) \subseteq \mathrm{A}_1\text{-GProj}^\perp$, and similarly we have $\mathrm{Add}(0, {}_\mathrm{B}\mathrm{M}, 0) \subseteq \mathrm{A}_2\text{-GProj}^\perp$. By [11, Proposition 6.1] or [7, Lemma 3.7], it follows from $\mathrm{Tor}_1^\Lambda(M, H) = 0$ for all $H \in \Lambda$-GProj that $\mathrm{Tor}_1^{A_2}((M_\Lambda, 0, 0), H') = 0$ for all $H' \in A_2$-GProj. Then, by the short exact sequence

$$0 \longrightarrow (M_\Lambda, 0, 0) \longrightarrow A_2 \longrightarrow A_0 \longrightarrow 0\ ,$$

we have $\mathrm{Tor}_1^{A_2}(A_0, H') = 0$ for all $H' \in A_2$-GProj. Actually, for every $H' \in A_2$-GProj, we have $\mathrm{Tor}_1^{A_2}(A_0, H') = \mathrm{Tor}_2^{A_2}(A_0, \mho H') = \mathrm{Tor}_1^{A_2}((M_\Lambda, 0, 0), \mho H') = 0$, where $\mho H'$ lies in the short exact sequence $0 \to H' \to P \to \mho H' \to 0$ taken from the complete projective resolution of H'. □

Corollary 4.1 extends both the construction of Gorenstein-projective modules over Morita context algebras in [5, Theorem 3.10(α)] and the sufficient conditions for a module to be Gorenstein-projective over Noetherian rings in [7, Proposition 3.14(2)] to the general rings. The conditions (1) and (2) in Corollary 4.1 can be realized by $\mathrm{flatdim} M_\Lambda < \infty$, $\mathrm{flatdim} N_B < \infty$, $\mathrm{projdim}_\Lambda N < \infty$, $\mathrm{projdim}_B M < \infty$.

Let $\Delta_{(0,0)}$ be the special Morita context rings where $\Lambda = N = M = B$ in $\Lambda_{(0,0)}$. By Corollary 4.1, we have the following, which extends the results in [5, Corollary 3.12], also in [10, Example 4.11].

Corollary 4.2. *Let* $U, V \in \Lambda$-GProj. *If there exist two* Λ*-modules* X *and* Y *and two short exact sequences*

$$0 \longrightarrow V \xrightarrow{g} X \xrightarrow{s} U \longrightarrow 0$$

and

$$0 \longrightarrow U \xrightarrow{f} Y \xrightarrow{t} V \longrightarrow 0$$

in Λ-Mod, *then* $(X, Y, tg, sf), (Y, X, sf, tg) \in \Delta_{(0,0)}$-Gproj.

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Conflict of interest

The author declares no conflicts of interest in this paper.

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