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Research article

Sharp inequalities for q-starlike functions associated with differential subordination and q-calculus

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Abstract: This paper employs differential subordination and quantum calculus to investigate a new class of q-starlike functions associated with an eight-like image domain. Our study laid a foundational understanding of the behavior of these q-starlike functions. We derived the results in first-order differential subordination. We established sharp inequalities for the initial Taylor coefficients and provided optimal estimates for solving the Fekete-Szegö problem and a second-order Hankel determinant applicable to all q-starlike functions in this class. Furthermore, we presented a series of corollaries that demonstrate the broader implications of our findings in geometric function theory.

Keywords: analytic functions; univalent functions; starlike functions; q-starlike functions; q-derivative operator; differential subordination
Mathematics Subject Classification: 05A30, 30C45

1. Introductions and research background

Quantum calculus (known as q-calculus) extends traditional calculus by introducing the parameter q, offering flexible and powerful tools for analyzing mathematical functions. This framework has attracted growing attention due to its applications in various fields, including mathematics, engineering, and physics. Recent advancements have amplified its relevance, with notable applications in q-transform analysis, q-difference and q-integral equations, optimal control, and fractional calculus.

The q-derivative operator plays a central role in the theory of special functions, enabling the definition and in-depth analysis of various subclasses of analytic functions. For instance, Mahmood et al. [1] investigated the coefficients of q-starlike functions associated with conic domains, Ahmad et al. [2] explored the Hankel determinants and Zalcman conjecture for q-starlike functions associated with the balloon-shaped domain, and Shi et al. [3] studied the properties of generalized integral operators in the lemniscate domain of Bernoulli. Ahmad et al. [4] extended these ideas by exploring Janowski functions to establish q-analogues of differential subordination results. The q-derivative operator also underpins the q-extension of starlike functions, as discussed in [5].

Recent research highlights the significant role of q-calculus in advancing contemporary mathematical theory. Studies such as [6–8] deepen our understanding of analytic and symmetric functions through q-analytic techniques. This body of work demonstrates the broad applicability of q-calculus, with potential implications across quantum theory, number theory, and statistical mechanics. For further exploration of fractional derivatives, including Caputo and conformable fractional derivatives, readers may consult [9, 10] and their references.

We present some basic definitions related to our work before moving on to our primary findings.

Let $\mathcal{H}(\mathfrak{D})$ denote the class of all analytic functions f defined in the open unit disk

$$\mathfrak{D} = \{\varepsilon \in \mathbb{C} : \quad |\varepsilon| < 1\},$$

where $\mathbb{C}$ represents the set of complex numbers. Let $\mathcal{A} \subseteq \mathcal{H}(\mathfrak{D})$ be the subclass of all analytic functions f having the Taylor series representation:

$$f(\varepsilon) = \varepsilon + \sum_{k=2}^{\infty} \xi_k \varepsilon^k \qquad (\varepsilon \in \mathfrak{D}). \tag{1.1}$$

Suppose that $\mathcal{P}$ represents the Carathéodory class of analytic functions h normalized by

$$h(\varepsilon) = 1 + \sum_{n=1}^{\infty} c_n \varepsilon^n, \tag{1.2}$$

such that the real part is positive:

$$\Re(h(\varepsilon)) > 0.$$

For each given analytic function $f \in \mathcal{A}$, the r^{th} Hankel determinant $\mathcal{H}_{r,n}$ is defined in [11] as follows,

$$\mathcal{H}_{r,n}(f) = \begin{vmatrix} \xi_n & \xi_{n+1} & \cdots & \xi_{n+r-1} \\ \xi_{n+1} & \xi_{n+2} & \cdots & \xi_{n+r} \\ \vdots & \vdots & \cdots & \vdots \\ \xi_{n+r-1} & \xi_{n+r} & \cdots & \xi_{n+2r-2} \end{vmatrix},$$

where $r, n \in \mathbb{N}$ and $\xi_1 = 1$. The following are two second Hankel determinants.

$$\mathcal{H}_{2,1}(f) = \begin{vmatrix} 1 & \xi_2 \\ \xi_2 & \xi_3 \end{vmatrix} = \xi_3 - \xi_2^2, \ \mathcal{H}_{2,2}(f) = \begin{vmatrix} \xi_2 & \xi_3 \\ \xi_3 & \xi_4 \end{vmatrix} = \xi_2\xi_4 - \xi_3^2. \tag{1.3}$$

In recent years, considerable attention has been devoted to investigating the upper bounds of the expression $\left|H_{2,2}(f)\right|$ across various subclasses of analytic functions. Key contributions in this domain have been made by researchers such as Noonan and Thomas [12], Hayman [13], Ohran et al. [14], and Shi et al. [15]. Babalola [16] initiated the study of bounds for the third Hankel determinant, further enriching the field.

For a deeper exploration of this subject, recent studies provide valuable insights and are discussed in references [17–19]. This growing body of research highlights the importance of Hankel determinants in analytic function theory, offering the potential for discoveries and advancing our understanding of this mathematical area.

For the functions f and $g \in \mathcal{H}(\mathfrak{D})$, we say that f is subordinated to g, written as

$$f(\varepsilon) \prec g(\varepsilon),$$

if there exists a Schwarz function $\omega(\varepsilon)$, which is an analytic function in $\mathfrak{D}$ with $\omega(0) = 0$ and $|\omega(\varepsilon)| \leq 1$, such that

$$f(\varepsilon) = g(\omega(\varepsilon)).$$

Furthermore, if two functions f and g are analytic in $\mathfrak{D}$ and g is univalent, then f is subordinated to g if and only if

$$f(\mathfrak{D}) \subseteq g(\mathfrak{D}) \quad \text{and} \quad f(0) = g(0).$$

There are some applications of subordination below; see [20] for more applications.

Definition 1.1. *A function $p \in \mathcal{P}[A, B]$ if*

$$p(\varepsilon) \prec \frac{1 + A\varepsilon}{1 + B\varepsilon} \qquad (-1 \leq B < A \leq 1).$$

Equivalently,

$$\left|\frac{p(\varepsilon) - 1}{A - Bp(\varepsilon)}\right| < 1.$$

In particular, f is a Janowski starlike function if $\frac{\varepsilon f'(\varepsilon)}{f(\varepsilon)} \in \mathcal{P}[A, B]$. See [21] for more details about Janowski starlike functions.

We take into consideration a class of functions in the domain bounded by a tangent function. All functions h will belong to such a class if they fulfill

$$h(\varepsilon) \prec 1 + \frac{1}{2}\tan(\varepsilon). \tag{1.4}$$

The images of these functions lie in the right-half plane and the geometrical representation is like an eight-shape domain. Simple computations allow the above (1.4) to be expressed as

$$\left|\tan^{-1}(2h(\varepsilon) - 2)\right| < 1. \tag{1.5}$$

In parallel comparison to starlike functions, Khan et al. [22] introduced the following class of Janowski-type starlike functions along with some properties.

$$\mathcal{S}^*_{\tan} = \left\{f(\varepsilon) \in \mathcal{A} : \frac{\varepsilon f'(\varepsilon)}{f(\varepsilon)} \prec 1 + \frac{1}{2}\tan(\varepsilon)\right\}. \tag{1.6}$$

Thus, by the relation of (1.4) and (1.5),

$$\mathcal{S}^*_{\tan} = \left\{f(\varepsilon) \in \mathcal{A} : \left|\tan^{-1}\left(2\frac{\varepsilon f'(\varepsilon)}{f(\varepsilon)} - 2\right)\right| < 1\right\}.$$

Definition 1.2. *[23] For a function f, the q-derivative (also known as the q-difference operator) is defined by*

$$\left(D_q f\right)(\varepsilon) = \frac{f(\varepsilon) - f(q\varepsilon)}{(1-q)\,\varepsilon}, \tag{1.7}$$

where $\varepsilon \neq 0$ and $0 < q < 1$.

For example, for $n \in \mathbb{N}$ and $\varepsilon \in \mathfrak{D}$, we have

$$D_q \left\{ \sum_{n=1}^{\infty} \xi_n \varepsilon^n \right\} = \sum_{n=1}^{\infty} [n]_q \, \xi_n \varepsilon^{n-1}, \tag{1.8}$$

where the q-number is defined by

$$[n]_q = \frac{1-q^n}{1-q} = 1 + \sum_{l=1}^{n-1} q^l \quad \text{and} \quad [0]_q = 0.$$

Now, we introduce a new class $\mathcal{S}^*_{\tan}(q)$ of Janowski-type q-starlike functions associated with the eight-shaped image domain. Several classes of Janowski-type q-starlike functions have been investigated previously (see [24]).

Definition 1.3. *A function f in $\mathcal{A}$ is said to belong to $\mathcal{S}^*_{\tan}(q)$ if the following holds*

$$\frac{\varepsilon D_q f(\varepsilon)}{f(\varepsilon)} \prec 1 + \frac{1}{2}\tan(\varepsilon). \tag{1.9}$$

Remark 1.1. *One can see that*

$$\lim_{q \to 1^-} \mathcal{S}^*_{\tan}(q) = \mathcal{S}^*_{\tan}.$$

The graphological representation for the class $\mathcal{S}^*_{\tan}(q)$ is given in the following Figure 1.

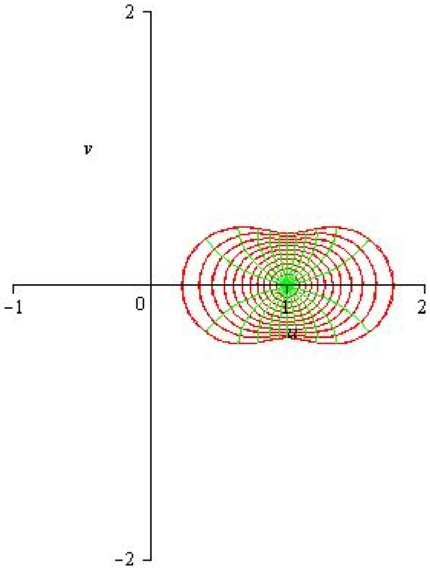

Figure 1. Graph of $\frac{2+\tan(\varepsilon)}{2}$ under open unit disc $\mathfrak{D}$.

In our research, we set out to explore and characterize a novel class of q-starlike functions associated with an eight-like image domain. We employed differential subordination and quantum calculus techniques to achieve this goal. Our primary aim is to establish a fundamental understanding of the behavior of these q-starlike functions, focusing on deriving first-order differential subordination results. Additionally, we sought to determine sharp inequalities for initial Taylor coefficients and provide optimal estimates for the Fekete-Szegö problem and a second-order Hankel determinant applicable to all q-starlike functions within this newly defined class. Through this work, we intend to contribute to the broader field of geometric function theory, demonstrating the wider implications of our findings through a series of corollaries.

2. A set of lemmas

The following lemmas are essential to investigate our main results.

Lemma 2.1. *[25] (q-Jack's Lemma) Let $\omega(\varepsilon) \in \mathcal{H}(\mathfrak{D})$ with $\omega(0) = 0$. If $|\omega(\varepsilon)|$ achieves the maximum value on the circle $|\varepsilon| = r$ at a point ε_0, then*

$$\varepsilon_0 \left(D_q \omega\right)(\varepsilon_0) = m\omega(\varepsilon_0),$$

where $0 < q < 1$ and $m \geq 1$.

Lemma 2.2. *[26] Let $h \in \mathcal{P}$ have the series of the form (1.2). Then the following inequalities hold true:*

$$|c_t| \leq 2 \qquad (t \geq 1), \tag{2.1}$$

$$|c_{t+k} - \nu c_t c_k| < 2 \qquad (0 \leq \nu \leq 1). \tag{2.2}$$

Lemma 2.3. *[27] Let $h \in \mathcal{P}$ be represented by* (1.2)*, and then the following inequality holds.*

$$\left|p_2 - \lambda p_1^2\right| \le 2\ \max\{1; |2\lambda - 1|\} \quad (\lambda \in \mathbb{C}).$$

Lemma 2.4. *Let $h \in \mathcal{P}$ be given in* (1.2)*, and then there exist k and $\delta \in \mathfrak{D}$ such that*

$$c_2 = \frac{1}{2}\left(c_1^2 + k\left(4 - c_1^2\right)\right), \tag{2.3}$$

$$c_3 = \frac{1}{4}\left(c_1^3 + 2c_1 k\left(4 - c_1^2\right) - \left(4 - c_1^2\right)c_1 k^2 + 2\left(4 - c_1^2\right)\left(1 - |k|^2\right)\delta\right). \tag{2.4}$$

The values given in (2.3) and (2.4) are due to [26] and [28], respectively.

Lemma 2.5. *[28] If $h \in \mathcal{P}$ has the form* (1.2)*, then*

$$|\alpha_1 c_1^3 - 2\alpha_2 c_1 c_2 + c_3| \le 2, \tag{2.5}$$

where

$$0 \le \alpha_1 \le 1 \quad \textit{and} \quad \alpha_1(2\alpha_1 - 1) \le \alpha_2 \le \alpha_1.$$

Lemma 2.6. *[29] Suppose that*

$$\max\left\{\left|P + Qx + Rx^2\right| + 1 - |x|^2\right\} = \chi(P, Q, R), \tag{2.6}$$

where P, Q, and R are real numbers, and $x \in \overline{\mathfrak{D}} = \{\varepsilon \in \mathbb{C} : |\varepsilon| \le 1\}$.
If $PR \ge 0$, then

$$\chi(P, Q, R) = \begin{cases} |P| + |Q| + |R|, & |Q| \ge 2(1 - |R|), \\ 1 + |P| + \frac{Q^2}{4(1-|R|)}, & |Q| < 2(1 - |R|). \end{cases}$$

3. First-order differential subordination results

In this section, we start with the results of differential subordination. In the next section, we derive the sharp constraints for the first three unknown coefficients, the sharp Fekete-Szegö problem, and the sharp estimate of the second-order Hankel determinant for the newly defined class of q-starlike functions. In addition, the consequences of these results are given in the form of corollaries. In the last section, we establish the sufficient criteria for functions belonging to the class $\mathcal{S}^*_{\tan}(q)$.

Theorem 3.1. *Let $f(\varepsilon) \in \mathcal{A}$ and $h(\varepsilon) \in \mathcal{H}(\mathfrak{D})$. Suppose that*

$$-1 \le B < \frac{\sec h^2(1)}{\sec^2(1)} < A \le 1,$$

$$|\delta| \ge \frac{2(A - B)}{(\sec h^2(1) - |B|\sec^2(1))}. \tag{3.1}$$

If the following condition holds:

$$1 + \delta\varepsilon D_q h(\varepsilon) \prec \frac{1 + A\varepsilon}{1 + B\varepsilon}. \tag{3.2}$$

Then

$$h(\varepsilon) \prec \frac{2 + \tan(\varepsilon)}{2}.$$

Proof. Suppose that

$$p(\varepsilon) = 1 + \delta\varepsilon D_q h(\varepsilon). \tag{3.3}$$

Also, for $\omega(\varepsilon) \in \mathcal{H}(\mathfrak{D})$, we consider

$$h(\varepsilon) = 1 + \frac{\tan(\omega(\varepsilon))}{2}. \tag{3.4}$$

It is sufficient to demonstrate that $|\omega(\varepsilon)| < 1$ in order to yield the required result. Using (3.3) and (3.4), we have

$$p(\varepsilon) = 1 + \delta\frac{\sec^2(\omega(\varepsilon))\,\varepsilon D_q\omega(\varepsilon)}{2},$$

and hence,

$$\begin{aligned}\left|\frac{p(\varepsilon)-1}{A-Bp(\varepsilon)}\right| &= \left|\frac{\delta\sec^2(\omega(\varepsilon))\,\varepsilon D_q\omega(\varepsilon)}{A-B\left(1+\frac{\delta\sec^2(\omega(\varepsilon))\varepsilon D_q\omega(\varepsilon)}{2}\right)}\right| \\ &= \left|\frac{\delta\sec^2(\omega(\varepsilon))\,\varepsilon D_q\omega(\varepsilon)}{2(A-B)-B\delta\sec^2(\omega(\varepsilon))\,\varepsilon D_q\omega(\varepsilon)}\right|.\end{aligned}$$

If $\omega(\varepsilon)$ achieves, at some point $\varepsilon = \varepsilon_0$, its maximum value $|\omega(\varepsilon_0)| = 1$, i.e., $\omega(\varepsilon_0) = e^{i\theta}$, $\theta \in [-\pi, \pi]$, then, by Lemma 2.1, for $m \geq 1$,

$$\varepsilon_0 D_q\omega(\varepsilon_0) = m\omega(\varepsilon_0).$$

Thus,

$$\begin{aligned}&\left|\frac{p(\varepsilon_0)-1}{A-Bp(\varepsilon_0)}\right| \\ &= \left|\frac{\delta\sec^2\left(e^{i\theta}\right)m\omega(\varepsilon_0)}{2(A-B)-B\delta\sec^2(e^{i\theta})\,m\omega(\varepsilon_0)}\right| \\ &\geq \frac{|\delta|\,m\left|\sec\left(e^{i\theta}\right)\right|^2}{2(A-B)+|B|\,|\delta|\,m\left|\sec(e^{i\theta})\right|^2}.\end{aligned} \tag{3.5}$$

A direct computation gives that

$$\begin{aligned}\left|\sec\left(e^{i\theta}\right)\right|^2 &= \frac{1}{|\cos(\cos\theta)\cosh(\sin\theta) - i\sin(\cos\theta)\sinh(\sin\theta)|^2} \\ &= \frac{1}{\cosh^2(\sin\theta)+\cos^2(\cos\theta)-1} := \varphi(\theta).\end{aligned}$$

Since $\varphi(-\theta) = \varphi(\theta)$, for $\theta \in [0, \pi]$, then

$$\begin{aligned}\min\{\varphi(\theta)\} &= \varphi\left(\frac{\pi}{2}\right) = \sec h^2(1), \\ \max\{\varphi(\theta)\} &= \varphi(0) = \varphi(\pi) = \sec^2(1).\end{aligned}$$

Therefore,

$$\sec h^2(1) \leq \left|\sec\left(e^{i\theta}\right)\right|^2 \leq \sec^2(1). \tag{3.6}$$

Putting values of (3.6) into (3.5) , we get

$$\left|\frac{p(\varepsilon_0)-1}{A-Bp(\varepsilon_0)}\right| \geq \frac{|\delta| m \sec h^2(1)}{2(A-B)+|B||\delta| m \sec^2(1)}. \tag{3.7}$$

Let

$$\phi(m) = \frac{|\delta| m \sec h^2(1)}{2(A-B)+|B||\delta| m \sec^2(1)}.$$

Then

$$\phi'(m) = \frac{2(A-B)\sec h^2(1)|\delta|}{\left(2(A-B)+|B||\delta| m \sec^2(1)\right)^2} > 0,$$

which shows the increasing behavior of $\phi(m)$, so the maximum of $\phi(m)$ will be obtained at $m = 1$. It follows that

$$\left|\frac{p(\varepsilon_0)-1}{A-Bp(\varepsilon_0)}\right| \geq \frac{|\delta| \sec h^2(1)}{2(A-B)+|B||\delta| \sec^2(1)}.$$

From (3.1),

$$\left|\frac{p(\varepsilon_0)-1}{A-Bp(\varepsilon_0)}\right| \geq 1,$$

which contradicts (3.2) , thus $|\omega(\varepsilon)| < 1$ and we achieve the intended outcome. □

By taking $h(\varepsilon) = \frac{\varepsilon D_q f(\varepsilon)}{f(\varepsilon)}$, we deduce the following result.

Corollary 3.1. *Let $f(\varepsilon) \in \mathcal{A}$ and $h(\varepsilon) = \frac{\varepsilon D_q f(\varepsilon)}{f(\varepsilon)}$. Suppose that*

$$-1 \leq B < \frac{\sec h^2(1)}{\sec^2(1)} < A \leq 1,$$

$$|\delta| \geq \frac{2(A-B)}{\left(\sec h^2(1) - |B|\sec^2(1)\right)}. \tag{3.8}$$

If the following condition holds:

$$1 + \delta\varepsilon D_q\left(\frac{\varepsilon\left(D_q f\right)(\varepsilon)}{f(\varepsilon)}\right) \prec \frac{1+A\varepsilon}{1+B\varepsilon}. \tag{3.9}$$

Then

$$\frac{\varepsilon D_q f(\varepsilon)}{f(\varepsilon)} \prec \frac{2+\tan(\varepsilon)}{2}.$$

Theorem 3.2. *Let $h(\varepsilon) \in \mathcal{H}(\mathfrak{D})$ with $h(0) = 1$. Suppose that*

$$-1 \leq B < \frac{\sec h^2(1)}{\sec^2(1)} < A \leq 1,$$

$$|\delta| \geq \frac{(A - B)(2 + \tan(1))}{2(\sec h^2(1) - |B| \sec^2(1))}. \tag{3.10}$$

If the following subordination criteria hold:

$$1 + \delta\left(\frac{\varepsilon D_q h(\varepsilon)}{h(\varepsilon)}\right) \prec \frac{1 + A\varepsilon}{1 + B\varepsilon}. \tag{3.11}$$

Then the following subordination holds:

$$h(\varepsilon) \prec \frac{2 + \tan(\varepsilon)}{2}.$$

Proof. Define

$$p(\varepsilon) = 1 + \delta\left(\frac{\varepsilon D_q h(\varepsilon)}{h(\varepsilon)}\right). \tag{3.12}$$

Let $\omega(\varepsilon) \in \mathcal{H}(\mathfrak{D})$, and consider

$$h(\varepsilon) = \frac{2 + \tan(\omega(\varepsilon))}{2}. \tag{3.13}$$

We need to show that $|\omega(\varepsilon)| \leq 1$. Using logarithmic differentiation on (3.13), we obtain from (3.12) that

$$p(\varepsilon) = 1 + \delta \frac{2\sec^2(\omega(\varepsilon))\, \varepsilon D_q \omega(\varepsilon)}{2 + \tan(\omega(\varepsilon))},$$

and so

$$\left|\frac{p(\varepsilon) - 1}{A - Bp(\varepsilon)}\right| = \left|\frac{\delta \frac{2\sec^2(\omega(\varepsilon))\varepsilon D_q \omega(\varepsilon)}{2+\tan(\omega(\varepsilon))}}{A - B\left(1 + \delta \frac{2\sec^2(\omega(\varepsilon))\varepsilon D_q \omega(\varepsilon)}{2+\tan(\omega(\varepsilon))}\right)}\right|$$
$$= \left|\frac{2\delta \sec^2(\omega(\varepsilon))\, \varepsilon D_q \omega(\varepsilon)}{(A - B)(2 + \tan(\omega(\varepsilon))) - 2\delta B \sec^2(\omega(\varepsilon))\, \varepsilon D_q \omega(\varepsilon)}\right|.$$

If at some $\varepsilon = \varepsilon_0$, $\omega(\varepsilon)$ attains its maximum value for example $|\omega(\varepsilon_0)| = 1$, then, by Lemma 2.1,

$$\left|\frac{p(\varepsilon_0) - 1}{A - Bp(\varepsilon_0)}\right|$$
$$= \left|\frac{2\delta \sec^2\left(e^{i\theta}\right) m\omega(\varepsilon_0)}{(A - B)(2 + \tan(e^{i\theta})) - 2\delta B \sec^2(e^{i\theta})\, m\omega(\varepsilon_0)}\right|$$
$$\geq \frac{2m|\delta| \left|\sec^2\left(e^{i\theta}\right)\right|}{(A - B)(2 + |\tan(e^{i\theta})|) + 2m|\delta| B \left|\sec^2(e^{i\theta})\right|}. \tag{3.14}$$

Now, a direct, simple calculation gives us

$$\left|\tan\left(e^{i\theta}\right)\right|^2 = \left|\frac{\sin(\cos(\theta))\cosh(\sin(\theta)) + i\cos(\cos(\theta))\sinh(\sin(\theta))}{\cos(\cos(\theta))\cosh(\sin(\theta)) - i\sin(\cos(\theta))\sinh(\sin(\theta))}\right|^2$$

$$= \frac{\cosh^2(\sin(\theta)) - \cos^2(\sin(\theta))}{\cos^2(\cos(\theta)) + \cosh^2(\sin(\theta)) - 1} := \varphi_1(\theta).$$

Since $\varphi_1(-\theta) = \varphi_1(\theta)$, consider $\theta \in [0, \pi]$. Then

$$\min\{\varphi_1(\theta)\} = \varphi_1\left(\frac{\pi}{2}\right) = \tanh^2(1),$$

$$\max\{\varphi_1(\theta)\} = \varphi_1(0) = \varphi_1(\pi) = \tan^2(1).$$

Therefore,

$$\tanh(1) \leq \left|\tan\left(e^{i\theta}\right)\right| \leq \tan(1). \tag{3.15}$$

Now set (3.6) and (3.15) in Eq (3.14), and we get

$$\phi(m) = \frac{2m|\delta|\sec h^2(1)}{(A-B)(2+\tan(1)) + 2m|\delta||B|\sec^2(1)},$$

$$\phi'(m) = \frac{2|\delta|(A-B)(2+\tan(1))|B|\sec h^2(1)}{((A-B)(2+\tan(1)) + 2m|\delta||B|\sec^2(1))^2} > 0.$$

Clearly, one can observe that $\phi(m)$ is increasing in nature so its maximum value is obtained at $m = 1$, thus

$$\left|\frac{p(\varepsilon_0) - 1}{A - Bp(\varepsilon_0)}\right| \geq \frac{2|\delta|\sec h^2(1)}{(A-B)(2+\tan(1)) + 2|\delta||B|\sec^2(1)}.$$

By (3.10), we have

$$\left|\frac{p(\varepsilon_0) - 1}{A - Bp(\varepsilon_0)}\right| \geq 1,$$

which contradicts (3.11), therefore $|\omega(\varepsilon)| < 1$ and so the desired result is obtained. □

By taking $h(\varepsilon) = \frac{\varepsilon D_q f(\varepsilon)}{f(\varepsilon)}$, we deduce the following corollary.

Corollary 3.2. *Let $f \in \mathcal{A}$. Suppose that*

$$-1 \leq B < \frac{\sec h^2(1)}{\sec^2(1)} < A \leq 1,$$

$$|\delta| \geq \frac{(A-B)(2+\tan(1))}{2\left(\sec h^2(1) - |B|\sec^2(1)\right)}. \tag{3.16}$$

If the following subordination holds:

$$1 + \delta\varepsilon\left(\frac{f(\varepsilon)}{\varepsilon D_q(f(\varepsilon))}\right) D_q\left(\frac{\varepsilon D_q f(\varepsilon)}{f(\varepsilon)}\right) \prec \frac{1 + A\varepsilon}{1 + B\varepsilon}, \tag{3.17}$$

then we have

$$\frac{\varepsilon D_q f(\varepsilon)}{f(\varepsilon)} \prec \frac{2 + \tan(\varepsilon)}{2}.$$

Theorem 3.3. *Let $h(\varepsilon) \in \mathcal{H}(\mathfrak{D})$ with $h(0) = 1$. Suppose that*

$$|\delta| \geq \frac{(A-B)(2+|\tan(1)|)^2}{2^2\left(\left|\sec h^2(1)\right| - |B|\,|\sec 1|\right)}, \quad for\ -1 \leq B < \frac{\sec h^2(1)}{\sec^2(1)} < A \leq 1. \tag{3.18}$$

If the following subordination criteria are fulfilled:

$$1 + \delta\frac{\varepsilon D_q h(\varepsilon)}{(h(\varepsilon))^2} \prec \frac{1+A\varepsilon}{1+B\varepsilon},$$

then we have

$$h(\varepsilon) \prec \frac{2+\tan(\varepsilon)}{2}.$$

Proof. We define a function:

$$p(\varepsilon) = 1 + \delta\frac{\varepsilon D_q h(\varepsilon)}{(h(\varepsilon))^2}.$$

If there exists $\omega(\varepsilon) \in \mathcal{H}(\mathfrak{D})$ such that

$$h(\varepsilon) = \frac{2+\tan(\omega(\varepsilon))}{2},$$

then, we obtain that

$$p(\varepsilon) = 1 + \delta\frac{4\sec^2(\omega(\varepsilon))\,\varepsilon D_q\omega(\varepsilon)}{(2+\tan(\omega(\varepsilon)))^2},$$

and hence,

$$\left|\frac{p(\varepsilon)-1}{A-Bp(\varepsilon)}\right| = \left|\frac{\delta\frac{4\sec^2(\omega(\varepsilon))\varepsilon D_q\omega(\varepsilon)}{(2+\tan(\omega(\varepsilon)))^2}}{A - B\left(1+\delta\frac{4\sec^2(\omega(\varepsilon))\varepsilon D_q\omega(\varepsilon)}{(2+\tan(\omega(\varepsilon)))^2}\right)}\right|$$
$$= \left|\frac{4\delta\sec^2(\omega(\varepsilon))\,\varepsilon D_q\omega(\varepsilon)}{(A-B)(2+\tan(\omega(\varepsilon)))^2 - B\left(4\delta\sec^2(\omega(\varepsilon))\,\varepsilon D_q\omega(\varepsilon)\right)}\right|.$$

If $\omega(\varepsilon)$ attains its maximum value at some point $\varepsilon = \varepsilon_0$, which is $|\omega(\varepsilon_0)| = 1 : \omega(\varepsilon_0) = e^{i\theta}$, for some $\theta \in [-\pi, \pi]$, then, by Lemma 2.1,

$$\varepsilon_0 D_q\omega(\varepsilon_0) = m\omega(\varepsilon_0), \quad for\ m \geq 1.$$

Thus, we have

$$\left|\frac{p(\varepsilon_0)-1}{A-Bp(\varepsilon_0)}\right|$$
$$= \left|\frac{4\delta\sec^2\left(e^{i\theta}\right)m\omega(\varepsilon_0)}{(A-B)(2+\tan(e^{i\theta}))^2 - B\delta\left(4\sec^2(e^{i\theta})\,m\omega(\varepsilon_0)\right)}\right|$$

$$\geq \frac{4m\,|\delta|\left|\sec^2\left(e^{i\theta}\right)\right|}{(A-B)\,(2+|\tan(e^{i\theta})|)^2+4m\,|\delta|\,|B|\left|\sec^2(e^{i\theta})\right|}. \tag{3.19}$$

Applying (3.6) and (3.15) in Eq (3.19) ,

$$\phi(m) = \frac{4m\,|\delta|\sec h^2(1)}{(A-B)\,(2+\tan(1))^2+4m\,|\delta|\,|B|\sec^2(1)}$$
$$\Rightarrow \phi'(m) = \frac{4\,|\delta|\,(A-B)\,(2+|\tan(1)|)^2\sec h^2(1)}{\left((A-B)\,(2+\tan(1))^2+4m\,|\delta|\,|B|\sec^2(1)\right)^2} > 0.$$

Thus, the function $\phi(m)$ is increasing; hence, it has its maximum value at $m = 1$. Now, we have

$$\left|\frac{p(\varepsilon_0)-1}{A-Bp(\varepsilon_0)}\right| \geq \frac{4\,|\delta|\sec h^2(1)}{(A-B)\,(2+\tan(1))^2+4\,|\delta|\,|B|\sec^2(1)}.$$

By (3.18) ,

$$\left|\frac{p(\varepsilon_0)-1}{A-Bp(\varepsilon_0)}\right| \geq 1.$$

It is a contradiction to

$$p(\varepsilon) \prec \frac{1+A\varepsilon}{1+B\varepsilon}.$$

Therefore, $|\omega(\varepsilon)| < 1$, and the required result is obtained. □

By taking $h(\varepsilon) = \frac{\varepsilon\left(D_q f\right)(\varepsilon)}{(f(\varepsilon))^2}$, we deduce the following corollay.

Corollary 3.3. *Let $f \in \mathcal{A}$. Suppose that*

$$-1 \leq B < \frac{\sec h^2(1)}{\sec^2(1)} < A \leq 1,$$

$$|\delta| \geq \frac{(A-B)\,(2+\tan(1))^2}{4\left(\sec h^2(1)-|B|\sec^2(1)\right)}. \tag{3.20}$$

If the following holds:

$$1+\delta\varepsilon\left(\frac{f(\varepsilon)}{\varepsilon\left(D_q f\right)(\varepsilon)}\right)^2 D_q\left(\frac{\varepsilon\left(D_q f\right)(\varepsilon)}{f(\varepsilon)}\right) \prec \frac{1+A\varepsilon}{1+B\varepsilon}, \tag{3.21}$$

then, we have

$$\frac{\varepsilon\left(D_q f\right)(\varepsilon)}{(f(\varepsilon))^2} \prec \frac{2+\tan(\varepsilon)}{2}.$$

Theorem 3.4. *Let $h(\varepsilon) \in \mathcal{H}(\mathfrak{D})$ with $h(0) = 1$. Suppose that*

$$|\delta| \geq \frac{(A-B)\left(2+\left|\tan\left(e^{i\theta}\right)\right|\right)^3}{2^3\left(\left|\sec^2(e^{i\theta})\right|-|B|\left|\sec^2(e^{i\theta})\right|\right)}, \quad \textit{for } -1 \leq B < \frac{\sec h^2(1)}{\sec^2(1)} < A \leq 1. \tag{3.22}$$

If the following subordination holds:

$$1+\delta\frac{\varepsilon\left(D_{q}h\right)(\varepsilon)}{(h(\varepsilon))^{3}}\prec\frac{1+A\varepsilon}{1+B\varepsilon},\tag{3.23}$$

then,

$$h(\varepsilon)\prec\frac{2+\tan(\varepsilon)}{2}.$$

Proof. Suppose that

$$p(\varepsilon)=1+\delta\frac{\varepsilon D_{q}h(\varepsilon)}{(h(\varepsilon))^{3}}.$$

Now if

$$h(\varepsilon)=\frac{2+\tan(\omega(\varepsilon))}{2},$$

then we can easily obtain

$$p(\varepsilon)=1+\delta\frac{8\varepsilon D_{q}\omega(\varepsilon)\sec^{2}(\omega(\varepsilon))}{(2+\tan(\omega(\varepsilon)))^{3}},$$

and so,

$$\begin{aligned}\left|\frac{p(\varepsilon)-1}{A-Bp(\varepsilon)}\right| &= \left|\frac{\delta\frac{8\varepsilon D_{q}\omega(\varepsilon)\sec^{2}(\omega(\varepsilon))}{(2+\tan(\omega(\varepsilon)))^{3}}}{A-B\left(1+\delta\frac{8\varepsilon D_{q}\omega(\varepsilon)\sec^{2}(\omega(\varepsilon))}{(2+\tan(\omega(\varepsilon)))^{3}}\right)}\right| \\ &= \left|\frac{8\delta\sec^{2}(\omega(\varepsilon))\,\varepsilon D_{q}\omega(\varepsilon)}{(A-B)(2+\tan(\omega(\varepsilon)))^{3}-8B\delta\sec^{2}(\omega(\varepsilon))\,\varepsilon D_{q}\omega(\varepsilon)}\right|.\end{aligned}$$

If the function ω achieves its maximum value, $|\omega(\varepsilon_{0})|=1$, at some point $\varepsilon=\varepsilon_{0}$, applying Lemma 2.1 gives us,

$$\begin{aligned}&\left|\frac{p(\varepsilon_{0})-1}{A-Bp(\varepsilon_{0})}\right| \\ &= \left|\frac{8\delta\sec^{2}(\omega(\varepsilon_{0}))\,m\omega(\varepsilon_{0})}{(A-B)(2+\tan(\omega(\varepsilon_{0})))^{3}-8\delta B\sec^{2}(\omega(\varepsilon_{0}))\,m\omega(\varepsilon_{0})}\right| \\ &\geq \frac{8m\delta\left|\sec^{2}\left(e^{i\theta}\right)\right|}{(A-B)(2+|\tan(e^{i\theta})|)^{3}+8m|\delta||B|\left|\sec^{2}(e^{i\theta})\right|}.\end{aligned}\tag{3.24}$$

Substituting (3.6) and (3.15) into Eq (3.24) , we have

$$\phi(m)=\frac{8m|\delta|\sec h^{2}(1)}{(A-B)(2+\tan(1))^{3}-8m|\delta||B|\sec^{2}(1)},$$

$$\phi'(m)=\frac{8|\delta|(A-B)(2+\tan(1))^{3}\sec h^{2}(1)}{\left((A-B)(2+\tan(1))^{3}+8m|\delta||B|\sec^{2}(1)\right)^{2}}>0.$$

It demonstrates that the function $\phi(m)$ is increasing. So, its maximum value is obtained at $m = 1$. Thus,

$$\left|\frac{p(\varepsilon_0)-1}{A-Bp(\varepsilon_0)}\right| \geq \frac{8|\delta|\left|\sec^2\left(e^{i\theta}\right)\right|}{(A-B)\left(2+\left|\tan\left(e^{i\theta}\right)\right|\right)^3+8|\delta||B|\left|\sec^2\left(e^{i\theta}\right)\right|},$$

and hence

$$\left|\frac{p(\varepsilon_0)-1}{A-Bp(\varepsilon_0)}\right| \geq 1.$$

By (3.23) , a contradiction occurs. We must have $|\omega(\varepsilon)| < 1$, so we obtain the needed outcome. □

By taking $h(\varepsilon) = \frac{\varepsilon D_q f(\varepsilon)}{(f(\varepsilon))^3}$, we deduce the following result.

Corollary 3.4. *Let $f \in \mathcal{A}$. Suppose that*

$$-1 \leq B < \frac{\sec h^2(1)}{\sec^2(1)} < A \leq 1,$$

$$|\delta| \geq \frac{(A-B)(2+\tan(1))^3}{2^3\left(\sec h^2(1)-|B|\sec^2(1)\right)}. \tag{3.25}$$

If the following condition holds:

$$1+\delta\varepsilon\left(\frac{f(\varepsilon)}{\varepsilon D_q(f(\varepsilon))}\right)^3 D_q\left(\frac{\varepsilon D_q f(\varepsilon)}{f(\varepsilon)}\right) \prec \frac{1+A\varepsilon}{1+B\varepsilon}, \tag{3.26}$$

then,

$$\frac{\varepsilon D_q f(\varepsilon)}{(f(\varepsilon))^3} \prec \frac{2+\tan(\varepsilon)}{2}.$$

Theorem 3.5. *Let $h(\varepsilon) \in \mathcal{H}(\mathfrak{D})$ with $h(0) = 1$. Suppose that*

$$|\delta| \geq \frac{(A-B)\left(2+\left|\tan\left(e^{i\theta}\right)\right|\right)^n}{2^n\left(\left|\sec^2\left(e^{i\theta}\right)\right|-|B|\left|\sec^2\left(e^{i\theta}\right)\right|\right)}, \quad \text{for } -1 \leq B < \frac{\sec h^2(1)}{\sec^2(1)} < A \leq 1. \tag{3.27}$$

If the following subordination is provided:

$$1+\delta\frac{\varepsilon D_q h(\varepsilon)}{(h(\varepsilon))^n} \prec \frac{1+A\varepsilon}{1+B\varepsilon}, \tag{3.28}$$

then,

$$h(\varepsilon) \prec \frac{2+\tan(\varepsilon)}{2}.$$

Proof. Suppose

$$p(\varepsilon) = 1+\delta\frac{\varepsilon D_q h(\varepsilon)}{(h(\varepsilon))^n}.$$

Now consider

$$h(\varepsilon) = \frac{2+\tan(\omega(\varepsilon))}{2}.$$

We can easily obtain that

$$p(\varepsilon) = 1 + \delta \frac{2^n \varepsilon D_q \omega(\varepsilon) \sec^2(\omega(\varepsilon))}{(2 + \tan(\omega(\varepsilon)))^n},$$

and hence,

$$\begin{aligned}\left|\frac{p(\varepsilon)-1}{A-Bp(\varepsilon)}\right| &= \left|\frac{\delta \frac{2^n \varepsilon D_q \omega(\varepsilon)\sec^2(\omega(\varepsilon))}{(2+\tan(\omega(\varepsilon)))^n}}{A - B\left(1+\delta\frac{2^n \varepsilon D_q \omega(\varepsilon)\sec^2(\omega(\varepsilon))}{(2+\tan(\omega(\varepsilon)))^n}\right)}\right| \\ &= \left|\frac{2^n \delta \varepsilon D_q \omega(\varepsilon)\sec^2(\omega(\varepsilon))}{(A-B)(2+\tan(\omega(\varepsilon)))^n - 2^n B \delta \varepsilon D_q \omega(\varepsilon)\sec^2(\omega(\varepsilon))}\right|.\end{aligned}$$

If the function ω accomplishes its maximum value, $|\omega(\varepsilon_0)| = 1$, at some point $\varepsilon = \varepsilon_0$, then, utilizing Lemma 2.1, we have

$$\begin{aligned}&\left|\frac{p(\varepsilon_0)-1}{A-Bp(\varepsilon_0)}\right| \\ &= \left|\frac{2^n \delta \varepsilon D_q \omega(\varepsilon)\sec^2(\omega(\varepsilon))}{(A-B)(2+\tan(\omega(\varepsilon)))^n - 2^n B \delta \varepsilon D_q \omega(\varepsilon)\sec^2(\omega(\varepsilon))}\right| \\ &\geq \frac{2^n m \delta \left|\sec^2\left(e^{i\theta}\right)\right|}{(A-B)(2+|\tan(e^{i\theta})|)^n + 2^n m |\delta| |B| \left|\sec^2(e^{i\theta})\right|}.\end{aligned} \tag{3.29}$$

Now set (3.6) and (3.15) in (3.29) , and we get

$$\begin{aligned}\phi(m) &= \frac{2^n m \delta \left|\sec^2\left(e^{i\theta}\right)\right|}{(A-B)(2+|\tan(e^{i\theta})|)^n + 2^n m |\delta| |B| \left|\sec^2(e^{i\theta})\right|} \\ \Rightarrow \phi'(m) &= \frac{2^n |\delta| (A-B)(2+\tan(1))^n \sec h^2(1)}{\left((A-B)(2+|\tan(e^{i\theta})|)^n + 2^n m |\delta| |B| \left|\sec^2(e^{i\theta})\right|\right)^2} > 0.\end{aligned}$$

It demonstrates that $\phi(m)$ increases, achieving the maximum value at $m = 1$. Thus,

$$\left|\frac{p(\varepsilon_0)-1}{A-Bp(\varepsilon_0)}\right| \geq \frac{2^n \delta \left|\sec^2\left(e^{i\theta}\right)\right|}{(A-B)(2+|\tan(e^{i\theta})|)^n + 2^n |\delta| |B| \left|\sec^2(e^{i\theta})\right|},$$

and hence

$$\left|\frac{p(\varepsilon_0)-1}{A-Bp(\varepsilon_0)}\right| \geq 1,$$

which contradicts the assumption (3.28) , so $|\omega(\varepsilon)| < 1$. We complete the proof. □

By taking $h(\varepsilon) = \frac{\varepsilon D_q f(\varepsilon)}{(f(\varepsilon))^n}$, we deduce the following corollary.

Corollary 3.5. *Let $f \in \mathcal{A}$. Suppose that*

$$-1 \leq B < \frac{\sec h^2(1)}{\sec^2(1)} < A \leq 1,$$

$$|\delta| \geq \frac{(A-B)(2+\tan(1))^n}{2^n\left(\sec h^2(1)-|B|\sec^2(1)\right)}. \tag{3.30}$$

If the following is given:

$$1+\delta\varepsilon\left(\frac{f(\varepsilon)}{\varepsilon D_q(f(\varepsilon))}\right)^n D_q\left(\frac{\varepsilon D_q f(\varepsilon)}{f(\varepsilon)}\right) \prec \frac{1+A\varepsilon}{1+B\varepsilon}, \tag{3.31}$$

then,

$$\frac{\varepsilon D_q f(\varepsilon)}{(f(\varepsilon))^n} \prec \frac{2+\tan(\varepsilon)}{2}.$$

Remark 3.1. *One can obtain new results for the class $\mathcal{S}^*_{\tan}$ by taking the limit $q \to 1^-$ in our results.*

4. Coefficient estimates for class $\mathcal{S}^*_{\tan}(q)$

Theorem 4.1. *If $f \in \mathcal{S}^*_{\tan}(q)$ is of the form* (1.1)*, then*

$$|\xi_2| \leq \frac{1}{2q} \qquad (0<q<1),$$
$$|\xi_3| \leq \frac{1}{2q(q+1)} \qquad (0.5<q<1),$$
$$|\xi_4| \leq \frac{1}{2q(q^2+q+1)} \qquad (0.41310<q<0.57708).$$

All these estimates are extreme for a function defined below:

$$\begin{aligned}\frac{\varepsilon D_q f(\varepsilon)}{f(\varepsilon)} &= \frac{2+\tan(\varepsilon^n)}{2} \\ &= 1+\frac{1}{2}\varepsilon^n+\cdots \qquad (n=1,2,3).\end{aligned} \tag{4.1}$$

Proof. Suppose $f \in \mathcal{S}^*_{\tan}(q)$, and then there exists a Schwarz function $\omega(\varepsilon)$ such that

$$\frac{\varepsilon\left(D_q f\right)(\varepsilon)}{f(\varepsilon)} = \frac{2+\tan(\omega(\varepsilon))}{2},$$

where $\omega(\varepsilon) = \frac{p(\varepsilon)-1}{p(\varepsilon)+1}$. If $p(\varepsilon)$ follows the form of (1.2), then

$$\omega(\varepsilon) = \frac{c_1\varepsilon + c_2\varepsilon^2 + c_3\varepsilon^3 + c_4\varepsilon^4 + \cdots}{2 + c_1\varepsilon + c_2\varepsilon^2 + c_3\varepsilon^3 + c_4\varepsilon^4 + \cdots}.$$

Using this, one can easily find

$$\begin{aligned}\frac{2+\tan(\omega(\varepsilon))}{2} &= 1+\frac{1}{4}c_1\varepsilon+\left(\frac{1}{4}c_2-\frac{1}{8}c_1^2\right)\varepsilon^2 \\ &\quad+\left(\frac{1}{12}c_1^3-\frac{1}{4}c_2c_1+\frac{1}{4}c_3\right)\varepsilon^3\end{aligned}$$

$$+\left(-\frac{1}{16}c_1^4+\frac{1}{4}c_1^2c_2-\frac{1}{4}c_3c_1-\frac{1}{8}c_2^2+\frac{1}{4}c_4\right)\varepsilon^4+\cdots. \tag{4.2}$$

Then

$$\begin{aligned}\frac{\varepsilon D_q f(\varepsilon)}{f(\varepsilon)} &= 1+q\xi_2\varepsilon+\left[q(q+1)\xi_3-q\xi_2^2\right]\varepsilon^2\\ &+\left[q\left(\sum_{n=0}^{2}q^n\right)\xi_4-q(q+2)\xi_2\xi_3+q\xi_2^3\right]\varepsilon^3\\ &+q\left[q^3\xi_5+(\xi_5-\xi_2\xi_4)q^2+\left(\xi_5-\xi_3^2+\xi_2^2\xi_3-\xi_2\xi_4\right)q\right.\\ &\left.+\xi_5+3\xi_2^2\xi_3-2\xi_2\xi_4-\xi_2^4-\xi_3^2\right]\varepsilon^4+\cdots. \end{aligned} \tag{4.3}$$

Comparing (4.2) and (4.3) , we obtain

$$\xi_2=\frac{1}{4q}c_1, \tag{4.4}$$

$$\xi_3=\frac{1}{4q(q+1)}\left(c_2-\frac{2q-1}{4q}c_1^2\right), \tag{4.5}$$

$$\xi_4=\frac{1}{4q(1+q+q^2)}\left(\left(\frac{16q^3+10q^2-12q+3}{48q^2(q+1)}\right)c_1^3-\frac{\left(4q^2+3q-2\right)}{4q(q+1)}c_1c_2+c_3\right). \tag{4.6}$$

Applying (2.1) to (4.4) , and we have

$$|\xi_2|\leq\frac{1}{2q}.$$

To find the bound of ξ_3, apply (2.2) to (4.5) with $0.5<q<1$, and we have

$$|\xi_3|\leq\frac{1}{2q(q+1)}.$$

From Lemma 2.5,

$$\alpha_1=\frac{16q^3+10q^2-12q+3}{48q^2(q+1)},$$

and

$$\alpha_2=\frac{\left(4q^2+3q-2\right)}{2q(q+1)},$$

so the conditions $0\leq\alpha_2\leq1$ and $\alpha_2(2\alpha_2-1)\leq\alpha_1\leq\alpha_2$ are satisfied for $0.41310<q<0.57708$. Applying Lemma 2.5, we get

$$|\xi_4|\leq\frac{1}{2q(1+q+q^2)}.$$

□

Corollary 4.1. *If $f\in\mathcal{S}^*_{\tan}$, then*

$$|\xi_2|\leq\frac{1}{2},$$

$$|\xi_3| \leq \frac{1}{4},$$
$$|\xi_4| \leq \frac{1}{6}.$$

The above three bounds are sharp for the function defined below:

$$\frac{\varepsilon f'(\varepsilon)}{f(\varepsilon)} = \frac{2+\tan(\varepsilon^n)}{2} = 1 + \frac{1}{2}\varepsilon^n + \cdots \qquad (n = 1, 2, 3). \tag{4.7}$$

Theorem 4.2. *If $f \in \mathcal{S}^*_{\tan}(q)$, then*

$$\left|\xi_3 - \lambda\xi_2^2\right| \leq \frac{1}{2q(q+1)} \max\left\{1, \left|\frac{(1+q)\xi - 1}{2q}\right|\right\}.$$

The result is sharp for the function defined in (4.1) *, for $n = 2$.*

Proof. From (4.4) and (4.5) , we have

$$\left|\xi_3 - \lambda\xi_2^2\right| = \frac{1}{4q(q+1)} \left|c_2 - \frac{(a(1+q)+2q-1)}{4q} c_1^2 c_1^2\right|.$$

Applying Lemma 2.3 to the preceding equation yields the desired outcome. □

Corollary 4.2. *If $f \in \mathcal{S}^*_{\tan}$, then*

$$\left|\xi_3 - \lambda\xi_2^2\right| \leq \frac{1}{4} \max\left\{1, \left|\frac{2\lambda - 1}{2}\right|\right\}.$$

The result is sharp for the function defined in (4.7) *, for $n = 2$.*

Theorem 4.3. *If $f \in \mathcal{S}^*_{\tan}(q)$, then*

$$|\xi_2\xi_3 - \xi_4| \leq \frac{1}{2q(1+q+q^2)} \qquad (0.145\,27 < q < 0.222\,65).$$

The outcome is sharp for the function defined in (4.1) *, for $n = 3$.*

Proof. From (4.4)–(4.6), we have

$$|\xi_2\xi_3 - \xi_4| = \frac{1}{4q(1+q+q^2)} \left|\frac{9-22q}{48q} c_1^3 - \frac{(1-5q)}{4q} c_1 c_2 + c_3\right|.$$

Comparing with Lemma 2.5, we have

$$\alpha_1 = \frac{9-22q}{48q} \quad \text{and} \quad \alpha_2 = \frac{(1-5q)}{2q},$$

and the conditions $0 \leq \alpha_2 \leq 1$ and $\alpha_2(2\alpha_2 - 1) \leq \alpha_1 \leq \alpha_2$ are satisfied for $0.14527 < q < 0.22265$, so Lemma 2.5 is valid to apply. Hence

$$|\xi_2\xi_3 - \xi_4| \leq \frac{1}{2q(1+q+q^2)}.$$

□

Corollary 4.3. *If $f \in \mathcal{S}^{*}_{\tan}$, then*

$$|\xi_2\xi_3 - \xi_4| \leq \frac{1}{6}.$$

The result is sharp for the function defined in (4.7) *, for* $n = 3$.

Theorem 4.4. *Let $f \in \mathcal{S}^{*}_{\tan}(q)$ be given in the form* (1.1)*, and then*

$$\left|\xi_2\xi_4 - \xi_3^2\right| \leq \frac{1}{4\left(1+q+q^2\right)^2}, \quad \text{for } q \in (0.8382, 1). \tag{4.8}$$

Proof. From (4.4)–(4.6), we have

$$\begin{aligned}\left|\xi_2\xi_4 - \xi_3^2\right| &= \left|\frac{(1+q)}{16\left(1+q+q^2\right)\left(1+q+q^2+q^3\right)}c_1c_3 - \frac{1}{16\left(1+q+q^2\right)^2}c_2^2\right.\\ &\quad - \frac{\left(q^5+q^4+8q^3-5q^2+10q-5\right)}{64\left(1+q\right)\left(1+q+q^2\right)^2\left(1+q+q^2+q^3\right)}c_1^2c_2\\ &\quad \left.- \frac{1}{1024}\frac{\left(q^8+4q^7+14q^6-40q^5+11q^4-116q^3+186q^2-60q+88\right)}{\left(1+q\right)^3\left(1+q+q^2\right)^2\left(1+q+q^2+q^3\right)}c_1^4\right|.\end{aligned}$$

Using Lemma 2.4 for $c_1 = c$, we have

$$\begin{aligned}\left|\xi_2\xi_4 - \xi_3^2\right| = &\left|-\frac{\left(q^8+12q^7+38q^6+32q^5+43q^4-148q^3+202q^2-76q+48\right)}{1024\left(q+1\right)^3\left(q^2+q+1\right)^2\left(q^3+q^2+q+1\right)}c^4\right.\\ &- \frac{\left(q^5+q^4+4q^3-13q^2+6q-5\right)}{128\left(q+1\right)\left(q^2+q+1\right)^2\left(q^3+q^2+q+1\right)}c^2\left(4-c^2\right)x\\ &- \left(\frac{\left(1+q+q^2+q^3\right)\left(4-c^2\right)+\left(1+q\right)\left(1+q+q^2\right)c^2}{64\left(1+q+q^2\right)^2\left(1+q+q^2+q^3\right)}\right)\left(4-c^2\right)x^2\\ &\left.+\frac{(1+q)}{32\left(1+q+q^2\right)\left(1+q+q^2+q^3\right)}\left(4-c^2\right)c\left(1-|x|^2\right)\varepsilon\right|.\end{aligned}$$

For $c \in [0, 2]$, the simple calculation gives

$$\begin{aligned}\left|\xi_2\xi_4 - \xi_3^2\right| = &\frac{(1+q)}{32\left(1+q+q^2\right)\left(1+q+q^2+q^3\right)}\\ &\left|-\frac{\left(q^8+12q^7+38q^6+32q^5+43q^4-148q^3+202q^2-76q+48\right)}{32\left(q+1\right)^4\left(q^2+q+1\right)}c^4\right.\\ &- \frac{\left(q^5+q^4+4q^3-13q^2+6q-5\right)}{4\left(q+1\right)^2\left(q^2+q+1\right)}c^2\left(4-c^2\right)x\\ &- \left(\frac{\left(1+q+q^2+q^3\right)\left(4-c^2\right)+\left(1+q\right)\left(1+q+q^2\right)c^2}{2\left(1+q\right)\left(1+q+q^2\right)}\right)\left(4-c^2\right)x^2\end{aligned}$$

$$+\left(4-c^2\right)c\left(1-|x|^2\right)\varepsilon\Big|,$$

where x and δ satisfy $|x| \leq 1$ and $|\delta| \leq 1$.
Next, we will find the maximum value of $\left|\xi_2\xi_4-\xi_3^2\right|$ for $c \in [0,2]$.
Case 1. When $c = 0$, we have

$$\left|\xi_2\xi_4-\xi_3^2\right| \leq \frac{1}{4\left(1+q+q^2\right)^2} \qquad q \in (0,1).$$

Case 2. When $c = 2$, we get

$$\begin{aligned}\left|a_2a_4-a_3^2\right| &= \frac{1}{4\left(1+q+q^2\right)^2} \\ &\quad \cdot\frac{\left(q^8+12q^7+38q^6+32q^5+43q^4-148q^3+202q^2-76q+48\right)}{16\left(q+1\right)^3\left(1+q+q^2+q^3\right)} \\ &\leq \frac{1}{4\left(1+q+q^2\right)^2} \qquad q \in (0,1).\end{aligned}$$

Case 3. Assume $c \in (0,2)$. Then, by the above equation and the triangular inequality,

$$\begin{aligned}\left|\xi_2\xi_4-\xi_3^2\right| \leq{}& \frac{(1+q)\left(4-c^2\right)c}{32\left(1+q+q^2\right)\left(1+q+q^2+q^3\right)} \\ & \Bigg|-\frac{\left(q^8+12q^7+38q^6+32q^5+43q^4-148q^3+202q^2-76q+48\right)}{32\left(q+1\right)^4\left(q^2+q+1\right)\left(4-c^2\right)}c^3 \\ & -\frac{\left(q^5+q^4+4q^3-13q^2+6q-5\right)c}{4\left(q+1\right)^2\left(q^2+q+1\right)}x \\ & -\left(\frac{\left(1+q+q^2+q^3\right)\left(4-c^2\right)+(1+q)\left(1+q+q^2\right)c^2}{2c(1+q)\left(1+q+q^2\right)}\right)x^2+\left(1-|x|^2\right)\Bigg|.\end{aligned}$$

By Lemma 2.6, we can write it as

$$\left|\xi_2\xi_4-\xi_3^2\right| \leq \frac{(1+q)\left(4-c^2\right)c}{32\left(1+q+q^2\right)\left(1+q+q^2+q^3\right)}\psi(P,Q,R),$$

where

$$P = -\frac{\left(q^8+12q^7+38q^6+32q^5+43q^4-148q^3+202q^2-76q+48\right)}{32\left(q+1\right)^4\left(q^2+q+1\right)\left(4-c^2\right)}c^3,$$

$$Q = \frac{\left(-q^5-q^4-4q^3+13q^2-6q+5\right)c}{4\left(q+1\right)^2\left(q^2+q+1\right)},$$

and

$$R = -\frac{\left(\left(1+q+q^2+q^3\right)\left(4-c^2\right)+(1+q)\left(1+q+q^2\right)c^2\right)}{2c(1+q)\left(1+q+q^2\right)}.$$

Clearly, for all $q \in (0,1)$ and $c \in (0,2)$, we have

$$PR = \frac{\left(c^2q + 4q^2 + 4\right)\left(q^8 + 12q^7 + 38q^6 + MD\right)}{64\,(4 - c^2)\,(q+1)^4\,(q^2+q+1)^2}c^2 > 0,$$

where

$$MD = 32q^5 + 43q^4 - 148q^3 + 202q^2 - 76q + 48.$$

Now we consider $|Q| - 2\,(1 - |R|)$. Let

$$\varphi(c) = 8\,(q+1)^2\left(q^2+q+1\right)c\,\{|Q| - 2\,(1-|R|)\},$$

which implies

$$\begin{aligned}\varphi(c) = {} & 2\left(5 - 2q + 21q^2 - q^4 - q^5\right)c^2 \\ & - 16\,(q+1)^2\left(q^2+q+1\right)c + 32\left(q^2+1\right)(q+1)^2.\end{aligned}$$

Differentiating $\varphi(c)$ twice, we have

$$\varphi'(c) = 4\left(5 - 2q + 21q^2 - q^4 - q^5\right)c - 16\,(q+1)^2\left(q^2+q+1\right),$$

$$\varphi''(c) = 4\left(5 - 2q + 21q^2 - q^4 - q^5\right) > 0 \qquad q \in (0,1).$$

This shows that $\varphi'(c)$ is an increasing function and

$$\begin{aligned}\max\varphi'(c) = \varphi'(2) = -8q^5 - 24q^4 - 48q^3 \\ +104q^2 - 64q + 24 < 0 \qquad q \in (0.8382, 1).\end{aligned}$$

It follows that

$$\varphi'(c) < 0,\ c \in (0,2),\ q \in (0.8382, 1).$$

Hence, $\varphi(c)$ is a decreasing function and

$$\begin{aligned}\min\varphi(c) = \varphi(2) = -8q^5 - 8q^4 - 32q^3 \\ +104q^2 - 48q + 40 \geq 0 \quad q \in (0,1).\end{aligned}$$

This implies $|Q| - 2\,(1 - |R|) > 0$, and then by Lemma 2.6,

$$\left|\xi_2\xi_4 - \xi_3^2\right| \leq \frac{(1+q)\left(4-c^2\right)c}{32\,(1+q+q^2)\,(1+q+q^2+q^3)}\,(|P| + |Q| + |R|) = h(c),$$

where

$$h(c) = -\frac{Mc^4 + Nc^2 - 256\left(q^2+1\right)(q+1)^4}{1024\,(q+1)^3\,(q^2+q+1)^2\,(1+q+q^2+q^3)}.$$

By differentiating, we have

$$h'(c) = \frac{1}{1024\,(q+1)^3\,(q^2+q+1)^2\,(1+q+q^2+q^3)}\left(Mc^3 + Nc\right),$$

where

$$M=-\left(-q^{8}-20q^{7}-62q^{6}-72q^{5}+53q^{4}+372q^{3}-90q^{2}+124q-8\right)\leq 0$$
$$\text{for } q\in(0.0669,1),$$

and

$$N=-\left(32q^{7}+160q^{6}+416q^{5}+64q^{4}-384q^{3}+64q-96\right)\leq 0 \text{ for } q\in(0.8382,1),$$

which implies

$$h'(c)\leq 0.$$

We conclude that the function $h(c)$ is a decreasing function and

$$h(c)\leq h(0)=\frac{1}{4(q^{2}+q+1)^{2}}.$$

From all of the above discussion, we conclude that

$$\left|\xi_2\xi_4-\xi_3^2\right|\leq\frac{1}{4(1+q+q^{2})^{2}},\quad \text{for } q\in(0.8382,1).$$

□

5. Sufficiency criteria for class $\mathcal{S}^{*}_{\tan}(q)$

Theorem 5.1. *A function $f\in\mathcal{S}^{*}_{\tan}(q)$ if and only if*

$$\frac{1}{\varepsilon}\left[f(\varepsilon)*\frac{H\varepsilon^{2}-G\varepsilon}{2(1-\varepsilon)(1-q\varepsilon)}\right]\neq 0, \tag{5.1}$$

where

$$G=2+\tan(e^{i\theta}),\ H=\tan(e^{i\theta}),\ \ \text{or } G=H=1. \tag{5.2}$$

Proof. If a function $f\in\mathcal{S}^{*}_{\tan}(q)$, then f is analytic in $\mathfrak{D}$, and hence $\frac{1}{\varepsilon}f(\varepsilon)\neq 0$ for all ε in $\mathfrak{D}^{*}=\mathfrak{D}-\{0\}$. Thus, we have the Eq (5.1) for $G=H=1$.
Now, by using (1.9) along with the principle of subordination, there exists a Schwarz function ω such that

$$\frac{\varepsilon\left(D_q f\right)(\varepsilon)}{f(\varepsilon)}=1+\frac{\tan(\omega(\varepsilon))}{2}.$$

Taking into consideration $\omega(\varepsilon)=e^{i\theta}$, for $0\leq\theta\leq 2\pi$, then the above expression becomes

$$\frac{\varepsilon\left(D_q f\right)(\varepsilon)}{f(\varepsilon)}\neq 1+\frac{\tan\left(e^{i\theta}\right)}{2}, \tag{5.3}$$

$$\varepsilon\left(D_q f\right)(\varepsilon)-\left(\frac{2+\tan(e^{i\theta})}{2}\right)f(\varepsilon)\neq 0. \tag{5.4}$$

Now, by using the relations

$$f(\varepsilon) = f(\varepsilon) * \frac{\varepsilon}{1-\varepsilon} \text{ and } \varepsilon\left(D_q f\right)(\varepsilon) = f(\varepsilon) * \frac{\varepsilon}{(1-\varepsilon)(1-q\varepsilon)},$$

the Eq (5.4) becomes

$$f(\varepsilon) * \frac{\varepsilon}{(1-\varepsilon)(1-q\varepsilon)} - \left(\frac{2+\tan(e^{i\theta})}{2}\right)\left(f(\varepsilon) * \frac{\varepsilon}{1-\varepsilon}\right) \neq 0.$$

After some simple calculations, we get

$$\begin{aligned} f(\varepsilon) * \left(\frac{\left(\left(2+\tan\left(e^{i\theta}\right)\right) q\varepsilon^2 - \tan\left(e^{i\theta}\right)\varepsilon\right)}{2(1-\varepsilon)(1-q\varepsilon)}\right) &\neq 0, \\ \frac{1}{2\varepsilon}\left(f(\varepsilon) * \frac{H\varepsilon^2 - G\varepsilon}{(1-\varepsilon)(1-q\varepsilon)}\right) &\neq 0, \end{aligned}$$

where H and G are given in (5.2), and thus the the necessary condition holds.
Conversely, assume that the condition in (5.1) satisfies, then $\frac{1}{\varepsilon} f(\varepsilon) \neq 0$, for all $\varepsilon \in \mathfrak{D}$. Let $K(\varepsilon) = \frac{\varepsilon\left(D_q f\right)(\varepsilon)}{f(\varepsilon)}$, which is regular in $\mathfrak{D}$ and $K(0) = 1$.
Also, assume that $f(\varepsilon) = 1 + \frac{\tan(\varepsilon)}{2}$, and from (5.3), $f(\partial\mathfrak{D}) \cap K(\varepsilon) = \emptyset$.
Therefore, the connected component $\mathbb{C} - f(\partial\mathfrak{D})$ containing the domain $K(\varepsilon)$ is connected as well. Given the univalence of "K" and the supposition that $f(0) = K(0) = 1$, it is evident that $K \prec f$, indicating that $f \in \mathcal{S}^*_{\tan}(q)$. □

6. Conclusions

In conclusion, leveraging the framework of q-calculus, we have introduced a novel class of q-starlike functions associated with the eight-shaped image domain, offering new insights into their geometric behavior. Our research successfully established results in differential subordination and derived sharp inequalities for the first three unknown coefficients of the Taylor series. Additionally, we provided precise solutions to the Fekete-Szegö problem and second-order Hankel determinants for this newly defined class, with the broader implications demonstrated through a series of corollaries.

The scope of these findings extends beyond the specific class of functions examined. Our methodology offers a flexible foundation for analyzing other image domains and subclasses within geometric function theory. This opens avenues for future work that could explore more general families of functions, potentially leading to new results in related areas, such as higher-order Hankel determinants, multi-variable quantum calculus, and further applications of q-differential operators. The results enhance the current understanding of q-starlike functions and provide a robust platform for future investigations into the more profound applications of quantum calculus in geometric and analytic function theory.

Author contributions

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Conflict of interest

The authors declare that they have no competing interests.

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