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Research article

Numerical investigation of systems of fractional partial differential equations by new transform iterative technique

Mariam Sultana[1], Muhammad Waqar[1], Ali Hasan Ali[2,3,4,*], Alina Alb Lupaş[5,*], F. Ghanim[6] and Zaid Ameen Abduljabbar[7]

[1] Department of Mathematical Sciences, Federal Urdu University of Arts, Sciences and Technology, Gulshan-e-Iqbal Campus, University Road, Karachi-75300, Karachi, Pakistan
[2] Department of Mathematics, College of Education for Pure Sciences, University of Basrah, Basrah 61001, Iraq
[3] Department of Business Management, Al-imam University College, 34011 Balad, Iraq
[4] Technical Engineering College, Al-Ayen University, Dhi Qar 64001, Iraq
[5] Department of Mathematics and Computer Science, University of Oradea, 1 Universitatii Street, 410087 14 Oradea, Romania
[6] Department of Mathematics, College of Sciences, University of Sharjah, Sharjah, United Arab Emirates
[7] Department of Computer Science, College of Education for Pure Sciences, University of Basrah, Basrah 61001, Iraq

* **Correspondence:** Email: ali.hasan@science.unideb.hu, alblupas@gmail.com.

Abstract: This research introduced a new method, the Aboodh Tamimi Ansari transform method ($(AT)^2$ method), for solving systems of linear and nonlinear fractional partial differential equations. The method combined the Aboodh transform method and the Tamimi Ansari method, allowing for the simultaneous solution of linear and nonlinear terms without restrictions. The Caputo sense was considered for fractional derivatives. The effectiveness of the proposed method was demonstrated through numerical solutions, graphical representations, and tabular data, showing strong agreement with exact solutions. The approach was deemed precise, easy to apply, and could be extended to address further challenges in fractional-order problems. Computational tasks were carried out using Mathematica 13.

Keywords: fractional calculus; Caputo derivative; linear and nonlinear; system of partial differential equations; Aboodh transform; Tamimi Ansari method
Mathematics Subject Classification: 65R10, 34A08, 35R11

1. Introduction

Partial differential equations of fractional order (FPDEs) have been widely employed and developed in fluid mechanics, engineering, and physics domains throughout the past few decades. In contrast to integer-order PDEs, FPDEs are better suited for representing complex phenomena and processes. Hence, the resolution of FPDEs becomes a significant issue. Currently, various approaches exist for solving FPDEs. Numerous analytical and numerical techniques are employed to get approximate solutions for FPDEs. A few of these techniques are the multistep generalised differential transform method [1], the Mikusiński operational calculus [2], the iterative Laplace transform (LT) method [3], the Fourier transform method [4], the Adomian decomposition method [5], the variational iteration method [6], the Homotopy analysis method [7], the spectral method with the finite difference method and the Galerkin finite element method [8], reduce differential transform method [9], the homotopy pertubation method [10], operational wavelet matrix method [11], iterative reproducing kernel [12], Elzaki transform method [13], double integral transform [14], Fourier sine transform [15], and others [16–18].

The Temimi and Ansari method (TAM), introduced in 2011, is a method that provides accurate results in various equations [19]. The TAM technique aims to integrate the several tools available to problem solvers, including analytical, symbolic, and numerical computation. Essentially, the method aims to simplify the issue by transforming it into a linear form. It then employs an iterative approach that relies on both analytical and numerical computations. It originated from the homotopy perturbation method (HAM) [20] and has been used in solving various types of equations. TAM was also influenced by the renowned fixed point iteration method, as utilized by the authors of [21], within the realm of function space. This approach can be characterized as a quasi-linear iterative method that utilizes the fixed point iteration method, implemented in function space. For details, see [21]. Al-Jawary et al. [22] used TAM to solve the Duffing equation with damping and undamping type equations, Tamimi et al. [21] solved coupled nonlinear differential equations, and Hussien et al. [23] solved Korteweg–De Vries (KDV) equations. Overall, TAM's capability to provide accurate results is evident in its applications. For more details, see [24–26].

The Aboodh transform (AT) [27–30] is a unique integral transform, similar to well-recognized transforms such as the LT and other integral transforms that are specified in the time domain for $t \geq 0$. The AT introduces a distinctive mathematical method that provides a different viewpoint and technique for dealing with mathematical expressions and issues in the time domain. This transform is currently recognized as a basic approach for solving linear differential equations but lacks the capability to handle nonlinear equations. Combining the AT with the Tamimi Ansari method allows for the resolution of both linear and nonlinear issues. The fractional derivative used in this study is in the Caputo scene. Caputo fractional derivatives are valuable tools in various scientific and engineering disciplines due to their unique advantages. These advantages include incorporating memory effects, having non-singular kernels, offering physical interpretations in applications like viscoelasticity, being essential for modeling fractional order systems, possessing good mathematical properties, facilitating the solution of fractional differential equations, generalizing traditional derivatives, and finding applications in control systems for improved performance. It makes it possible to include initial and boundary conditions in the problem formulation. It adheres to the classical calculus condition that the constant's derivative is zero. The derivative does not disappear at zero.

In order to provide an approximate solution to the system for PDEs with fractional order, Tamimi and Ansari's iterative methodology and the AT are used in this study. In this paper, we deal with both linear and nonlinear fractional order systems using the method mentioned above to suggest its potential applicability to other FPDEs. The TAM technique is a simple and effective way to find power series solutions for FPDEs. Unlike other series solution methods, it does not require linearization, perturbation, or discretization. The benefit of this approach is that it is easy to implement with minimal computing work. According to the authors of this article, no one has worked on this hybrid version of the TAM with the AT.

The research is organized as follows: The first part introduces and establishes the backdrop for the investigation. The second portion will provide fundamental definitions and properties from fractional calculus theory and the AT that will be used throughout the article. The third section introduces the $(AT)^2$ method for a system of fractional order as a basic principle for predicting solutions to FPDEs. The fourth part discusses the convergence of the suggested approach. In Section 5, the study utilizes the suggested approach to address linear and nonlinear systems of fractional order. Section 6 provides a summary of the study's findings and implications in the concluding remarks.

2. Preliminaries

Definition 2.1. *The fractional order derivative of a function $\rho(\zeta,t)$ in the Caputo sense, denoted by u, can be expressed as explained in reference [31]*

$$D_t^u\rho(\zeta,t) = \frac{1}{\Gamma(p-u)}\int_0^t (t-\mu)^{p-u-1}\rho^{(p)}(\zeta,\mu)d\mu \quad (p-1 < u \leq p).$$

Definition 2.2. *The LT of $\theta(t), t > 0$ is [32]*

$$L[\theta(t)] = H(s) = \int_0^\infty e^{-st}\theta(t)dt, \tag{2.1}$$

where s is real or complex.

Definition 2.3. *The inverse of LT of $\theta(t)$ is described as*

$$L^{-1}[H(s)] = \theta(t) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} e^{ts}H(s)ds, \tag{2.2}$$

where a is a real number.

Definition 2.4. *The AT of a function $\phi(t)$ of exponential order over the set of functions is described as [32]*

$$G = \left\{\phi : \phi(t) < Me^{p_j|t|} \text{ if } t \in (-1)^j \times [0,\infty), j = 1,2; (M, p_1, p_2 > 0)\right\},$$

is written as

$$\varkappa[\phi(t)] = \lambda(\varepsilon),$$

and defined as

$$\varkappa[\phi(t)] = \frac{1}{\varepsilon}\int_0^\infty e^{-\varepsilon t}\phi(t)dt = \lambda(\varepsilon), \qquad p_1 \leq \varepsilon \leq p_2, \tag{2.3}$$

from Eqs (2.1) and (2.3). It is clear that if s and ε are equal to unity, then (2.1) and (2.3) are identical; otherwise, the relationship between (2.1) and (2.3) is symmetrical.

Definition 2.5. *The inverse AT of $\phi(t), t \in (0, \infty)$ is described as [32, 33]*

$$\varkappa^{-1}[\lambda(\varepsilon)] = \phi(t) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \varepsilon e^{\varepsilon t} \lambda(\varepsilon) d\varepsilon.$$

Theorem 2.6. *Given that the linearity property of the AT is given by [33], let the AT of the functions $\phi_1(t)$ and $\phi_2(t)$ be $\lambda_1(\varepsilon)$ and $\lambda_2(\varepsilon)$, respectively. Then, for any real numbers δ_1 and δ_2, we have*

$$\varkappa\left[\delta_1\phi_1(t) \pm \delta_2\phi_2(t)\right] = \varkappa[\delta_1\phi_1(t)] \pm \varkappa[\delta_2\phi_2(t)] = \delta_1\lambda_1(\varepsilon) \pm \delta_2\lambda_2(\varepsilon),$$

where $\varkappa$ denotes the AT operator.

Theorem 2.7. *If $\phi(t)$ belongs to G, then AT of $\phi(t)$ is $\lambda(\varepsilon)$, and the LT of $L[\phi(t)]$ is $\theta(\varepsilon)$, then the following condition will be held [32]:*

$$\lambda(\varepsilon) = \frac{1}{\varepsilon}\theta(\varepsilon).$$

Theorem 2.8. *Let the AT of a function $\phi(t)$ in Caputo's sense be given as [31]:*

$$\varkappa[D_t^u\phi(t); \varepsilon] = \varepsilon^u \varkappa[\phi(t)] - \sum_{l=0}^{n-1} \frac{\phi(t)^l|_{t=0}}{\varepsilon^{2-u+l}}, \quad n-1 < u \leq n, n \in \mathbb{N},$$

then the AT of some elementary functions can be described as listed below in Table 1.

Table 1. AT and inverse AT of some elementary functions.

$\phi(t)$	$\varkappa[\phi(t)] = \lambda(\varepsilon)$	$\varkappa^{-1}[\lambda(\varepsilon)]$	$\phi(t)$
1	$\frac{1}{\varepsilon^2}$	$\frac{1}{\varepsilon^2}$	1
t	$\frac{1}{\varepsilon^3}$	$\frac{1}{\varepsilon^3}$	t
t^u	$\frac{\Gamma(u+1)}{\varepsilon^{u+2}}$	$\frac{\Gamma(u+1)}{\varepsilon^{u+2}}$	t^u
Sin(at)	$\frac{a}{\varepsilon(\varepsilon^2+a^2)}$	$\frac{a}{\varepsilon(\varepsilon^2+a^2)}$	Sin(at)

3. Methodology of Aboodh Tamimi Ansari transform method

In this section, we will discuss the idea of the $(AT)^2$ method to find the general solution of the FPDEs. The derivative is involved in the Caputo sense

$$D_t^u\phi(\zeta, \omega, t) + M\phi(\zeta, \omega, t) + N\phi(\zeta, \omega, t) = h(\zeta, \omega, t), \tag{3.1}$$

where $D_t^u = \frac{\partial^u}{\partial t^u}$ is the Caputo operator, M and N are linear and nonlinear differential operators, and h is the source function. If $u \in (0, 1]$, then $n = 1$. The initial condition is given below:

$$\phi(\zeta, \omega, 0) = k(\zeta, \omega, 0).$$

To obtain the following response, utilize the AT on Eq (3.1), followed by the application of the inverse AT

$$\phi(\zeta,\omega,t) = k(\zeta,\omega,0) + \varkappa^{-1}\left[\frac{1}{\varepsilon^{u}}\varkappa[h(\zeta,\omega,t)]\right] - \varkappa^{-1}\left[\frac{1}{\varepsilon^{u}}\left[\varkappa[M\phi(\zeta,\omega,t) + N\phi(\zeta,\omega,t)]\right]\right].$$

The solutions to the aforementioned equations can be determined using the iterative method of the TAM technique. So, the initial approximation is:

$$\phi_0(\zeta,\omega,t) = k(\zeta,\omega,0) + \varkappa^{-1}\left[\frac{1}{\varepsilon^{u}}\varkappa[h_(\zeta,\omega,t)]\right].$$

Similarly, in the same manner, the remaining iteration can be calculated for $n \geq 1$, and each iteration represents the approximate solution of the problem. After a similar computation, we get

$$\phi_n(\zeta,\omega,t) = \phi_0(\zeta,\omega,t) - \varkappa^{-1}\left[\frac{1}{\varepsilon^{u}}\left[\varkappa[M\phi_{n-1}(\zeta,\omega,t) + N\phi_{n-1}(\zeta,\omega,t)]\right]\right]. \tag{3.2}$$

When $n \to \infty$, then $\phi_n(\zeta,\omega,t)$ converges to the exact solution of Eq (3.1).

4. Convergence of the method

To show the convergence analysis of the $(AT)^2$ method, let us begin by introducing the following process for our proposed iterative method. We have the terms in these forms [34]

$$\begin{cases} z_0 = \phi_0(\zeta,\omega,t), \\ z_1 = \mathcal{F}[z_0], \\ z_2 = \mathcal{F}[z_0 + z_1], \\ . \\ . \\ . \\ z_{n+1} = \mathcal{F}[z_0 + z_1 + z_2 + \cdots + z_n]. \end{cases} \tag{4.1}$$

The operator $\mathcal{F}$ can be defined by

$$\mathcal{F}[z_k] = S_k - \sum_{i=0}^{k-1} z_i(\zeta,\omega,t), k = 1, 2 \ldots \tag{4.2}$$

The term S_k is the solution that occurred from the $(AT)^2$ method. From the Eq (3.2), we have $\phi(\zeta,\omega,t) = \lim_{n\to\infty}\phi_n(\zeta,\omega,t) = \sum_{i=0}^{\infty} z_n$. So by using Eq (4.2), we can get the following solution in a series form:

$$\phi(\zeta,\omega,t) = \sum_{i=0}^{\infty} z_i(\zeta,\omega,t). \tag{4.3}$$

According to this procedure for the $(AT)^2$ method, the sufficient conditions for convergence of this iterative technique are presented. The main results are stated in the following theorems.

Theorem 4.1. *Consider an operator $\mathcal{F}$ defined in (4.2) from a Hilbert space J to J. Then, the infinite series solution computed by the iterative procedure defined in Eq (3.2) converges to the exact solution of Eq (3.1), if there exists $0 < \beta < 1$ such that $\|\mathcal{F}[z_0 + z_1 + \cdots + z_{i+1}]\| \leq \beta \|\mathcal{F}[z_0 + z_1 + \cdots + z_i]\|$ (such that $\|z_{i+1}\| \leq \beta \|z_i\|$) $\forall i = 0, 1, 2, 3 \ldots$, [34, 35].*

This theorem is just a special case of Banach's fixed point theorem, which is a sufficient condition to study the convergence of the $(AT)^2$ method.

Proof. Define the sequence $\{S_n\}_{n=1}^{\infty}$ as

$$\begin{cases} S_0 = z_0 \\ S_1 = z_0 + z_1, \\ S_2 = z_0 + z_1 + z_2 \\ . \\ . \\ . \\ S_n = z_0 + z_1 + z_2 + \cdots + z_n, \end{cases} \tag{4.4}$$

and we show that $\{S_n\}_{n=1}^{\infty}$ is a Cauchy sequence in the Hilbert space J to J. For this purpose, consider

$$\|S_{n+1} - S_n\| = \|z_{n+1}\| \leq \beta \|z_n\| \leq \beta^2 \|z_{n-1}\| \leq \cdots \leq \beta^{n+1} \|z_0\|.$$

For every $n, l \in N$, $l \leq n$,

$$\begin{aligned} \|S_n - S_l\| &= \|(S_n - S_{n-1}) + (S_{n-1} - S_{n-2}) + \cdots + (S_{l+1} - S_l)\| \\ &\leq \|(S_n - S_{n-1})\| + \|(S_{n-1} - S_{n-2})\| + \cdots + \|(S_{l+1} - S_l)\| \\ &\leq \beta^n \|z_0\| + \beta^{n-1} \|z_0\| + \cdots + \beta^{l+1} \|z_0\| \\ &= \frac{1 - \beta^{n-l}}{1 - \beta} \beta^{l+1} \|z_0\|, \end{aligned} \tag{4.5}$$

and since $0 < \beta < 1$, we get

$$\lim_{n,l \to \infty} \|S_n - S_l\| = 0.$$

Therefore, $\{S_n\}_{n=1}^{\infty}$ is a Cauchy sequence in the Hilbert space J and it implies that series solution $\phi(\zeta, \omega, t) = \sum_{i=0}^{\infty} z_i(\zeta, \omega, t)$ is converged. This completes the proof of Theorem 4.1.

Theorem 4.2. *If the series solution $\phi(\zeta, \omega, t) = \sum_{i=0}^{\infty} z_i(\zeta, \omega, t)$ is convergent, then this series will represent the exact solution of the current problem [34, 35].*

Proof. See [35].

Theorem 4.3. *Suppose that the series solution $\sum_{i=0}^{\infty} z_i(\zeta, \omega, t)$ computed by the iterative procedure defined in Eq (3.2), converges to the exact solution of Eq (3.1). If the truncated series $\sum_{i=0}^{l} z_i(\zeta, \omega, t)$ is used as an approximation to the solution of the current problem, then the maximum error $E_l(\zeta, \omega, t)$ is estimated by [34, 35].*

$$E_l(\zeta, \omega, t) \leq \frac{1}{\beta - 1} \beta^{l+1} \|z_0\|.$$

Proof. From inequality (4.5), we have

$$\|S_n - S_l\| \leq \frac{1-\beta^{n-l}}{1-\beta}\beta^{l+1}\|z_0\|,$$

for $l \leq n$. Now, as $n \to \infty$, then $S_n \to \phi(\zeta, \omega, t)$. So,

$$\left\|\phi(\zeta, \omega, t) - \sum_{i=0}^{l} z_i\right\| \leq \frac{1-\beta^{n-l}}{1-\beta}\beta^{l+1}\|z_0\|.$$

Also, since $0 < \beta < 1$, we have $(1-\beta^{n-l}) < 1$. Therefore, the above inequality becomes,

$$\left\|\phi(\zeta, \omega, t) - \sum_{i=0}^{l} z_i\right\| \leq \frac{1}{1-\beta}\beta^{l+1}\|z_0\|.$$

This completes the proof of Theorem 4.3.

In short, Theorems 4.1 and 4.2 state that the solution obtained by the $(AT)^2$ method computed by the iterative procedure defined in Eq (3.2) converges to the exact solution under the condition: There exists $0 < \beta < 1$ such that $\|\mathcal{F}[z_0 + z_1 + \ldots z_{i+1}]\| \leq \beta\|\mathcal{F}[z_0 + z_1 + \ldots z_i]\|$, that is, $\|z_{i+1}\| \leq \beta\|z_i\|)$ $\forall i = 0, 1, 2, \ldots$. In another meaning, for each i, if we define the parameters,

$$\gamma_k = \begin{cases} \frac{\|z_{k+1}\|}{\|z_k\|}, & \|z_k\| \neq 0 \\ 0, & \|z_k\| = 0 \end{cases}$$

then the series solution $\sum_{i=0}^{\infty} z_i(\zeta, \omega, t)$ converges to the exact solution of (3.1), when $0 \leq \gamma_k < 1, \forall k = 0, 1, 2, \ldots$. Also, as in Theorem 4.3, the maximum truncation error is estimated to be $\left\|\phi(\zeta, \omega, t) - \sum_{i=0}^{l} z_i\right\| \leq \frac{1}{1-\gamma}\gamma^{l+1}\|z_0\|$, where $\gamma = \max\{\gamma_k, k = 0, 1, 2, \ldots, n\}$.

5. Implementation of the method

In this section, we discuss the efficiency of the iterative method by solving some linear and nonlinear systems of FPDEs.

Example 5.1. Consider the nonlinear system of FPDEs [36],

$$\begin{aligned} D_t^u\phi(\zeta, \omega, t) - \theta(\zeta, \omega, t)_\zeta\rho(\zeta, \omega, t)_\omega &= 1, \\ D_t^v\theta(\zeta, \omega, t) - \rho(\zeta, \omega, t)_\zeta\phi(\zeta, \omega, t)_\omega &= 5, \\ D_t^q\rho(\zeta, \omega, t) - \phi(\zeta, \omega, t)_\zeta\theta(\zeta, \omega, t)_\omega &= 5, \end{aligned} \tag{5.1}$$

with the initial conditions

$$\begin{aligned} \phi(\zeta, \omega, 0) &= \zeta + 2\omega, \\ \theta(\zeta, \omega, 0) &= \zeta - 2\omega, \\ \rho(\zeta, \omega, 0) &= -\zeta + 2\omega. \end{aligned}$$

To generate the following outcome, apply the AT to Eq (5.1), and then apply the inverse AT

$$\begin{aligned}
\phi(\zeta,\omega,t) &= \zeta + 2\omega + \frac{t^u}{\Gamma(u+1)} + \varkappa^{-1}\left[\frac{1}{\varepsilon^u}[\varkappa[\theta(\zeta,\omega,t)_\zeta \rho(\zeta,\omega,t)_\omega]]\right],\\
\theta(\zeta,\omega,t) &= \zeta - 2\omega + \frac{5t^v}{\Gamma(v+1)} + \varkappa^{-1}\left[\frac{1}{\varepsilon^v}[\varkappa[\rho(\zeta,\omega,t)_\zeta \phi(\zeta,\omega,t)_\omega]]\right],\\
\rho(\zeta,\omega,t) &= -\zeta + 2\omega + \frac{5t^q}{\Gamma(q+1)} + \varkappa^{-1}\left[\frac{1}{\varepsilon^q}[\varkappa[\phi(\zeta,\omega,t)_\zeta \theta(\zeta,\omega,t)_\omega]]\right].
\end{aligned}$$

By utilizing the TAM technique, we obtain

$$\begin{aligned}
\phi_0(\zeta,\omega,t) &= \zeta + 2\omega + \frac{t^u}{\Gamma(u+1)},\\
\theta_0(\zeta,\omega,t) &= \zeta - 2\omega + \frac{5t^v}{\Gamma(v+1)},\\
\rho_0(\zeta,\omega,t) &= -\zeta + 2\omega + \frac{5t^q}{\Gamma(q+1)}.
\end{aligned}$$

Similarly, in the same manner, the remaining iteration can be calculated for $n \geq 1$. Each iteration represents the approximate solution of the problem.

$$\begin{aligned}
\phi_n(\zeta,\omega,t) &= \zeta + 2\omega + \frac{t^u}{\Gamma(u+1)} + \varkappa^{-1}\left[\frac{1}{\varepsilon^u}[\varkappa[\theta_{n-1}(\zeta,\omega,t)_\zeta \rho_{n-1}(\zeta,\omega,t)_\omega]]\right],\\
\theta_n(\zeta,\omega,t) &= \zeta - 2\omega + \frac{5t^v}{\Gamma(u+1)} + \varkappa^{-1}\left[\frac{1}{\varepsilon^v}[\varkappa[\rho_{n-1}(\zeta,\omega,t)_\zeta \phi_{n-1}(\zeta,\omega,t)_\omega]]\right],\\
\rho_n(\zeta,\omega,t) &= -\zeta + 2\omega + \frac{5t^q}{\Gamma(q+1)} + \varkappa^{-1}\left[\frac{1}{\varepsilon^q}[\varkappa[\phi_{n-1}(\zeta,\omega,t)_\zeta \theta_{n-1}(\zeta,\omega,t)_\omega]]\right].
\end{aligned}$$

When $n \to \infty$, then $\phi_n(\zeta,\omega,t)$, $\theta_n(\zeta,\omega,t)$, and $\rho_n(\zeta,\omega,t)$ converge to the exact solution of Example 5.1.

$$\begin{aligned}
\phi_1(\zeta,\omega,t) &= \zeta + 2\omega + \frac{3t^u}{\Gamma(u+1)},\\
\theta_1(\zeta,\omega,t) &= \zeta - 2\omega + \frac{3t^v}{\Gamma(v+1)},\\
\rho_1(\zeta,\omega,t) &= -\zeta + 2\omega + \frac{3t^q}{\Gamma(q+1)},\\
\phi_2(\zeta,\omega,t) &= 0,\\
\theta_2(\zeta,\omega,t) &= 0,\\
\rho_2(\zeta,\omega,t) &= 0,
\end{aligned}$$

for u, v, and $q = 1$. The exact solution of the above system is

$$\begin{aligned}
\phi(\zeta,\omega,t) &= \zeta + 2\omega + 3t,\\
\theta(\zeta,\omega,t) &= \zeta - 2\omega + 3t,\\
\rho(\zeta,\omega,t) &= -\zeta + 2\omega + 3t.
\end{aligned}$$

Example 5.2. Consider the following system of linear FPDEs, [37],

$$D_t^u\phi(\zeta,t)-\theta(\zeta,t)_\zeta+\theta(\zeta,t)+\phi(\zeta,t)=0,$$
$$D_t^v\theta(\zeta,t)-\phi(\zeta,t)_\zeta+\theta(\zeta,t)+\phi(\zeta,t)=0, \tag{5.2}$$

with the initial conditions,

$$\phi(\zeta,0)=Sinh(\zeta),$$
$$\theta(\zeta,0)=Cosh(\zeta).$$

After applying AT and inverse AT, we get

$$\phi(\zeta,t)=Sinh(\zeta)-\varkappa^{-1}\left[\frac{1}{\varepsilon^u}[\varkappa[-\theta(\zeta,t)_\zeta+\theta(\zeta,t)+\phi(\zeta,t)]]\right],$$
$$\theta(\zeta,t)=Cosh(\zeta)-\varkappa^{-1}\left[\frac{1}{\varepsilon^v}[\varkappa[-\phi(\zeta,t)_\zeta+\theta(\zeta,t)+\phi(\zeta,t)]]\right].$$

According to the TAM technique, the solutions to the above equations can be found using the iterative procedure

$$\phi_0(\zeta,t)=Sinh(\zeta),$$
$$\theta_0(\zeta,t)=Cos(\zeta).$$

Likewise, the remaining iteration may be determined for $n\geq 1$. Each iteration corresponds to the approximate solution of the problem, and then the n^{th} approximate solution of the problem is given:

$$\phi_n(\zeta,t)=Sinh(\zeta)-\varkappa^{-1}\left[\frac{1}{\varepsilon^u}[\varkappa[-\theta_{n-1}(\zeta,t)_\zeta+\theta_{n-1}(\zeta,t)+\phi_{n-1}(\zeta,t)]]\right],$$
$$\theta_n(\zeta,t)=Cosh(\zeta)-\varkappa^{-1}\left[\frac{1}{\varepsilon^v}[\varkappa[-\phi_{n-1}(\zeta,t)_\zeta+\theta_{n-1}(\zeta,t)+\phi_{n-1}(\zeta,t)]]\right].$$

When $n\to\infty$, the nth approximate solutions of Example 5.2 for particular values of $u,v=1$, converge to the exact solution $\phi(\zeta,t)=Sinh(\zeta-t)$, $\theta(\zeta,t)=Cosh(\zeta-t)$.

$$\phi_1(\zeta,t)=Sinh(\zeta)-\frac{t^uCosh(\zeta)}{\Gamma(1+u)},$$
$$\theta_1(\zeta,t)=Cosh(\zeta)-\frac{t^vSinh(\zeta)}{\Gamma(1+v)},$$
$$\phi_2(\zeta,t)=Sinh(\zeta)+\frac{t^{u+v}Sinh(\zeta)}{\Gamma(1+u+v)}-\frac{t^uCosh(\zeta)}{\Gamma(1+u)}+\frac{t^{2u}Cosh(\zeta)}{\Gamma(1+2u)}-\frac{t^{u+v}Cosh(\zeta)}{\Gamma(1+u+v)},$$
$$\theta_2(\zeta,t)=Cosh(\zeta)+\frac{t^{u+v}Cosh(\zeta)}{\Gamma(1+u+v)}-\frac{t^vSinh(\zeta)}{\Gamma(1+v)}-\frac{t^{u+v}Sinh(\zeta)}{\Gamma(1+u+v)}+\frac{2t^{2v}Sinh(\zeta)}{\Gamma(1+2v)}.$$
$$\begin{aligned}\phi_3(\zeta,t)=&Sinh(\zeta)+\frac{t^{u+v}Sinh(\zeta)}{\Gamma(1+u+v)}+\frac{t^{2u+v}Sinh(\zeta)}{\Gamma(1+2u+v)}-\frac{t^{u+2v}Sinh(\zeta)}{\Gamma(1+u+2v)}-\frac{t^uCosh(\zeta)}{\Gamma(1+u)}\\&+\frac{t^{2u}Cosh(\zeta)}{\Gamma(1+2u)}-\frac{t^{3u}Cosh(\zeta)}{\Gamma(1+3u)}-\frac{t^{u+v}Cosh(\zeta)}{\Gamma(1+u+v)}-\frac{t^{2u+v}Cosh(\zeta)}{\Gamma(1+2u+v)}+\frac{t^{u+2v}Cosh(\zeta)}{\Gamma(1+u+2v)},\end{aligned}$$

$$\theta_3(\zeta,t) = Cosh(\zeta) + \frac{t^{u+v}Cosh(\zeta)}{\Gamma(1+u+v)} - \frac{t^{2u+v}Cosh(\zeta)}{\Gamma(1+2u+v)} + \frac{t^{u+2v}Cosh(\zeta)}{\Gamma(1+u+2v)} - \frac{t^{v}S\,inh(\zeta)}{\Gamma(u+v)}$$
$$- \frac{t^{u+v}S\,inh(\zeta)}{\Gamma(1+u+v)} + \frac{t^{2u+v}S\,inh(\zeta)}{q(1+2u+v)} + \frac{t^{2v}S\,inh(\zeta)}{\Gamma(1+2v)} - \frac{t^{u+2v}S\,inh(\zeta)}{\Gamma(1+u+2v)} - \frac{t^{3v}S\,inh(\zeta)}{\Gamma(1+3v)}.$$

Moreover, the remaining iteration for $n \geq 4$ can be found in the same manner. So, the 3^{rd} iteration represents the approximate solutions of Example 5.2.

Example 5.3. Consider the following system of nonlinear FPDEs, [37],

$$\begin{aligned} &D_t^u\phi(\zeta,\omega,t) + \theta(\zeta,\omega,t)_\zeta\rho(\zeta,\omega,t)_\omega - \theta(\zeta,\omega,t)_\omega\rho(\zeta,\omega,t)_\zeta = -\phi(\zeta,\omega,t),\\ &D_t^v\theta(\zeta,\omega,t) + \phi(\zeta,\omega,t)_\zeta\rho(\zeta,\omega,t)_\omega + \phi(\zeta,\omega,t)_\omega\rho(\zeta,\omega,t)_\zeta = \theta(\zeta,\omega,t),\\ &D_t^q\rho(\zeta,\omega,t) + \phi(\zeta,\omega,t)_\zeta\theta(\zeta,\omega,t)_\omega + \theta(\zeta,\omega,t)_\omega\theta(\zeta,\omega,t)_\zeta = \rho(\zeta,\omega,t), \end{aligned} \tag{5.3}$$

with the initial conditions

$$\begin{aligned} \phi(\zeta,\omega,0) &= e^{\zeta+\omega},\\ \theta(\zeta,\omega,0) &= e^{\zeta-\omega},\\ \rho(\zeta,\omega,0) &= e^{-\zeta+\omega}. \end{aligned}$$

Applying AT, and using the differential property of AT, and also applying inverse AT, we get

$$\begin{aligned} \phi(\zeta,\omega,t) &= e^{\zeta+\omega} + \varkappa^{-1}\left[\frac{1}{\varepsilon^u}[\varkappa[-\phi(\zeta,\omega,t) - \theta(\zeta,\omega,t)_\zeta\rho(\zeta,\omega,t)_\omega + \theta(\zeta,\omega,t)_\omega\rho(\zeta,\omega,t)_\zeta]]\right],\\ \theta(\zeta,\omega,t) &= e^{\zeta-\omega} + \varkappa^{-1}\left[\frac{1}{\varepsilon^v}[\varkappa[\theta(\zeta,\omega,t) - \phi(\zeta,\omega,t)_\zeta\rho(\zeta,\omega,t)_\omega - \phi(\zeta,\omega,t)_\omega\rho(\zeta,\omega,t)_\zeta]]\right],\\ \rho(\zeta,\omega,t) &= e^{-\zeta+\omega} + \varkappa^{-1}\left[\frac{1}{\varepsilon^q}[\varkappa[\rho(\zeta,\omega,t) - \phi(\zeta,\omega,t)_\zeta\rho(\zeta,\omega,t)_\omega - \phi(\zeta,\omega,t)_\omega\theta(\zeta,\omega,t)_\zeta]]\right]. \end{aligned}$$

According to the TAM approach, the solution of the aforementioned system may be obtained by an iterative procedure:

$$\begin{aligned} \phi_0(\zeta,\omega,t) &= e^{\zeta+\omega},\\ \theta_0(\zeta,\omega,t) &= e^{\zeta-\omega},\\ \rho_0(\zeta,\omega,t) &= e^{-\zeta+\omega}. \end{aligned}$$

Similarly, in the same manner, the remaining iteration can be calculated for $n \geq 1$. Each iteration represents the approximate solution of the problem

$$\begin{aligned} &\phi_n(\zeta,\omega,t) = e^{\zeta-\omega} +\\ &\varkappa^{-1}\left[\frac{1}{\varepsilon^u}\varkappa[-\phi_{n-1}(\zeta,\omega,t) - \theta_{n-1}(\zeta,\omega,t)_\zeta\rho_{n-1}(\zeta,\omega,t)_\omega + \theta_{n-1}(\zeta,\omega,t)_\omega\rho_{n-1}(\zeta,\omega,t)_\zeta]\right],\\ &\theta_n(\zeta,\omega,t) = e^{\zeta-\omega} +\\ &\varkappa^{-1}\left[\frac{1}{\varepsilon^v}\varkappa[\theta_{n-1}(\zeta,\omega,t) - \phi_{n-1}(\zeta,\omega,t)_\zeta\rho_{n-1}(\zeta,\omega,t)_\omega - \phi_{n-1}(\zeta,\omega,t)_\omega\rho_{n-1}(\zeta,\omega,t)_\zeta]\right], \end{aligned}$$

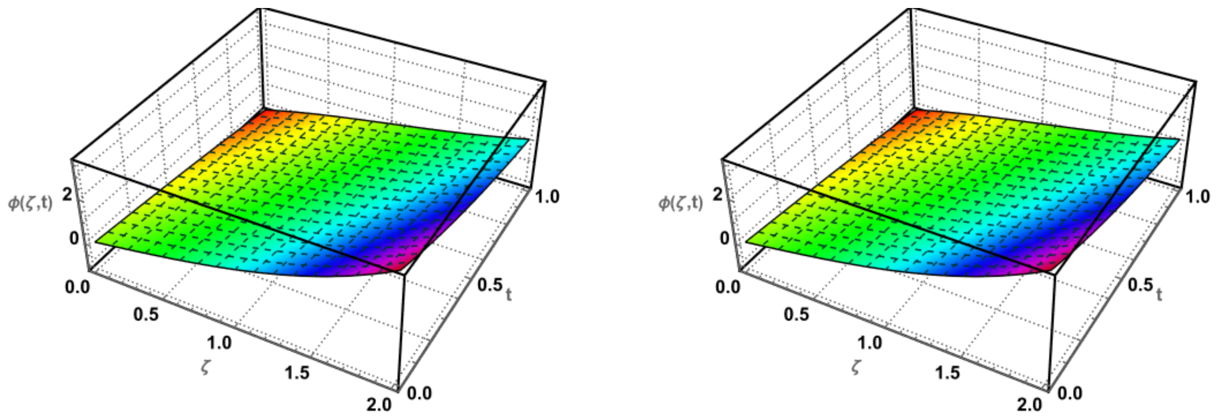
$$\rho_n(\zeta,\omega,t) = e^{-\zeta+\omega}+$$
$$\varkappa^{-1}\left[\frac{1}{\varepsilon^q}\varkappa[\rho_{n-1}(\zeta,\omega,t) - \phi_{n-1}(\zeta,\omega,t)_\zeta\rho_{n-1}(\zeta,\omega,t)_\omega - \phi_{n-1}(\zeta,\omega,t)_\omega\theta_{n-1}(\zeta,\omega,t)_\zeta]\right].$$

When $n \to \infty$, the nth iterative approximate solutions of Example 5.3 converge to exact solutions for the special case of u, v, and $q = 1$. $\phi(\zeta,\omega,t) = e^{\zeta+\omega-t}$, $\theta(\zeta,\omega,t) = e^{\zeta-\omega+t}$, and $\rho(\zeta,\omega,t) = e^{-\zeta+\omega+t}$.

$$\begin{aligned}
\phi_1(\zeta,\omega,t) &= e^{\zeta+\omega} - \frac{e^{\zeta+\omega}t^u}{\Gamma(1+u)},\\
\theta_1(\zeta,\omega,t) &= e^{\zeta-\omega} + \frac{e^{\zeta-\omega}t^v}{\Gamma(1+v)},\\
\rho_1(\zeta,\omega,t) &= e^{-\zeta+\omega} + \frac{e^{-\zeta+\omega}t^q}{\Gamma(1+q)},\\
\phi_2(\zeta,\omega,t) &= e^{\zeta+\omega} - \frac{e^{\zeta+\omega}t^u}{\Gamma(1+u)} + \frac{e^{\zeta+\omega}t^{2u}}{\Gamma(1+2u)},\\
\theta_2(\zeta,\omega,t) &= e^{\zeta-\omega} + \frac{e^{\zeta-\omega}t^v}{\Gamma(1+v)} + \frac{e^{\zeta-\omega}t^{2v}}{\Gamma(1+2v)},\\
\rho_2(\zeta,\omega,t) &= e^{-\zeta+\omega} + \frac{e^{-\zeta+\omega}t^q}{\Gamma(1+q)} + \frac{e^{-\zeta+\omega}t^{2q}}{\Gamma(1+2q)},\\
\phi_3(\zeta,\omega,t) &= e^{\zeta+\omega} - \frac{e^{\zeta+\omega}t^u}{\Gamma(1+u)} + \frac{e^{\zeta+\omega}t^{2u}}{\Gamma(1+2u)} - \frac{e^{\zeta+\omega}t^{3u}}{\Gamma(1+3u)},\\
\theta_3(\zeta,\omega,t) &= e^{\zeta-\omega} + \frac{e^{\zeta-\omega}t^v}{\Gamma(1+v)} + \frac{e^{\zeta-\omega}t^{2v}}{\Gamma(1+2v)} + \frac{e^{\zeta-\omega}t^{3v}}{\Gamma(1+3v)},\\
\rho_3(\zeta,\omega,t) &= e^{-\zeta+\omega} + \frac{e^{-\zeta+\omega}t^q}{\Gamma(1+q)} + \frac{e^{-\zeta+\omega}t^{2q}}{q(1+2q)} + \frac{e^{-\zeta+\omega}t^{3q}}{\Gamma(1+3q)}.
\end{aligned}$$

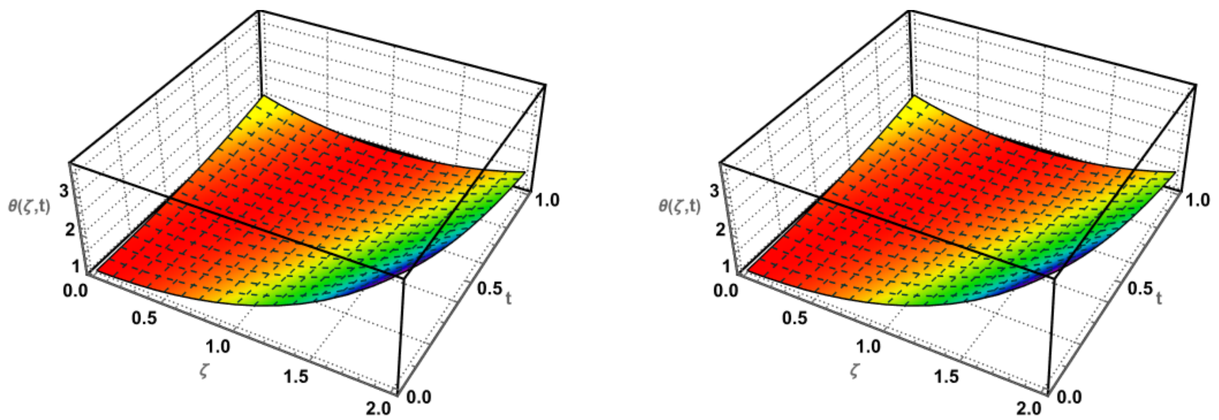
Furthermore, the remaining iteration for $n \geq 4$ may be found in the same way. The 3^{rd} iteration of the $(AT)^2$ method is used to illustrate the approximate answer of Example 5.3.

Figures 1 and 2 show the exact and estimated solutions of Example 5.2 in 3D graphs. The data suggests a strong correlation between the estimated approximate solution and the exact answer. Figures 3 and 4 show the 3D and 2D plots achieved during the 6^{th} iteration of the fractional order solution in Example 5.2 at various values of u and v. Plot the non-integer curve solutions in the ζ, t plane for the 6^{th} iterative solution of Example 5.2 at $t = 0.25$, varying values of u and v. These examples demonstrate that a non-integer curve in the ζ, t plane will converge to an integer curve. Figure 5 displays 3D error plots of $\phi(\zeta,t)$ and $\theta(\zeta,t)$ for ζ, t values ranging from 0 to 1. Figure 6 exhibits surface plots comparing the exact and 6^{th} iterative estimated solutions of $\phi(\zeta,\omega,t)$ at u, v, and $q = 1$, demonstrating the efficiency and convergence of the $(AT)^2$ technique. Increasing the number of iterations may improve the estimated answer's accuracy, as seen in these graphs. Figures 6–8 display 3D surface plots comparing the exact and 6^{th} iterative estimated solutions of $\phi(\zeta,\omega,t)$, $\theta(\zeta,\omega,t)$, $\rho(\zeta,\omega,t)$ at u, v, and $q = 1$. Figures 9 and 10 show the 3D and 2D plots achieved during the 6^{th} iteration of the fractional order solution in Example 5.3 at various values of u, v, and q. Figure 11 shows the 3D error plot of Example 5.3. Itillustrates the efficiency and convergence of the $(AT)^2$ approach. Table 1 displays the AT of some simple functions. The point-wise error between the analytical solution and the 6^{th} iteration calculated for Examples 5.2 and 5.3 at $t = 0.2$ is displayed in Tables 2 and 3. A comparison with the solution derived using the HAM is also provided. Table 4 shows some important abbreviations which are used in this article.



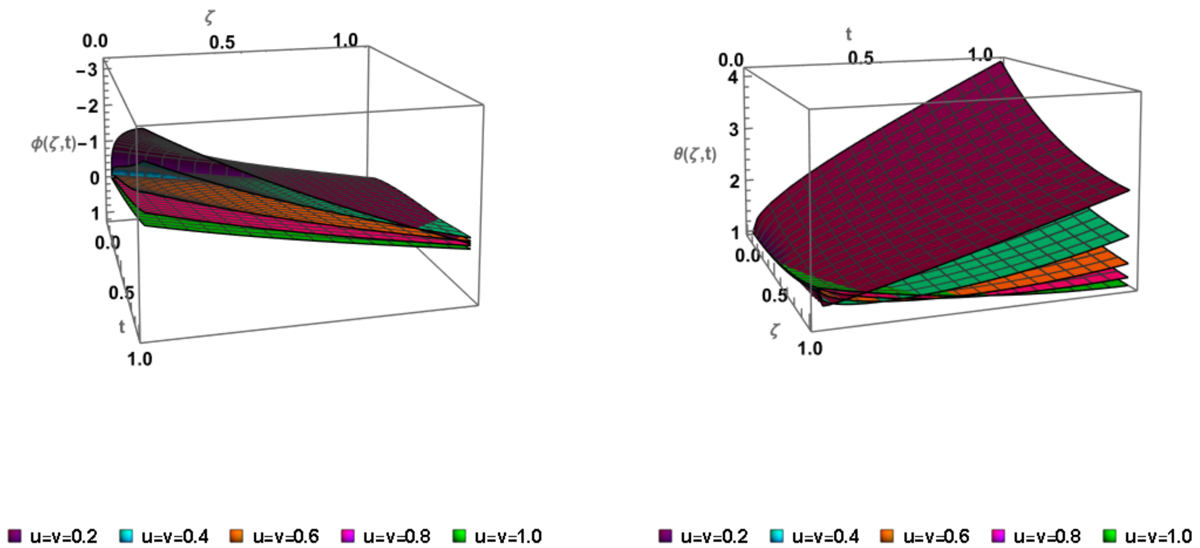
(a) 3D Plot of exact solution **(b)** 3D approximate plot for $n = 6$

Figure 1. Exact and approximate solution of $\phi(\zeta, t)$ by $(AT)^2$ method.



(a) 3D Plot of exact solution **(b)** 3D approximate plot for $n = 6$

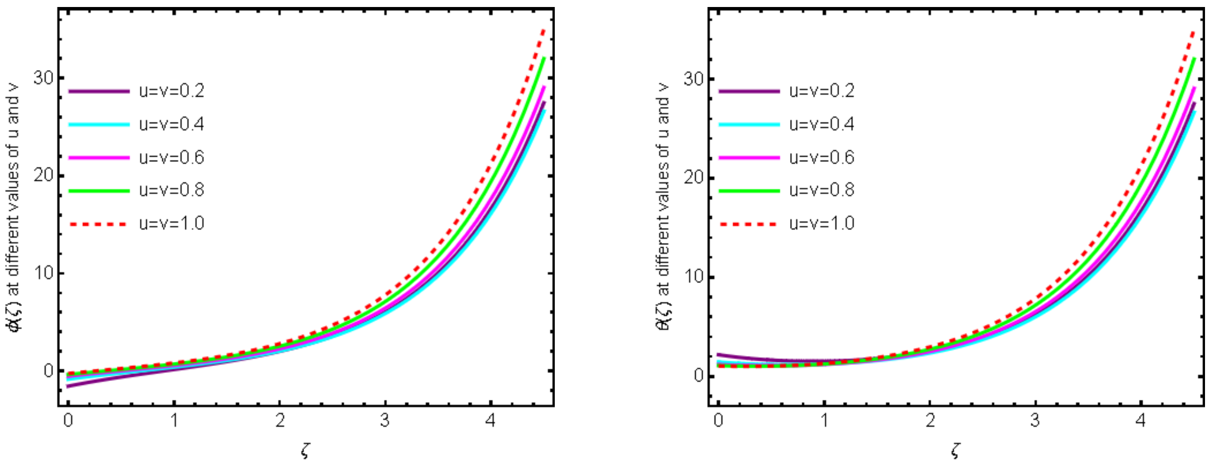
Figure 2. Exact and approximate solution of $\theta(\zeta, t)$ by $(AT)^2$ method.



(a) 3D graph of $\phi(\zeta, t)$ at different values of u and v for $n = 6$

(b) 3D graph of $\theta(\zeta, t)$ at different values of u and v for $n = 6$

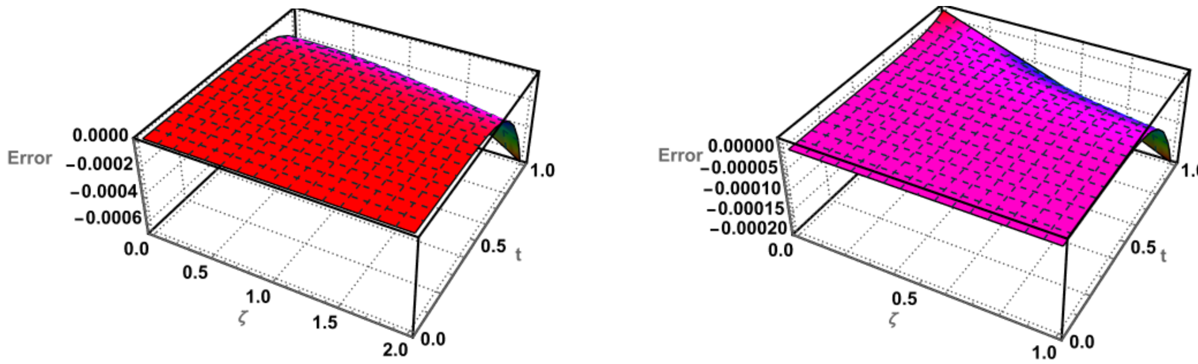
Figure 3. The non-integer 3D plots of the 6^{th} iterative estimated solution of Example 5.2.



(a) 2D graph of $\phi(\zeta)$ at different values of u and v for $n = 6$

(b) 2D graph of $\theta(\zeta)$ at different values of u and v for $n = 6$

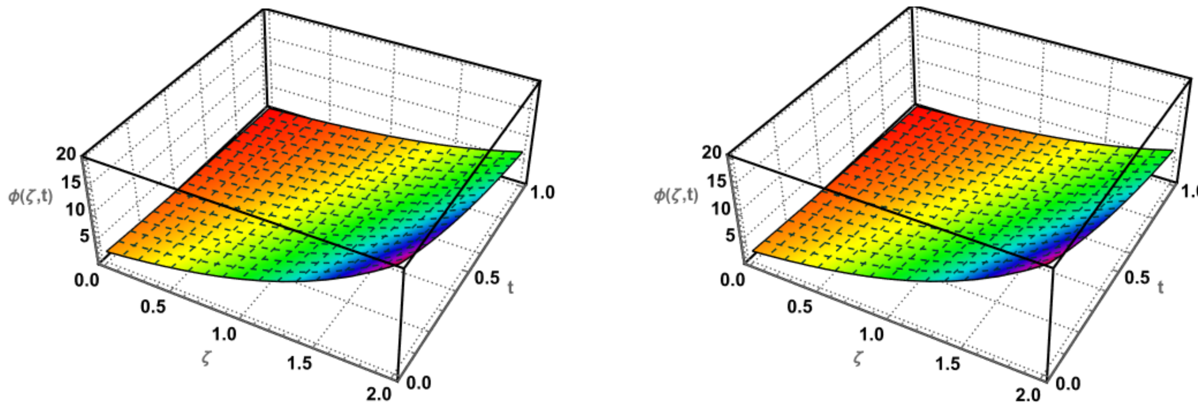
Figure 4. The non-integer curve solutions in ζ, t plane of the 6^{th} iterative solution of Example 5.2 at t=0.25 for different values of u and v. These also illustrate that a non-integer curve in the ζ, t plane converges to an integer curve.



(a) Error plot of $\phi(\zeta, t)$ for $n = 6$

(b) Error plot of $\theta(\zeta, t)$ for $n = 6$

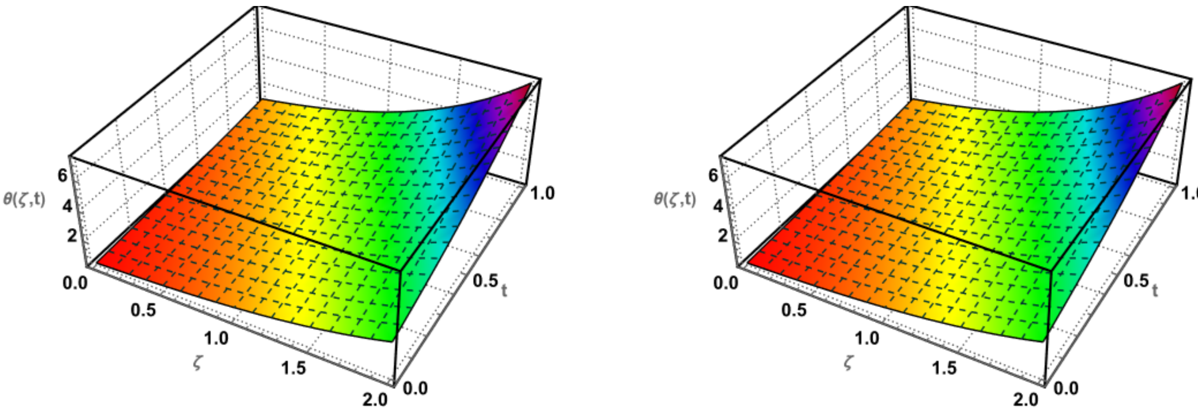
Figure 5. The 3D error plots of $\phi(\zeta, t)$, $\theta(\zeta, t)$ and when $\zeta, t \in [0, 1]$ for Example 5.2.



(a) 3D Plot of exact solution

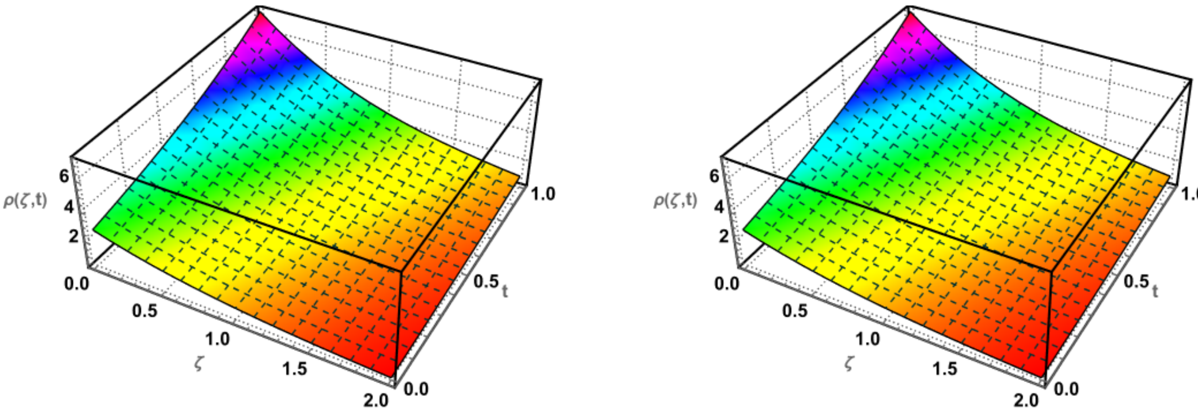
(b) 3D approximate plot for $n = 6$

Figure 6. The 3D surface plots of exact and 6^{th} iterative estimated solutions of $\phi(\zeta, \omega, t)$ at u, v, and $q = 1$, which also shows the efficiency and convergence of the $(AT)^2$ method.



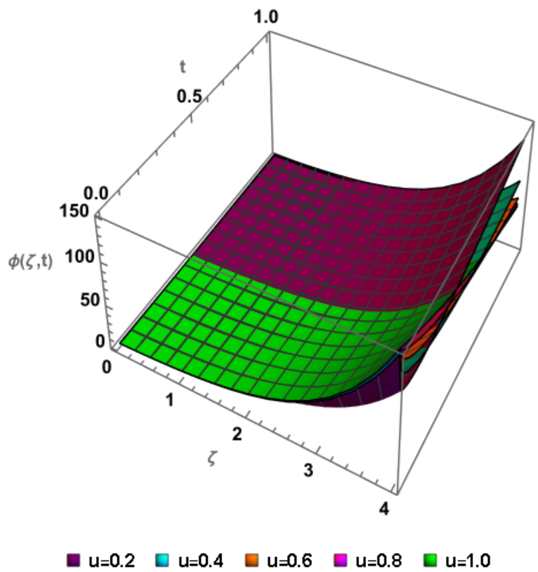
(a) 3D Plot of exact solution **(b)** 3D approximate plot for $n = 6$

Figure 7. The 3D surface plots of exact and 6^{th} iterative estimated solutions of $\theta(\zeta, \omega, t)$ at u, v, and $q = 1$, which also shows the efficiency and convergence of the $(AT)^2$ method.

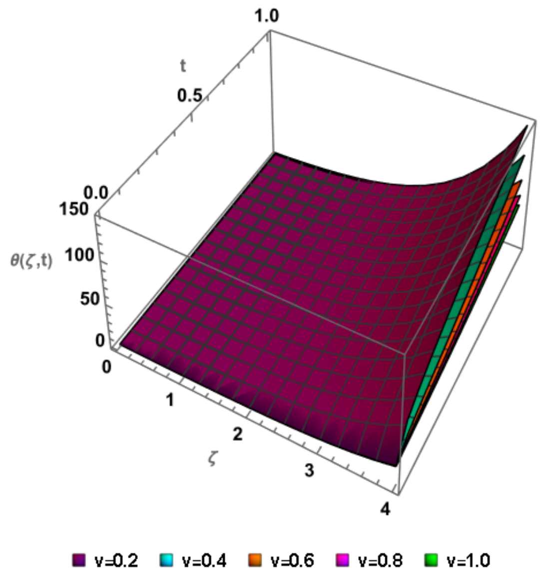


(a) 3D Plot of exact solution **(b)** 3D approximate plot for $n = 6$

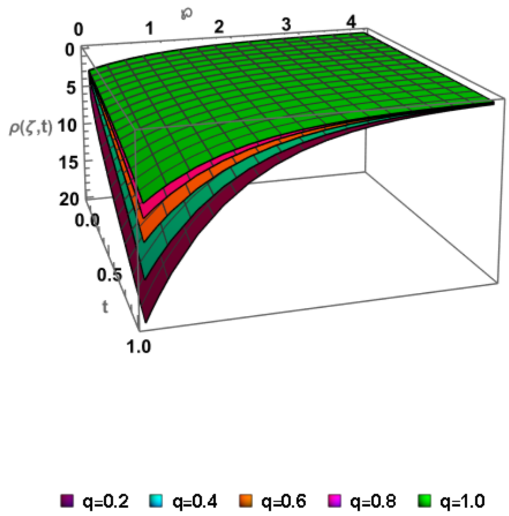
Figure 8. The 3D surface plots of exact and 6^{th} iterative estimated solutions of $\rho(\zeta, \omega, t)$ at u, v, and $q = 1$, which also shows the efficiency and convergence of the $(AT)^2$ method.



(a) 3D graph of $\phi(\zeta, t)$ at different values of u for $n = 6$

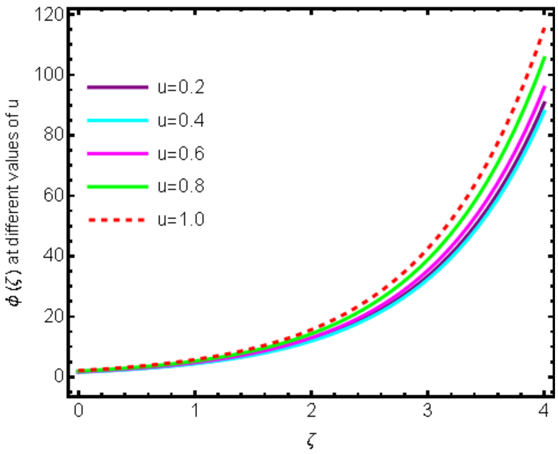


(b) 3D graph of $\theta(\zeta, t)$ at different values of v for $n = 6$

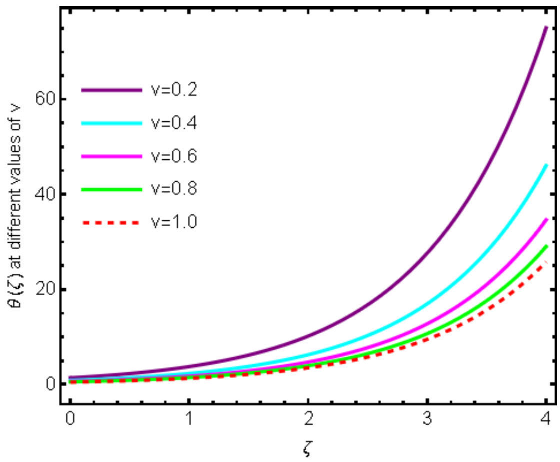


(c) 3D graph of $\rho(\zeta, t)$ at different values of q for $n = 6$

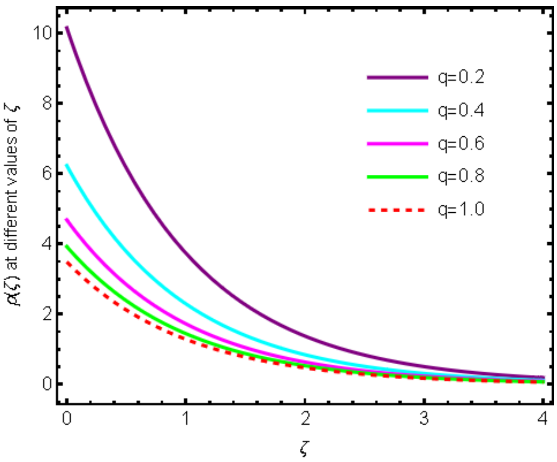
Figure 9. The non-integer order surface plots of the 6^{th} iterative estimated solution of Example 5.3 at different values of u, v, and q.



(a) 2D graph of $\phi(\zeta)$ at different values of u for $n = 6$

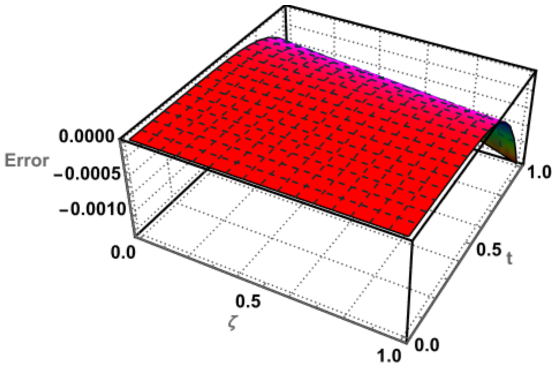


(b) 2D graph of $\theta(\zeta)$ at different values of v for $n = 6$

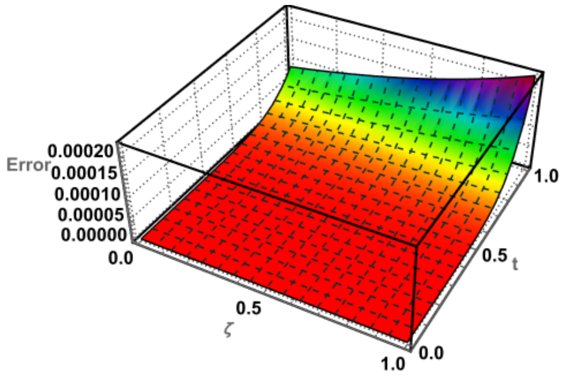


(c) 2D graph of $\rho(\zeta)$ at different values of q for $n = 6$

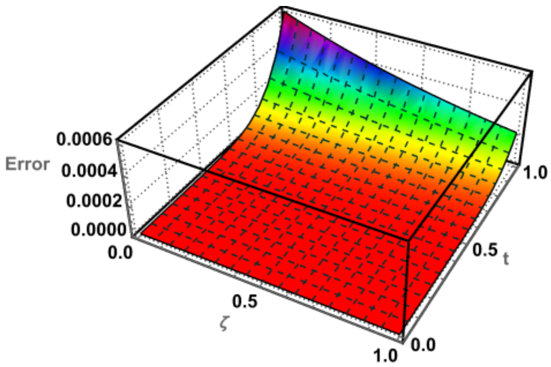
Figure 10. The non-integer order solutions in ζ t plane of the 6^{th} iterative approximate solution of Example 5.3, at $t = 0.25$ for different values u, v, and q. These also illustrate that a non-integer curve in the ζ, t plane converges to an integer curve.



(a) Error plot of $\phi(\zeta, t)$ for $n = 6$



(b) Error plot of $\theta(\zeta, t)$ for $n = 6$



(c) Error plot of $\rho(\zeta, t)$ for $n = 6$

Figure 11. The 3D error plots of Example 5.3.

Table 2. The point-wise error of the 6^{th} iterative estimated and exact solution of Example 5.2 at $t = 0.2$ also shows the comparison with the solution obtained by the HAM [38].

ζ	$\phi(\zeta, t)$	HAM [38]	$\theta(\zeta, t)$	HAM [38]
0.0	2.5410×10^{-9}	1.21207×10^{-8}	6.3520×10^{-11}	8.89524×10^{-8}
0.1	2.5474×10^{-9}	2.18216×10^{-8}	1.9069×10^{-10}	8.91432×10^{-8}
0.2	2.5793×10^{-9}	5.59822×10^{-8}	4.4681×10^{-10}	9.02258×10^{-8}
0.3	2.6369×10^{-9}	9.07031×10^{-8}	7.0741×10^{-10}	9.22116×10^{-8}
0.4	2.7210×10^{-9}	1.26332×10^{-8}	9.7509×10^{-10}	9.51202×10^{-8}
0.5	2.832×10^{-9}	1.63225×10^{-8}	1.2525×10^{-9}	9.89809×10^{-8}
0.6	2.9719×10^{-9}	2.01752×10^{-7}	1.5425×10^{-9}	1.03832×10^{-7}
0.7	3.1413×10^{-9}	2.42298×10^{-7}	1.8474×10^{-9}	1.09723×10^{-7}
0.8	3.3421×10^{-9}	2.85268×10^{-7}	2.1718×10^{-9}	1.16711×10^{-7}
0.9	3.5764×10^{-9}	3.31094×10^{-7}	2.5174×10^{-9}	1.24868×10^{-7}
1.0	3.8464×10^{-9}	3.80234×10^{-7}	2.8882×10^{-9}	1.34274×10^{-7}

Table 3. The point-wise error of the 6^{th} iterative estimated and exact solution of Example 5.3 at $t = 0.2$ also shows the comparison with the solution obtained by the HAM [38].

ζ	$\phi(\zeta,t)$	HAM [38]	$\theta(\zeta,t)$	HAM [38]	$\rho(\zeta,t)$	HAM [38]
0.1	7.44×10^{-9}	2.59×10^{-7}	1.05×10^{-10}	6.40×10^{-9}	2.25×10^{-7}	2.48×10^{-7}
0.2	8.22×10^{-9}	2.86×10^{-7}	1.17×10^{-9}	3.71×10^{-8}	5.79×10^{-9}	2.03×10^{-7}
0.3	9.09×10^{-9}	3.17×10^{-7}	1.29×10^{-9}	4.11×10^{-8}	5.24×10^{-9}	1.84×10^{-7}
0.4	1.00×10^{-8}	3.50×10^{-7}	1.42×10^{-9}	4.54×10^{-8}	4.74×10^{-9}	1.66×10^{-7}
0.5	1.11×10^{-8}	3.87×10^{-7}	1.57×10^{-9}	5.02×10^{-8}	4.29×10^{-9}	1.50×10^{-7}
0.6	1.22×10^{-8}	4.27×10^{-7}	1.74×10^{-9}	5.54×10^{-8}	3.88×10^{-9}	1.36×10^{-7}
0.7	1.35×10^{-8}	4.73×10^{-7}	1.92×10^{-9}	6.13×10^{-8}	3.51×10^{-9}	1.23×10^{-7}
0.8	1.49×10^{-8}	5.22×10^{-7}	2.13×10^{-9}	6.77×10^{-8}	3.18×10^{-9}	1.11×10^{-7}
0.9	1.65×10^{-8}	5.77×10^{-7}	2.35×10^{-9}	7.49×10^{-8}	2.87×10^{-9}	1.01×10^{-7}
1.0	1.83×10^{-8}	6.38×10^{-7}	2.60×10^{-9}	9.14×10^{-8}	8.27×10^{-8}	2.6×10^{-9}

Table 4. Some important abbreviations.

$(AT)^2$ method	Aboodh Tamimi Ansari transform method
FPDE	Fractional Partial Differential Equation
TAM	Tamimi and Ansai method
HPM	Homotopy Perturbation method
LT	Laplace transform
AT	Aboodh transform
HAM	Homotopy Analysis method

6. Conclusions

The convergence and efficiency of the new iterative technique Aboodh Tamimi Ansari Transform Method ($(AT)^2$ method) is shown for solving three systems of FPDEs, and the results are shown in the form of tables and graphs. Our findings demonstrate that this approach effectively decreases the computing workload in comparison to traditional methods while still preserving the precision of the numerical outcomes. Furthermore, it is evident that this method possesses a distinct advantage over the Adomian decomposition and homotopy analysis approaches in the context of solving nonlinear problems. This advantage stems from the fact that the Aboodh Tamimi Ansari Transform Method does not require the preliminary computation of any polynomials. Therefore, it can be inferred that the $(AT)^2$ method is a valuable enhancement to the current numerical methodologies and holds the potential for extensive utilization.

Author contributions

Mariam Sultana and Muhammad Waqar: Conceptualization; Muhammad Waqar and Zaid Ameen Abduljabbar: Data curation; Alina Alb Lupaş and F. Ghanim: Formal analysis; Alina Alb Lupaş:

Funding acquisition; Mariam Sultana and Ali Hasan Ali: Methodology; Ali Hasan Ali: Project administration, software, visualization; Mariam Sultana and F. Ghanim: Supervision; Alina Alb Lupaş and Zaid Ameen Abduljabbar: Validation; Mariam Sultana, Muhammad Waqar and Ali Hasan Ali: Writing–original draft; Alina Alb Lupaş, F. Ghanim and Zaid Ameen Abduljabbar: Writing–review and editing. All authors have read and approved the final version of the manuscript for publication.

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Conflict of interest

The authors declare no conflict of interest.

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