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Research article

Lie (Jordan) $\sigma-$centralizer at the zero products on generalized matrix algebra

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Abstract: Given a unital commutative ring $\mathscr{R}$, $(\mathscr{A},\mathscr{B})$ and $(\mathscr{B},\mathscr{A})$ are bimodules of $\mathscr{M}$ and $\mathscr{N}$, respectively, where $\mathscr{A},\mathscr{B}$ are unitals $\mathscr{R}-$algebras. The $\mathscr{R}-$algebra $\mathscr{G} = \mathscr{G}(\mathscr{A},\mathscr{M},\mathscr{N},\mathscr{B})$ is a generalized matrix algebra described by the Morita context $(\mathscr{A},\mathscr{B},\mathscr{M},\mathscr{N},\zeta_{\mathscr{M}\mathscr{N}},\chi_{\mathscr{N}\mathscr{M}})$. The present study investigated the structure of Lie (Jordan) $\sigma-$centralizers at the zero products on order two generalized matrix algebra and established that each Jordan $\sigma-$centralizer at the zero products is a $\sigma-$centralizer at the zero product on order two generalized matrix algebra. We also provided sufficient and necessary conditions under which a Lie $\sigma-$centralizer at the zero product is proper on an order two generalized matrix algebra.

Keywords: generalized matrix algebra; Lie (Jordan) $\sigma-$centralizer; $\sigma-$centralizer
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1. Brief historical development and motivation

Since the mid-20^{th} century, the researchers have shifted their interest from classical matrix algebra to more general algebraic structure. They started exploring the properties of matrices in terms of general algebraic structures such as nonassociative algebras and rings. This resulted in the establishment of generalized matrix algebras with a binary operation (zero product) instead of the usual matrix multiplication. The study of Lie (Jordan) centralizers at zero products in generalized matrix algebras has its roots in the broader field of matrix theory and algebraic structures, as well as Lie algebras, operator theory, and nonassociative algebras.

The zero product operation introduces new challenges in understanding matrix behavior, leading the researchers to extend the concept of Lie (Jordan) centralizers to generalized matrix algebras, operator algebras, and alternative algebras. The motivation behind the study of Lie (Jordan) centralizers at zero products reflects the broader evolution of algebraic structures and their applications. The exploration of these centralizers has contributed to a deeper understanding of generalized matrix algebra. The analysis of centralizers and related concepts is essential for knowing the structure and properties of noncommutative multiplication, while Lie (Jordan) centralizers play a role in characterizing certain types of sub-algebras and their properties (see [22, 23, 28, 29]). The Lie (Jordan) centralizer at zero products provides a framework for studying the algebraic properties and structures of matrices in generalized matrix algebras (see [16–19]).

Numerous researchers have been diligently investigating over the past few years the behavior of Lie (Jordan) centralizers for matrix algebras, triangular rings, nest algebras, and alternating algebras (see [1, 11–14, 26]). In 2021, Jabeen [10] investigated Lie centralizers on generalized matrix algebra and obtained the necessary and sufficient criteria for suitable Lie centralizers. Foŝner and Xing [8] explored the relevancy of Lie centralizers on nest algebras and triangular rings. Liu [21] introduced nonlinear Lie centralizers for a particular class of generalized matrix algebra. Recently, many algebraists [4,5,25] examined certain specific Lie centralizers/$\sigma-$centralizers on generalized matrix algebras and triangular algebras. In 2023, Fadaee et al. [7] derived Jabeen's ideas to Lie triple centralizers on generalized matrix algebras. In 2023, Ashraf and Ansari [2] described Lee (Jordan) $\sigma-$centralizers of generalized matrix algebra. They demonstrated that each Lie $\sigma-$centralizer of a generalized matrix algebra can be expressed as a sum of centralizers and center-valued mappings under particular situations. Furthermore, in 2023, Ashraf and Ansari [3] argued that any Jordan $\sigma-$centralizer of a triangular algebra is a centralizer.

Motivated by recent works, the key objective of this manuscript is to investigate the structure of Lie (Jordan) $\sigma-$centralizers on generalized matrix algebra at zero products and to present the relationship between Lie (Jordan) $\sigma-$centralizers and generalized matrix algebra's centralizers.

2. Key content and notations

Let's keep in mind that $\mathscr{R}$ be a unital commutative ring, $\mathscr{A}$ be $\mathscr{R}-$algebra and $Z(\mathscr{A})$ be the center of $\mathscr{A}$. Readers are referred to basic definitions and related characteristics [6, 15, 20]. Let us review numerous definitions and properties utilized throughout the article. A Morita context consists of two $\mathscr{R}-$algebras $\mathscr{A}$ and $\mathscr{B}$, two bimodules ${}_{\mathscr{A}}\mathscr{M}_{\mathscr{B}}$ and ${}_{\mathscr{B}}\mathscr{N}_{\mathscr{A}}$, and two bimodule homomorphisms called the pairings $\zeta_{\mathscr{M}\mathscr{N}} : \mathscr{M} \bigotimes_{\mathscr{B}} \mathscr{N} \to \mathscr{A}$ and $\chi_{\mathscr{N}\mathscr{M}} : \mathscr{N} \bigotimes_{\mathscr{A}} \mathscr{M} \to \mathscr{B}$ satisfying the following commutative diagrams:

$$
\begin{array}{ccc}
\mathscr{M} \underset{\mathscr{B}}{\otimes} \mathscr{N} \underset{\mathscr{A}}{\otimes} \mathscr{M} & \xrightarrow{\zeta_{\mathscr{M}\mathscr{N}} \otimes \mathscr{I}_{\mathscr{M}}} & \mathscr{A} \underset{\mathscr{A}}{\otimes} \mathscr{M} \\
{\scriptstyle \mathscr{I}_{\mathscr{M}} \otimes \chi_{\mathscr{N}\mathscr{M}}} \Big\downarrow & & \Big\downarrow {\scriptstyle \cong} \\
\mathscr{M} \underset{\mathscr{B}}{\otimes} \mathscr{B} & \xrightarrow{\cong} & \mathscr{M}
\end{array},
$$

and

$$\begin{array}{ccc}
\mathcal{N} \underset{\mathcal{A}}{\otimes} \mathcal{M} \underset{\mathcal{B}}{\otimes} \mathcal{N} & \xrightarrow{\chi_{\mathcal{N}\mathcal{M}} \otimes \mathcal{I}_{\mathcal{N}}} & \mathcal{B} \underset{\mathcal{B}}{\otimes} \mathcal{N} \\
\Big\downarrow {\scriptstyle \mathcal{I}_{\mathcal{N}} \otimes \zeta_{\mathcal{M}\mathcal{N}}} & & \Big\downarrow {\scriptstyle \cong} \\
\mathcal{N} \underset{\mathcal{A}}{\otimes} \mathcal{A} & \xrightarrow{\cong} & \mathcal{N}
\end{array} .$$

Let's call this Morita context $(\mathcal{A}, \mathcal{B}, \mathcal{M}, \mathcal{N}, \zeta_{\mathcal{M}\mathcal{N}}, \chi_{\mathcal{N}\mathcal{M}})$. For additional information on Morita contexts, look through [24]. If $(\mathcal{A}, \mathcal{B}, \mathcal{M}, \mathcal{N}, \zeta_{\mathcal{M}\mathcal{N}}, \chi_{\mathcal{N}\mathcal{M}})$ is a Morita context, then the collection

$$\begin{pmatrix} \mathcal{A} & \mathcal{M} \\ \mathcal{N} & \mathcal{B} \end{pmatrix} = \left\{ \begin{pmatrix} a & m \\ n & b \end{pmatrix} \middle| \, m \in \mathcal{M}, n \in \mathcal{N}, a \in \mathcal{A}, b \in \mathcal{B} \right\}$$

is an algebra when standard matrix operations are applied. An $\mathcal{R}-$algebra of this type is known as a generalized matrix algebra of order two, and it is represented by $\mathcal{G} = \mathcal{G}(\mathcal{A}, \mathcal{M}, \mathcal{N}, \mathcal{B})$. $\mathcal{G}$ is referred to be a trivial generalized matrix algebra if $\zeta_{\mathcal{M}\mathcal{N}}$ and $\chi_{\mathcal{N}\mathcal{M}}$ are zero. Such a form of the algebra was initially proposed by Sands in [27]. Whenever $\mathcal{M} = 0$ or $\mathcal{N} = 0$, $\mathcal{G}$ immediately reduces to the identified triangular algebra. Generalized matrix algebras are isomorphic to all associative algebras with nontrivial idempotents. Natural generalized matrix algebra, semi-hereditary algebra, complete matrix algebra, and nested algebra are some typical examples of generalized matrix algebra [8]. A generalized matrix algebra is a unit algebra $\mathcal{A}$ over all matrices of rank $n \times n$ [9]. In case $\mathcal{N} = 0$, $\mathcal{G}$ is referred to as triangular algebra and denoted by $\mathrm{Tri}(\mathcal{A},\mathcal{M},\mathcal{B}) = \begin{bmatrix} \mathcal{A} & \mathcal{M} \\ 0 & \mathcal{B} \end{bmatrix}$. Additionally, $\mathcal{G} = \mathcal{G}(\mathcal{A}, \mathcal{M}, \mathcal{N}, \mathcal{B})$ has unity if and only if $\mathcal{R}-$algebra of $\mathcal{A}$ and $\mathcal{B}$ has unity $1_{\mathcal{A}}$ and $1_{\mathcal{B}}$, respectively. Therefore, $(\mathcal{A},\mathcal{B})-$bimodule $\mathcal{M}$ is unital such that $1_{\mathcal{A}}m = m = m1_{\mathcal{B}}$ for every $m \in \mathcal{M}$, and in the same manner, $(\mathcal{B},\mathcal{A})-$bimodule $\mathcal{N}$ is also unital [9]. Thus, the identity of generalized matrix algebra $\mathcal{G}$ is $I = \begin{bmatrix} 1_{\mathcal{A}} & 0 \\ 0 & 1_{\mathcal{B}} \end{bmatrix}$. On the other hand, it is easy to show that the triangular algebra is unital if and only if $\mathcal{R}-$algebra of $\mathcal{A}$ and $\mathcal{B}$ are unital as well as the $(\mathcal{A},\mathcal{B})-$bimodule $\mathcal{M}$ is unital [9]. The center of $\mathcal{G}$ is $Z(\mathcal{G}) = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \middle\| na = bn, am = mb, \text{ for every } a \in Z(\mathcal{A}),\ b \in Z(\mathcal{B}), m \in \mathcal{M}, n \in \mathcal{N} \right\}$, where $\mathcal{G}$ is of order two.

Remark 2.1. Consider two projections $\pi_{\mathcal{B}} : \mathcal{G} \to \mathcal{B}$ and $\pi_{\mathcal{A}} : \mathcal{G} \to \mathcal{A}$, which is defined as $\pi_{\mathcal{B}}\left(\begin{bmatrix} a & m \\ n & b \end{bmatrix}\right) = b$ and $\pi_{\mathcal{A}}\left(\begin{bmatrix} a & m \\ n & b \end{bmatrix}\right) = a$. Additionally, $\pi_{\mathcal{B}}(Z(\mathcal{G})) \subseteq Z(\mathcal{B})$, $\pi_{\mathcal{A}}(Z(\mathcal{G})) \subseteq Z(\mathcal{A})$, and there is one, and only one, algebraic isomorphism $\tau : \pi_{\mathcal{A}}(Z(\mathcal{G})) \to \pi_{\mathcal{B}}(Z(\mathcal{G}))$ in which $am = m\tau(a)$, $na = \tau(a)n$ for every $a \in \pi_{\mathcal{A}}(Z(\mathcal{G})), m \in \mathcal{M}$, and $n \in \mathcal{N}$.

3. Lie (Jordan) $\sigma-$centralizers at the zero products

Theorem 3.1. *Suppose $\Phi_c : \mathcal{G} \to \mathcal{G}$ is a linear mapping satisfying $\Phi_c(xy) = \Phi_c(x)\sigma(y) = \sigma(x)\Phi_c(y)$ for every $x, y \in \mathcal{G}$, then*

$$\Phi_c\left(\begin{bmatrix} a & m \\ n & b \end{bmatrix}\right) = \begin{bmatrix} \Theta_1(a) & \Theta_1(a)m_0 + \mho_2(m) - m_0U_4(b) \\ n_0\Theta_1(a) + V_3(n) - U_4(b)n_0 & U_4(b) \end{bmatrix},$$

where $\Theta_1 : \mathscr{A} \to \mathscr{A}$, $\mho_2 : \mathscr{M} \to \mathscr{M}$, $\mathtt{U}_4 : \mathscr{B} \to \mathscr{B}$, *and* $\mathtt{V}_3 : \mathscr{N} \to \mathscr{N}$ *are linear mappings such that* $\Theta_1(mn) = \mho_2(m)\nu(n)$ *and* $\mathtt{U}_4(nm) = \mathtt{V}_3(n)\mu(m)$ *are satisfied.*

Proof. Assume the map Φ_c on $\mathscr{G}$ takes the form

$$\Phi_c\left(\begin{bmatrix} a & m \\ n & b \end{bmatrix}\right) = \begin{bmatrix} \Theta_1(a) + \Theta_2(m) + \Theta_3(n) + \Theta_4(b) & \mho_1(a) + \mho_2(m) + \mho_3(n) + \mho_4(b) \\ \mathtt{V}_1(a) + \mathtt{V}_2(m) + \mathtt{V}_3(n) + \mathtt{V}_4(b) & \mathtt{U}_1(a) + \mathtt{U}_2(m) + \mathtt{U}_3(n) + \mathtt{U}_4(b) \end{bmatrix},$$

where $\mho_1 : \mathscr{A} \to \mathscr{M}$, $\mho_2 : \mathscr{M} \to \mathscr{M}$, $\mho_3 : \mathscr{N} \to \mathscr{M}$, $\mho_4 : \mathscr{B} \to \mathscr{M}$, $\Theta_1 : \mathscr{A} \to \mathscr{A}$, $\Theta_2 : \mathscr{M} \to \mathscr{A}$, $\Theta_3 : \mathscr{N} \to \mathscr{A}$, $\Theta_4 : \mathscr{B} \to \mathscr{A}$, $\mathtt{V}_1 : \mathscr{A} \to \mathscr{N}$, $\mathtt{V}_2 : \mathscr{M} \to \mathscr{N}$, $\mathtt{V}_3 : \mathscr{N} \to \mathscr{N}$, $\mathtt{V}_4 : \mathscr{B} \to \mathscr{N}$, and $\mathtt{U}_1 : \mathscr{A} \to \mathscr{B}$, $\mathtt{U}_2 : \mathscr{M} \to \mathscr{B}$, $\mathtt{U}_3 : \mathscr{N} \to \mathscr{B}$, $\mathtt{U}_4 : \mathscr{B} \to \mathscr{B}$ are all linear mappings with automorphism σ.

If a and b are in $\mathscr{A}$ and $\mathscr{B}$, respectively, then $\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix} = 0$. Additonally,

$$0 = \Phi_c\left(\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\right)\sigma\left(\begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}\right) = \sigma\left(\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\right)\Phi_c\left(\begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}\right). \tag{3.1}$$

This equation implies

$$\begin{aligned} 0 &= \begin{bmatrix} \Theta_1(a) & \mho_1(a) \\ \mathtt{V}_1(a) & \mathtt{U}_1(a) \end{bmatrix}\begin{bmatrix} 0 & -m_0\delta(b) \\ -\delta(b)n_0 & \delta(b) \end{bmatrix} \\ &= \begin{bmatrix} -\mho_1(a)\delta(b)n_0 & -\Theta_1(a)m_0\delta(b) + \mho_1(a)\delta(b) \\ -\mathtt{U}_1(a)\delta(b)n_0 & -\mathtt{V}_1(a)m_0\delta(b) + \mathtt{U}_1(a)\delta(b) \end{bmatrix}, \end{aligned}$$

and, hence,

$$\begin{aligned} 0 &= -\mho_1(a)\delta(b)n_0, \\ 0 &= -\Theta_1(a)m_0\delta(b) + \mho_1(a)\delta(b), \\ 0 &= -\mathtt{U}_1(a)\delta(b)n_0, \\ 0 &= -\mathtt{V}_1(a)m_0\delta(b) + \mathtt{U}_1(a)\delta(b). \end{aligned}$$

If we set $b = 1_{\mathscr{B}}$, then $\mho_1(a)n_0 = 0$, $\mho_1(a) = \Theta_1(a)m_0$, and $\mathtt{U}_1(a) = \mathtt{V}_1(a)m_0$. Again by (3.1), we have

$$\begin{aligned} 0 &= \begin{bmatrix} \gamma(a) & \gamma(a)m_0 \\ n_0\gamma(a) & 0 \end{bmatrix}\begin{bmatrix} \Theta_4(b) & \mho_4(b) \\ \mathtt{V}_4(b) & \mathtt{U}_4(b) \end{bmatrix} \\ &= \begin{bmatrix} \gamma(a)\Theta_4(b) + \gamma(a)m_0\mathtt{V}_4(b) & \gamma(a)\mho_4(b) + \gamma(a)m_0\mathtt{U}_4(b) \\ n_0\gamma(a)\Theta_4(b) & n_0\gamma(a)\mho_4(b) \end{bmatrix}. \end{aligned}$$

Now taking $a = 1_{\mathscr{A}}$ in the above, we obtain $\Theta_4(b) = -m_0\mathtt{V}_4(b)$, $\mho_4(b) = -m_0\mathtt{U}_4(b)$, and $n_0\mho_4(b) = 0$.

Again, applying a similar calculative procedure with

$$0 = \Phi_c\left(\begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}\right)\sigma\left(\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\right) = \sigma\left(\begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}\right)\Phi_c\left(\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\right), \tag{3.2}$$

we get $\Theta_4(b) = -\mho_4(b)n_0$, $\mathtt{V}_4(b) = -\mathtt{U}_4(b)n_0$, $\mathtt{V}_4(b)m_0 = 0$, $m_0\mathtt{V}_1(a) = 0$, $\mathtt{V}_1(a) = n_0\Theta_1(a)$, and $\mathtt{U}_1(a) = -n_0\mho_1(a)$. Therefore, by (3.1), (3.2), and [4, Proposition 2.1], we get $\mathtt{U}_1(a) = 0$ and $\Theta_4(b) = 0$.

If $a_1, a_2 \in \mathscr{A}$ with $a_1a_2 = 0$, we have $\begin{bmatrix} a_1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a_2 & 0 \\ 0 & 0 \end{bmatrix} = 0$. Thus,

$$0 = \Phi_c\left(\begin{bmatrix} a_1 & 0 \\ 0 & 0 \end{bmatrix}\right)\sigma\left(\begin{bmatrix} a_2 & 0 \\ 0 & 0 \end{bmatrix}\right) = \sigma\left(\begin{bmatrix} a_1 & 0 \\ 0 & 0 \end{bmatrix}\right)\Phi_c\left(\begin{bmatrix} a_2 & 0 \\ 0 & 0 \end{bmatrix}\right). \tag{3.3}$$

Then, $\Theta_1(a_1a_2) = 0 = \Theta_1(a_1)\gamma(a_2) + \mho_1(a_1)n_0\gamma(a_2) = \gamma(a_1)\Theta_1(a_2) + \gamma(a_1)m_0\mathtt{V}_1(a_2)$. Hence, by (3.1) and (3.2), we have $\Theta_1(a_1a_2) = 0 = \Theta_1(a_1)\gamma(a_2) = \gamma(a_1)\Theta_1(a_2)$. Therefore, Θ_1 is a $\sigma-$centralizer at the zero product on $\mathscr{A}$.

Similarly, if $b_1, b_2 \in \mathscr{B}$ with $b_1b_2 = 0$, we have $\begin{bmatrix} 0 & 0 \\ 0 & b_1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & b_2 \end{bmatrix} = 0$. Then, we get $\mathtt{U}_4(b_1b_2) = 0 = -\mathtt{V}_4(b_1)m_0\delta(b_2) + \mathtt{U}_4(b_1)\delta(b_2) = -\delta(b_1)n_0\mho_4(b_2) + \delta(b_1) + \delta(b_1)\mathtt{U}_4$. Hence, by (3.1) and (3.2), we have $\mathtt{U}_4(b_1b_2) = 0 = \mathtt{U}_4(b_1)\delta(b_2) = \delta(b_1)+\delta(b_1)\mathtt{U}_4$. Therefore, $\mathtt{U}_4$ is a $\sigma-$centralizer at the zero product on $\mathscr{B}$.

If m and a are in $\mathscr{M}$ and $\mathscr{A}$, respectively, then $\begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} = 0$. Furthermore,

$$0 = \Phi_c\left(\begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix}\right)\sigma\left(\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\right) = \sigma\left(\begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix}\right)\Phi_c\left(\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\right). \tag{3.4}$$

This leads to

$$\begin{aligned} 0 &= \Theta_2(m)\gamma(a) + \mho_2(m)n_0\gamma(a) = \mu(m)\mathtt{V}_1(a), \\ 0 &= \Theta_2(m)\gamma(a)m_0 = \mu(m)\mathtt{U}_1(a), \\ 0 &= \mathtt{V}_2(m)\gamma(a) + \mathtt{U}_2(m)n_0\gamma(a), \\ 0 &= \mathtt{V}_2(m)\gamma(a)m_0. \end{aligned}$$

Put $a = 1_{\mathscr{A}}$, and we have $\Theta_2(m) = -\mho_2(m)n_0$ and $\mathtt{V}_2(m) = -\mathtt{U}_2(m)n_0$. Similarly, $\begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}\begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix} = 0$. Then,

$$0 = \Phi_c\left(\begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}\right)\sigma\left(\begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix}\right) = \sigma\left(\begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}\right)\Phi_c\left(\begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix}\right). \tag{3.5}$$

From the above, we find that $\mathtt{V}_2(m) = n_0\delta_2(m)$ and $\mathtt{U}_2(m) = n_0\mho_2(m)$.

We recognize that $\begin{bmatrix} -1_{\mathscr{A}} & m \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & m \\ 0 & 1_{\mathscr{B}} \end{bmatrix} = 0$ for every $m \in \mathscr{M}$. Therefore,

$$0 = \Phi_c\left(\begin{bmatrix} -1_{\mathscr{A}} & m \\ 0 & 0 \end{bmatrix}\right)\sigma\left(\begin{bmatrix} 0 & m \\ 0 & 1_{\mathscr{B}} \end{bmatrix}\right) = \sigma\left(\begin{bmatrix} -1_{\mathscr{A}} & m \\ 0 & 0 \end{bmatrix}\right)\Phi_c\left(\begin{bmatrix} 0 & m \\ 0 & 1_{\mathscr{B}} \end{bmatrix}\right). \tag{3.6}$$

The above equation implies that

$$\begin{aligned} 0 &= \Phi_c\left(\begin{bmatrix} -1_{\mathscr{A}} & m \\ 0 & 0 \end{bmatrix}\right)\sigma\left(\begin{bmatrix} 0 & m \\ 0 & 1_{\mathscr{B}} \end{bmatrix}\right) \\ &= \begin{bmatrix} -\Theta_1(1_{\mathscr{A}}) + \Theta_2(m) & -\mho_1(1_{\mathscr{A}}) + \mho_2(m) \\ -\mathtt{V}_1(1_{\mathscr{A}}) + \mathtt{V}_2(m) & \mathtt{U}_2(m) \end{bmatrix}\begin{bmatrix} 0 & -m_0 + \mu(m) \\ -n_0 & -1_{\mathscr{B}} \end{bmatrix}. \end{aligned}$$

Hence, we get $\mho_2(m)n_0 = 0$, $\mho_2(m) = \Theta_1(1_{\mathscr{A}})\mu(m) - \Theta_2(m)\mu(m)$, and $\mathtt{U}_2(m)n_0 = 0$. By (3.4) and concerning that $m \in \mathscr{M}$, we get $\Theta_2(m) = 0$ and $\mathtt{V}_2(m) = 0$. Also since we know Θ_1 is a $\sigma-$centralizer at the zero product on $\mathscr{A}$, then $\mho_2(am) = \Theta_1(1_{\mathscr{A}})\mu(am) = \Theta_1(1_{\mathscr{A}})\gamma(a)\mu(m) = \Theta_1(a)\mu(m)$ for any $m \in \mathscr{M}$ with $a = 0$. Therefore, $\mathscr{M}$ is a left faithful $\mathscr{A}-$module.

Again by (3.6), we have $0 = \begin{bmatrix} -1_{\mathscr{A}} & -m_0 + \mu(m) \\ -n_0 & 0 \end{bmatrix}\begin{bmatrix} 0 & \mho_2(m) + \mho_4(\mathscr{B}) \\ \mathtt{V}_4(1_{\mathscr{B}}) & \mathtt{U}_2(m) + \mathtt{U}_4(1_{\mathscr{B}}) \end{bmatrix}$. Then, we find $\mho_2(m) = \mu(m)\mathtt{U}_4(1_{\mathscr{B}})$ and $n_0\mho_2(m) = 0$. Hence, by (3.4) and concerning that $m \in \mathscr{M}$, we have $\mathtt{U}_2(m) = 0$. Also, by (3.3) we have that $\mathtt{U}_4$ is a $\sigma-$centralizer at the zero product on $\mathscr{B}$, then $\mho_2(mb) = \mu(mb)\mathtt{U}_4(1_{\mathscr{B}}) = \mu(m)\delta(b)\mathtt{U}_4(1_{\mathscr{B}}) = \mu(m)\mathtt{U}_4(b)$ for every $m \in \mathscr{M}$, where $b = 0$ implies that $\mathscr{M}$ is a right faithful $\mathscr{B}-$module. Since $\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ n & 0 \end{bmatrix} = 0$, for any $a \in \mathscr{A}$ and $n \in \mathscr{N}$, we have

$$0 = \Phi_c\left(\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\right)\sigma\left(\begin{bmatrix} 0 & 0 \\ n & 0 \end{bmatrix}\right) = \sigma\left(\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\right)\Phi_c\left(\begin{bmatrix} 0 & 0 \\ n & 0 \end{bmatrix}\right). \tag{3.7}$$

By (3.5), we deduce that

$$0 = \begin{bmatrix} \mho_1(a)\nu(n) & 0 \\ \mathtt{U}_1(a)\nu(n) & 0 \end{bmatrix} = \begin{bmatrix} \gamma(a)\Theta_3(n) + \gamma(a)m_0\mathtt{V}_3(n) & \gamma(a)\mho_3(n) + \gamma(a)m_0\mathtt{U}_3(n) \\ n_0\gamma(a)\Theta_3(n) & n_0\gamma(a)\mho_3(n) \end{bmatrix}.$$

So, we get

$$\begin{aligned} 0 &= \mho_1(a)\nu(n) = \gamma(a)\Theta_3(n) + \gamma(a)m_0\mathtt{V}_3(n), \\ 0 &= \gamma(a)\mho_3(n) + \gamma(a)m_0\mathtt{U}_3(n), \\ 0 &= \mathtt{U}_1(a)\nu(n) = n_0\gamma(a)\Theta_3(n), \\ 0 &= n_0\gamma(a)\mho_3(n). \end{aligned}$$

Put $a = 1_{\mathscr{A}}$, and we find that $\Theta_3(n) = -m_0\mathtt{V}_3(n)$ and $\mho_3(n) = -m_0\mathtt{U}_3(n)$. Similarly, $\begin{bmatrix} 0 & 0 \\ n & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix} = 0$, for every $n \in \mathscr{N}$ and $b \in \mathscr{B}$, we have

$$0 = \Phi_c\left(\begin{bmatrix} 0 & 0 \\ n & 0 \end{bmatrix}\right)\sigma\left(\begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}\right) = \sigma\left(\begin{bmatrix} 0 & 0 \\ n & 0 \end{bmatrix}\right)\Phi_c\left(\begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}\right). \tag{3.8}$$

Hence, we find that $\mho_3(n) = \Theta_3(n)m_0$ and $\mathtt{U}_3(n) = \mathtt{V}_3(n)m_0$. For any $n \in \mathscr{N}$, $\begin{bmatrix} 0 & 0 \\ n & 1_{\mathscr{B}} \end{bmatrix}\begin{bmatrix} -1_{\mathscr{A}} & 0 \\ n & 0 \end{bmatrix} = 0$, we have

$$0 = \Phi_c\left(\begin{bmatrix} 0 & 0 \\ n & 1_{\mathscr{B}} \end{bmatrix}\right)\sigma\left(\begin{bmatrix} -1_{\mathscr{A}} & 0 \\ n & 0 \end{bmatrix}\right) = \sigma\left(\begin{bmatrix} 0 & 0 \\ n & 1_{\mathscr{B}} \end{bmatrix}\right)\Phi_c\left(\begin{bmatrix} -1_{\mathscr{A}} & 0 \\ n & 0 \end{bmatrix}\right). \tag{3.9}$$

Therefore, for any $n \in \mathscr{N}$, we find these results: $\Theta_3(n)m_0 = 0$, $\mathtt{V}_3(n)m_0 = 0$, $m_0\mathtt{V}_3(n) = 0$, $\mathtt{V}_3(n) = \mathtt{U}_4(1_{\mathscr{B}})\nu(n)$, and $\mathtt{V}_3(n) = \nu(n)\Theta_1(1_{\mathscr{A}})$. Hence, from (3.7) and (3.8), we get $\mho_3(n) = 0$, $\mathtt{U}_3(n) = 0$, and $\Theta_3(n) = 0$. Also, in the case of Θ_1 and $\mathtt{U}_4$ are $\sigma-$centralizers on $\mathscr{A}$ and $\mathscr{B}$, respectively, we have the $(\mathscr{B}, \mathscr{A})-$bimodule $\mathscr{N}$ as a faithful $(\mathscr{B}, \mathscr{A})-$bimodule.

If m and n are in $\mathscr{M}$ and $\mathscr{N}$, respectively, then $\begin{bmatrix} mn & m \\ 0 & b \end{bmatrix}\begin{bmatrix} -1 & 0 \\ n & 0 \end{bmatrix} = 0$. Then,

$$0 = \Phi_c\left(\begin{bmatrix} mn & m \\ 0 & 0 \end{bmatrix}\right)\sigma\left(\begin{bmatrix} -1 & 0 \\ n & 0 \end{bmatrix}\right) = \sigma\left(\begin{bmatrix} mn & m \\ 0 & 0 \end{bmatrix}\right)\Phi_c\left(\begin{bmatrix} -1 & 0 \\ n & 0 \end{bmatrix}\right). \tag{3.10}$$

By (3.10), we obtain

$$\begin{aligned} 0 &= \begin{bmatrix} \Theta_1(mn) & \mho_1(mn) + \mho_2(m) \\ \mathtt{V}_1(mn) & 0 \end{bmatrix} \begin{bmatrix} -1 & -m_0 \\ -n_0 + \nu(n) & 0 \end{bmatrix} \\ &= \begin{bmatrix} \gamma(mn) & \gamma(mn)m_0 + \mu(m) \\ n_0\gamma(mn) & 0 \end{bmatrix} \begin{bmatrix} \Theta_1(-1) & 1(-1) \\ \mathtt{V}_1(-1) + \mathtt{V}_3(n) & 0 \end{bmatrix}. \end{aligned}$$

This equation leads to $\Theta_1(mn) = \mho_2(m)\nu(n)$. Similarly, with $\begin{bmatrix} 0 & 0 \\ n & nm \end{bmatrix}\begin{bmatrix} 0 & m \\ 0 & -1 \end{bmatrix} = 0$, we have $\mathtt{V}_3(n)\mu(m) = \mathtt{U}_4(nm)$. □

Theorem 3.2. *Suppose $\mathscr{G}$ is a generalized matrix algebra of 2−torsion free and a linear mapping Φ_{Jc}: $\mathscr{G} \to \mathscr{G}$ satisfies $\Phi_c(x \circ y) = \Phi_c(x) \circ \sigma(y) = \sigma(x) \circ \Phi_c(y)$ for every $x, y \in \mathscr{G}$, such that*

$$\Phi_{Jc}\left(\begin{bmatrix} a & m \\ n & b \end{bmatrix}\right) = \begin{bmatrix} \rho_1(a) & \rho_1(a)m_0 + \mathtt{r}_2(m) - m_0\mathtt{g}_4(b) \\ n_0\rho_1(a) + \mathtt{s}_3(n) - \mathtt{g}_4(b)n_0 & \mathtt{g}_4(b) \end{bmatrix},$$

where $\rho_1 : \mathscr{A} \to \mathscr{A}$, $\mathtt{r}_2 : \mathscr{M} \to \mathscr{M}$, $\mathtt{s}_3 : \mathscr{N} \to \mathscr{N}$, and $\mathtt{g}_4 : \mathscr{B} \to \mathscr{B}$ are linear mappings that hold

(i) $\mathtt{r}_2(am) = \gamma(a)\mathtt{r}_2(m) = \rho_1(a)\mu(m)$ *and* $\mathtt{r}_2(mb) = \mathtt{r}_2(m)\delta(b) = \mu(m)\mathtt{g}_4(b)$.
(ii) $\mathtt{s}_3(na) = \mathtt{s}_3(n)\gamma(a) = \nu(n)\rho_1(a)$ *and* $\mathtt{s}_3(bn) = \delta(b)\mathtt{s}_3(n) = \mathtt{g}_4(b)\nu(n)$.
(iii) $\rho_1(mn) = \mu(m)\mathtt{s}_3(n) = \mathtt{r}_2(m)\nu(n)$ *and* $\mathtt{g}_4(nm) = \mathtt{s}_3(n)\mu(m) = \nu(n)\mathtt{r}_2(m)$.

Proof. Assume the map Φ_{Jc} on $\mathscr{G}$ takes the form

$$\Phi_{Jc}\left(\begin{bmatrix} a & m \\ n & b \end{bmatrix}\right) = \begin{bmatrix} \rho_1(a) + \rho_2(m) + \rho_3(n) + \rho_4(b) & \mathtt{r}_1(a) + \mathtt{r}_2(m) + \mathtt{r}_3(n) + \mathtt{r}_4(b) \\ \mathtt{s}_1(a) + \mathtt{s}_2(m) + \mathtt{s}_3(n) + \mathtt{s}_4(b) & \mathtt{g}_1(a) + \mathtt{g}_2(m) + \mathtt{g}_3(n) + \mathtt{g}_4(b) \end{bmatrix},$$

where $\rho_1 : \mathscr{A} \to \mathscr{A}$, $\rho_2 : \mathscr{M} \to \mathscr{A}$, $\rho_3 : \mathscr{N} \to \mathscr{A}$, $\rho_4 : \mathscr{B} \to \mathscr{A}$, $\mathtt{r}_1 : \mathscr{A} \to \mathscr{M}$, $\mathtt{r}_2 : \mathscr{M} \to \mathscr{M}$, $\mathtt{r}_3 : \mathscr{N} \to \mathscr{M}$, $\mathtt{r}_4 : \mathscr{B} \to \mathscr{M}$, $\mathtt{s}_1 : \mathscr{A} \to \mathscr{N}$, $\mathtt{s}_2 : \mathscr{M} \to \mathscr{N}$, $\mathtt{s}_3 : \mathscr{N} \to \mathscr{N}$, $\mathtt{s}_4 : \mathscr{B} \to \mathscr{N}$, and $\mathtt{g}_1 : \mathscr{A} \to \mathscr{B}$, $\mathtt{g}_2 : \mathscr{M} \to \mathscr{B}$, $\mathtt{g}_3 : \mathscr{N} \to \mathscr{B}$, $\mathtt{g}_4 : \mathscr{B} \to \mathscr{B}$ are all linear mappings with automorphism σ.

If a and b are in $\mathscr{A}$ and $\mathscr{B}$, respectively, then $\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix} = 0$. Moreover,

$$0 = \Phi_{Jc}\left(\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\right) \circ \sigma\left(\begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}\right) = \sigma\left(\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\right) \circ \Phi_{Jc}\left(\begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}\right). \tag{3.11}$$

This implies that

$$0 = \begin{bmatrix} \rho_1(a) & \mathtt{r}_1(a) \\ \mathtt{s}_1(a) & \mathtt{g}_1(a) \end{bmatrix} \circ \begin{bmatrix} 0 & -m_0\delta(b) \\ -\delta(b)n_0 & \delta(b) \end{bmatrix},$$

which is rewritten as

$$\begin{bmatrix} -\mathtt{r}_1(a)\delta(b)n_0 - m_0\delta(b)\mathtt{s}_1(a) & -\rho_1(a)m_0\delta(b) + \mathtt{r}_1(a)\delta(b) - m_0\delta(b)\mathtt{g}_1(a) \\ -\mathtt{g}_1(a)\delta(b)n_0 - \delta(b)n_0\rho_1(a) + \delta(b)\mathtt{s}_1(a) & -\mathtt{s}_1(a)m_0\delta(b) + \mathtt{g}_1(a)\delta(b) - \delta(b)n_0\mathtt{r}_1(a) + \delta(b)\mathtt{g}_1(a) \end{bmatrix},$$

and gives

$$0 = -\mathtt{r}_1(a)\delta(b)n_0 - m_0\delta(b)\mathtt{s}_1(a),$$

$$
\begin{aligned}
0 &= -\rho_1(a)m_0\delta(b) + \mathtt{r}_1(a)\delta(b) - m_0\delta(b)\mathtt{g}_1(a),\\
0 &= -\mathtt{g}_1(a)\delta(b)n_0 - \delta(b)n_0\rho_1(a) + \delta(b)\mathtt{s}_1(a),\\
0 &= -\mathtt{s}_1(a)m_0\delta(b) + \mathtt{g}_1(a)\delta(b) - \delta(b)n_0\mathtt{r}_1(a) + \delta(b)\mathtt{g}_1(a).
\end{aligned}
$$

If we set $b = 1_{\mathscr{B}}$, then $\mathtt{r}_1(a) = \rho_1(a)m_0$, $\mathtt{s}_1(a) = n_0\rho_1(a)$, and $\mathtt{g}_1(a) = 0$. Again with (3.9), the following matrix

$$
\begin{bmatrix}
\gamma(a)\rho_4(b) + \gamma(a)m_0\mathtt{s}_4(b) + \rho_4(b)\gamma(a) + \mathtt{r}_4(b)n_0\gamma(a) & \gamma(a)\mathtt{r}_4(b) + \gamma(a)m_0\mathtt{g}_4(b) + \rho_4(b)\gamma(a)m_0\\
n_0\gamma(a)\rho_4(b) + \mathtt{s}_4(b)\gamma(a) + \mathtt{g}_4(b)n_0\gamma(a) & n_0\gamma(a)\mathtt{r}_4(b) + \mathtt{s}_4(b)\gamma(a)m_0
\end{bmatrix},
$$

is equal to zero. Evaluating with $a = 1_{\mathscr{A}}$, we obtain $\rho_4(b) = 0$, $\mathtt{r}_4(b) = -m_0\mathtt{g}_4(b)$, and $\mathtt{s}_4(b) = -\mathtt{g}_4(b)n_0$.

If m and a are in $\mathscr{M}$ and $\mathscr{A}$, respectively, then $\begin{bmatrix}0 & m\\0 & 0\end{bmatrix}\begin{bmatrix}a & 0\\0 & 0\end{bmatrix} = 0$. Therefore,

$$
\begin{aligned}
\Phi_{Jc}\left(\begin{bmatrix}0 & am\\0 & 0\end{bmatrix}\right) &= \Phi_{Jc}\left(\begin{bmatrix}0 & m\\0 & 0\end{bmatrix}\right)\circ\sigma\left(\begin{bmatrix}a & 0\\0 & 0\end{bmatrix}\right)\\
&= \sigma\left(\begin{bmatrix}0 & m\\0 & 0\end{bmatrix}\right)\circ\Phi_{Jc}\left(\begin{bmatrix}a & 0\\0 & 0\end{bmatrix}\right).
\end{aligned}
\tag{3.12}
$$

This implies that

$$
\begin{bmatrix}\rho_2(am) & \mathtt{r}_2(am)\\ \mathtt{s}_2(am) & \mathtt{g}_2(am)\end{bmatrix} = \begin{bmatrix}\rho_2(m) & \mathtt{r}_2(m)\\ \mathtt{s}_2(m) & \mathtt{g}_2(m)\end{bmatrix}\circ\begin{bmatrix}\gamma(a) & \gamma(a)m_0\\ n_0\gamma(a) & 0\end{bmatrix} = \begin{bmatrix}0 & \mu(m)\\ 0 & 0\end{bmatrix}\circ\begin{bmatrix}\rho_1(a) & \mathtt{r}_1(a)\\ \mathtt{s}_1(a) & \mathtt{g}_1(a)\end{bmatrix},
$$

and, hence,

$$
\begin{aligned}
\rho_2(am) &= \rho_2(m)\gamma(a) + \mathtt{r}_2(m)n_0\gamma(a) + \gamma(a)\rho_2(m) + \gamma(a)m_0\mathtt{s}_2(m) = \mu(m)\mathtt{s}_1(a),\\
\mathtt{r}_2(am) &= \rho_2(m)\gamma(a)m_0 + \gamma(a)\mathtt{r}_2(m) + \gamma(a)m_0\mathtt{g}_2(m) = \rho_1(a)\mu(m) + \mu(m)\mathtt{g}_1(a),\\
\mathtt{s}_2(am) &= \mathtt{s}_2(m)\gamma(a) + \mathtt{g}_2(m)n_0\gamma(a) + n_0\gamma(a)\rho_2(m) = 0,\\
\mathtt{g}_2(am) &= \mathtt{s}_2(m)\gamma(a)m_0 + n_0\gamma(a)\mathtt{r}_2(m) = \mathtt{s}_1(a)\mu(m).
\end{aligned}
$$

On taking $a = 1_{\mathscr{A}}$ in the above, we see that $\rho_2(m) = 0$, $\mathtt{s}_2(m) = 0$, $\mathtt{g}_2(m) = 0$, and $\mathtt{r}_2(am) = \gamma(a)\mathtt{r}_2(m)$. Similar with $\begin{bmatrix}0 & 0\\0 & b\end{bmatrix}\begin{bmatrix}0 & m\\0 & 0\end{bmatrix} = 0$, we find that $\mathtt{r}_2(mb) = \rho_4(b)\mu(m) + \mu(m)\mathtt{g}_4(b) = \mathtt{r}_2(m)\delta(b)$.

Since for any $a \in \mathscr{A}$ and $n \in \mathscr{N}$, $\begin{bmatrix}a & 0\\0 & 0\end{bmatrix}\begin{bmatrix}0 & 0\\n & 0\end{bmatrix} = 0$, we obtain

$$
\begin{aligned}
\Phi_{Jc}\left(\begin{bmatrix}0 & 0\\na & 0\end{bmatrix}\right) &= \Phi_{Jc}\left(\begin{bmatrix}a & 0\\0 & 0\end{bmatrix}\right)\circ\sigma\left(\begin{bmatrix}0 & 0\\n & 0\end{bmatrix}\right)\\
&= \sigma\left(\begin{bmatrix}a & 0\\0 & 0\end{bmatrix}\right)\circ\Phi_{Jc}\left(\begin{bmatrix}0 & 0\\n & 0\end{bmatrix}\right).
\end{aligned}
\tag{3.13}
$$

By (3.12), we deduce that

$$
\begin{bmatrix}\rho_3(na) & \mathtt{r}_3(na)\\ \mathtt{s}_3(na) & \mathtt{g}_3(na)\end{bmatrix} = \begin{bmatrix}\mathtt{r}_1(a)\nu(n) & 0\\ \mathtt{g}_1(a)\nu(n) + \nu(n)\rho_1(a) & \nu(n)\mathtt{r}_1(a)\end{bmatrix},
$$

which is rewritten as

$$\begin{bmatrix} \gamma(a)\rho_3(n)+\gamma(a)m_0\mathtt{s}_3(n)+\rho_3(n)\gamma(a)+\mathtt{r}_3(n)n_0\gamma(a) & \gamma(a)\mathtt{r}_3(n)+\gamma(a)m_0\mathtt{g}_3(n)+\rho_3(n)\gamma(a)m_0 \\ n_0\gamma(a)\rho_3(n)+\mathtt{s}_3(n)\gamma(a)+\mathtt{g}_3(n)n_0\gamma(a) & n_0\gamma(a)\mathtt{r}_3(n)+\mathtt{s}_3(n)\gamma(a)m_0 \end{bmatrix}.$$

This leads to

$$\begin{aligned} \rho_3(na) &= \mathtt{r}_1(a)\nu(n) = \gamma(a)\rho_3(n)+\gamma(a)m_0\mathtt{s}_3(n)+\rho_3(n)\gamma(a)+\mathtt{r}_3(n)n_0\gamma(a), \\ \mathtt{r}_3(na) &= 0 = \gamma(a)\mathtt{r}_3(n)+\gamma(a)m_0\mathtt{g}_3(n)+\rho_3(n)\gamma(a)m_0, \\ \mathtt{s}_3(na) &= \mathtt{g}_1(a)\nu(n)+\nu(n)\rho_1(a) = n_0\gamma(a)\rho_3(n)+\mathtt{s}_3(n)\gamma(a)+\mathtt{g}_3(n)n_0\gamma(a), \\ \mathtt{g}_3(na) &= \nu(n)\mathtt{r}_1(a) = n_0\gamma(a)\mathtt{r}_3(n)+\mathtt{s}_3(n)\gamma(a)m_0. \end{aligned}$$

Set $a = 1_{\mathscr{A}}$, and we find $\rho_3(n) = 0$, $\mathtt{r}_3(n) = 0$, $\mathtt{g}_3(n) = 0$, and $\mathtt{s}_3(na) = \mathtt{g}_1(a)\nu(n)+\nu(n)\rho_1(a) = \mathtt{s}_3(n)\gamma(a)$. Follow the similar steps with $\begin{bmatrix} 0 & 0 \\ n & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix} = 0$, and we get $\mathtt{s}_3(bn) = \delta(b)\mathtt{s}_3(n) = \nu(n)\rho_4(b)+\mathtt{g}_4(b)\nu(n)$.

For any $m \in \mathscr{M}$ and $n \in \mathscr{N}$, $\begin{bmatrix} mn & m \\ 0 & 0 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ n & 0 \end{bmatrix} = 0$, and we have

$$\begin{aligned} \Phi_{Jc}\left(\begin{bmatrix} -mn & -m \\ nmn & nm \end{bmatrix}\right) &= \Phi_{Jc}\left(\begin{bmatrix} mn & m \\ 0 & 0 \end{bmatrix}\right)\circ\sigma\left(\begin{bmatrix} -1 & 0 \\ n & 0 \end{bmatrix}\right) \\ &= \sigma\left(\begin{bmatrix} mn & m \\ 0 & 0 \end{bmatrix}\right)\circ\Phi_{Jc}\left(\begin{bmatrix} -1 & 0 \\ n & 0 \end{bmatrix}\right). \end{aligned} \tag{3.14}$$

By (3.14), we obtain

$$\begin{aligned} &\begin{bmatrix} -\rho_1(mn)-\rho_2(m)+\rho_3(nmn)+\rho_4(nm) & -\mathtt{r}_1(mn)-\mathtt{r}_2(m)+\mathtt{r}_3(nmn)+\mathtt{r}_4(nm) \\ -\mathtt{s}_1(mn)-\mathtt{s}_2(m)+\mathtt{s}_3(nmn)+\mathtt{s}_4(nm) & -\mathtt{g}_1(mn)-\mathtt{g}_2(m)+\mathtt{g}_3(nmn)+\mathtt{g}_4(nm) \end{bmatrix} \\ &\qquad = \begin{bmatrix} \rho_1(mn)+\rho_2(m) & \mathtt{r}_1(mn)+\mathtt{r}_2(m) \\ \mathtt{s}_1(mn)+\mathtt{s}_2(m) & \mathtt{g}_1(mn)+\mathtt{g}_2(m) \end{bmatrix}\circ\begin{bmatrix} -1 & -m_0 \\ -n_0+\nu(n) & 0 \end{bmatrix}. \end{aligned}$$

This equation gives $\rho_1(mn) = \mathtt{r}_2(m)\nu(n)$ and $\mathtt{g}_4(nm) = \nu(n)\mathtt{r}_2(m)$. Similarly for $\begin{bmatrix} 0 & 0 \\ n & nm \end{bmatrix}\begin{bmatrix} 0 & m \\ 0 & -1 \end{bmatrix} = 0$, we have $\rho_1(mn) = \mu(m)\mathtt{s}_3(n)$ and $\mathtt{g}_4(nm) = \mathtt{s}_3(n)\mu(m)$. □

From the above two results it is easy to conclude that:

Theorem 3.3. *Every Jordan $\sigma-$centralizer is a $\sigma-$centralizer at zero products on generalized matrix algebras.*

Theorem 3.4. *Suppose $\Phi_{lc} : \mathscr{G} \to \mathscr{G}$ is a linear mapping satisfying $\Phi_{lc}([x,y]) = [\Phi_{lc}(x),\sigma(y)] = [\sigma(x),\Phi_{lc}(y)]$, then*

$$\Phi_{lc}\left(\begin{bmatrix} a & m \\ n & b \end{bmatrix}\right) = \begin{bmatrix} \rho_1(a)+\rho_4(b) & \mathtt{r}_1(a)+\mathtt{r}_2(m)+\mathtt{r}_4(b) \\ \mathtt{s}_1(a)+\mathtt{s}_3(n)+\mathtt{s}_4(b) & \mathtt{g}_1(a)+\mathtt{g}_4(b) \end{bmatrix},$$

where $\rho_1 : \mathscr{A} \to \mathscr{A}$, $\rho_4 : \mathscr{B} \to \mathscr{A}$, $\mathtt{r}_1 : \mathscr{A} \to \mathscr{M}$. $\mathtt{r}_2 : \mathscr{M} \to \mathscr{M}$, $\mathtt{r}_4 : \mathscr{B} \to \mathscr{M}$, $\mathtt{s}_1 : \mathscr{A} \to \mathscr{N}$, $\mathtt{s}_3 : \mathscr{N} \to \mathscr{N}$, $\mathtt{s}_4 : \mathscr{B} \to \mathscr{N}$, $\mathtt{g}_1 : \mathscr{A} \to \mathscr{B}$ and $\mathtt{g}_4 : \mathscr{B} \to \mathscr{B}$ are linear mappings such that *(i)* $\mathtt{r}_1(a) = \rho_1(a)m_0 - m_0\mathtt{g}_1(a)$ *and* $\mathtt{r}_4(b) = -m_0\mathtt{g}_4(b) + \rho_4(b)m_0$.

(ii) $\mathtt{s}_1(a) = -\mathtt{g}_1(a)n_0 + n_0\rho_1(a)$ *and* $\mathtt{s}_4(b) = n_0\rho_4(b) - \mathtt{g}_4(b)n_0$.

(iii) $\mathtt{r}_2(am) = \gamma(a)\mathtt{r}_2(m) = \rho_1(a)\mu(m) - \mu(m)\mathtt{g}_1(a)$ *and* $\mathtt{r}_2(mb) = \mu(m)\mathtt{g}_4(b) - \rho_4(b)\mu(m) = \mathtt{r}_2(m)\delta(b)$.

(iv) $\mathtt{s}_3(na) = \nu(n)\rho_1(a) - \mathtt{g}_1(a)\nu(n) = \mathtt{s}_3(n)\gamma(a)$ *and* $\mathtt{s}_3(bn) = \delta(b)\mathtt{s}_3(n) = \mathtt{g}_4(b)\nu(n) - \nu(n)\rho_4(b)$.

(v) $\rho_1(mn) - \rho_4(nm) = \mu(m)\mathtt{s}_3(n) = \mathtt{r}_2(m)\nu(n)$ *and* $\mathtt{g}_1(mn) - \mathtt{g}_4(nm) = -\mathtt{s}_3(n)\mu(m) = -\nu(n)\mathtt{r}_2(m)$.

Proof. Assume the map Φ_c on $\mathscr{G}$ takes the form

$$\Phi_{lc}\left(\begin{bmatrix} a & m \\ n & b \end{bmatrix}\right) = \begin{bmatrix} \rho_1(a)+\rho_2(m)+\rho_3(n)+\rho_4(b) & \mathtt{r}_1(a)+\mathtt{r}_2(m)+\mathtt{r}_3(n)+\mathtt{r}_4(b) \\ \mathtt{s}_1(a)+\mathtt{s}_2(m)+\mathtt{s}_3(n)+\mathtt{s}_4(b) & \mathtt{g}_1(a)+\mathtt{g}_2(m)+\mathtt{g}_3(n)+\mathtt{g}_4(b) \end{bmatrix},$$

where $\rho_1 : \mathscr{A} \to \mathscr{A}$, $\rho_2 : \mathscr{M} \to \mathscr{A}$, $\rho_3 : \mathscr{N} \to \mathscr{A}$, $\rho_4 : \mathscr{B} \to \mathscr{A}$, $\mathtt{r}_1 : \mathscr{A} \to \mathscr{M}$, $\mathtt{r}_2 : \mathscr{M} \to \mathscr{M}$, $\mathtt{r}_3 : \mathscr{N} \to \mathscr{M}$, $\mathtt{r}_4 : \mathscr{B} \to \mathscr{M}$, $\mathtt{s}_1 : \mathscr{A} \to \mathscr{N}$, $\mathtt{s}_2 : \mathscr{M} \to \mathscr{N}$, $\mathtt{s}_3 : \mathscr{N} \to \mathscr{N}$, $\mathtt{s}_4 : \mathscr{B} \to \mathscr{N}$, and $\mathtt{g}_1 : \mathscr{A} \to \mathscr{B}$, $\mathtt{g}_2 : \mathscr{M} \to \mathscr{B}$, $\mathtt{g}_3 : \mathscr{N} \to \mathscr{B}$, $\mathtt{g}_4 : \mathscr{B} \to \mathscr{B}$ are all linear mappings with automorphism σ. If a and b are in $\mathscr{A}$ and $\mathscr{B}$, respectively, then $\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix} = 0$. Therefore,

$$0 = \left[\Phi_{lc}\left(\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\right), \sigma\left(\begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}\right)\right] = \left[\sigma\left(\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\right), \Phi_{lc}\left(\begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}\right)\right]. \tag{3.15}$$

This equation implies

$$0 \quad = \quad \left[\begin{bmatrix} \rho_1(a) & \mathtt{r}_1(a) \\ \mathtt{s}_1(a) & \mathtt{g}_1(a) \end{bmatrix}, \begin{bmatrix} 0 & -m_0\delta(b) \\ -\delta(b)n_0 & \delta(b) \end{bmatrix}\right],$$

which is rewritten as

$$\begin{bmatrix} -\mathtt{r}_1(a)\delta(b)n_0 + m_0\delta(b)\mathtt{s}_1(a) & -\rho_1(a)m_0\delta(b) + \mathtt{r}_1(a)\delta(b) + m_0\delta(b)\mathtt{g}_1(a) \\ -\mathtt{g}_1(a)\delta(b)n_0 + \delta(b)n_0\rho_1(a) - \delta(b)\mathtt{s}_1(a) & -\mathtt{s}_1(a)m_0\delta(b) + \mathtt{g}_1(a)\delta(b) + \delta(b)n_0\mathtt{r}_1(a) - \delta(b)\mathtt{g}_1(a) \end{bmatrix},$$

and gives

$$\begin{aligned} 0 &= -\mathtt{r}_1(a)\delta(b)n_0 + m_0\delta(b)\mathtt{s}_1(a), \\ 0 &= -\rho_1(a)m_0\delta(b) + \mathtt{r}_1(a)\delta(b) + m_0\delta(b)\mathtt{g}_1(a), \\ 0 &= -\mathtt{g}_1(a)\delta(b)n_0 + \delta(b)n_0\rho_1(a) - \delta(b)\mathtt{s}_1(a), \\ 0 &= -\mathtt{s}_1(a)m_0\delta(b) + \mathtt{g}_1(a)\delta(b) + \delta(b)n_0\mathtt{r}_1(a) - \delta(b)\mathtt{g}_1(a). \end{aligned}$$

If we set $b = 1_{\mathscr{B}}$, then

$$\begin{aligned} \mathtt{r}_1(a) &= \rho_1(a)m_0 - m_0\mathtt{g}_1(a), \\ \mathtt{s}_1(a) &= -\mathtt{g}_1(a)n_0 + n_0\rho_1(a). \end{aligned}$$

Again, by the other part of (3.15), we have

$$\begin{aligned} \mathtt{r}_4(b) &= -m_0\mathtt{g}_4(b) + \rho_4(b)m_0, \\ \mathtt{s}_4(b) &= n_0\rho_4(b) - \mathtt{g}_4(b)n_0. \end{aligned}$$

If m and a are in $\mathscr{M}$ and $\mathscr{A}$, respectively, then $\begin{bmatrix}0 & m\\0 & 0\end{bmatrix}\begin{bmatrix}a & 0\\0 & 0\end{bmatrix} = 0$. Additionally,

$$\begin{aligned}\Phi_{lc}\left(\begin{bmatrix}0 & -am\\0 & 0\end{bmatrix}\right) &= \left[\Phi_{lc}\left(\begin{bmatrix}0 & m\\0 & 0\end{bmatrix}\right), \sigma\left(\begin{bmatrix}a & 0\\0 & 0\end{bmatrix}\right)\right]\\ &= \left[\sigma\left(\begin{bmatrix}0 & m\\0 & 0\end{bmatrix}\right), \Phi_{Jc}\left(\begin{bmatrix}a & 0\\0 & 0\end{bmatrix}\right)\right]. \end{aligned}\tag{3.16}$$

This implies that

$$\begin{aligned}\begin{bmatrix}-\rho_2(am) & -\mathtt{r}_2(am)\\-\mathtt{s}_2(am) & -\mathtt{g}_2(am)\end{bmatrix} &= \left[\begin{bmatrix}\rho_2(m) & \mathtt{r}_2(m)\\ \mathtt{s}_2(m) & \mathtt{g}_2(m)\end{bmatrix}, \begin{bmatrix}\gamma(a) & \gamma(a)m_0\\ n_0\gamma(a) & 0\end{bmatrix}\right]\\ &= \left[\begin{bmatrix}0 & \mu(m)\\0 & 0\end{bmatrix}, \begin{bmatrix}\rho_1(a) & \mathtt{r}_1(a)\\ \mathtt{s}_1(a) & \mathtt{g}_1(a)\end{bmatrix}\right].\end{aligned}$$

It follows that

$$\begin{aligned}\rho_2(am) &= -\rho_2(m)\gamma(a) - \mathtt{r}_2(m)n_0\gamma(a) + \gamma(a)\rho_2(m) + \gamma(a)m_0\mathtt{s}_2(m) = -\mu(m)\mathtt{s}_1(a),\\ \mathtt{r}_2(am) &= -\rho_2(m)\gamma(a)m_0 + \gamma(a)\mathtt{r}_2(m) + \gamma(a)m_0\mathtt{g}_2(m) = -\mu(m)\mathtt{g}_1(a) + \rho_1(a)\mu(m),\\ \mathtt{s}_2(am) &= -\mathtt{s}_2(m)\gamma(a) - \mathtt{g}_2(m)n_0\gamma(a) + n_0\gamma(a)\rho_2(m) = 0,\\ \mathtt{g}_2(am) &= -\mathtt{s}_2(m)\gamma(a)m_0 + n_0\gamma(a)\mathtt{r}_2(m) = \mathtt{s}_1(a)\mu(m).\end{aligned}$$

Taking $a = 1_{\mathscr{A}}$ in the above, we see that $\rho_2(m) = 0$, $\mathtt{s}_2(m) = 0$, $\mathtt{g}_2(m) = 0$, and $\mathtt{r}_2(am) = \gamma(a)\mathtt{r}_2(m) = -\mu(m)\mathtt{g}_1(a)+\rho_1(a)\mu(m)$. Likewise with $\begin{bmatrix}0 & 0\\0 & b\end{bmatrix}\begin{bmatrix}0 & m\\0 & 0\end{bmatrix} = 0$, we have $\mathtt{r}_2(mb) = -\rho_4(b)\mu(m)+\mu(m)\mathtt{g}_4(b) = \mathtt{r}_2(m)\delta(b)$.

If a and n are in $\mathscr{A}$ and $\mathscr{N}$, respectively, then $\begin{bmatrix}a & 0\\0 & 0\end{bmatrix}\begin{bmatrix}0 & 0\\n & 0\end{bmatrix} = 0$, and we have

$$\begin{aligned}\Phi_{lc}\left(\begin{bmatrix}0 & 0\\-na & 0\end{bmatrix}\right) &= \left[\Phi_{lc}\left(\begin{bmatrix}a & 0\\0 & 0\end{bmatrix}\right), \sigma\left(\begin{bmatrix}0 & 0\\n & 0\end{bmatrix}\right)\right]\\ &= \left[\sigma\left(\begin{bmatrix}a & 0\\0 & 0\end{bmatrix}\right), \Phi_{lc}\left(\begin{bmatrix}0 & 0\\n & 0\end{bmatrix}\right)\right]. \end{aligned}\tag{3.17}$$

By (3.17), we deduce that

$$\begin{bmatrix}\rho_3(na) & \mathtt{r}_3(na)\\ \mathtt{s}_3(na) & \mathtt{g}_3(na)\end{bmatrix} = \begin{bmatrix}-\mathtt{r}_1(a)\nu(n) & 0\\ -\mathtt{g}_1(a)\nu(n) + \nu(n)\rho_1(a) & \nu(n)\mathtt{r}_1(a)\end{bmatrix},$$

which is rewritten as

$$\begin{bmatrix}-\gamma(a)\rho_3(n) - \gamma(a)m_0\mathtt{s}_3(n) + \rho_3(n)\gamma(a) + \mathtt{r}_3(n)n_0\gamma(a) & -\gamma(a)\mathtt{r}_3(n) - \gamma(a)m_0\mathtt{g}_3(n) + \rho_3(n)\gamma(a)m_0\\ -n_0\gamma(a)\rho_3(n) + \mathtt{s}_3(n)\gamma(a) + \mathtt{g}_3(n)n_0\gamma(a) & -n_0\gamma(a)\mathtt{r}_3(n) + \mathtt{s}_3(n)\gamma(a)m_0\end{bmatrix},$$

and gives

$$\rho_3(na) = -\mathtt{r}_1(a)\nu(n) = -\gamma(a)\rho_3(n) - \gamma(a)m_0\mathtt{s}_3(n) + \rho_3(n)\gamma(a) + \mathtt{r}_3(n)n_0\gamma(a),$$

$$\begin{aligned}
\mathtt{r}_3(na) &= 0 = -\gamma(a)\mathtt{r}_3(n) - \gamma(a)m_0\mathtt{g}_3(n) + \rho_3(n)\gamma(a)m_0,\\
\mathtt{s}_3(na) &= -\mathtt{g}_1(a)\nu(n) + \nu(n)\rho_1(a) = -n_0\gamma(a)\rho_3(n) + \mathtt{s}_3(n)\gamma(a) + \mathtt{g}_3(n)n_0\gamma(a),\\
\mathtt{g}_3(na) &= \nu(n)\mathtt{r}_1(a) = -n_0\gamma(a)\mathtt{r}_3(n) + \mathtt{s}_3(n)\gamma(a)m_0.
\end{aligned}$$

Set $a = 1_{\mathscr{A}}$, and we find $\rho_3(n) = 0$, $\mathtt{r}_3(n) = 0$, $\mathtt{g}_3(n) = 0$, and $\mathtt{s}_3(na) = -\mathtt{g}_1(a)\nu(n) + \nu(n)\rho_1(a) = \mathtt{s}_3(n)\gamma(a)$. Similar with $\begin{bmatrix} 0 & 0 \\ n & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix} = 0$, we have $\mathtt{s}_3(bn) = \delta(b)\mathtt{s}_3(n) = -\nu(n)\rho_4(b) + \mathtt{g}_4(b)\nu(n)$.

Next, if m and n are in $\mathscr{M}$ and $\mathscr{N}$, respectively, then $\begin{bmatrix} mn & m \\ 0 & 0 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ n & 0 \end{bmatrix} = 0$. Therefore,

$$\begin{aligned}
\Phi_{lc}\left(\begin{bmatrix} mn & m \\ -nmn & -nm \end{bmatrix}\right) &= \left[\Phi_{lc}\left(\begin{bmatrix} mn & m \\ 0 & 0 \end{bmatrix}\right), \sigma\left(\begin{bmatrix} -1 & 0 \\ n & 0 \end{bmatrix}\right)\right]\\
&= \left[\sigma\left(\begin{bmatrix} mn & m \\ 0 & 0 \end{bmatrix}\right), \Phi_{lc}\left(\begin{bmatrix} -1 & 0 \\ n & 0 \end{bmatrix}\right)\right]. \qquad (3.18)
\end{aligned}$$

By (3.18), we obtain

$$\begin{aligned}
&\begin{bmatrix} \rho_1(mn) + \rho_2(m) - \rho_3(nmn) - \rho_4(nm) & \mathtt{r}_1(mn) + \mathtt{r}_2(m) - \mathtt{r}_3(nmn) - \mathtt{r}_4(nm) \\ \mathtt{s}_1(mn) + \mathtt{s}_2(m) - \mathtt{s}_3(nmn) - \mathtt{s}_4(nm) & \mathtt{g}_1(mn) + \mathtt{g}_2(m) - \mathtt{g}_3(nmn) - \mathtt{g}_4(nm) \end{bmatrix}\\
&\qquad = \left[\begin{bmatrix} \rho_1(mn) + \rho_2(m) & \mathtt{r}_1(mn) + \mathtt{r}_2(m) \\ \mathtt{s}_1(mn) + \mathtt{s}_2(m) & \mathtt{g}_1(mn) + \mathtt{g}_2(m) \end{bmatrix}, \begin{bmatrix} -1 & -m_0 \\ -n_0 + \nu(n) & 0 \end{bmatrix}\right].
\end{aligned}$$

This equation gives $\rho_1(mn) - \rho_4(nm) = \mathtt{r}_2(m)\nu(n)$ and $\mathtt{g}_1(mn) - \mathtt{g}_4(nm) = -\nu(n)\mathtt{r}_2(m)$. Likewise, for any $m \in \mathscr{M}$ and $n \in \mathscr{N}$, we have $\begin{bmatrix} 0 & 0 \\ n & nm \end{bmatrix}\begin{bmatrix} 0 & m \\ 0 & -1 \end{bmatrix} = 0$. Thus,

$$\begin{aligned}
\Phi_{lc}\left(\begin{bmatrix} -mn & -mnm \\ n & nm \end{bmatrix}\right) &= \left[\Phi_{lc}\left(\begin{bmatrix} 0 & 0 \\ n & nm \end{bmatrix}\right), \sigma\left(\begin{bmatrix} 0 & m \\ 0 & -1 \end{bmatrix}\right)\right]\\
&= \left[\sigma\left(\begin{bmatrix} 0 & 0 \\ n & nm \end{bmatrix}\right), \Phi_{lc}\left(\begin{bmatrix} 0 & m \\ 0 & -1 \end{bmatrix}\right)\right]. \qquad (3.19)
\end{aligned}$$

By (3.19), we deduce that $\rho_1(mn) - \rho_4(nm) = \mu(m)\mathtt{s}_3(n)$ and $\mathtt{g}_1(mn) - \mathtt{g}_4(nm) = -\mathtt{s}_3(n)\mu(m)$. □

The next theorem provides sufficient and necessary conditions under which a Lie σ−centralizer at the zero product is proper on an order two generalized matrix algebra.

Theorem 3.5. *Let* $(\mathscr{B}, \mathscr{A})$*−bimodule* $\mathscr{N}$ *and* $(\mathscr{A}, \mathscr{B})$*−bimodule* $\mathscr{M}$ *on generalized matrix algebra* $\mathscr{G}$ *have the weaker condition as follows:*

$$a \in \mathscr{A}, \quad \mathscr{N}a = 0 \ \text{ and } \ a\mathscr{M} = 0 \Rightarrow a = 0$$

$$b \in \mathscr{B}, \quad b\mathscr{N} = 0 \ \text{ and } \ \mathscr{M}b = 0 \Rightarrow b = 0$$

and let $\Phi_{lc} : \mathscr{G} \to \mathscr{G}$ *be a Lie* σ*−centralizer at the zero product on* $\mathscr{G}$ *such that* $\Phi_{lc}\left(\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} * & * \\ * & \mathtt{g}_1(a) \end{bmatrix}$ *and* $\Phi_{lc}\left(\begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}\right) = \begin{bmatrix} \rho_4(b) & * \\ * & * \end{bmatrix}$ *for every* $a \in \mathscr{A}$ *and* $b \in \mathscr{B}$*. Then, the following arguments are identical:*

(i) $\phi_{lc}(S) = \lambda\sigma(S) + \tau(S)$ *for any* $S \subset \mathscr{G}$*, where* $\lambda \in Z(\mathscr{G})$ *and* $\tau : \mathscr{G} \to Z(\mathscr{G})$ *is a linear map in which* $\tau([G_1,G_2]) = 0$*, for any* $G_1, G_2 \in \mathscr{G}$ *with* $G_1G_2 = 0$.

(ii) $\mathtt{g}_1(\mathscr{A}) \subseteq \pi_{\mathscr{B}}(Z(\sigma(\mathscr{G})))$ *and* $\rho_4(\mathscr{B}) \subseteq \pi_{\mathscr{A}}(Z(\sigma(\mathscr{G})))$.

(iii) $\mathtt{g}_1(1_{\mathscr{A}}) \in \pi_{\mathscr{B}}(Z(\sigma(\mathscr{G})))$ *and* $\rho_4(1_{\mathscr{B}}) \in \pi_{\mathscr{A}}(Z(\sigma(\mathscr{G})))$.

Proof. By Theorem 3.4, the form of ϕ_{lc} is as follows

$$\Phi_{lc}\left(\begin{bmatrix} a & m \\ n & b \end{bmatrix}\right) = \begin{bmatrix} \rho_1(a) + \rho_4(b) & \mathtt{r}_1(a) + \mathtt{r}_2(m) + \mathtt{r}_4(b) \\ \mathtt{s}_1(a) + \mathtt{s}_3(n) + \mathtt{s}_4(b) & \mathtt{g}_1(a) + \mathtt{g}_4(b) \end{bmatrix},$$

where $\rho_1, \rho_4, \mathtt{r}_1, \mathtt{r}_2, \mathtt{r}_3, \mathtt{s}_1, \mathtt{s}_3, \mathtt{s}_4, \mathtt{g}_1$, and $\mathtt{g}_4$ are all linear maps and have properties in Theorem 3.4.

(i) ⇒ (ii) Let $m \in \mathscr{M}$, $n \in \mathscr{N}$, and $a \in \mathscr{A}$ be arbitrary elements. Take $G_1 = \begin{bmatrix} 0 & am \\ na & 0 \end{bmatrix}$, and we have $\sigma(G_1) = \begin{bmatrix} 0 & \gamma(a)\mu(m) \\ \nu(n)\gamma(a) & 0 \end{bmatrix}$ and

$$\phi_{lc}(G_1) = \begin{bmatrix} 0 & \mathtt{r}_2(am) \\ \mathtt{s}_3(na) & 0 \end{bmatrix} = \begin{bmatrix} 0 & \rho_1(a)\mu(m) - \mu(m)\mathtt{g}_1(a) \\ \nu(n)\rho_1(a) - \mathtt{g}_1(a)\nu(n) & 0 \end{bmatrix}.$$

Let $\lambda = \begin{bmatrix} a_1 & 0 \\ 0 & \eta(a_1) \end{bmatrix}$, where $a_1 \in \pi_{\mathscr{A}}(Z(\sigma(\mathscr{G})))$, $\tau(G_1) = \begin{bmatrix} a_2 & 0 \\ 0 & \eta(a_2) \end{bmatrix}$, and $a_2 \in \pi_{\mathscr{A}}(Z(\sigma(\mathscr{G})))$. By assumption, we get

$$\phi_{lc}(G_1) = \lambda\sigma(G_1) + \tau(G_1) = \begin{bmatrix} a_1 & 0 \\ 0 & \eta(a_1) \end{bmatrix}\begin{bmatrix} 0 & \gamma(a)\mu(m) \\ \nu(n)\gamma(a) & 0 \end{bmatrix} + \begin{bmatrix} a_2 & 0 \\ 0 & \eta(a_2) \end{bmatrix}$$

$$= \begin{bmatrix} a_2 & a_1\gamma(a)\mu(m) \\ \eta(a_1)\nu(n)\gamma(a) & \eta(a_2) \end{bmatrix},$$

and comparing the last two results of ϕ_{lc}, we get

$$\rho_1(a)\mu(m) - \mu(m)\mathtt{g}_1(a) = a_1\gamma(a)\mu(m),$$

$$\nu(n)\rho_1(a) - \mathtt{g}_1(a)\nu(n) = \eta(a_1)\nu(n)\gamma(a) = \nu(n)a_1\gamma(a).$$

Therefore,

$$(\rho_1(a) - a_1\gamma(a))\mu(m) = \mu(m)\mathtt{g}_1(a),$$

$$\nu(n)(\rho_1(a) - a_1\gamma(a)) = \mathtt{g}_1(a)\nu(n).$$

As $m \in \mathscr{M}$, $n \in \mathscr{N}$ are arbitrary, focusing on Remark 2.1, we get $\mathtt{g}_1(a) \in \pi_{\mathscr{B}}(Z(\sigma(\mathscr{G})))$ for each $a \in \mathscr{A}$. Now, we will use the same argument for arbitrary elements $b \in \mathscr{B}$, $m \in \mathscr{M}$, and $n \in \mathscr{N}$. Upon taking $\lambda = \begin{bmatrix} a_1 & 0 \\ 0 & \eta(a_1) \end{bmatrix}$, $G_2 = \begin{bmatrix} 0 & mb \\ bn & 0 \end{bmatrix}$, and $\tau(G_2) = \begin{bmatrix} a_2 & 0 \\ 0 & \eta(a_2) \end{bmatrix}$, we can conclude from hypothesis and Theorem 3.4 that

$$\phi_{lc}(G_2) = \begin{bmatrix} 0 & \mathtt{r}_2(mb) \\ \mathtt{s}_3(bn) & 0 \end{bmatrix} = \begin{bmatrix} 0 & \mu(m)\mathtt{g}_4(b) - \rho_4(b)\mu(m) \\ \mathtt{g}_4(b)\nu(n) - \nu(n)\rho_4(b) & 0 \end{bmatrix},$$

and

$$\phi_{lc}(G_2) = \lambda\sigma(G_2) + \tau(G_2) = \begin{bmatrix} a_1 & 0 \\ 0 & \eta(a_1) \end{bmatrix}\begin{bmatrix} 0 & \mu(m)\delta(b) \\ \delta(b)\nu(n) & 0 \end{bmatrix} + \begin{bmatrix} a_2 & 0 \\ 0 & \eta(a_2) \end{bmatrix}$$

$$= \begin{bmatrix} a_2 & a_1\mu(m)\delta(b) \\ \eta(a_1)\delta(b)\nu(n) & \eta(a_2) \end{bmatrix}.$$

From the last two relations, we get

$$\mu(m)\mathtt{g}_4(b) - \rho_4(b)\mu(m) = a_1\mu(m)\delta(b) = \mu(m)\eta(a_1)\delta(b),$$

$$\mathtt{g}_4(b)\nu(n) - \nu(n)\rho_4(b)) = \eta(a_1)\delta(b)\nu(n).$$

Hence,

$$\mu(m)(\mathtt{g}_4(b) - \eta(a_1)\delta(b)) = \rho_4(b)\mu(m),$$

$$(\mathtt{g}_4(b) - \eta(a_1)\delta(b))\nu(n) = \nu(n)\rho_4(b)).$$

Now, for $b \in \mathscr{B}$, $m \in \mathscr{M}$, and $n \in \mathscr{N}$, in view of Remark 2.1, we obtain $\rho_4(b) \in \pi_{\mathscr{A}}(Z(\sigma(\mathscr{G})))$.

(ii) ⇒ (i). We can define the following well-defined functions based on the hypothesis:

$$\mathtt{p}' : \mathscr{A} \to \mathscr{A}; \qquad \mathtt{p}'(a) = \rho_1(a) - \eta^{-1}(\mathtt{g}_1(a)),$$

$$\mathtt{q}' : \mathscr{B} \to \mathscr{B}; \qquad \mathtt{q}'(b) = \mathtt{g}_4(b) - \eta(\rho_4(b)),$$

where the maps $\mathtt{p}'$ and $\mathtt{q}'$ are linear. Applied to Theorem 3.4 (iii), we obtain

$$\mathtt{r}_2(am) = \gamma(a)\mathtt{r}_2(m) = \mathtt{p}'(a)\mu(m),$$

and

$$\mathtt{r}_2(mb) = \mathtt{r}_2(m)\delta(b) = \mu(m)\mathtt{q}'(b),$$

for every $m \in \mathscr{M}$, $a \in \mathscr{A}$, and $b \in \mathscr{B}$. Therefore,

$$\mathtt{r}_2(m) = \mathtt{p}'(1_{\mathscr{A}})\mu(m) = \mu(m)\mathtt{q}'(1_{\mathscr{B}}),$$

for every $m \in \mathcal{M}$.

Now, consider $a \in \mathcal{A}$, $b \in \mathcal{B}$, $n \in \mathcal{N}$, and by Theorem 3.4 (iii), we get

$$\mathtt{s}_3(na) = \mathtt{s}_3(n)\gamma(a) = \nu(n)\mathtt{p}'(a),$$

$$\mathtt{s}_3(bn) = \delta(b)\mathtt{s}_3(n) = \mathtt{q}'(b)\nu(n),$$

and

$$\mathtt{s}_3(n) = \nu(n)\mathtt{p}'(1_{\mathcal{A}}) = \mathtt{q}'(1_{\mathcal{B}})\nu(n).$$

By Remark 2.1, we find $\mathtt{p}'(1_{\mathcal{A}}) \in \pi_{\mathcal{A}}(Z(\sigma(\mathcal{G})))$, $\mathtt{q}'(1_{\mathcal{B}}) \in \pi_{\mathcal{B}}(Z(\sigma(\mathcal{G})))$, and $\eta(\mathtt{p}'(1_{\mathcal{A}})) = \mathtt{q}'(1_{\mathcal{B}})$.

Next, consider $a, a' \in \mathcal{A}$, $m \in \mathcal{M}$, $n \in \mathcal{N}$, and by Theorem 3.4 (iii) and (iv), we have

$$\mathtt{r}_2(aa'm) = \gamma(a)\mathtt{r}_2(a'm) = \gamma(a)\gamma(a')\mathtt{r}_2(m) = \mathtt{p}'(aa')\mu(m),$$

and

$$\mathtt{r}_2(aa'm) = \gamma(a)\mathtt{r}_2(a'm) = \gamma(a)\mathtt{p}'(a')\mu(m).$$

Therefore, $(\mathtt{p}'(aa') - \gamma(a)\mathtt{p}'(a'))\mu(m) = 0$ and, hence, $(\mathtt{p}'(aa') - \gamma(a)\mathtt{p}'(a'))\mathcal{M} = 0$. Also, $\mathtt{s}_3(naa') = \nu(n)\mathtt{p}'(aa')$ and $\mathtt{r}_2(aa'm) = \nu(m)\gamma(a)\mathtt{p}'(a')$. Thus, $\nu(n)(\mathtt{p}'(aa') - \gamma(a)\mathtt{p}'(a')) = 0$. Since ν is an isomorphism, we have $\mathcal{N}(\mathtt{p}'(aa') - \gamma(a)\mathtt{p}'(a')) = 0$. Now, by assumption we get $\mathtt{p}'(aa') = \gamma(a)\mathtt{p}(a')$ for any $a, a' \in \mathcal{A}$. So, $\mathtt{p}'(a) = \mathtt{p}'(a1_{\mathcal{A}}) = \gamma(a)\mathtt{p}'(1_{\mathcal{A}})$, and since $\mathtt{p}'(1_{\mathcal{A}}) \in \pi_{\mathcal{A}}(Z(\sigma(\mathcal{G})))$, it follows that $\mathtt{p}'(a) = \gamma(a)\mathtt{p}'(1_{\mathcal{A}}) = \mathtt{p}'(1_{\mathcal{A}})\gamma(a)$ for any $a \in \mathcal{A}$. Similarly, it follows $\mathtt{q}'(b) = \mathtt{q}'(1_{\mathcal{B}})\delta(b) = \delta(b)\mathtt{q}'(1_{\mathcal{B}})$ for any $b \in \mathcal{B}$.

If m and n are in $\mathcal{M}$ and $\mathcal{N}$, respectively, then $\rho_1(mn) - \rho_4(nm) = \mathtt{r}_2(m)\nu(n) = \mathtt{p}'(1_{\mathcal{A}})\mu(m)\nu(n) = \mathtt{p}'(mn) = \rho_1(mn) - \eta^{-1}(\mathtt{g}_1(mn))$. Moreover, $\rho_4(nm) = \eta^{-1}(\mathtt{g}_1(mn))$ and $\rho_1(mn) - \eta^{-1}(\mathtt{g}_1(mn)) = \mathtt{r}_2(m)\nu(n)$. Thus, $\mathtt{p}'(mn) = \mathtt{r}_2(m)\nu(n)$ for any $m \in \mathcal{M}$ and $n \in \mathcal{N}$. Using the same procedures as above and Theorem 3.4, we arrive to the conclusion that $\mathtt{q}'(nm) = \mathtt{s}_3(n)\mu(m)$. Set $\lambda = \begin{bmatrix} \mathtt{p}'(1_{\mathcal{A}}) & 0 \\ 0 & \mathtt{q}'(1_{\mathcal{B}}) \end{bmatrix}$, since $\eta(\mathtt{p}'1_{\mathcal{A}}) = \mathtt{q}'(1_{\mathcal{B}})$; consequently, $\lambda \in Z(\sigma(\mathcal{G}))$.

Now, consider the linear map $\tau : \mathcal{G} \to \mathcal{G}$ such that

$$\tau\left(\begin{bmatrix} a & m \\ n & b \end{bmatrix}\right) = \begin{bmatrix} \eta^{-1}(\mathtt{g}_1(mn)) + \rho_4(b) & 0 \\ 0 & \mathtt{g}_1(a) + \eta(\rho_4(b)) \end{bmatrix}.$$

We can conclude from our hypothesis $\tau(G) \in Z(\sigma(\mathcal{G}))$ for every $G \in Z(\sigma(\mathcal{G}))$.

Now, for any $G = \begin{bmatrix} a & m \\ n & b \end{bmatrix} \in \mathcal{G}$, we obtain $\phi_{lc}(G) = \lambda\sigma(G) + \tau(G)$. Finally, by using Lie σ-centralizer at the zero product and the above results for any $G_1, G_2 \in \mathcal{G}$ where $G_1G_2 = 0$, we have $\tau([G_1, G_2]) = \phi_{lc}([G_1, G_2]) - \lambda\sigma([G_1, G_2]) = 0$. Next, (ii) $\Rightarrow$ (iii) is obvious.

(iii) $\Rightarrow$ (ii) Let $a_0 = \rho_1(1_{\mathcal{A}}) - \eta^{-1}(\mathtt{g}_1(1_{\mathcal{A}}))$ and $b_0 = \mathtt{g}_4(1_{\mathcal{B}}) - \eta^{-1}(P_4(1_{\mathcal{B}}))$. By assumption and Theorem 3.4 for any $a \in \mathcal{A}$, $m \in \mathcal{M}$, and $n \in \mathcal{N}$, we have $\mathtt{r}_2(m) = a_0\mu(m)$. Therefore, $\gamma(a)\mathtt{r}_2(m) = \gamma(a)a_0\mu(m) = \rho_1(a)\mu(m) - \mu(m)\mathtt{g}_1(a)$. So, $(\rho_1(a) - \gamma(a)a_0)\mu(m) = \mu(m)\mathtt{g}_1(a)$. Similarly, $s_3(m) = \nu(n)a_0$ and $\mathtt{s}_3(n)\gamma(a) = \nu(n)a_0\gamma(a) = \nu(n)\rho_1(a) - \mathtt{g}_1(a)\nu(n)$. Hence, $\nu(n)(\rho_1(a) - a_0\gamma(a)) = \mathtt{g}_1(a)\nu(n)$. From Remark 2.1, we get $\mathtt{g}_1(a) \in \pi_{\mathcal{B}}(Z(\sigma(\mathcal{G})))$ and $\eta^{-1}(\mathtt{g}_1(a)) = \rho_1(a) - a_0\gamma(a)$, for any $a \in \mathcal{A}$. Similarly, by using the same argument and Theorem 3.4 for any $b \in \mathcal{B}$, $m \in \mathcal{M}$, and $n \in \mathcal{N}$, we see that $\mu(m)(\mathtt{g}_4(b) - b_0\delta(b)) = \rho_4(b)\mu(m)$. Also, $(\mathtt{g}_4(b) - b_0\delta(b))\nu(n) = \nu(n)\rho_4(b)$. By Remark 2.1, it follows that $\rho_4(b) \in \pi_{\mathcal{A}}(Z(\sigma(\mathcal{G})))$ and $\eta(\rho_4(b)) = \mathtt{g}_4(b) - \delta(b)b_0$, for any $b \in \mathcal{B}$. □

4. Conclusions

We studied the structure of Lie and Jordan $\sigma-$centralizers at zero products in order two generalized matrix algebras. We considered the case where $(\mathscr{A},\mathscr{B})$ and $(\mathscr{B},\mathscr{A})$ are bimodules of $\mathscr{M}$ and $\mathscr{N}$, respectively, with $\mathscr{A},\mathscr{B}$ being unital $\mathscr{R}-$algebras. The $\mathscr{R}-$algebra $\mathscr{G} = \mathscr{G}(\mathscr{A},\mathscr{M},\mathscr{N},\mathscr{B})$ is a generalized matrix algebra defined by the Morita context $(\mathscr{A},\mathscr{B},\mathscr{M},\mathscr{N},\zeta_{\mathscr{M}\mathscr{N}},\chi_{\mathscr{N}\mathscr{M}})$. This study based on the present structure of generalized matrix algebras has produced numerous substantial results that enhance our understanding of these algebraic structures. One of the key results is that every Jordan $\sigma-$centralizer at the zero products on the order two generalized matrix algebra $\mathscr{G}$ is indeed a $\sigma-$centralizer at the zero product. This result establishes a fundamental connection between two classes of algebraic objects, shedding light on their connection within the context of generalized matrix algebras. Furthermore, we provided necessary and sufficient conditions under which a Lie $\sigma-$centralizer at the zero product is proper on the order two generalized matrix algebra $\mathscr{G}$. This characterization is valuable, as it allows for a deeper understanding of the algebraic properties and structure of these Lie $\sigma-$centralizers. This work extends the theory of generalized matrix algebras and provides novel insights into the characteristics of Lie and Jordan $\sigma-$centralizers in this algebraic scenario.

Author contributions

Mohd Arif Raza: Conceptualization, Methodology, Supervision, Writing-review & editing; Huda Eid Almehmadi: Writing-original draft, Methodology. All the authors have read and approved the final version of the manuscript for publication.

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Conflict of interest

The authors disclose that they have no conflicts of interest.

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