Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 9(10): 26602–26630.
DOI: 10.3934/math.20241294
Received: 05 July 2024
Revised: 29 August 2024
Accepted: 10 September 2024
Published: 13 September 2024

Research article

Besicovitch almost periodic solutions for a stochastic generalized Mackey-Glass hematopoietic model

Xianying Huang and Yongkun Li*

Department of Mathematics, Yunnan University, Kunming, Yunnan 650091, China

* **Correspondence:** Email: yklie@ynu.edu.cn.

Abstract: This article aimed to investigate the existence and stability of Besicovitch almost periodic (B_{ap}) positive solutions for a stochastic generalized Mackey-Glass hematopoietic model. To begin with, we used stochastic analysis theory, inequality techniques, and fixed point theorems to prove the existence and uniqueness of $\mathcal{L}^p$-bounded and $\mathcal{L}^p$-uniformly continuous positive solutions for the model under consideration. Then, we used definitions to prove that this unique positive solution is also a B_{ap} solution in finite-dimensional distributions. In addition, we established the global exponential stability of the B_{ap} positive solution using reduction to absurdity. Finally, we provided a numerical example to verify the effectiveness of our conclusions.

Keywords: stochastic Mackey-Glass hematopoietic model; Besicovitch almost periodic solutions; finite-dimensional distributions; stability
Mathematics Subject Classification: 34K14, 34K50, 92D25

1. Introduction

In 1977, in order to study the regulation and control mechanisms of human diseases, Mackey and Glass [1] put forward a hematopoiesis model with a single-humped production rate, which is governed by the following differential equation with delay:

$$\frac{dN(t)}{dt} = -\alpha N(t) + \frac{\beta N(t-\tau)}{1 + N^m(t-\tau)}, \quad m > 1 \tag{1.1}$$

and is now called the Mackey-Glass hematopoietic model. Biologically, the state variable $N(t)$ signifies the density of mature circulating cells at time t, in which the cells are assumed to be lost from the circulation at a rate α and they are believed to be generated at a rate β, and τ indicates the time delay required to manufacture mature white blood cells in the bone marrow.

Model (1.1) has attracted much attention in the past decades because of its simple representation, complex and diverse dynamic properties, and wide practical significance. Moreover, the generalized models in various senses of this model have been constantly proposed and studied. For instance, recently, in Refs. [2,3], generalized Mackey-Glass models with iterative terms were studied, in Ref. [4], a fractional order Mackey-Glass model was considered, and in Ref. [5], a Mackey-Glass model with diffusion terms was investigated. Also, considering the influence of changing environmental factors, model (1.1) has been modified to a nonautonomous system. For nonautonomous systems, there is generally no constant equilibrium solution. As a consequence, the existence of periodic solutions [6–12], almost periodic solutions [13–19], and pseudo almost periodic solutions [20] in the nonautonomous form of model (1.1) and its generalized forms have been extensively studied. For instance, in [17], the existence of Bohr almost periodic solutions of the following generalized hematopoietic model:

$$x'(t) = -a(t)x(t) + \sum_{j=1}^{n} \frac{b_j(t)x^{\alpha}(t-\tau(t))}{1+x^{\beta}(t-\tau(t))}$$

was studied by utilizing the fixed point theorem in a cone; and in [18,19], the existence of Bohr almost periodic solutions of the following generalized hematopoietic models:

$$x'(t) = -a(t)x(t) + \sum_{j=1}^{n} \beta_j \gamma_j(t) \frac{x^{m_j}\left(t-\tau_j(t)\right)}{1+x^{n_j}\left(t-\tau_j(t)\right)}$$

and

$$x'(t) = -a(t)x(t) + \sum_{j=1}^{n} \beta_j \gamma_j(t) \frac{x^{m}\left(t-\tau_j(t)\right)}{1+x^{n}\left(t-\tau_j(t)\right)}$$

were investigated by using the fixed point theorem of a mixed monotone operator, respectively.

On the one hand, it is well-known that the B_{ap} concept embodies the Bohr almost periodic, Stepanov almost periodic, and Weyl almost periodic concepts as its special cases, which describes a more complicated phenomenon of recurrent motions [21,22]. However, there have been no published papers on the B_{ap} oscillations of hematopoietic models so far.

On the other hand, due to the fact that any real system is subject to various random factors, considering a stochastic hematopoietic model is more practical.

Moreover, for stochastic differential equations, studying almost periodic solutions in the distribution sense is more rational than studying almost periodic solutions in the mean square sense [23].

Inspired by the analysis of the above aspects, in the present paper, we consider the following stochastic generalized hematopoietic model with multiple delays:

$$dx(t) = \left[-a(t)x(t) + \sum_{j=1}^{n} \frac{b_j(t)x^{\alpha_j}\left(t-\tau_j(t)\right)}{1+x^{\beta_j}\left(t-\tau_j(t)\right)}\right]dt + \sum_{j=1}^{n} c_j(t)\sigma_j\left(x(t-\eta_j(t))\right)d\omega_j(t), \tag{1.2}$$

where x indicates the density of the mature blood cells circulating in the bloodstream; $a(t)$ signifies the rate of cells loss during the blood circulation at time t; $\tau_j(t)$ and $\eta_j(t)$ denote time delays; $\alpha_j \in (0,1), \beta_j \in \mathbb{R}^+$ are constants; $\omega_j(t)$ indicates a Brownian motion defined on a complete probability space; $c_j(t)$ denotes the noise intensity; and $\sigma_j : \mathbb{R} \to \mathbb{R}$ is a Borel measurable function.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathrm{P})$ signify a complete probability space, where $\{\mathcal{F}_t\}_{t\geq 0}$ is a natural filtration. Denote by $C^b_{\mathcal{F}_0}([-\vartheta, 0], \mathbb{R})$ the family of all bounded, $\mathcal{F}_0$-measurable, and real-valued random variables.

System (1.2) is supplemented with the following initial value condition:

$$x(s) = \varphi(s), \quad \varphi(t_0) > 0, \quad s \in [t_0 - \vartheta, t_0], \tag{1.3}$$

in which $\varphi \in C^b_{\mathcal{F}_0}([-\vartheta, 0], \mathbb{R}^+)$, $\vartheta = \max_{1\leq j\leq n}\{\sup_{t\in\mathbb{R}} \tau_j(t), \sup_{t\in\mathbb{R}} \eta_j(t)\}$.

Our primary objective of this article is to investigate the existence and stability of B_{ap}-solutions in finite-dimensional distributions to system (1.2). This is the first article to study the B_{ap} solutions of stochastic population systems. The main difficulty of this article is that the space composed of B_{ap} random processes is not a complete space, so the existence of B_{ap} solutions cannot be directly proved using the fixed point theorem. At the same time, the presence of fractional nonlinearity in the model poses difficulties for related estimation. To overcome these difficulties, we first establish the existence of $\mathcal{L}^p$-bounded and $\mathcal{L}^p$-uniformly continuous solutions to system (1.2), and then use the definition to verify that this solution is also a B_{ap} solution. At the same time, we overcome the difficulties caused by the fractional nonlinearity using the mean value theorem and inequality techniques.

The structure of the remaining part of this article is as follows: In Section 2, we review some basic definitions of almost periodic functions and almost periodic stochastic processes, as well as an important inequality that will be repeatedly used in this paper. In Section 3, we present the main results and proof of this paper. In Section 4, we provide a numerical example to verify the effectiveness of our results. Finally, in Section 5, we present a brief conclusion to conclude this paper.

2. Preliminaries

For a random variable $Z : (\Omega, \mathcal{F}, P) \to \mathbb{R}$, we will use $law(Z) := P \circ Z^{-1}$ to signify its distribution and EZ to stand for its expectation. Let $\mathcal{L}^p(\Omega, \mathbb{R})$ be the space of all real-valued random variables Z with $E|Z|^p = \int_\Omega |Z|^p dP < \infty$, where $1 \leq p < \infty$.

Definition 2.1. [24] *A stochastic process $Z : \mathbb{R} \to \mathcal{L}^p(\Omega, \mathbb{R})$ is called $\mathcal{L}^p$-continuous if for every $s \in \mathbb{R}$,*

$$\lim_{t\to s} E|Z(t) - Z(s)|^p = 0.$$

It is $\mathcal{L}^p$-bounded if $\sup\limits_{t\in\mathbb{R}} E|Z(t)|^p < \infty$.

Let $\mathcal{L}^p_{loc}(\mathbb{R}, \mathbb{R})$ be the space of all the locally pth integrable functions from $\mathbb{R} \to \mathbb{R}$. For $g \in \mathcal{L}^\infty_{loc}(\mathbb{R}, \mathbb{R})$, its Besicovitch semi-norm is defined as follows:

$$\|g\|_{B^p} = \left(\limsup_{l\to+\infty} \frac{1}{2l} \int_{-l}^{l} |g(t)|^p dt\right)^{\frac{1}{p}}.$$

Definition 2.2. [22] *A continuous function $h : \mathbb{R} \to \mathbb{R}$ is said to be (Bohr) almost periodic if for any $\varepsilon > 0$ there exists a positive number l, such that every interval with length l contains a number τ such that for all $t \in \mathbb{R}$ it holds*

$$|h(t + \tau) - h(t)| < \varepsilon.$$

We will signify the set of all such functions by $AP(\mathbb{R}, \mathbb{R})$.

Definition 2.3. [22] *A function $h \in \mathcal{L}^p_{loc}(\mathbb{R}, \mathbb{R})$ is said to be pth B_{ap} if for every $\varepsilon > 0$ there exists a positive number l, such that every interval of length l contains a number τ such that*

$$\|h(\cdot + \tau) - h(\cdot)\|_{B^p} < \varepsilon.$$

Denote by $B^p(\mathbb{R}, \mathbb{R})$ the set of all such functions.

Consider the h-dimensional Euclidean metric space $(\mathbb{R}^h, d_h)$, where $d_h(u, v) = \max_{1\le i\le h}\{|x_i - y_i|\}$ for $u = (x_1, x_2, \cdots, x_h), v = (y_1, y_2, \cdots, y_h) \in \mathbb{R}^h$. Denote by $P(\mathbb{R}^h)$ the collection of Borel probability measures on $\mathbb{R}^h$. Let $BC(\mathbb{R}^h, \mathbb{R})$ be the space of all bounded and continuous functions from $\mathbb{R}^h$ to $\mathbb{R}$ with the norm $\|g\|_U := \sup_{u\in\mathbb{R}^h} |g(u)| < \infty$.

For $g \in BC(\mathbb{R}^h, \mathbb{R}), \zeta, \xi \in P(\mathbb{R}^h)$, we define

$$\|g\|_A = \sup_{u\neq v} \frac{|g(u) - g(v)|}{d_h(u, v)}, \quad \|g\|_B = \max\{\|g\|_U, \|g\|_A\}, \quad d_C(\zeta, \xi) = \sup_{\|g\|_B\le 1} \left| \int_{\mathbb{R}^h} g d(\zeta - \xi) \right|.$$

According to [25], the space $(P(\mathbb{R}^h), d_C)$ is a separable complete metric space.

Definition 2.4. [26] *A stochastic process $Y : \mathbb{R} \to \mathcal{L}^p(\Omega, \mathbb{R})$ is said to be a B_{ap} stochastic process in finite-dimensional distributions if for every finite points $t_1, t_2, \cdots, t_h \in \mathbb{R}$ and every $\varepsilon > 0$, there exists an $l(\varepsilon) > 0$ such that every interval of length l contains a number τ such that*

$$\limsup_{l\to+\infty} \left(\frac{1}{2l} \int_{-l}^{l} d_C^p \left(G_Y(t + \tau), G_Y(t)\right) dt \right)^{\frac{1}{p}} \le \varepsilon,$$

where the mapping $G_Y : \mathbb{R} \to \left(P(\mathbb{E}^h), d_C(\cdot, \cdot)\right)$ is defined by

$$G_Y(t) = law(Y(t + t_1), Y(t + t_2), \cdots, Y(t + t_h)).$$

Lemma 2.1. [27] *If $f \in \mathcal{L}^2(J, \mathbb{R})$, $p > 2$, $B(t)$ is a Brownian motion, then*

$$E\left(\sup_{t\in J} \left| \int_{t_0}^{t} f(s) dB(s) \right|^p \right) \le C_p E \left(\int_{t_0}^{T} |f(s)|^2 ds \right)^{\frac{p}{2}},$$

where $C_p = \left(\dfrac{p^{p+1}}{2(p-1)^{p-1}} \right)^{\frac{p}{2}}$.

Let $\mathcal{L}^\infty(\mathbb{R}, \mathbb{R})$ indicate the set of all essentially bounded and measurable functions from $\mathbb{R}$ to $\mathbb{R}$ with the norm $\|x\|_\infty = \inf\{r \ge 0 : |x(t)| \le r\ a.e.\ t \in \mathbb{R}\}$, where $x \in \mathcal{L}^\infty(\mathbb{R}, \mathbb{R})$.

In the rest part of this paper, we will use the following symbols:

$$\underline{a} = \inf_{t\in\mathbb{R}} a(t), \quad a^+ = \sup_{t\in\mathbb{R}} a(t), \quad \hat{b}_j = \|b_j\|_\infty, \quad c_j^+ = \sup_{t\in\mathbb{R}} c_j(t),$$
$$\tau_j^+ = \sup_{t\in\mathbb{R}} \tau_j(t), \quad \dot{\tau}_j^+ = \sup_{t\in\mathbb{R}} \dot{\tau}_j(t), \quad \eta_j^+ = \sup_{t\in\mathbb{R}} \eta_j(t), \quad \dot{\eta}_j^+ = \sup_{t\in\mathbb{R}} \dot{\eta}_j(t),$$

and need the following conditions:

(H_1) Functions $a(t), c_j(t) \in AP(\mathbb{R}, \mathbb{R}^+)$ with $\underline{a} > 0$, $\tau_j(t), \eta_j(t) \in AP(\mathbb{R}, \mathbb{R}^+) \cap C^1(\mathbb{R}, \mathbb{R})$ satisfying $1 - \dot{\tau}_j^+ > 0$ and $1 - \dot{\eta}_j^+ > 0$, and $b_j(t) \in B^p(\mathbb{R}, \mathbb{R}^+) \cap \mathcal{L}^\infty(\mathbb{R}, \mathbb{R})$, where $j = 1, 2, \ldots$.

(H_2) For $j = 1, 2, \cdots, n$, $\sigma_j \in C(\mathbb{R}, \mathbb{R}^+)$, and there exist constants $L_j^\sigma > 0$ such that for all $x, y \in \mathbb{R}$,

$$|\sigma_j(y) - \sigma_j(x)| \le L_j^\sigma |y - x|.$$

(H_3) $p \ge 2$, $\dfrac{1}{p} + \dfrac{1}{q} = 1$, and there exists a constant $\kappa > 0$ such that

$$\begin{aligned}
\varsigma_1 :=& 2^{p-1}\Big[\Big(\frac{p}{q\underline{a}}\Big)^{\frac{p}{q}} \frac{q}{p\underline{a}}\Big(\sum_{j=1}^n \big(\hat{b}_j\big)^q\Big)^{\frac{p}{q}} n + 2^{p-1} C_p\Big(\frac{p-2}{2\underline{a}}\Big)^{\frac{p-2}{2}} \frac{q}{p\underline{a}}\Big(\sum_{j=1}^n \big(c_j^+\big)^q\Big)^{\frac{p}{q}} \\
&\times \sum_{j=1}^n \big(L_j^\sigma\big)^p\Big]\kappa^p + 4^{p-1} C_p\Big(\frac{p-2}{2\underline{a}}\Big)^{\frac{p-2}{2}} \frac{q}{p\underline{a}}\Big(\sum_{j=1}^n \big(c_j^+\big)^q\Big)^{\frac{p}{q}} \sum_{j=1}^n \sigma_j^p(0) \le \kappa^p, (p > 2), \\
\varsigma_2 :=& 2\Big\{\frac{1}{(\underline{a})^2} \sum_{j=1}^n \big(\hat{b}_j\big)^2 n + \frac{1}{\underline{a}} \sum_{j=1}^n \big(c_j^+\big)^2 \sum_{j=1}^n \big(L_j^\sigma\big)^2\Big\}\kappa^2 \\
&+ \frac{2}{\underline{a}} \sum_{j=1}^n \big(c_j^+\big)^2 \sum_{j=1}^n \sigma_j^2(0) \le \kappa^2, (p = 2).
\end{aligned}$$

(H_4) $p \ge 2$, $\dfrac{1}{p} + \dfrac{1}{q} = 1$, and

$$\begin{aligned}
\Theta_1 :=& 2^{p-1}\Big\{2^{p-1}\Big(\frac{p}{q\underline{a}}\Big)^{\frac{p}{q}} \frac{q}{p\underline{a}}\Big(\sum_{j=1}^n \big(\hat{b}_j\big)^q\Big)^{\frac{p}{q}} \sum_{j=1}^n ((2\kappa)^{(\alpha_j-1)p} + (2\kappa)^{(\alpha_j+\beta_j-1)p}) \\
&+ C_p\Big(\frac{p-2}{2\underline{a}}\Big)^{\frac{p-2}{2}} \frac{q}{p\underline{a}}\Big(\sum_{j=1}^n (c_j^+)^q\Big)^{\frac{p}{q}} \sum_{j=1}^n \big(L_j^\sigma\big)^p\Big\} < 1, (p > 2), \\
\Theta_2 :=& 4\Big(\frac{1}{\underline{a}}\Big)^2 \sum_{j=1}^n \big(\hat{b}_j\big)^2 \sum_{j=1}^n ((2\kappa)^{2(\alpha_j-1)} + (2\kappa)^{2(\alpha_j+\beta_j-1)}) + \frac{1}{\underline{a}} \\
&\times \sum_{j=1}^n \big(c_j^+\big)^2 \sum_{j=1}^n \big(L_j^\sigma\big)^2 < 1, (p = 2),
\end{aligned}$$

in which κ is the same as that in (H_3).

(H_5) $p \ge 2$, $\dfrac{1}{p} + \dfrac{1}{q} = 1$, and

$$\begin{aligned}
\varpi_1 :=& 12^{p-1}\frac{q}{p\underline{a}}\Big\{2^{p-1}\Big(\frac{p}{q\underline{a}}\Big)^{\frac{p}{q}}\Big(\sum_{j=1}^n \big(\hat{b}_j\big)^q\Big)^{\frac{p}{q}} \sum_{j=1}^n ((2\kappa)^{(\alpha_j-1)p} + (2\kappa)^{(\alpha_j+\beta_j-1)p}) \\
&\times \frac{e^{\frac{p\underline{a}}{q}\tau_j^+}}{1 - \dot{\tau}_j^+} + C_p\Big(\frac{p-2}{2\underline{a}}\Big)^{\frac{p-2}{2}}\Big(\sum_{j=1}^n \big(c_j^+\big)^q\Big)^{\frac{p}{q}} \sum_{j=1}^n \big(L_j^\sigma\big)^p \frac{e^{\frac{p}{q}\underline{a}\eta_j^+}}{1 - \dot{\eta}_j^+}\Big\} < 1, (p > 2),
\end{aligned}$$

$$\varpi_2 = 12\frac{1}{\underline{a}}\Big\{2\frac{1}{\underline{a}}\sum_{j=1}^{n}\left(\hat{b}_j\right)^2\sum_{j=1}^{n}((2\kappa)^{(\alpha_j-1)2}+(2\kappa)^{(\alpha_j+\beta_j-1)2})\frac{e^{\underline{a}\tau_j^+}}{1-\dot{\tau}_j^+}$$
$$+\sum_{j=1}^{n}\left(c_j^+\right)^2\sum_{j=1}^{n}\left(L_j^\sigma\right)^2\frac{e^{2\underline{a}\eta_j^+}}{1-\dot{\eta}_j^+}\Big\}<1,(p=2),$$

in which κ is the same as that in (H_3).

3. Main results

Let $\mathbb{Y}$ be the space of all $\mathcal{L}^p$-bounded and uniformly $\mathcal{L}^p$-continuous functions from $\mathbb{R}$ to $\mathcal{L}^p(\Omega,\mathbb{R})$, then with the norm $\|x\|_{\mathbb{Y}} = \sup_{t\in\mathbb{R}}(E|x(t)|^p)^{\frac{1}{p}}$ for $x\in\mathbb{Y}$, it is a Banach space.

Definition 3.1. *An $\mathcal{F}_t$-progressively measurable stochastic process $x(t)$ is called a solution of system* (1.2)*, if $x(t)$ satisfies initial value condition* (1.3) *for $t\in[t_0-\vartheta,t_0]$ and the following stochastic process:*

$$x(t) = x(t_0)e^{-\int_{t_0}^t a(u)du}+\int_{t_0}^t e^{-\int_s^t a(u)du}\sum_{j=1}^{n}\frac{b_j(s)x^{\alpha_j}\left(s-\tau_j(s)\right)}{1+x^{\beta_j}\left(s-\tau_j(s)\right)}ds$$
$$+\int_{t_0}^t e^{-\int_s^t a(u)du}\sum_{j=1}^{n}c_j(s)\sigma_j\left(x(s-\eta_j(s))\right)d\omega_j(s),\ \ t\ge t_0. \tag{3.1}$$

Let $t_0\to-\infty$, and we obtain

$$x(t) = \int_{-\infty}^t e^{-\int_s^t a(u)du}\sum_{j=1}^{n}\frac{b_j(s)x^{\alpha_j}\left(s-\tau_j(s)\right)}{1+x^{\beta_j}\left(s-\tau_j(s)\right)}ds$$
$$+\int_{-\infty}^t e^{-\int_s^t a(u)du}\sum_{j=1}^{n}c_j(s)\sigma_j\left(x(s-\eta_j(s))\right)d\omega_j(s), \tag{3.2}$$

which is also a solution of system (1.2).

Theorem 3.1. *Assume that assumptions (H_1)–(H_4) hold. Then system* (1.2) *has a unique $\mathcal{L}^p$-bounded and uniformly $\mathcal{L}^p$-continuous solution x in $\mathbb{Y}^*=\{\upsilon|\upsilon\in\mathbb{Y},\upsilon(t)\ge 0,\|\upsilon\|_{\mathbb{Y}}\le\kappa\}$ that solves* (3.2)*, where κ is mentioned in (H_3).*

Proof. We define an operator Ψ: $\mathbb{Y}^*\to\mathbb{Y}^*$ as follows:

$$(\Psi\varphi)(t) = \int_{-\infty}^t e^{-\int_s^t a(u)du}\sum_{j=1}^{n}\frac{b_j(s)\varphi^{\alpha_j}\left(s-\tau_j(s)\right)}{1+\varphi^{\beta_j}\left(s-\tau_j(s)\right)}ds$$
$$+\int_{-\infty}^t e^{-\int_s^t a(u)du}\sum_{j=1}^{n}c_j(s)\sigma_j\left(\varphi(s-\eta_j(s))\right)d\omega_j(s), \tag{3.3}$$

where $\varphi \in \mathbb{Y}^*$, $t \in \mathbb{R}$. To begin with, we will verify that Ψ maps $\mathbb{Y}^*$ into $\mathbb{Y}^*$. For any $\varphi \in \mathbb{Y}^*$, it holds

$$\begin{aligned}\|\Psi\varphi\|_{\mathbb{Y}}^p \le & 2^{p-1}\sup_{t\in\mathbb{R}}\Big\{E\Big|\int_{-\infty}^t e^{-\int_s^t a(u)du}\sum_{j=1}^n \frac{b_j(s)\varphi^{\alpha_j}\left(s-\tau_j(s)\right)}{1+\varphi^{\beta_j}\left(s-\tau_j(s)\right)}ds\Big|^p\Big\} \\ & + 2^{p-1}\sup_{t\in\mathbb{R}}\Big\{E\Big|\int_{-\infty}^t e^{-\int_s^t a(u)du}\sum_{j=1}^n c_j(s)\sigma_j\left(\varphi(s-\eta_j(s))\right)d\omega_j(s)\Big|^p\Big\} := A_1 + A_2. \end{aligned} \tag{3.4}$$

By the Hölder inequality, we deduce that

$$\begin{aligned} A_1 \le & 2^{p-1}\sup_{t\in\mathbb{R}}\Big\{E\Big|\Big[\int_{-\infty}^t e^{-\frac{q}{p}\int_s^t a(u)du}ds\Big]^{\frac{p}{q}}\Big[\int_{-\infty}^t e^{-\frac{p}{q}\int_s^t a(u)du}\Big(\sum_{j=1}^n \frac{b_j(s)\varphi^{\alpha_j}\left(s-\tau_j(s)\right)}{1+\varphi^{\beta_j}\left(s-\tau_j(s)\right)}\Big)^p ds\Big]\Big|\Big\} \\ \le & 2^{p-1}\sup_{t\in\mathbb{R}}\Big\{\Big(\frac{p}{q\underline{a}}\Big)^{\frac{p}{q}}\Big[\int_{-\infty}^t e^{-\frac{p}{q}\underline{a}(t-s)}\Big(\sum_{j=1}^n\left(\hat{b}_j\right)^q\Big)^{\frac{p}{q}}\sum_{j=1}^n E\Big|\frac{\varphi^{\alpha_j}\left(s-\tau_j(s)\right)}{1+\varphi^{\beta_j}\left(s-\tau_j(s)\right)}\Big|^p ds\Big]\Big\} \\ \le & 2^{p-1}\Big\{\Big(\frac{p}{q\underline{a}}\Big)^{\frac{p}{q}}\frac{q}{p\underline{a}}\Big(\sum_{j=1}^n\left(\hat{b}_j\right)^q\Big)^{\frac{p}{q}} n\Big\}\kappa^p. \end{aligned} \tag{3.5}$$

When $p > 2$, in view of Lemma 2.1 and the Hölder inequality, it holds

$$\begin{aligned} A_2 \le & 2^{p-1}C_p\sup_{t\in\mathbb{R}}\Big\{E\Big[\int_{-\infty}^t e^{-2\int_s^t a(u)du}\Big|\sum_{j=1}^n c_j(s)\sigma_j(\varphi(s-\eta_j(s)))\Big|^2 ds\Big]^{\frac{p}{2}}\Big\} \\ \le & 2^{p-1}C_p\sup_{t\in\mathbb{R}}\Big\{\Big(\frac{p-2}{2\underline{a}}\Big)^{\frac{p-2}{2}}\frac{q}{p\underline{a}}\Big(\sum_{j=1}^n\left(c_j^+\right)^q\Big)^{\frac{p}{q}}\sum_{j=1}^n E|\sigma_j(\varphi(s-\eta_j(s)))-\sigma_j(0)+\sigma_j(0)|^p\Big\} \\ \le & 4^{p-1}C_p\Big(\frac{p-2}{2\underline{a}}\Big)^{\frac{p-2}{2}}\frac{q}{p\underline{a}}\Big(\sum_{j=1}^n\left(c_j^+\right)^q\Big)^{\frac{p}{q}}\sum_{j=1}^n[\left(L_j^\sigma\right)^p\kappa^p+\sigma_j^p(0)]. \end{aligned} \tag{3.6}$$

When $p = 2$, by the Itô isometry and the Hölder inequality, we get

$$\begin{aligned} A_2 \le & 2\sup_{t\in\mathbb{R}}\Big\{\int_{-\infty}^t e^{-2\int_s^t a(u)du}E\Big|\sum_{j=1}^n c_j(t)\sigma_j(\varphi(s-\eta_j(s)))\Big|^2 ds\Big\} \\ \le & 2\sup_{t\in\mathbb{R}}\Big\{\int_{-\infty}^t e^{-2\int_s^t a(u)du}\sum_{j=1}^n(c_j^+)^2\sum_{j=1}^n E|\sigma_j(\varphi(s-\eta_j(s)))-\sigma_j(0)+\sigma_j(0)|^2 ds\Big\} \\ \le & 4\frac{1}{2\underline{a}}\sum_{j=1}^n\left(c_j^+\right)^2\sum_{j=1}^n[\left(L_j^\sigma\right)^2\kappa^2+\sigma_j^2(0)]. \end{aligned} \tag{3.7}$$

For the case of $p > 2$, from (3.4)–(3.6), it follows that

$$\begin{aligned} \|\Psi\varphi\|_{\mathbb{Y}}^p \le & 2^{p-1}\Big\{\Big(\frac{p}{q\underline{a}}\Big)^{\frac{p}{q}}\frac{q}{p\underline{a}}\Big(\sum_{j=1}^n\left(\hat{b}_j\right)^q\Big)^{\frac{p}{q}}n\kappa^p + 2^{p-1}C_p\Big(\frac{p-2}{2\underline{a}}\Big)^{\frac{p-2}{2}}\frac{q}{p\underline{a}}\Big(\sum_{j=1}^n\left(c_j^+\right)^q\Big)^{\frac{p}{q}} \\ & \times\sum_{j=1}^n[\left(L_j^\sigma\right)^p\kappa^p+\sigma_j^p(0)]\Big\}. \end{aligned} \tag{3.8}$$

For the case of $p=2$, from (3.4), (3.5), and (3.7), it follows that

$$\|\Psi\varphi\|_{\mathbb{Y}}^2 \leq 2\Big\{\frac{1}{(\underline{a})^2}\sum_{j=1}^{n}\big(\hat{b}_j\big)^2 n\kappa^2 + \frac{2}{\underline{a}}\sum_{j=1}^{n}\big(c_j^+\big)^2\sum_{j=1}^{n}[\big(L_j^\sigma\big)^2\kappa^2+\sigma_j^2(0)]\Big\}. \tag{3.9}$$

The inequalities (3.8) and (3.9) combined with condition (H_3) imply that $\|\Psi\varphi\|_{\mathbb{Y}} \leq \kappa$.

By (H_1), we have $a(t), b_j(t), c_j(t) \geq 0$ for $t \in \mathbb{R}$. Hence, it is easy to see that $\Psi\varphi \geq 0$ for any $\varphi \in \mathbb{Y}^*$.

For every $\varphi \in \mathbb{Y}^*$, we show the uniformly $\mathcal{L}^p$-continuous of $\Psi\varphi$. For any $t \in \mathbb{R}$ and $h>0$, we have

$$\begin{aligned}
&E|(\Psi\varphi)(t+h)-(\Psi\varphi)(t)|^p\\
=&E\Big|\int_{-\infty}^{t+h} e^{-\int_s^{t+h}a(u)\mathrm{d}u}\sum_{j=1}^{n}\frac{b_j(s)\varphi^{\alpha_j}(s-\tau_j(s))}{1+\varphi^{\beta_j}(s-\tau_j(s))}\mathrm{d}s\\
&+\int_{-\infty}^{t+h} e^{-\int_s^{t+h}a(u)\mathrm{d}u}\sum_{j=1}^{n}c_j(s)\sigma_j(\varphi(s-\eta_j(s)))\mathrm{d}\omega_j(s)\\
&-\int_{-\infty}^{t} e^{-\int_s^{t}a(u)\mathrm{d}u}\sum_{j=1}^{n}\frac{b_j(s)\varphi^{\alpha_j}(s-\tau_j(s))}{1+\varphi^{\beta_j}(s-\tau_j(s))}\mathrm{d}s\\
&-\int_{-\infty}^{t} e^{-\int_s^{t}a(u)\mathrm{d}u}\sum_{j=1}^{n}c_j(s)\sigma_j(\varphi(s-\eta_j(s)))\mathrm{d}\omega_j(s)\Big|^p\\
=&E\Big|\int_{-\infty}^{t} e^{-\int_s^{t+h}a(u)\mathrm{d}u}\sum_{j=1}^{n}\frac{b_j(s)\varphi^{\alpha_j}(s-\tau_j(s))}{1+\varphi^{\beta_j}(s-\tau_j(s))}\mathrm{d}s\\
&+\int_{t}^{t+h} e^{-\int_s^{t+h}a(u)\mathrm{d}u}\sum_{j=1}^{n}\frac{b_j(s)\varphi^{\alpha_j}(s-\tau_j(s))}{1+\varphi^{\beta_j}(s-\tau_j(s))}\mathrm{d}s\\
&+\int_{-\infty}^{t} e^{-\int_s^{t+h}a(u)\mathrm{d}u}\sum_{j=1}^{n}c_j(s)\sigma_j(\varphi(s-\eta_j(s)))\mathrm{d}\omega_j(s)\\
&+\int_{t}^{t+h} e^{-\int_s^{t+h}a(u)\mathrm{d}u}\sum_{j=1}^{n}c_j(s)\sigma_j(\varphi(s-\eta_j(s)))\mathrm{d}\omega_j(s)\\
&-\int_{-\infty}^{t} e^{-\int_s^{t}a(u)\mathrm{d}u}\sum_{j=1}^{n}\frac{b_j(s)\varphi^{\alpha_j}(s-\tau_j(s))}{1+\varphi^{\beta_j}(s-\tau_j(s))}\mathrm{d}s\\
&-\int_{-\infty}^{t} e^{-\int_s^{t}a(u)\mathrm{d}u}\sum_{j=1}^{n}c_j(s)\sigma_j(\varphi(s-\eta_j(s)))\mathrm{d}\omega_j(s)\Big|^p\\
\leq&4^{p-1}E\Big|\int_{-\infty}^{t}\Big(e^{-\int_s^{t+h}a(u)du}-e^{-\int_s^{t}a(u)du}\Big)\sum_{j=1}^{n}\frac{b_j(s)\varphi^{\alpha_j}(s-\tau_j(s))}{1+\varphi^{\beta_j}(s-\tau_j(s))}ds\Big|^p\\
&+4^{p-1}E\Big|\int_{t}^{t+h} e^{-\int_s^{t+h}a(u)du}\sum_{j=1}^{n}\frac{b_j(s)\varphi^{\alpha_j}(s-\tau_j(s))}{1+\varphi^{\beta_j}(s-\tau_j(s))}ds\Big|^p\\
&+4^{p-1}E\Big|\int_{-\infty}^{t}\Big(e^{-\int_s^{t+h}a(u)du}-e^{-\int_s^{t}a(u)du}\Big)\sum_{j=1}^{n}c_j(s)\sigma_j(\varphi(s-\eta_j(s)))d\omega_j(s)\Big|^p
\end{aligned}$$

$$+\,4^{p-1}E\Big|\int_t^{t+h} e^{-\int_s^{t+h}a(u)du}\sum_{j=1}^n c_j(s)\sigma_j(\varphi(s-\eta_j(s)))d\omega_j(s)\Big|^p$$
$$:=A_3+A_4+A_5+A_6. \tag{3.10}$$

Since

$$\begin{aligned}\Big|e^{-\int_s^{t+h}a(u)du}-e^{-\int_s^{t}a(u)du}\Big| \le& e^{-\underline{a}(t-s)}\Big|\int_s^{t+h}a(u)du-\int_s^t a(u)du\Big|\\ \le& e^{-\underline{a}(t-s)}\Big|\int_t^{t+h}a(u)du\Big|\le e^{-\underline{a}(t-s)}a^+h,\end{aligned} \tag{3.11}$$

by the Hölder inequality, we deduce that

$$\begin{aligned}A_3 \le& 4^{p-1}\Big[\int_{-\infty}^t\Big|e^{-\int_s^{t+h}a(u)du}-e^{-\int_s^{t}a(u)du}\Big|^{\frac{q}{p}}ds\Big]^{\frac{p}{q}}\Big[\int_{-\infty}^t\Big|e^{-\int_s^{t+h}a(u)du}-e^{-\int_s^{t}a(u)du}\Big|^{\frac{p}{q}}\\ &\times E\Big|\sum_{j=1}^n\frac{b_j(s)\varphi^{\alpha_j}(s-\tau_j(s))}{1+\varphi^{\beta_j}(s-\tau_j(s))}\Big|^p ds\Big]\\ \le& 4^{p-1}\Big[\int_{-\infty}^t\Big(e^{-\underline{a}(t-s)}a^+h\Big)^{\frac{q}{p}}ds\Big]^{\frac{p}{q}}\Big[\int_{-\infty}^t\Big(e^{-\underline{a}(t-s)}a^+h\Big)^{\frac{p}{q}}\\ &\times\Big(\sum_{j=1}^n\big(\hat{b}_j\big)^q\Big)^{\frac{p}{q}}\sum_{j=1}^n E|\varphi^{\alpha_j}(s-\tau_j(s))|^p ds\Big]\\ \le& 4^{p-1}\Big(\frac{p}{q\underline{a}}\Big)^{\frac{p}{q}}(a^+h)^{\frac{p}{q}+1}\frac{q}{p\underline{a}}\Big(\sum_{j=1}^n\big(\hat{b}_j\big)^q\Big)^{\frac{p}{q}}\sum_{j=1}^n\kappa^{\alpha_j p}.\end{aligned} \tag{3.12}$$

About A_4, we have

$$\begin{aligned}A_4 \le& 4^{p-1}\Big[\int_t^{t+h}e^{-q\int_s^{t+h}a(u)du}ds\Big]^{\frac{p}{q}}\Big[\int_t^{t+h}E\Big|\sum_{j=1}^n\frac{b_j(s)\varphi^{\alpha_j}(s-\tau_j(s))}{1+\varphi^{\beta_j}(s-\tau_j(s))}\Big|^p ds\Big]\\ \le& 4^{p-1}\Big[\int_t^{t+h}1ds\Big]^{\frac{p}{q}}\Big[\int_t^{t+h}\Big(\sum_{j=1}^n\big(\hat{b}_j\big)^q\Big)^{\frac{p}{q}}\sum_{j=1}^n E|\varphi^{\alpha_j}|^p ds\Big]\\ \le& 4^{p-1}\Big(\sum_{j=1}^n\big(\hat{b}_j\big)^q\Big)^{\frac{p}{q}}h^{\frac{p}{q}+1}\sum_{j=1}^n\kappa^{\alpha_j p}.\end{aligned} \tag{3.13}$$

When $p>2$, by virtue of Lemma 2.1 and Hölder's inequality, we derive that

$$\begin{aligned}A_5 \le& 4^{p-1}C_pE\Big[\int_{-\infty}^t\Big(e^{-\int_s^{t+h}a(u)du}-e^{-\int_s^{t}a(u)du}\Big)^2\Big|\sum_{j=1}^n c_j(s)\sigma_j(\varphi(s-\eta_j(s)))\Big|^2 ds\Big]^{\frac{p}{2}}\\ \le& 4^{p-1}C_p\Big[\int_{-\infty}^t\Big(e^{-\int_s^{t+h}a(u)du}-e^{-\int_s^{t}a(u)du}\Big)^{2\times\frac{p}{p-2}\times\frac{1}{p}}ds\Big]^{\frac{p-2}{p}\times\frac{p}{2}}\\ &\times\Big[\int_{-\infty}^t\Big(e^{-\int_s^{t+h}a(u)du}-e^{-\int_s^{t}a(u)du}\Big)^{2\times\frac{1}{q}\times\frac{p}{2}}E\Big|\sum_{j=1}^n c_j(s)\sigma_j(\varphi(s-\eta_j(s)))\Big|^p ds\Big]\end{aligned}$$

$$
\begin{aligned}
&\leq 4^{p-1}C_p\Big[\int_{-\infty}^{t}\Big(e^{-\underline{a}(t-s)}a^+h\Big)^{\frac{2}{p-2}}ds\Big]^{\frac{p-2}{2}}\Big[\int_{-\infty}^{t}\Big(e^{-\underline{a}(t-s)}a^+h\Big)^{\frac{p}{q}}\Big(\sum_{j=1}^{n}\Big(c_j^+\Big)^q\Big)^{\frac{p}{q}}\\
&\quad\times\sum_{j=1}^{n}E|\sigma_j(\varphi(s-\eta_j(s)))-\sigma_j(0)+\sigma_j(0)|^p ds\Big]\\
&\leq 8^{p-1}C_p\Big[\int_{-\infty}^{t}\Big(e^{-\underline{a}(t-s)}a^+h\Big)^{\frac{2}{p-2}}ds\Big]^{\frac{p-2}{2}}\Big[\int_{-\infty}^{t}\Big(e^{-\underline{a}(t-s)}a^+h\Big)^{\frac{p}{q}}\Big(\sum_{j=1}^{n}\Big(c_j^+\Big)^q\Big)^{\frac{p}{q}}\\
&\quad\times\sum_{j=1}^{n}[E|\sigma_j(\varphi(s-\eta_j(s)))-\sigma_j(0)|^p+|\sigma_j(0)|^p]ds\Big]\\
&\leq 8^{p-1}C_p\Big[\int_{-\infty}^{t}\Big(e^{-\underline{a}(t-s)}a^+h\Big)^{\frac{2}{p-2}}ds\Big]^{\frac{p-2}{2}}\Big[\int_{-\infty}^{t}\Big(e^{-\underline{a}(t-s)}a^+h\Big)^{\frac{p}{q}}\Big(\sum_{j=1}^{n}\Big(c_j^+\Big)^q\Big)^{\frac{p}{q}}\\
&\quad\times\sum_{j=1}^{n}[(L_j^\sigma)^pE|\varphi|^p+|\sigma_j(0)|^p]ds\Big]\\
&\leq 8^{p-1}C_p\Big(\frac{p-2}{2\underline{a}}\Big)^{\frac{p-2}{2}}\frac{q}{p\underline{a}}(a^+h)^{\frac{p}{q}+1}\Big(\sum_{j=1}^{n}\Big(c_j^+\Big)^q\Big)^{\frac{p}{q}}\sum_{j=1}^{n}[(L_j^\sigma)^p\kappa^p+\sigma_j^p(0)],
\end{aligned}
\tag{3.14}
$$

$$
\begin{aligned}
A_6&\leq 4^{p-1}C_pE\Big[\int_t^{t+h}e^{-2\int_s^{t+h}a(u)du}\Big|\sum_{j=1}^{n}c_j(s)\sigma_j(\varphi(s-\eta_j(s)))\Big|^2ds\Big]^{\frac{p}{2}}\\
&\leq 4^{p-1}C_p\Big[\int_t^{t+h}e^{-\frac{2p}{p-2}\int_s^{t+h}a(u)du}ds\Big]^{\frac{p-2}{p}\times\frac{p}{2}}\Big[\int_t^{t+h}E\Big|\sum_{j=1}^{n}c_j(s)\sigma_j(\varphi(s-\eta_j(s)))\Big|^pds\Big]\\
&\leq 8^{p-1}C_p\Big[\int_t^{t+h}1ds\Big]^{\frac{p-2}{2}}\Big[\int_t^{t+h}\Big(\sum_{j=1}^{n}\Big(c_j^+\Big)^q\Big)^{\frac{p}{q}}\sum_{j=1}^{n}[\Big(L_j^\sigma\Big)^pE|\varphi|^p+|\sigma_j(0)|^p]ds\Big]\\
&\leq 8^{p-1}C_ph^{\frac{p-2}{2}+1}\Big(\sum_{j=1}^{n}\Big(c_j^+\Big)^q\Big)^{\frac{p}{q}}\sum_{j=1}^{n}[\Big(L_j^\sigma\Big)^p\kappa^p+\sigma_j^p(0)].
\end{aligned}
\tag{3.15}
$$

When $p = 2$, by using the Itô isometry and the Hölder inequality, there holds:

$$
\begin{aligned}
A_5&\leq 8\int_{-\infty}^{t}\Big|e^{-\int_s^{t+h}a(u)du}-e^{-\int_s^{t}a(u)du}\Big|^2\sum_{j=1}^{n}\Big(c_j^+\Big)^2\sum_{j=1}^{n}[\Big(L_j^\sigma\Big)^2E|\varphi|^2+\sigma_j^2(0)]ds\\
&\leq 8\int_{-\infty}^{t}(e^{-\underline{a}(t-s)}a^+h)^2ds\sum_{j=1}^{n}\Big(c_j^+\Big)^2\sum_{j=1}^{n}[\Big(L_j^\sigma\Big)^2\kappa^2+\sigma_j^2(0)]\\
&\leq 8\frac{1}{2\underline{a}}(a^+)^2h^2\sum_{j=1}^{n}\Big(c_j^+\Big)^2\sum_{j=1}^{n}[\Big(L_j^\sigma\Big)^2\kappa^2+\sigma_j^2(0)],
\end{aligned}
\tag{3.16}
$$

$$
\begin{aligned}
A_6&\leq 4\int_t^{t+h}E\Big|\sum_{j=1}^{n}c_j(s)\sigma_j(\varphi(s-\eta_j(s)))\Big|^2ds\\
&\leq 8\int_t^{t+h}\sum_{j=1}^{n}(c_j^+)^2\sum_{j=1}^{n}[(L_j^\sigma)^2E|\varphi|^2+\sigma_j^2(0)]ds
\end{aligned}
$$

$$\le 8\sum_{j=1}^{n}\left(c_j^+\right)^2\sum_{j=1}^{n}[\left(L_j^\sigma\right)^2\kappa^2+\sigma_j^2(0)]h. \tag{3.17}$$

Putting (3.12)–(3.17) into (3.10), for the case of $p>2$, we have

$$\begin{aligned}
&E|(\Psi\varphi)(t+h)-(\Psi\varphi)(t)|^p\\
\le&4^{p-1}\Big\{\Big(\frac{p}{q\underline{a}}\Big)^{\frac{p}{q}}(a^+)^{\frac{p}{q}+1}h^{\frac{p}{q}}\frac{q}{p\underline{a}}\Big(\sum_{j=1}^{n}\left(\hat{b}_j\right)^q\Big)^{\frac{p}{q}}\sum_{j=1}^{n}\kappa^{\alpha_j p}+\Big(\sum_{j=1}^{n}\left(\hat{b}_j\right)^q\Big)^{\frac{p}{q}}h^{\frac{p}{q}}\sum_{j=1}^{n}\kappa^{\alpha_j p}\\
&+2^{p-1}C_p\Big(\frac{p-2}{2\underline{a}}\Big)^{\frac{p-2}{2}}\frac{q}{p\underline{a}}(a^+)^{\frac{p}{q}+1}h^{\frac{p}{q}}\Big(\sum_{j=1}^{n}\left(c_j^+\right)^q\Big)^{\frac{p}{q}}\sum_{j=1}^{n}[\left(L_j^\sigma\right)^p\kappa^p+\sigma_j^p(0)]\\
&+2^{p-1}C_ph^{\frac{p-2}{2}}\Big(\sum_{j=1}^{n}\left(c_j^+\right)^q\Big)^{\frac{p}{q}}\sum_{j=1}^{n}[\left(L_j^\sigma\right)^p\kappa^p+\sigma_j^p(0)]\Big\}h,
\end{aligned}\tag{3.18}$$

and for the case of $p=2$, we gain

$$\begin{aligned}
&E|(\Psi\varphi)(t+h)-(\Psi\varphi)(t)|^2\\
\le&4\Big\{\Big(\frac{1}{\underline{a}}\Big)^2(a^+)^2h\sum_{j=1}^{n}\left(\hat{b}_j\right)^2\sum_{j=1}^{n}\kappa^{2\alpha_j}+\sum_{j=1}^{n}\left(\hat{b}_j\right)^2h\sum_{j=1}^{n}\kappa^{2\alpha_j}+2\frac{1}{2\underline{a}}(a^+)^2h\sum_{j=1}^{n}\left(c_j^+\right)^2\\
&\times\sum_{j=1}^{n}[\left(L_j^\sigma\right)^2\kappa^2+\sigma_j^2(0)]+2\sum_{j=1}^{n}\left(c_j^+\right)^2\sum_{j=1}^{n}[\left(L_j^\sigma\right)^2\kappa^2+\sigma_j^2(0)]\Big\}h.
\end{aligned}\tag{3.19}$$

By (3.18) and (3.19), letting $h\to 0^+$, one can get

$$E|(\Psi\varphi)(t+h)-(\Psi\varphi)(t)|^p\to 0,$$

which means that $\Psi\varphi$ is uniformly $\mathcal{L}^p$-continuous. Therefore, $\Psi(\mathbb{Y}^*)\subset\mathbb{Y}^*$.

Then, we will show that Ψ is a contraction mapping. In fact, note that

$$\begin{aligned}
&\left|\frac{\varphi^{\alpha_j}(s-\tau_j(s))}{1+\varphi^{\beta_j}(s-\tau_j(s))}-\frac{\psi^{\alpha_j}(s-\tau_j(s))}{1+\psi^{\beta_j}(s-\tau_j(s))}\right|\\
=&\Bigg|\frac{\varphi^{\alpha_j}(s-\tau_j(s))-\psi^{\alpha_j}(s-\tau_j(s))}{\left(1+\varphi^{\beta_j}(s-\tau_j(s))\right)\left(1+\varphi^{\beta_j}(s-\tau_j(s))\right)}\\
&+\frac{\varphi^{\alpha_j}(s-\tau_j(s))\left(\psi^{\beta_j}(s-\tau_j(s))-\varphi^{\beta_j}(s-\tau_j(s))\right)}{\left(1+\varphi^{\beta_j}(s-\tau_j(s))\right)\left(1+\varphi^{\beta_j}(s-\tau_j(s))\right)}\\
&+\frac{\varphi^{\beta_j}(s-\tau_j(s))\left(\varphi^{\alpha_j}(s-\tau_j(s))-\psi^{\alpha_j}(s-\tau_j(s))\right)}{\left(1+\varphi^{\beta_j}(s-\tau_j(s))\right)\left(1+\varphi^{\beta_j}(s-\tau_j(s))\right)}\Bigg|\\
\le&|\varphi^{\alpha_j}(s-\tau_j(s))-\psi^{\alpha_j}(s-\tau_j(s))|+|\varphi^{\alpha_j}(s-\tau_j(s))|\\
&\times|\psi^{\beta_j}(s-\tau_j(s))-\varphi^{\beta_j}(s-\tau_j(s))|.
\end{aligned}\tag{3.20}$$

For any $\varphi, \psi \in \mathbb{Y}^*$, one can obtain

$$
\begin{aligned}
& E|(\Psi\varphi)(t) - (\Psi\psi)(t)|^p \\
\leq & 2^{p-1} E\Big| \int_{-\infty}^{t} e^{-\int_s^t a(u)du} \sum_{j=1}^{n} b_j(s) \Big(\frac{\varphi^{\alpha_j}(s-\tau_j(s))}{1+\varphi^{\beta_j}(s-\tau_j(s))} - \frac{\psi^{\alpha_j}(s-\tau_j(s))}{1+\psi^{\beta_j}(s-\tau_j(s))} \Big) ds \Big|^p \\
& + 2^{p-1} E\Big| \int_{-\infty}^{t} e^{-\int_s^t a(u)du} \sum_{j=1}^{n} c_j(s) (\sigma_j\left(\varphi(s-\eta_j(s))\right) - \sigma_j\left(\psi(s-\eta_j(s))\right)) d\omega_j(s) \Big|^p \\
:= & A_{11} + A_{12}.
\end{aligned} \tag{3.21}
$$

By the Hölder inequality and the mean value theorem of calculus, we have

$$
\begin{aligned}
A_{11} \leq & 2^{p-1} \Big[\int_{-\infty}^{t} e^{-\frac{q}{p}\int_s^t a(u)du} ds \Big]^{\frac{p}{q}} \Big[\int_{-\infty}^{t} e^{-\frac{p}{q}\int_s^t a(u)du} E\Big(\sum_{j=1}^{n} b_j(s) (|\varphi^{\alpha_j}(s-\tau_j(s)) \\
& - \psi^{\alpha_j}(s-\tau_j(s))| + |\varphi^{\alpha_j}(s-\tau_j(s))||\psi^{\beta_j}(s-\tau_j(s)) - \varphi^{\beta_j}(s-\tau_j(s))|) \Big)^p ds \Big] \\
\leq & 2^{p-1} \Big[\int_{-\infty}^{t} e^{-\frac{q}{p}\int_s^t a(u)du} ds \Big]^{\frac{p}{q}} \Big[\int_{-\infty}^{t} e^{-\frac{p}{q}\int_s^t a(u)du} \Big(\sum_{j=1}^{n} \left(\hat{b}_j\right)^q \Big)^{\frac{p}{q}} \sum_{j=1}^{n} E(|\varphi^{\alpha_j}(s-\tau_j(s)) \\
& - \psi^{\alpha_j}(s-\tau_j(s))| + |\varphi^{\alpha_j}(s-\tau_j(s))||\psi^{\beta_j}(s-\tau_j(s)) - \varphi^{\beta_j}(s-\tau_j(s))|)^p ds \Big] \\
\leq & 2^{p-1} \Big[\int_{-\infty}^{t} e^{-\frac{q}{p}\int_s^t a(u)du} ds \Big]^{\frac{p}{q}} \Big[\int_{-\infty}^{t} e^{-\frac{p}{q}\int_s^t a(u)du} \Big(\sum_{j=1}^{n} \left(\hat{b}_j\right)^q \Big)^{\frac{p}{q}} \sum_{j=1}^{n} E(|\varphi_1^{\alpha_j-1}||\varphi(s-\tau_j(s)) \\
& - \psi(s-\tau_j(s))| + |\varphi_2^{\beta_j-1}||\varphi^{\alpha_j}(s-\tau_j(s))||\psi(s-\tau_j(s)) - \varphi(s-\tau_j(s))|)^p ds \Big] \\
\leq & 4^{p-1} \Big(\frac{p}{q\underline{a}} \Big)^{\frac{p}{q}} \frac{q}{p\underline{a}} \Big(\sum_{j=1}^{n} \left(\hat{b}_j\right)^q \Big)^{\frac{p}{q}} \sum_{j=1}^{n} ((2\kappa)^{(\alpha_j-1)p} + (2\kappa)^{(\alpha_j-1)p+\beta_j p}) \|\varphi - \psi\|_{\mathbb{Y}}^p,
\end{aligned} \tag{3.22}
$$

where φ_1 and φ_2 are between $\varphi(s-\tau_j(s))$ and $\psi(s-\tau_j(s))$; for this reason, we have used $\|\varphi_1\|_{\mathbb{Y}} \leq 2\kappa$ and $\|\varphi_2\|_{\mathbb{Y}} \leq 2\kappa$ in the derivation above.

For random term A_{12}, when $p > 2$, by Lemma 2.1 and the Hölder inequality, we deduce that

$$
\begin{aligned}
A_{12} \leq & 2^{p-1} C_p E\Big[\int_{-\infty}^{t} e^{-2\int_s^t a(u)du} \Big| \sum_{j=1}^{n} c_j(s) (\sigma_j\left(\varphi(s-\eta_j(s))\right) - \sigma_j\left(\psi(s-\eta_j(s))\right)) \Big|^2 ds \Big]^{\frac{p}{2}} \\
\leq & 2^{p-1} C_p \Big[\int_{-\infty}^{t} \Big(e^{-2\int_s^t a(u)du} \Big)^{\frac{1}{p}\times\frac{p}{p-2}} ds \Big]^{\frac{p-2}{p}\times\frac{p}{2}} \Big[\int_{-\infty}^{t} \Big(e^{-2\int_s^t a(u)du} \Big)^{\frac{1}{q}\times\frac{p}{2}} \\
& \times \Big(\sum_{j=1}^{n} (c_j^+)^q \Big)^{\frac{p}{q}} \sum_{j=1}^{n} \left(L_j^\sigma\right)^p E|\varphi - \psi| ds \Big] \\
\leq & 2^{p-1} C_p \Big(\frac{p-2}{2\underline{a}} \Big)^{\frac{p-2}{2}} \frac{q}{p\underline{a}} \Big(\sum_{j=1}^{n} (c_j^+)^q \Big)^{\frac{p}{q}} \sum_{j=1}^{n} \left(L_j^\sigma\right)^p \|\varphi - \psi\|_{\mathbb{Y}}^p.
\end{aligned} \tag{3.23}
$$

When $p = 2$, by the Itô isometry and the Hölder inequality, we get

$$\begin{aligned}A_{12} \leq & 2\int_{-\infty}^{t} e^{-2\underline{a}(t-s)} \sum_{j=1}^{n}\left(c_j^+\right)^2 \sum_{j=1}^{n}\left(L_j^\sigma\right)^2 E|\varphi-\psi|^2 ds\\ \leq & 2\frac{1}{2\underline{a}}\sum_{j=1}^{n}\left(c_j^+\right)^2 \sum_{j=1}^{n}\left(L_j^\sigma\right)^2 \|\varphi-\psi\|_{\mathbb{Y}}^2.\end{aligned} \tag{3.24}$$

Putting (3.22)–(3.24) into (3.21), we gain

$$\begin{aligned}&\|(\Psi\varphi)(t)-(\Psi\psi)(t)\|_{\mathbb{Y}}^p\\ \leq & 2^{p-1}\Big\{2^{p-1}\Big(\frac{p}{q\underline{a}}\Big)^{\frac{p}{q}}\frac{q}{p\underline{a}}\Big(\sum_{j=1}^{n}\left(\hat{b}_j\right)^q\Big)^{\frac{p}{q}}\sum_{j=1}^{n}((2\kappa)^{(\alpha_j-1)p}+(2\kappa)^{(\alpha_j+\beta_j-1)p})\\ &+C_p\Big(\frac{p-2}{2\underline{a}}\Big)^{\frac{p-2}{2}}\frac{q}{p\underline{a}}\Big(\sum_{j=1}^{n}(c_j^+)^q\Big)^{\frac{p}{q}}\sum_{j=1}^{n}\left(L_j^\sigma\right)^p\Big\}\|\varphi-\psi\|_{\mathbb{Y}}^p\\ =&\Theta_1\|\varphi-\psi\|_{\mathbb{Y}}^p,\quad (p>2),\\ &\|(\Psi\varphi)(t)-(\Psi\psi)(t)\|_{\mathbb{Y}}^2\\ \leq&\Big\{4\Big(\frac{1}{\underline{a}}\Big)^2\sum_{j=1}^{n}\left(\hat{b}_j\right)^2\sum_{j=1}^{n}((2\kappa)^{2(\alpha_j-1)}+(2\kappa)^{2(\alpha_j+\beta_j-1)})+\frac{1}{\underline{a}}\sum_{j=1}^{n}\left(c_j^+\right)^2\sum_{j=1}^{n}\left(L_j^\sigma\right)^2\Big\}\|\varphi-\psi\|_{\mathbb{Y}}^2\\ =&\Theta_2\|\varphi-\psi\|_{\mathbb{Y}}^2,\quad (p=2).\end{aligned}$$

From (H_4), we see that Ψ is a contraction mapping. Hence, Ψ has a unique point $x \in \mathbb{Y}^*$. As a consequence, system (1.2) possesses a unique solution x in $\mathbb{Y}^*$. This finishes the proof. □

Theorem 3.2. *Assume that assumptions* (H_1)–(H_5) *hold. Then, system* (1.2) *has a unique pth* B_{ap} *solution in finite-dimensional distributions in* $\mathbb{Y}^* = \{x | x \in \mathbb{Y}, x(t) \geq 0, \|x\|_{\mathbb{Y}} \leq \kappa\}$, *where* κ *is mentioned in* (H_3).

Proof. In the light of Theorem 3.1, we can assume that x is the unique solution to system (1.2) in $\mathbb{Y}^*$. Since $x \in \mathbb{Y}^*$, for given any $\varepsilon > 0$, there exists $\delta \in (0, \varepsilon)$. When $|h| < \delta$, we have $E|x(t + h) - x(t)|^p < \varepsilon$. Hence, we derive that $\frac{1}{2l}\int_{-l}^{l} E|x(t+h)-x(t)|^p dt < \varepsilon$. For the previously mentioned δ above, there exists $l > 0$ such that in every interval with length l of $\mathbb{R}$, we can find a number τ such that, for all $t \in \mathbb{R}$,

$$\begin{aligned}&|a(t+\tau)-a(t)|<\delta,\quad \|b_j(\cdot+\tau)-b_j(\cdot)\|_{B^p}<\delta,\quad |\tau_j(t+\tau)-\tau_j(t)|<\delta,\\ &|c_j(t+\tau)-c_j(t)|<\delta,\quad |\eta_j(t+\tau)-\eta_j(t)|<\delta.\end{aligned}$$

Since $|\tau_j(s + \tau) - \tau_j(s)| < \delta$, we further have

$$\frac{1}{2l}\int_{-l}^{l} E|x(s-\tau_j(s+\tau))-x(s-\tau_j(s))|^p dt < \varepsilon.$$

By (3.2), for any $t_i \in \mathbb{R}$, one can get

$$\begin{aligned}
&x(t+t_i+\tau)\\
=&\int_{-\infty}^{t+t_i} e^{-\int_s^{t+t_i} a(u+\tau)du} \sum_{j=1}^{n} \frac{b_j(s+\tau)x^{\alpha_j}\left(s+\tau-\tau_j(s+\tau)\right)}{1+x^{\beta_j}\left(s+\tau-\tau_j(s+\tau)\right)}ds\\
&+\int_{-\infty}^{t+t_i} e^{-\int_s^{t+t_i} a(u+\tau)du} \sum_{j=1}^{n} c_j(s+\tau)\sigma_j\left(x(s+\tau-\eta_j(s+\tau))\right)d\left[\omega_j(s+\tau)-\omega_j(s)\right],
\end{aligned}\tag{3.25}$$

where $\omega_j(s+\tau)-\omega_j(s)$ is a Brownian motion and it has the same distribution as $\omega_j(s)$.

Now, we consider the the following stochastic process:

$$\begin{aligned}
x(t+t_i+\tau)=&\int_{-\infty}^{t+t_i} e^{-\int_s^{t+t_i} a(u+\tau)du} \sum_{j=1}^{n} \frac{b_j(s+\tau)x^{\alpha_j}\left(s+\tau-\tau_j(s+\tau)\right)}{1+x^{\beta_j}\left(s+\tau-\tau_j(s+\tau)\right)}ds\\
&+\int_{-\infty}^{t+t_i} e^{-\int_s^{t+t_i} a(u+\tau)du} \sum_{j=1}^{n} c_j(s+\tau)\sigma_j\left(x(s+\tau-\eta_j(s+\tau))\right)d\omega_j(s),
\end{aligned}\tag{3.26}$$

then, we have

$$\begin{aligned}
&\frac{1}{2l}\int_{-l}^{l} E|x(t+t_i+\tau)-x(t+t_i)|^p dt\\
\le& 6^{p-1}\Big\{\frac{1}{2l}\int_{-l}^{l} E\Big|\int_{-\infty}^{t+t_i} e^{-\int_s^{t+t_i} a(u+\tau)du}\sum_{j=1}^{n} b_j(s+\tau)\Big(\frac{x^{\alpha_j}(s+\tau-\tau_j(s+\tau))}{1+x^{\beta_j}(s+\tau-\tau_j(s+\tau))}\\
&-\frac{x^{\alpha_j}(s-\tau_j(s))}{1+x^{\beta_j}(s-\tau_j(s))}\Big)ds\Big|^p dt+\frac{1}{2l}\int_{-l}^{l} E\Big|\int_{-\infty}^{t+t_i} e^{-\int_s^{t+t_i} a(u+\tau)du}\sum_{j=1}^{n}\left(b_j(s+\tau)-b_j(s)\right)\\
&\times\frac{x^{\alpha_j}(s-\tau_j(s))}{1+x^{\beta_j}(s-\tau_j(s))}ds\Big|^p dt+\frac{1}{2l}\int_{-l}^{l} E\Big|\int_{-\infty}^{t+t_i}\left(e^{-\int_s^{t+t_i} a(u+\tau)du}-e^{-\int_s^{t+t_i} a(u)du}\right)\\
&\times\sum_{j=1}^{n} b_j(s)\frac{x^{\alpha_j}(s-\tau_j(s))}{1+x^{\beta_j}(s-\tau_j(s))}ds\Big|^p dt+\frac{1}{2l}\int_{-l}^{l} E\Big|\int_{-\infty}^{t+t_i} e^{-\int_s^{t+t_i} a(u+\tau)du}\sum_{j=1}^{n} c_j(s+\tau)\\
&\times(\sigma_j(x(s+\tau-\eta_j(s+\tau)))-\sigma_j(x(s-\eta_j(s))))d\omega_j(s)\Big|^p dt\\
&+\frac{1}{2l}\int_{-l}^{l} E\Big|\int_{-\infty}^{t+t_i} e^{-\int_s^{t+t_i} a(u+\tau)du}\sum_{j=1}^{n}(c_j(s+\tau)-c_j(s))\sigma_j(x(s-\eta_j(s)))d\omega_j(s)\Big|^p dt\\
&+\frac{1}{2l}\int_{-l}^{l} E\Big|\int_{-\infty}^{t+t_i}\left(e^{-\int_s^{t+t_i} a(u+\tau)du}-e^{-\int_s^{t+t_i} a(u)du}\right)\sum_{j=1}^{n} c_j(s)\sigma_j(x(s-\eta_j(s)))d\omega_j(s)\Big|^p dt\Big\}\\
:=&B_1+B_2+B_3+B_4+B_5+B_6.
\end{aligned}\tag{3.27}$$

By the Hölder inequality and (3.20), we obtain

$$B_1 \le 6^{p-1}\frac{1}{2l}\int_{-l}^{l}\Big[\int_{-\infty}^{t+t_i} e^{-\frac{q}{p}\int_s^{t+t_i} a(u+\tau)du}ds\Big]^{\frac{p}{q}}\Big[\int_{-\infty}^{t+t_i} e^{-\frac{p}{q}\int_s^{t+t_i} a(u+\tau)du}E\Big(\sum_{j=1}^{n} b_j(s+\tau)$$

$$
\begin{aligned}
&\times \left(|x^{\alpha_j}\left(s+\tau-\tau_j(s+\tau)\right)-x^{\alpha_j}\left(s-\tau_j(s)\right)|+|x^{\alpha_j}\left(s+\tau-\tau_j(s+\tau)\right)|\right.\\
&\left.\times |x^{\beta_j}\left(s+\tau-\tau_j(s+\tau)\right)-x^{\beta_j}\left(s-\tau_j(s)\right)|\right)^p ds\Big]dt\\
\leq & 6^{p-1}\Big\{\Big(\frac{p}{q\underline{a}}\Big)^{\frac{p}{q}}\frac{1}{2l}\int_{-l}^{l}\int_{-\infty}^{t+t_i} e^{-\frac{p}{q}\int_s^{t+t_i}a(u+\tau)du}\Big(\sum_{j=1}^n\left(\hat{b}_j\right)^q\Big)^{\frac{p}{q}}\sum_{j=1}^n E\big(|x^{\alpha_j}\left(s+\tau-\tau_j(s+\tau)\right)\\
&-x^{\alpha_j}\left(s-\tau_j(s)\right)|+|x^{\alpha_j}\left(s+\tau-\tau_j(s+\tau)\right)|\\
&\times |x^{\beta_j}\left(s+\tau-\tau_j(s+\tau)\right)-x^{\beta_j}\left(s-\tau_j(s)\right)|\big)^p dsdt\Big\}\\
\leq & 12^{p-1}\Big\{\Big(\frac{p}{q\underline{a}}\Big)^{\frac{p}{q}}\Big(\sum_{j=1}^n\left(\hat{b}_j\right)^q\Big)^{\frac{p}{q}}\frac{1}{2l}\int_{-l}^{l}\int_{-\infty}^{t+t_i} e^{-\frac{p}{q}\underline{a}(t+t_i-s)}\sum_{j=1}^n E\Big(|x_1^{\alpha_j-1}|^p\\
&+|x^{\alpha_j}\left(s-\tau_j(s)\right)|^p|x_2^{\beta_j-1}|^p\Big)E|x\left(s+\tau-\tau_j(s+\tau)\right)-x\left(s-\tau_j(s)\right)|^p dsdt\Big\}\\
\leq & 24^{p-1}\Big\{\Big(\frac{p}{q\underline{a}}\Big)^{\frac{p}{q}}\Big(\sum_{j=1}^n\left(\hat{b}_j\right)^q\Big)^{\frac{p}{q}}\sum_{j=1}^n\left((2\kappa)^{(\alpha_j-1)p}+(2\kappa)^{(\alpha_j+\beta_j-1)p}\right)\\
&\times\frac{1}{2l}\int_{-l}^{l}\int_{-\infty}^{t+t_i} e^{-\frac{p}{q}\underline{a}(t+t_i-s)}\big(E|x\left(s+\tau-\tau_j(s+\tau)\right)-x(s-\tau_j(s+\tau))|^p\\
&+E|x(s-\tau_j(s+\tau))-x\left(s-\tau_j(s)\right)|^p\big)dsdt\Big\}\\
:=&B_{11}+B_{12},
\end{aligned}
\tag{3.28}
$$

where x_1 and x_2 take values between $x\left(s+\tau-\tau_j(s+\tau)\right)$ and $x\left(s-\tau_j(s)\right)$, hence, $\|x_1\|_{\mathbb{Y}}\leq 2\kappa, \|x_2\|_{\mathbb{Y}}\leq 2\kappa$.

Let $u=s-\tau_j(s+\tau)$, and making use of Fubini's theorem, we derive that

$$
\begin{aligned}
&\frac{1}{2l}\int_{-l}^{l}\int_{-\infty}^{t+t_i} e^{-\frac{p}{q}\underline{a}(t+t_i-s)}E|x\left(s+\tau-\tau_j(s+\tau)\right)-x\left(s-\tau_j(s+\tau)\right)|^p dsdt\\
\leq&\frac{e^{\frac{p}{q}\underline{a}\tau_j^+}}{1-\dot{\tau}_j^+}\frac{1}{2l}\int_{-l}^{l}\int_{-\infty}^{t+t_i} e^{-\frac{p}{q}\underline{a}(t+t_i-u)}E|x(u+\tau)-x(u)|^p dudt\\
\overset{t+t_i-u=w}{=\!=\!=}&\frac{e^{\frac{p}{q}\underline{a}\tau_j^+}}{1-\dot{\tau}_j^+}\int_0^{+\infty} e^{-\frac{p}{q}\underline{a}w}\frac{1}{2l}\int_{-l}^{l}E|x(t+t_i-w+\tau)-x(t+t_i-w)|^p dtdw\\
\overset{s=l-w}{=\!=\!=}&\frac{e^{\frac{p}{q}\underline{a}\tau_j^+}}{1-\dot{\tau}_j^+}\int_{-\infty}^{l} e^{-\frac{p}{q}\underline{a}(l-s)}\frac{1}{2l}\int_{-l}^{l}E|x(t+t_i+s-l+\tau)-x(t+t_i+s-l)|^p dtds\\
\overset{t+s-l=k}{=\!=\!=}&\frac{e^{\frac{p}{q}\underline{a}\tau_j^+}}{1-\dot{\tau}_j^+}\int_{-\infty}^{l} e^{-\frac{p}{q}\underline{a}(l-s)}\frac{1}{2l}\int_{s-2l}^{s}E|x(k+t_i+\tau)-x(k+t_i)|^p dkds\\
=&\frac{e^{\frac{p}{q}\underline{a}\tau_j^+}}{1-\dot{\tau}_j^+}\int_{-\infty}^{l} e^{-\frac{p}{q}\underline{a}(l-s)}\frac{1}{2l}\Big[\int_{s-2l}^{s}E|x(t+t_i+\tau)-x(t+t_i)|^p dt\Big]ds.
\end{aligned}
\tag{3.29}
$$

Based on the inequality above, we obtain

$$\begin{aligned}B_{11} \leq & 24^{p-1}\Big\{\Big(\frac{p}{q\underline{a}}\Big)^{\frac{p}{q}}\Big(\sum_{j=1}^{n}\big(\hat{b}_j\big)^{q}\Big)^{\frac{p}{q}}\sum_{j=1}^{n}((2\kappa)^{(\alpha_j-1)p}+(2\kappa)^{(\alpha_j+\beta_j-1)p})\frac{e^{\frac{p\underline{a}}{q}\tau_j^+}}{1-\dot{\tau}_j^+}\\ &\times\int_{-\infty}^{l}e^{-\frac{p}{q}\underline{a}(l-s)}\frac{1}{2l}\Big[\int_{s-2l}^{s}E|x(t+t_i+\tau)-x(t+t_i)|^p dt\Big]ds\Big\},\\ B_{12} \leq & 24^{p-1}\Big\{\Big(\frac{p}{q\underline{a}}\Big)^{\frac{p}{q}}\Big(\sum_{j=1}^{n}\big(\hat{b}_j\big)^{q}\Big)^{\frac{p}{q}}\sum_{j=1}^{n}((2\kappa)^{(\alpha_j-1)p}+(2\kappa)^{(\alpha_j+\beta_j-1)p})\\ &\times\frac{1}{2l}\int_{-l}^{l}\int_{-\infty}^{t+t_i}e^{-\frac{p}{q}\underline{a}(t+t_i-s)}\varepsilon^p ds dt\Big\}\\ \leq & 24^{p-1}\Big\{\Big(\frac{p}{q\underline{a}}\Big)^{\frac{p}{q}}\Big(\sum_{j=1}^{n}\big(\hat{b}_j\big)^{q}\Big)^{\frac{p}{q}}\frac{q}{p\underline{a}}\sum_{j=1}^{n}((2\kappa)^{(\alpha_j-1)p}+(2\kappa)^{(\alpha_j+\beta_j-1)p})\varepsilon^p\Big\}.\end{aligned}$$

Consequently, we arrive at

$$\begin{aligned}B_1 \leq & 24^{p-1}\Big(\frac{p}{q\underline{a}}\Big)^{\frac{p}{q}}\Big(\sum_{j=1}^{n}\big(\hat{b}_j\big)^{q}\Big)^{\frac{p}{q}}\sum_{j=1}^{n}((2\kappa)^{(\alpha_j-1)p}+(2\kappa)^{(\alpha_j+\beta_j-1)p})\Big\{\frac{e^{\frac{p\underline{a}}{q}\tau_j^+}}{1-\dot{\tau}_j^+}\int_{-\infty}^{l}e^{-\frac{p}{q}\underline{a}(l-s)}\\ &\times\frac{1}{2l}\Big[\int_{s-2l}^{s}E|x(t+t_i+\tau)-x(t+t_i)|^p dt\Big]ds+\frac{q}{p\underline{a}}\varepsilon^p\Big\}.\end{aligned}\tag{3.30}$$

By Hölder's inequality, Fubini's theorem, and Lebesgue's dominated convergence theorem, we infer that

$$\begin{aligned}B_2 \leq & 6^{p-1}\Big(\frac{p}{q\underline{a}}\Big)^{\frac{p}{q}}\frac{1}{2l}\int_{-l}^{l}\int_{-\infty}^{t+t_i}e^{-\frac{p}{q}\underline{a}(t+t_i-s)}\sum_{j=1}^{n}|b_j(s+\tau)-b_j(s)|^p\\ &\times\Big(\sum_{j=1}^{n}E\Big|\frac{x^{\alpha_j}\big(s-\tau_j(s)\big)}{1+x^{\beta_j}\big(s-\tau_j(s)\big)}\Big|^q ds dt\Big)^{\frac{p}{q}}\\ \leq & 6^{p-1}\Big(\frac{p}{q\underline{a}}\Big)^{\frac{p}{q}}\frac{1}{2l}\int_{-l}^{l}\int_{-\infty}^{t+t_i}e^{-\frac{p}{q}\underline{a}(t+t_i-s)}\sum_{j=1}^{n}|b_j(s+\tau)-b_j(s)|^p\\ &\times\Big(\sum_{j=1}^{n}E|x^{\alpha_j}\big(s-\tau_j(s)\big)|^q\Big)^{\frac{p}{q}}ds dt\\ \leq & 6^{p-1}\Big(\frac{p}{q\underline{a}}\Big)^{\frac{p}{q}}\Big(\sum_{j=1}^{n}\kappa^{\alpha_j q}\Big)^{\frac{p}{q}}\frac{1}{2l}\int_{-l}^{l}\int_{-\infty}^{t+t_i}e^{-\frac{p}{q}\underline{a}(t+t_i-s)}\sum_{j=1}^{n}|b_j(s+\tau)-b_j(s)|^p ds dt\\ \leq & 6^{p-1}\Big(\frac{p}{q\underline{a}}\Big)^{\frac{p}{q}}\Big(\sum_{j=1}^{n}\kappa^{\alpha_j q}\Big)^{\frac{p}{q}}\int_{0}^{+\infty}e^{-\frac{p}{q}\underline{a}u}du\\ &\times\frac{1}{2l}\int_{-l}^{l}\sum_{j=1}^{n}|b_j(t+t_i-u+\tau)-b_j(t+t_i-u)|^p dt\end{aligned}$$

$$
\begin{aligned}
&\le 6^{p-1}\Big(\frac{p}{q\underline{a}}\Big)^{\frac{p}{q}}\Big(\sum_{j=1}^{n}\kappa^{\alpha_j q}\Big)^{\frac{p}{q}}\int_0^{+\infty}e^{-\frac{p}{q}\underline{a}u}du\\
&\quad\times\frac{2(l+|t_i|+u)}{2l}\frac{1}{2(l+|t_i|+u)}\int_{-l-|t_i|-u}^{l+|t_i|+u}\sum_{j=1}^{n}|b_j(s+\tau)-b_j(s)|^p ds\\
&\le 6^{p-1}\Big\{\Big(\frac{p}{q\underline{a}}\Big)^{\frac{p}{q}}\frac{q}{p\underline{a}}n\varepsilon^p\sum_{j=1}^{n}\kappa^{\alpha_j p}\Big\}. \qquad (3.31)
\end{aligned}
$$

Since

$$
\begin{aligned}
&\int_{-\infty}^{t+t_i}\Big|e^{-\int_s^{t+t_i}a(u+\tau)du}-e^{-\int_s^{t+t_i}a(u)du}\Big|^{\frac{q}{p}}ds\\
\le&\int_{-\infty}^{t+t_i}e^{-\frac{q}{p}\underline{a}(t+t_i-s)}\Big(\int_s^{t+t_i}|a(u+\tau)-a(u)|du\Big)^{\frac{q}{p}}ds\\
\le&\Gamma\Big(\frac{p+q}{p}\Big)\Big(\frac{p}{q\underline{a}}\Big)^{\frac{q+p}{p}}\varepsilon^{\frac{q}{p}}, \qquad (3.32)
\end{aligned}
$$

and by virtue of the Hölder inequality, we infer that

$$
\begin{aligned}
B_3\le&6^{p-1}\Big\{\frac{1}{2l}\int_{-l}^{l}\Big[\int_{-\infty}^{t+t_i}\Big(e^{-\int_s^{t+t_i}a(u+\tau)du}-e^{-\int_s^{t+t_i}a(u)du}\Big)^{\frac{q}{p}}ds\Big]^{\frac{p}{q}}\\
&\times\Big[\int_{-\infty}^{t+t_i}\Big(e^{-\int_s^{t+t_i}a(u+\tau)du}-e^{-\int_s^{t+t_i}a(u)du}\Big)^{\frac{p}{q}}E\Big|\sum_{j=1}^{n}b_j(s)\frac{x^{\alpha_j}\big(s-\tau_j(s)\big)}{1+x^{\beta_j}\big(s-\tau_j(s)\big)}\Big|^p ds\Big]dt\Big\}\\
\le&6^{p-1}\Big\{\frac{1}{2l}\int_{-l}^{l}\Big[\int_{-\infty}^{t+t_i}\Big(e^{-\int_s^{t+t_i}a(u+\tau)du}-e^{-\int_s^{t+t_i}a(u)du}\Big)^{\frac{q}{p}}ds\Big]^{\frac{p}{q}}\\
&\times\Big[\int_{-\infty}^{t+t_i}\Big(e^{-\int_s^{t+t_i}a(u+\tau)du}-e^{-\int_s^{t+t_i}a(u)du}\Big)^{\frac{p}{q}}\Big(\sum_{j=1}^{n}\big(\hat{b}_j\big)^q\Big)^{\frac{p}{q}}\sum_{j=1}^{n}E|x^{\alpha_j}\big(s-\tau_j(s)\big)|^p ds\Big]dt\Big\}\\
\le&6^{p-1}\Big\{\Big(\Gamma\Big(\frac{p+q}{p}\Big)\Big)^{\frac{p}{q}}\Big(\frac{1}{\underline{a}}\Big)^{\frac{2(q+p)}{q}}\Gamma\Big(\frac{p+q}{q}\Big)\varepsilon^{\frac{p}{q}+1}\Big(\sum_{j=1}^{n}\big(\hat{b}_j\big)^q\Big)^{\frac{p}{q}}\sum_{j=1}^{n}\kappa^{\alpha_j p}\Big\}. \qquad (3.33)
\end{aligned}
$$

For random term B_4, when $p>2$, by Lemma 2.1 and Fubini's theorem, we have

$$
\begin{aligned}
B_4\le&6^{p-1}C_p\frac{1}{2l}\int_{-l}^{l}\Big[\int_{-\infty}^{t+t_i}\Big(e^{-2\int_s^{t+t_i}a(u+\tau)du}\Big)^{\frac{1}{p}\times\frac{p}{p-2}}ds\Big]^{\frac{p-2}{p}\times\frac{p}{2}}\Big[\int_{-\infty}^{t+t_i}\Big(e^{-2\int_s^{t+t_i}a(u+\tau)du}\Big)^{\frac{1}{q}\times\frac{p}{2}}\\
&\times E\Big|\sum_{j=1}^{n}c_j(s+\tau)\big[\sigma_j(x(s+\tau-\eta_j(s+\tau)))-\sigma_j(x(s-\eta_j(s)))\big]\Big|^p ds\Big]dt\\
\le&12^{p-1}C_p\Big(\frac{p-2}{2\underline{a}}\Big)^{\frac{p-2}{2}}\Big(\sum_{j=1}^{n}\big(c_j^+\big)^q\Big)^{\frac{p}{q}}\sum_{j=1}^{n}\big(L_j^\sigma\big)^p\frac{1}{2l}\int_{-l}^{l}\int_{-\infty}^{t+t_i}e^{-\frac{p}{q}\underline{a}(t+t_i-s)}\big(E|x(s+\tau\\
&-\eta_j(s+\tau))-x(s-\eta_j(s+\tau))|^p+E|x(s-\eta_j(s+\tau))-x(s-\eta_j(s))|^p\big)dsdt\\
\le&12^{p-1}C_p\Big(\frac{p-2}{2\underline{a}}\Big)^{\frac{p-2}{2}}\Big(\sum_{j=1}^{n}\big(c_j^+\big)^q\Big)^{\frac{p}{q}}\sum_{j=1}^{n}\big(L_j^\sigma\big)^p\Big\{\frac{e^{\frac{p}{q}\underline{a}\eta_j^+}}{1-\dot{\eta}_j^+}\int_{-\infty}^{l}e^{-\frac{p}{q}\underline{a}(l-s)}
\end{aligned}
$$

$$\times \frac{1}{2l}\Big[\int_{s-2l}^{s} E|x(t+t_i+\tau)-x(t+t_i)|^p dt\Big]ds + \frac{q}{p\underline{a}}\varepsilon^p\Big\}. \tag{3.34}$$

When $p = 2$, in view of the Itô isometry and Fubini's theorem, we get

$$\begin{aligned} B_4 \leq & 12\sum_{j=1}^{n}\left(c_j^+\right)^2\sum_{j=1}^{n}\left(L_j^\sigma\right)^2\frac{1}{2l}\int_{-l}^{l}\int_{-\infty}^{t+t_i} e^{-2\underline{a}(t+t_i-s)}\Big(E|x(s+\tau-\eta_j(s+\tau))-x(s-\eta_j(s+\tau))|^2 \\ & +E|x(s-\eta_j(s+\tau))-x(s-\eta_j(s))|^2\Big)dsdt \\ \leq & 12\sum_{j=1}^{n}\left(c_j^+\right)^2\sum_{j=1}^{n}\left(L_j^\sigma\right)^2\Big\{\frac{e^{2\underline{a}\eta_j^+}}{1-\dot{\eta}_j^+}\int_{-\infty}^{l} e^{-2\underline{a}(l-s)}\frac{1}{2l}\Big[\int_{s-2l}^{s} E|x(t+t_i+\tau)-x(t+t_i)|^2 dt\Big]ds \\ & +\frac{1}{2\underline{a}}\varepsilon^2\Big\}. \end{aligned} \tag{3.35}$$

For random term B_5, when $p > 2$, by Lemma 2.1 and the Hölder inequality, we have

$$\begin{aligned} B_5 \leq & 12^{p-1}C_p\frac{1}{2l}\int_{-l}^{l}\Big[\Big[\int_{-\infty}^{t+t_i}\Big(e^{-2\underline{a}(t+t_i-s)}\Big)^{\frac{1}{p-2}}ds\Big]^{\frac{p-2}{2}}\Big[\int_{-\infty}^{t+t_i}\Big(e^{-2\underline{a}(t+t_i-s)}\Big)^{\frac{1}{q}\times\frac{p}{2}} \\ & \times\Big(\sum_{j=1}^{n}\left|c_j(s+\tau)-c_j(s)\right|^q\Big)^{\frac{p}{q}}\sum_{j=1}^{n}[(L_j^\sigma)^pE|x(s-\eta_j(s))|^p+\sigma_j^p(0)]ds\Big]dt \\ \leq & 12^{p-1}C_p\Big(\frac{p-2}{2\underline{a}}\Big)^{\frac{p-2}{2}}\frac{q}{p\underline{a}}(n\varepsilon^q)^{\frac{p}{q}}\sum_{j=1}^{n}[\left(L_j^\sigma\right)^p\kappa^p+\sigma_j^p(0)]. \end{aligned} \tag{3.36}$$

When $p = 2$, by the Itô isometry and the Hölder inequality, we obtain

$$\begin{aligned} B_5 \leq & 12\frac{1}{2l}\int_{-l}^{l}\int_{-\infty}^{t+t_i} e^{-2\underline{a}(t+t_i-s)}\sum_{j=1}^{n}\left|c_j(s+\tau)-c_j(s)\right|^2 \\ & \times\sum_{j=1}^{n}[(L_j^\sigma)^2E|x(s-\eta_j(s))|^2+\sigma_j^2(0)]dsdt \\ \leq & 12\frac{1}{2\underline{a}}n\varepsilon^2\sum_{j=1}^{n}[\left(L_j^\sigma\right)^2\kappa^2+\sigma_j^2(0)]. \end{aligned} \tag{3.37}$$

When $p > 2$, by Lemma 2.1, the Hölder inequality, and (3.32), one can obtain

$$\begin{aligned} B_6 \leq & 6^{p-1}C_p\frac{1}{2l}\int_{-l}^{l}E\Big[\int_{-\infty}^{t+t_i}\Big(e^{-\int_s^{t+t_i}a(u+\tau)du}-e^{-\int_s^{t+t_i}a(u)du}\Big)^{2\times\frac{1}{p}\times\frac{p}{p-2}}ds\Big]^{\frac{p}{2}\times\frac{p-2}{p}} \\ & \times\Big[\Big(e^{-\int_s^{t+t_i}a(u+\tau)du}-e^{-\int_s^{t+t_i}a(u)du}\Big)^{2\times\frac{1}{q}\times\frac{p}{2}}E\Big|\sum_{j=1}^{n}c_j(s)\sigma_j(x(s-\eta_j(s)))\Big|^p ds\Big]dt \\ \leq & 12^{p-1}C_p\Big(\Gamma\Big(\frac{p}{p-2}\Big)\Big)^{\frac{p-2}{2}}\Big(\frac{p-2}{2\underline{a}}\Big)^{\frac{p}{2}}\Gamma\Big(\frac{p+q}{q}\Big)\Big(\frac{q}{p\underline{a}}\Big)^{\frac{p+q}{q}}\varepsilon^{\frac{p}{q}+1}\Big(\sum_{j=1}^{n}\left(c_j^+\right)^q\Big)^{\frac{p}{q}} \\ & \times\sum_{j=1}^{n}[\left(L_j^\sigma\right)^p\kappa^p+\sigma_j^p(0)]. \end{aligned} \tag{3.38}$$

When $p = 2$, by the Itô isometry and (3.32), we get

$$\begin{aligned}B_6 \le& 12\sum_{j=1}^{n}\left(c_j^+\right)^2\sum_{j=1}^{n}\Big(\left(L_j^\sigma\right)^2\kappa^2+\sigma_j^2(0)\Big)\frac{1}{2l}\int_{-l}^{l}\int_{-\infty}^{t+t_i}e^{-2\underline{a}(t+t_i-s)}\varepsilon^2\left(t+t_i-s\right)^2 dsdt\\ \le& 12\Gamma(3)\Big(\frac{1}{2\underline{a}}\Big)^2\varepsilon^2\sum_{j=1}^{n}\left(c_j^+\right)^2\sum_{j=1}^{n}\left(\left(L_j^\sigma\right)^2\kappa^2+\sigma_j^2(0)\right).\end{aligned} \tag{3.39}$$

Substituting (3.30)–(3.39) into (3.27), for $p > 2$, we have

$$\begin{aligned}&\frac{1}{2l}\int_{-l}^{l}E|x(t+t_i+\tau)-x(t+t_i)|^p dt\\ \le& \varDelta_1\varepsilon+\varpi_1\int_{-\infty}^{l}e^{-\frac{p}{q}\underline{a}(l-s)}\frac{1}{2l}\Big[\int_{s-2l}^{s}E|x(t+t_i+\tau)-x(t+t_i)|^p dt\Big]ds,\end{aligned}$$

where ϖ_1 is defined in (H_5) and

$$\begin{aligned}\varDelta_1 =& 6^{p-1}\Big\{\Big(\frac{p}{q\underline{a}}\Big)^{\frac{p}{q}}\Big(4^{p-1}\Big(\sum_{j=1}^{n}\left(\hat{b}_j\right)^q\Big)^{\frac{p}{q}}\sum_{j=1}^{n}\left((2\kappa)^{(\alpha_j-1)p}+(2\kappa)^{(\alpha_j+\beta_j-1)p}\right)\frac{q}{p\underline{a}}\varepsilon^{p-1}\\ &+\frac{q}{p\underline{a}}n^{\frac{p}{q}}\varepsilon^{p-1}\sum_{j=1}^{n}\kappa^{\alpha_j p}\Big)+\Big(\Gamma\Big(\frac{p+q}{p}\Big)\Big)^{\frac{p}{q}}\Big(\frac{1}{\underline{a}}\Big)^{\frac{2(q+p)}{q}}\Gamma\Big(\frac{p+q}{q}\Big)\varepsilon^{\frac{p}{q}}\Big(\sum_{j=1}^{n}\left(\hat{b}_j\right)^q\Big)^{\frac{p}{q}}\sum_{j=1}^{n}\kappa^{\alpha_j p}\\ &+2^{p-1}C_p\Big(\Big(\frac{p-2}{2\underline{a}}\Big)^{\frac{p-2}{2}}\Big[\Big(\sum_{j=1}^{n}\left(c_j^+\right)^q\Big)^{\frac{p}{q}}\sum_{j=1}^{n}\left(L_j^\sigma\right)^p\frac{q}{p\underline{a}}\varepsilon^{p-1}+\frac{q}{p\underline{a}}n^{\frac{p}{q}}\varepsilon^{p-1}\\ &\times\sum_{j=1}^{n}[\left(L_j^\sigma\right)^p\kappa^p+\sigma_j^p(0)]\Big]+\Big(\Gamma\Big(\frac{p}{p-2}\Big)\Big)^{\frac{p-2}{2}}\Big(\frac{p-2}{2\underline{a}}\Big)^{\frac{p}{2}}\Gamma\Big(\frac{p+q}{q}\Big)\Big(\frac{q}{p\underline{a}}\Big)^{\frac{p+q}{q}}\\ &\times\varepsilon^{\frac{p}{q}}\Big(\sum_{j=1}^{n}\left(c_j^+\right)^q\Big)^{\frac{p}{q}}\sum_{j=1}^{n}[\left(L_j^\sigma\right)^p\kappa^p+\sigma_j^p(0)]\Big)\Big\}.\end{aligned}$$

By (H_5), we derive that $\varpi_1 < \frac{p\underline{a}}{q}$. Thus, it follows from the variant of Gronwall's lemma (Lemma 3.3 in [28]) that

$$\frac{1}{2l}\int_{-l}^{l}E|x(t+t_i+\tau)-x(t+t_i)|^p dt \le \varDelta_1\varepsilon\frac{p\underline{a}}{p\underline{a}-q\varpi_1}. \tag{3.40}$$

Notice that

$$\begin{aligned}&d_C\left(G_x(t+\tau),G_x(t)\right)\\ =&\sup_{\|f\|_A\le1}\Big|\int_{\mathbb{R}^h}fd(G_x(t+\tau)-G_x(t))\Big|\\ =&\sup_{\|f\|_A\le1}\Big|\int_{\Omega^h}[f(x(t+\tau+t_1),x(t+\tau+t_2),\cdots,x(t+\tau+t_h))\\ &-f(x(t+t_1),x(t+t_2),\cdots,x(t+t_h))]d^hP\end{aligned}$$

$$\leq \int_{\Omega^h} \|(x(t+t_1), x(t+t_2), \cdots, x(t+t_h)) - (x(t+t_1), x(t+t_2), \cdots, x(t+t_h))\| d^h P$$
$$\leq \int_{\Omega} \max_{1\leq i\leq h}\{|x(t_i+t+\tau) - x(t_i+t)|\} dP$$
$$\leq \max_{1\leq i\leq h}\{(E|x(t_i+t+\tau) - x(t_i+t)|^p)^{\frac{1}{p}}\}. \tag{3.41}$$

We obtain that

$$d_C^p(G_x(t+\tau), G_X(t)) \leq \max_{1\leq i\leq h}\{E|x(t+t_i+\tau) - x(t+t_i)|^p\} \leq \varDelta_1\varepsilon\frac{p\underline{a}}{p\underline{a} - q\varpi_1}. \tag{3.42}$$

For $p = 2$, we have

$$\frac{1}{2l}\int_{-l}^{l} E|x(t+t_i+\tau) - x(t+t_i)|^2 dt$$
$$\leq \varDelta_2\varepsilon + \varpi_2\int_{-\infty}^{l} e^{-\underline{a}(l-s)}\frac{1}{2l}\Big[\int_{s-2l}^{s} E|x(t+t_i+\tau) - x(t+t_i)|^2 dt\Big] ds,$$

where ϖ_2 is mentioned in (H_5) and

$$\begin{aligned}\varDelta_2 =& 6\Big\{4\Big(\frac{1}{\underline{a}}\Big)^2\sum_{j=1}^{n}\big(\hat{b}_j\big)^2\sum_{j=1}^{n}((2\kappa)^{(\alpha_j-1)2} + (2\kappa)^{(\alpha_j+\beta_j-1)2})\varepsilon + \Big(\frac{1}{\underline{a}}\Big)^2 n\varepsilon\sum_{j=1}^{n}\kappa^{\alpha_j 2} \\ &+ \Gamma(2)^2\Big(\frac{1}{\underline{a}}\Big)^4\varepsilon\sum_{j=1}^{n}\big(\hat{b}_j\big)^2\sum_{j=1}^{n}\kappa^{\alpha_j 2} + \frac{1}{\underline{a}}\Big(\sum_{j=1}^{n}\big(c_j^+\big)^2\sum_{j=1}^{n}\big(L_j^\sigma\big)^2\varepsilon + n\varepsilon \\ &\times\sum_{j=1}^{n}[\big(L_j^\sigma\big)^2\kappa^2 + \sigma_j^2(0)] + \Gamma(3)\frac{1}{2\underline{a}}\varepsilon^2\sum_{j=1}^{n}\big(c_j^+\big)^2\sum_{j=1}^{n}[\big(L_j^\sigma\big)^2\kappa^2 + \sigma_j^2(0)]\Big)\Big\}.\end{aligned}$$

By (H_5), we derive that $\varpi_2 < \underline{a}$. Thus, again from Lemma 3.3 in [28], it follows that

$$\frac{1}{2l}\int_{-l}^{l} E|x(t+t_i+\tau) - x(t+t_i)|^2 dt \leq \varDelta_2\varepsilon\frac{\underline{a}}{\underline{a} - \varpi_1}. \tag{3.43}$$

Hence, by virtue of (3.41), we arrive at

$$d_C^2(G_x(t+\tau), G_x(t)) \leq \max_{1\leq i\leq h}\{E|x(t+t_i+\tau) - x(t+t_i)|^2\} \leq \varDelta_2\varepsilon\frac{\underline{a}}{\underline{a} - \varpi_1}. \tag{3.44}$$

From (3.42) and (3.44), according to Definition 2.4, we can conclude that $x(t)$ is a pth B_{ap} solution in finite-dimensional distributions. □

Theorem 3.3. *Suppose that conditions* (H_1)–(H_5) *are fulfilled. Let* $x(t)$ *be the* B_{ap} *solution in finite-dimensional distributions of system* (1.2) *with initial value* φ *and let* $y(t)$ *be an arbitrary solution of system* (1.2) *with initial value* ψ*, then there exist constants* $\lambda > 0$ *and* $M > 0$ *such that*

$$E|y(t) - x(t)|^p \leq M\|\psi - \varphi\|_1\, e^{-\frac{p}{q}\lambda(t-t_0)},\ \ t > t_0,$$

where $\|\psi - \varphi\|_1 = \sup_{s\in[t_0-\vartheta, t_0]} E|\psi(s) - \varphi(s)|$*. That is, the solution* $x(t)$ *of system* (1.2) *is globally exponentially stable.*

Proof. Let $z(t) = y(t) - x(t)$, then by (1.2), one has

$$dz(t) =\Big[-a(t)z(t) + \sum_{j=1}^{n} b_j(t)\Big(\frac{y^{\alpha_j}\left(t-\tau_j(t)\right)}{1+y^{\beta_j}\left(t-\tau_j(t)\right)} - \frac{x^{\alpha_j}\left(t-\tau_j(t)\right)}{1+x^{\beta_j}\left(t-\tau_j(t)\right)}\Big)\Big]dt$$
$$+ \sum_{j=1}^{n} c_j(t)(\sigma_j\left(y(t-\eta_j(t))\right) - \sigma_j\left(x(t-\eta_j(t))\right))d\omega_j(t),$$

then

$$z(t) =z(t_0)e^{-\int_{t_0}^{t} a(u)du} + \int_{t_0}^{t} e^{-\int_{s}^{t} a(u)du} \sum_{j=1}^{n} b_j(s)\Big(\frac{y^{\alpha_j}\left(s-\tau_j(s)\right)}{1+y^{\beta_j}\left(s-\tau_j(s)\right)} - \frac{x^{\alpha_j}\left(s-\tau_j(s)\right)}{1+x^{\beta_j}\left(s-\tau_j(s)\right)}\Big)ds$$
$$+ \int_{t_0}^{t} e^{-\int_{s}^{t} a(u)du} \sum_{j=1}^{n} c_j(s)(\sigma_j\left(y(s-\eta_j(s))\right) - \sigma_j\left(x(s-\eta_j(s))\right))d\omega_j(s). \tag{3.45}$$

Define continuous functions $\Gamma_1(\varrho), \Gamma_2(\varrho) \in \mathbb{C}(\mathbb{R}^+, \mathbb{R})$ as follows:

$$\Gamma_1(\varrho) = \frac{p\left(\underline{a}-\varrho\right)}{q} - 3^{p-1}\Lambda_1(\varrho) \text{ and } \Gamma_2(\varrho) = \left(\underline{a}-\varrho\right) - 3\Lambda_2(\varrho),$$

where

$$\Lambda_1(\varrho) :=2^{p-1}\Big(\frac{p}{q\underline{a}}\Big)^{\frac{p}{q}}\Big(\sum_{j=1}^{n}\left(\hat{b}_j\right)^q\Big)^{\frac{p}{q}} \sum_{j=1}^{n}\left((2\kappa)^{(\alpha_j-1)p} + (2\kappa)^{(\alpha_j-1+\beta_j)p}\right)$$
$$\times \frac{e^{\frac{p}{q}\underline{a}\tau_j^+}}{1-\dot{\tau}_j^+}e^{\frac{p}{q}\varrho\tau_j^+} + C_p\Big(\frac{p-2}{2\underline{a}}\Big)^{\frac{p-2}{2}}\Big(\sum_{j=1}^{n}\left(c_j^+\right)^q\Big)^{\frac{p}{q}} \sum_{j=1}^{n}\left(L_j^\sigma\right)^p \frac{e^{\frac{p}{q}\underline{a}\eta_j^+}}{1-\dot{\eta}_j^+}e^{\frac{p}{q}\varrho\eta_j^+},$$
$$\Lambda_2(\varrho) :=\frac{2}{\underline{a}}\sum_{j=1}^{n}\left(\hat{b}_j\right)^2 \sum_{j=1}^{n}\left((2\kappa)^{(\alpha_j-1)2} + (2\kappa)^{(\alpha_j-1+\beta_j)2}\right)\frac{e^{\underline{a}\tau_j^+}}{1-\dot{\tau}_j^+}e^{\varrho\tau_j^+}$$
$$+ \sum_{j=1}^{n}(c_j^+)^2 \sum_{j=1}^{n}\left(L_j^\sigma\right)^2 \frac{e^{2\underline{a}\eta_j^+}}{1-\dot{\eta}_j^+}e^{\varrho\eta_j^+}.$$

By virtue of (H_4), we have $\Gamma_1(0) > 0$ and $\Gamma_2(0) > 0$. Since $\Gamma_1(\varrho)$ and $\Gamma_2(\varrho)$ are continuous on $[0\ +\infty)$, and $\Gamma_1(\varrho) \to -\infty$ and $\Gamma_2(\varrho) \to -\infty$ as $\varrho \to \infty$, there exist $\xi_1 > 0, \xi_2 > 0$ such that $\Gamma_1(\xi_2) = 0, \Gamma_2(\xi_2) = 0, \Gamma_1(\varrho) > 0, \Gamma_2(\varrho) > 0$ for $\varrho \in (0, \min\{\xi_1, \xi_2\})$. So, we can take a positive constant $0 < \lambda < \min\{\underline{a}, \xi_1, \xi_2\}$ such that $\Gamma_1(\lambda) > 0, \Gamma_2(\lambda) > 0$, which implies that

$$\chi_1 := \frac{q3^{p-1}\Lambda_1(\lambda)}{p\left(\underline{a}-\lambda\right)} < 1, (p > 2) \text{ and } \chi_2 := \frac{3\Lambda_2(\lambda)}{\left(\underline{a}-\lambda\right)} < 1, (p = 2).$$

Take $M^1 = \dfrac{p\underline{a}}{3^{p-1}q\Lambda_1(0)}, M^2 = \dfrac{\underline{a}}{3\Lambda_2(0)}$, and $M = \max\{M^1, M^2\}$, then by (H_5), we know $M^1, M^2 > 1$. Hence, we can deduce that

$$\frac{1}{M^1} - \chi_1 \le 0 \text{ and } \frac{1}{M^2} - \chi_2 \le 0.$$

For any $\varepsilon > 0$, it is easy to see that

$$E|z(t_0)|^p \leq \|\psi - \varphi\|_1 + \varepsilon$$

and for any $t \in [t_0 - \vartheta, t_0]$,

$$E|z(t)|^p \leq (\|\psi - \varphi\|_1 + \varepsilon)\, e^{-\frac{p}{q}\lambda(t-t_0)} \leq M\,(\|\psi - \varphi\|_1 + \varepsilon)\, e^{-\frac{p}{q}\lambda(t-t_0)}.$$

We claim that

$$E|z(t)|^p \leq M\,(\|\psi - \varphi\|_1 + \varepsilon)\, e^{-\frac{p}{q}\lambda(t-t_0)},\ t > t_0. \tag{3.46}$$

In the contrary case, there exists some $t_1 > t_0$ such that

$$E|z(t_1)|^p = M\,(\|\psi - \varphi\|_1 + \varepsilon)\, e^{-\frac{p}{q}\lambda(t_1-t_0)}, \tag{3.47}$$

$$E|z(t)|^p < M\,(\|\psi - \varphi\|_1 + \varepsilon)\, e^{-\frac{p}{q}\lambda(t-t_0)},\ t_0 - \vartheta \leq t < t_1. \tag{3.48}$$

Based on (3.45), we derive

$$\begin{aligned}
&E|z(t_1)|^p \qquad (3.49)\\
\leq &3^{p-1}E|z(t_0)e^{-\int_{t_0}^{t_1} a(u)du}|^p + 3^{p-1}E\Big|\int_{t_0}^{t_1} e^{-\int_s^{t_1} a(u)du} \sum_{j=1}^{n} b_j(s)\Big(\frac{y^{\alpha_j}\big(s-\tau_j(s)\big)}{1+y^{\beta_j}\big(s-\tau_j(s)\big)}\\
&-\frac{x^{\alpha_j}\big(s-\tau_j(s)\big)}{1+x^{\beta_j}\big(s-\tau_j(s)\big)}\Big)\Big|^p + 3^{p-1}E\Big|\int_{t_0}^{t_1} e^{-\int_s^{t_1} a(u)du} \sum_{j=1}^{n} c_j(s)(\sigma_j\big(y(s-\eta_j(s))\big)\\
&-\sigma_j\big(x(s-\eta_j(s))\big))d\omega_j(s)\Big|^p := T_1 + T_2 + T_3.
\end{aligned} \tag{3.50}$$

It is easy to see that

$$T_1 \leq 3^{p-1}\,(\|\psi - \varphi\|_1 + \varepsilon)\, e^{-p\underline{a}(t_1-t_0)}. \tag{3.51}$$

By the Hölder inequality and (3.48), we infer that

$$\begin{aligned}
T_2 \leq &6^{p-1}\Big(\frac{p}{q\underline{a}}\Big)^{\frac{p}{q}}\Big(\sum_{j=1}^{n}\big(\hat{b}_j\big)^q\Big)^{\frac{p}{q}} \sum_{j=1}^{n}\Big((2\kappa)^{(\alpha_j-1)p} + (2\kappa)^{(\alpha_j+\beta_j-1)p}\Big)\int_{t_0}^{t_1} e^{-\frac{p}{q}\int_s^{t_1} a(u)du}\\
&\times E|y(s-\tau_j(s)) - x(s-\tau_j(s))|^p ds.
\end{aligned}$$

Letting $\rho = s - \tau_j(s)$ and $\nu = \rho + \tau_j(s)$, we have

$$\begin{aligned}
T_2 \leq &6^{p-1}\Big(\frac{p}{q\underline{a}}\Big)^{\frac{p}{q}}\Big(\sum_{j=1}^{n}\big(\hat{b}_j\big)^q\Big)^{\frac{p}{q}} \sum_{j=1}^{n}\Big((2\kappa)^{(\alpha_j-1)p} + (2\kappa)^{(\alpha_j+\beta_j-1)p}\Big)\frac{1}{1-\dot{\tau}_j^+}\\
&\times \int_{t_0-\tau_j(s)}^{t_1-\tau_j(s)} e^{-\frac{p}{q}\underline{a}(t_1-\rho-\tau_j(s))} M\,(\|\psi - \varphi\|_1 + \varepsilon)\, e^{-\frac{p}{q}\lambda(t_1-t_0)} d\rho
\end{aligned}$$

$$\leq 6^{p-1}\Big(\frac{p}{q\underline{a}}\Big)^{\frac{p}{q}}\Big(\sum_{j=1}^{n}\big(\hat{b}_j\big)^q\Big)^{\frac{p}{q}}\sum_{j=1}^{n}\Big((2\kappa)^{(\alpha_j-1)p}+(2\kappa)^{(\alpha_j+\beta_j-1)p}\Big)\frac{e^{\frac{p}{q}\underline{a}\tau_j^+}}{1-\dot{\tau}_j^+}$$
$$\times\int_{t_0}^{t_1}e^{-\frac{p}{q}(\underline{a}-\lambda)(t_1-\nu+\tau_j(s))}M\left(\|\psi-\varphi\|_1+\varepsilon\right)e^{-\frac{p}{q}\lambda(t_1-t_0)}d\nu$$
$$\leq 6^{p-1}\Big(\frac{p}{q\underline{a}}\Big)^{\frac{p}{q}}\Big(\sum_{j=1}^{n}\big(\hat{b}_j\big)^q\Big)^{\frac{p}{q}}\sum_{j=1}^{n}\Big((2\kappa)^{(\alpha_j-1)p}+(2\kappa)^{(\alpha_j+\beta_j-1)p}\Big)\frac{e^{\frac{p}{q}\underline{a}\tau_j^+}}{1-\dot{\tau}_j^+}e^{\frac{p}{q}\lambda\tau_j^+}$$
$$\times M\left(\|\psi-\varphi\|_1+\varepsilon\right)e^{-\frac{p}{q}\lambda(t_1-t_0)}\frac{q}{p(\underline{a}-\lambda)}\Big(1-e^{-\frac{p}{q}(\underline{a}-\lambda)(t_1-t_0)}\Big). \tag{3.52}$$

Similarly, when $p>2$, by Lemma 2.1, we deduce that

$$T_3\leq 3^{p-1}C_p\Big[\int_{t_0}^{t_1}\Big(e^{-2\int_s^{t_1}a(u)du}\Big)^{\frac{1}{p}\times\frac{p}{p-2}}ds\Big]^{\frac{p-2}{p}\times\frac{p}{2}}\Big[\int_{t_0}^{t_1}\Big(e^{-2\int_s^{t_1}a(u)du}\Big)^{\frac{1}{q}\times\frac{p}{2}}$$
$$\times E\Big|\sum_{j=1}^{n}c_j(s)(\sigma_j\big(y(s-\eta_j(s))\big)-\sigma_j\big(x(s-\eta_j(s))\big))\Big|^p ds\Big]$$
$$\leq 3^{p-1}C_p\Big(\frac{p-2}{2\underline{a}}\Big)^{\frac{p-2}{2}}\Big(\sum_{j=1}^{n}\big(c_j^+\big)^q\Big)^{\frac{p}{q}}\sum_{j=1}^{n}\big(L_j^\sigma\big)^p\int_{t_0}^{t_1}e^{-\frac{p}{q}\int_s^{t_1}a(u)du}$$
$$\times E|y(s-\eta_j(s))-x(s-\eta_j(s))|^p ds$$
$$\leq 3^{p-1}C_p\Big(\frac{p-2}{2\underline{a}}\Big)^{\frac{p-2}{2}}\Big(\sum_{j=1}^{n}\big(c_j^+\big)^q\Big)^{\frac{p}{q}}\sum_{j=1}^{n}\big(L_j^\sigma\big)^p\frac{e^{\frac{p}{q}\underline{a}\eta_j^+}}{1-\dot{\eta}_j^+}e^{\frac{p}{q}\lambda\eta_j^+}$$
$$\times\int_{t_0}^{t_1}e^{-\frac{p}{q}(\underline{a}-\lambda)(t_1-s)}M\left(\|\psi-\varphi\|_1+\varepsilon\right)e^{-\frac{p}{q}\lambda(t_1-t_0)}ds$$
$$\leq 3^{p-1}C_p\Big(\frac{p-2}{2\underline{a}}\Big)^{\frac{p-2}{2}}\Big(\sum_{j=1}^{n}\big(c_j^+\big)^q\Big)^{\frac{p}{q}}\sum_{j=1}^{n}\big(L_j^\sigma\big)^p\frac{e^{\frac{p}{q}\underline{a}\eta_j^+}}{1-\dot{\eta}_j^+}e^{\frac{p}{q}\lambda\eta_j^+}$$
$$\times M\left(\|\psi-\varphi\|_1+\varepsilon\right)e^{-\frac{p}{q}\lambda(t_1-t_0)}\frac{q}{p(\underline{a}-\lambda)}\Big(1-e^{-\frac{p}{q}(\underline{a}-\lambda)(t_1-t_0)}\Big) \tag{3.53}$$

and when $p=2$, by the Itô isometry, we infer that

$$T_3\leq 3\sum_{j=1}^{n}\big(c_j^+\big)^2\sum_{j=1}^{n}\big(L_j^\sigma\big)^2\int_{t_0}^{t_1}e^{-2\int_s^{t_1}a(u)du}E|y(s-\eta_j(s))-x(s-\eta_j(s))|^2 ds$$
$$\leq 3\sum_{j=1}^{n}\big(c_j^+\big)^2\sum_{j=1}^{n}\big(L_j^\sigma\big)^2\frac{e^{2\underline{a}\eta_j^+}}{1-\dot{\eta}_j^+}e^{\lambda\eta_j^+}M\left(\|\psi-\varphi\|_1+\varepsilon\right)e^{-\lambda(t_1-t_0)}\frac{1}{\underline{a}-\lambda}\Big(1-e^{-(\underline{a}-\lambda)(t_1-t_0)}\Big). \tag{3.54}$$

Substituting (3.51)–(3.54) into (3.49), for $p>2$, we derive

$$E|z(t_1)|^p$$
$$\leq 3^{p-1}M\left(\|\psi-\varphi\|_1+\varepsilon\right)e^{-\frac{p}{q}\lambda(t_1-t_0)}\Big\{\frac{e^{\left(\frac{p}{q}\lambda-p\underline{a}\right)(t_1-t_0)}}{M}+\frac{q}{p(\underline{a}-\lambda)}\left(1-e^{\frac{p}{q}(\lambda-\underline{a})(t_1-t_0)}\right)$$

$$\times\Big(2^{p-1}\Big(\frac{p}{q\underline{a}}\Big)^{\frac{p}{q}}\Big(\sum_{j=1}^{n}\big(\hat{b}_j\big)^q\Big)^{\frac{p}{q}}\sum_{j=1}^{n}\Big((2\kappa)^{(\alpha_j-1)p}+(2\kappa)^{(\alpha_j+\beta_j-1)p}\Big)\frac{e^{\frac{p}{q}\underline{a}\tau_j^+}}{1-\dot{\tau}_j^+}e^{\frac{p}{q}\lambda\tau_j^+}$$

$$+C_p\Big(\frac{p-2}{2\underline{a}}\Big)^{\frac{p-2}{2}}\Big(\sum_{j=1}^{n}\big(c_j^+\big)^q\Big)^{\frac{p}{q}}\sum_{j=1}^{n}\big(L_j^\sigma\big)^p\frac{e^{\frac{p}{q}\underline{a}\eta_j^+}}{1-\dot{\eta}_j^+}e^{\frac{p}{q}\lambda\eta_j^+}\Big)\Big\}$$

$$\le 3^{p-1}M\left(\|\psi-\varphi\|_1+\varepsilon\right)e^{-\frac{p}{q}\lambda(t_1-t_0)}\Big\{\frac{e^{\left(\frac{p}{q}\lambda-p\underline{a}\right)(t_1-t_0)}}{M}-\frac{q}{p(\underline{a}-\lambda)}e^{\frac{p}{q}\left(\lambda-\underline{a}\right)(t_1-t_0)}$$

$$\times\Big(2^{p-1}\Big(\frac{p}{q\underline{a}}\Big)^{\frac{p}{q}}\Big(\sum_{j=1}^{n}\big(\hat{b}_j\big)^q\Big)^{\frac{p}{q}}\sum_{j=1}^{n}\Big((2\kappa)^{(\alpha_j-1)p}+(2\kappa)^{(\alpha_j+\beta_j-1)p}\Big)\frac{e^{\frac{p}{q}\underline{a}\tau_j^+}}{1-\dot{\tau}_j^+}e^{\frac{p}{q}\lambda\tau_j^+}$$

$$+C_p\Big(\frac{p-2}{2\underline{a}}\Big)^{\frac{p-2}{2}}\Big(\sum_{j=1}^{n}\big(c_j^+\big)^q\Big)^{\frac{p}{q}}\sum_{j=1}^{n}\big(L_j^\sigma\big)^p\frac{e^{\frac{p}{q}\underline{a}\eta_j^+}}{1-\dot{\eta}_j^+}e^{\frac{p}{q}\lambda\eta_j^+}\Big)+\frac{q}{p(\underline{a}-\lambda)}\Big(2^{p-1}\Big(\frac{p}{q\underline{a}}\Big)^{\frac{p}{q}}$$

$$\times\Big(\sum_{j=1}^{n}\big(\hat{b}_j\big)^q\Big)^{\frac{p}{q}}\sum_{j=1}^{n}\Big((2\kappa)^{(\alpha_j-1)p}+(2\kappa)^{(\alpha_j+\beta_j-1)p}\Big)\frac{e^{\frac{p}{q}\underline{a}\tau_j^+}}{1-\dot{\tau}_j^+}e^{\frac{p}{q}\lambda\tau_j^+}+C_p\Big(\frac{p-2}{2\underline{a}}\Big)^{\frac{p-2}{2}}$$

$$\times\Big(\sum_{j=1}^{n}\big(c_j^+\big)^q\Big)^{\frac{p}{q}}\sum_{j=1}^{n}\big(L_j^\sigma\big)^p\frac{e^{\frac{p}{q}\underline{a}\eta_j^+}}{1-\dot{\eta}_j^+}e^{\frac{p}{q}\lambda\eta_j^+}\Big)\Big\}$$

$$\le 3^{p-1}M\left(\|\psi-\varphi\|_1+\varepsilon\right)e^{-\frac{p}{q}\lambda(t_1-t_0)}\Big\{\frac{e^{\frac{p}{q}\left(\lambda-\underline{a}\right)(t_1-t_0)}}{M}-\frac{q}{p(\underline{a}-\lambda)}e^{\frac{p}{q}\left(\lambda-\underline{a}\right)(t_1-t_0)}$$

$$\times\Big(2^{p-1}\Big(\frac{p}{q\underline{a}}\Big)^{\frac{p}{q}}\Big(\sum_{j=1}^{n}\big(\hat{b}_j\big)^q\Big)^{\frac{p}{q}}\sum_{j=1}^{n}\Big((2\kappa)^{(\alpha_j-1)p}+(2\kappa)^{(\alpha_j+\beta_j-1)p}\Big)\frac{e^{\frac{p}{q}\underline{a}\tau_j^+}}{1-\dot{\tau}_j^+}e^{\frac{p}{q}\lambda\tau_j^+}$$

$$+C_p\Big(\frac{p-2}{2\underline{a}}\Big)^{\frac{p-2}{2}}\Big(\sum_{j=1}^{n}\big(c_j^+\big)^q\Big)^{\frac{p}{q}}\sum_{j=1}^{n}\big(L_j^\sigma\big)^p\frac{e^{\frac{p}{q}\underline{a}\eta_j^+}}{1-\dot{\eta}_j^+}e^{\frac{p}{q}\lambda\eta_j^+}\Big)+\frac{q}{p(\underline{a}-\lambda)}\Big(2^{p-1}\Big(\frac{p}{q\underline{a}}\Big)^{\frac{p}{q}}$$

$$\times\Big(\sum_{j=1}^{n}\big(\hat{b}_j\big)^q\Big)^{\frac{p}{q}}\sum_{j=1}^{n}\Big((2\kappa)^{(\alpha_j-1)p}+(2\kappa)^{(\alpha_j+\beta_j-1)p}\Big)\frac{e^{\frac{p}{q}\underline{a}\tau_j^+}}{1-\dot{\tau}_j^+}e^{\frac{p}{q}\lambda\tau_j^+}+C_p\Big(\frac{p-2}{2\underline{a}}\Big)^{\frac{p-2}{2}}$$

$$\times\Big(\sum_{j=1}^{n}\big(c_j^+\big)^q\Big)^{\frac{p}{q}}\sum_{j=1}^{n}\big(L_j^\sigma\big)^p\frac{e^{\frac{p}{q}\underline{a}\eta_j^+}}{1-\dot{\eta}_j^+}e^{\frac{p}{q}\lambda\eta_j^+}\Big)\Big\}$$

$$\le 3^{p-1}M\left(\|\psi-\varphi\|_1+\varepsilon\right)e^{-\frac{p}{q}\lambda(t_1-t_0)}\Big\{e^{\frac{p}{q}\left(\lambda-\underline{a}\right)(t_1-t_0)}\Big[\frac{1}{M}-\frac{q}{p(\underline{a}-\lambda)}\Big(2^{p-1}\Big(\frac{p}{q\underline{a}}\Big)^{\frac{p}{q}}$$

$$\times\Big(\sum_{j=1}^{n}\big(\hat{b}_j\big)^q\Big)^{\frac{p}{q}}\sum_{j=1}^{n}\Big((2\kappa)^{(\alpha_j-1)p}+(2\kappa)^{(\alpha_j+\beta_j-1)p}\Big)\frac{e^{\frac{p}{q}\underline{a}\tau_j^+}}{1-\dot{\tau}_j^+}e^{\frac{p}{q}\lambda\tau_j^+}+C_p\Big(\frac{p-2}{2\underline{a}}\Big)^{\frac{p-2}{2}}$$

$$\times\Big(\sum_{j=1}^{n}\big(c_j^+\big)^q\Big)^{\frac{p}{q}}\sum_{j=1}^{n}\big(L_j^\sigma\big)^p\frac{e^{\frac{p}{q}\underline{a}\eta_j^+}}{1-\dot{\eta}_j^+}e^{\frac{p}{q}\lambda\eta_j^+}\Big)\Big]+\frac{q}{p(\underline{a}-\lambda)}\Big(2^{p-1}\Big(\frac{p}{q\underline{a}}\Big)^{\frac{p}{q}}$$

$$\times\Big(\sum_{j=1}^{n}\big(\hat{b}_j\big)^q\Big)^{\frac{p}{q}}\sum_{j=1}^{n}\Big((2\kappa)^{(\alpha_j-1)p}+(2\kappa)^{(\alpha_j+\beta_j-1)p}\Big)\frac{e^{\frac{p}{q}\underline{a}\tau_j^+}}{1-\dot{\tau}_j^+}e^{\frac{p}{q}\lambda\tau_j^+}+C_p\Big(\frac{p-2}{2\underline{a}}\Big)^{\frac{p-2}{2}}$$

$$\times\Big(\sum_{j=1}^{n}\left(c_j^+\right)^q\Big)^{\frac{p}{q}}\sum_{j=1}^{n}\left(L_j^{\sigma}\right)^p\frac{e^{\frac{p}{q}\underline{a}\eta_j^+}}{1-\dot{\eta}_j^+}e^{\frac{p}{q}\lambda\eta_j^+}\Big)\Big\}$$
$$\leq M\left(\|\psi-\varphi\|_1+\varepsilon\right)e^{-\frac{p}{q}\lambda(t_1-t_0)}.$$

When $p = 2$, we have

$$\begin{aligned}
&E|z(t_1)|^p\\
\leq&3M\left(\|\psi-\varphi\|_1+\varepsilon\right)e^{-\lambda(t_1-t_0)}\Big\{\frac{e^{(\lambda-2\underline{a})(t_1-t_0)}}{M}-\frac{e^{(\lambda-\underline{a})(t_1-t_0)}}{\underline{a}-\lambda}\Big(\frac{2}{\underline{a}}\sum_{j=1}^{n}\left(\hat{b}_j\right)^2\sum_{j=1}^{n}\Big((2\kappa)^{(\alpha_j-1)2}\\
&+(2\kappa)^{(\alpha_j+\beta_j-1)2}\Big)\frac{e^{\underline{a}\tau_j^+}}{1-\dot{\tau}_j^+}e^{\lambda\tau_j^+}+\sum_{j=1}^{n}\left(c_j^+\right)^2\sum_{j=1}^{n}\left(L_j^{\sigma}\right)^2\frac{e^{2\underline{a}\eta_j^+}}{1-\dot{\eta}_j^+}e^{\lambda\eta_j^+}\Big)+\frac{1}{\underline{a}-\lambda}\Big(\frac{2}{\underline{a}}\sum_{j=1}^{n}\left(\hat{b}_j\right)^2\\
&\times\sum_{j=1}^{n}\Big((2\kappa)^{(\alpha_j-1)2}+(2\kappa)^{(\alpha_j+\beta_j-1)2}\Big)\frac{e^{\underline{a}\tau_j^+}}{1-\dot{\tau}_j^+}e^{\lambda\tau_j^+}+\sum_{j=1}^{n}\left(c_j^+\right)^2\sum_{j=1}^{n}\left(L_j^{\sigma}\right)^2\frac{e^{2\underline{a}\eta_j^+}}{1-\dot{\eta}_j^+}e^{\lambda\eta_j^+}\Big)\Big\}\\
\leq&3M\left(\|\psi-\varphi\|_1+\varepsilon\right)e^{-\lambda(t_1-t_0)}\Big\{e^{(\lambda-\underline{a})(t_1-t_0)}\Big[\frac{1}{M}-\frac{1}{\underline{a}-\lambda}\Big(\frac{2}{\underline{a}}\sum_{j=1}^{n}\left(\hat{b}_j\right)^2\sum_{j=1}^{n}\Big((2\kappa)^{(\alpha_j-1)2}\\
&+(2\kappa)^{(\alpha_j+\beta_j-1)2}\Big)\frac{e^{\underline{a}\tau_j^+}}{1-\dot{\tau}_j^+}e^{\lambda\tau_j^+}+\sum_{j=1}^{n}\left(c_j^+\right)^2\sum_{j=1}^{n}\left(L_j^{\sigma}\right)^2\frac{e^{2\underline{a}\eta_j^+}}{1-\dot{\eta}_j^+}e^{\lambda\eta_j^+}\Big)\Big]+\frac{1}{\underline{a}-\lambda}\Big(\frac{2}{\underline{a}}\sum_{j=1}^{n}\left(\hat{b}_j\right)^2\\
&\times\sum_{j=1}^{n}\Big((2\kappa)^{(\alpha_j-1)2}+(2\kappa)^{(\alpha_j+\beta_j-1)2}\Big)\frac{e^{\underline{a}\tau_j^+}}{1-\dot{\tau}_j^+}e^{\lambda\tau_j^+}+\sum_{j=1}^{n}\left(c_j^+\right)^2\sum_{j=1}^{n}\left(L_j^{\sigma}\right)^2\frac{e^{2\underline{a}\eta_j^+}}{1-\dot{\eta}_j^+}e^{\lambda\eta_j^+}\Big)\Big\}\\
\leq&M\left(\|\psi-\varphi\|_1+\varepsilon\right)e^{-\lambda(t_1-t_0)}.
\end{aligned}$$

Therefore, for $p \geq 2$, we can conclude that

$$E|z(t_1)|^p < M\left(\|\psi-\varphi\|_1+\varepsilon\right)e^{-\frac{p}{q}\lambda t_1},$$

which contradicts (3.47). Hence, (3.46) holds. Letting $\varepsilon \to 0^+$, from (3.46), we have

$$E|z(t_1)|^p \leq M\left(\|\psi-\varphi\|_1\right)e^{-\frac{p}{q}\lambda(t-t_0)},\ t > t_0.$$

This completes the proof. □

4. Numerical simulation

In this part, we give an example to illustrate the validity of the results obtained in Section 3.

Example 4.1. *In system* (1.2)*, let $n = 2$, and we consider the coefficients are as follows:*

$$a(t) = 5 + e^{2\sin t} + |\sin \pi t|,\ \alpha_1 = \frac{1}{19},\ \alpha_2 = \frac{1}{50},\ \beta_1 = \frac{1}{30},\ \beta_2 = \frac{1}{100},$$
$$b_1(t) = 0.08|e^{-|t|} + \sin t + 1|,\ b_2(t) = 0.02|\cos\frac{\pi}{2}t + \sin t + \frac{5}{17+t^2}|,$$

$$\tau_1(t) = 0.032(2 - 0.02\cos t - \cos\sqrt{2}t),\ \tau_2(t) = 0.05(2 - \cos\sqrt{3}t - 0.02\cos^2\sqrt{2}t),$$
$$c_1(t) = \frac{1}{45}|\sin\pi t + \sin^2 t|,\ c_2(t) = \frac{1}{250}|e^{\cos t} + \sin\sqrt{3}t|,$$
$$\eta_1(t) = 0.2(\sin\pi t + e^{\sin t} + 1),\ \eta_2(t) = 0.3(2 + 0.02\sin 2t + \cos\sqrt{3}t),$$
$$\sigma_1(x) = \frac{1}{100}\sin\sqrt{2}x + \frac{1}{100},\ \sigma_2(x) = \frac{1}{25}\sin x + \frac{1}{50},\ \kappa = 1.5.$$

By calculations, we obtain

$$\underline{a} = 4.3689,\ \hat{b}_1 = 0.1789,\ \hat{b}_2 = 0.0411,\ \tau_1^+ = 0.0966,\ \tau_2^+ = 0.15,\quad \dot{\tau}_1^+ = 0.0459,$$
$$\dot{\tau}_2^+ = 0.0873,\ c_1^+ = 0.0444,\ c_2^+ = 0.1487,\ \eta_1^+ = 0.94,\ \dot{\eta}_1^+ = 0.92,\ \eta_2^+ = 0.906,$$
$$\dot{\eta}_2^+ = 0.5316,\ L_1^\sigma = \frac{1}{100},\ L_2^\sigma = \frac{1}{25},\ \vartheta = 0.94.$$

When $p = \frac{21}{10}$, $q = \frac{21}{11}$, we have $\varsigma_1 = 0.0132 < \kappa^{\frac{21}{10}} = 2.3431$, $\Theta_1 = 0.0026 < 1$, $\varpi_1 = 0.0394 < 1$.
When $p = q = 2$, we have $\varsigma_2 = 0.0159 < \kappa^2 = 2.2500$, $\Theta_2 = 0.0033 < 1$, $\varpi_2 = 0.0647 < 1$.
So, conditions (H_1)–(H_5) are verified. Therefore, according to Theorem 3.3, system (1.2) admits a unique B_{ap} solution in finite-dimensional distributions and it is globally exponentially stable (see Figure 1).

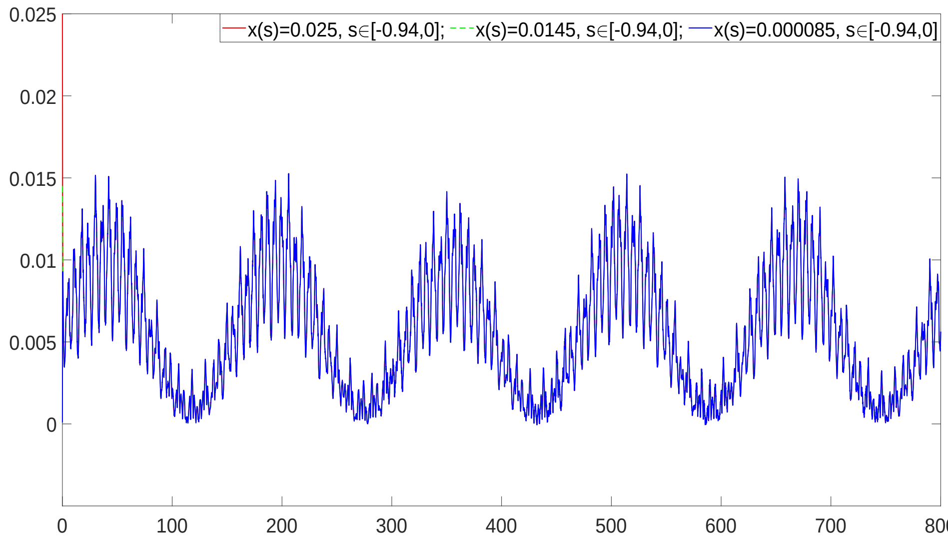

Figure 1. Global exponential stability of states x of (1.2) with different initial values.

5. Conclusions

This article established for the first time the existence and stability of positive B_{ap} solutions in finite-dimensional distributions of a stochastic generalized Mackey-Glass delayed hematopoietic model. The effectiveness of the obtained results was verified through a numerical example and computer simulations. As is known to all, B_{ap} oscillation is a more complex recurrent motion than

Bohr almost periodic oscillation, Stepanov almost periodic oscillation, and Weyl almost periodic oscillation. This further indicates that although the differential equation form describing the Mackey-Glass hematopoietic model is simple, it has very rich dynamics. Finally, it is worth mentioning that this article is also the first one on the B_{ap} oscillations in biological population systems.

Author contributions

Xianying Huang: Investigation, Writing-original draft, Software, Validation; Yongkun Li: Conceptualization, Methodology, Formal analysis, Investigation, Writing-original draft, Writing-review & editing. All authors have read and agreed to the published version of the manuscript.

Acknowledgments

This work is supported by the National Natural Science Foundation of China under Grant No. 12261098.

Conflict of interest

The authors declare that they have no conflicts of interest.

References

1. M. C. Mackey, L. Glass, Oscillation and chaos in physiological control systems, *Science*, **197** (1977), 287–289. https://doi.org/10.1126/science.267326
2. M. Khemis, A. Bouakkaz, R. Khemis, Positive periodic solutions of a leukopoiesis model with iterative terms, *Bol. Soc. Mat. Mex.*, **30** (2024), 1. https://doi.org/10.1007/s40590-023-00576-2
3. M. Khemis, A. Bouakkaz, R. Khemis, Positive periodic solutions for a delay model of erythropoiesis with iterative terms, *Appl. Anal.*, **103** (2023), 340–352. https://doi.org/10.1080/00036811.2023.2186862
4. S. Zhang, Y. Wang, H. Geng, W. Gao, E. Ilhan, H. M. Baskonus, On the analyzing of bifurcation properties of the one-dimensional Mackey-Glass model by using a generalized approach, *Math. Meth. Appl. Sci.*, 2024, 1–15. https://doi.org/10.1002/mma.10381
5. C. Huang, X. Ding, Existence of traveling wave fronts for a diffusive Mackey-Glass model with two delays, *Nonlinear Anal. Real World Appl.*, **76** (2024), 104024. https://doi.org/10.1016/j.nonrwa.2023.104024
6. Y. Yan, J. Sugie, Existence regions of positive periodic solutions for a discrete hematopoiesis model with unimodal production functions, *Appl. Math. Model.*, **68** (2019), 152–168. https://doi.org/10.1016/j.apm.2018.11.003
7. M. Adimy, P. Amster, J. Epstein, Periodic solutions for a nonautonomous mathematical model of hematopoietic stem cell dynamics, *Nonlinear Anal.*, **211** (2021), 112397. https://doi.org/10.1016/j.na.2021.112397

8. Y. Yan, H. Zhu, Number of positive periodic solutions for feedback-driven nonlinear differential equation: Application to hematopoietic process, *Monatsh. Math.*, **203** (2024), 523–542. https://doi.org/10.1007/s00605-023-01881-8

9. Y. Yan, Positive periodic solutions for discrete time-delay hematopoiesis model with impulses, *Open Math.*, **21** (2023), 20230613. https://doi.org/10.1515/math-2023-0613

10. Y. Yan, Multiplicity of positive periodic solutions for a discrete impulsive blood cell production model, *AIMS Mathematics*, **8** (2023), 26515–26531. https://doi.org/10.3934/math.20231354

11. A. Younsi, Exponential stability of periodic solutions for a hematopoiesis model with two time delays, *Stud. Eng. Exact Sci.*, **5** (2024), 3199–3226. https://doi.org/10.54021/seesv5n1-159

12. V. V. Alekseev, M. M. Preobrazhenskaia, Analysis of the asymptotic convergence of periodic solution of the Mackey-Glass equation to the solution of the limit relay equation, *Theor. Math. Phys.*, **220** (2024), 1241–1261. https://doi.org/10.1134/S0040577924080014

13. H. Zhou, W. Wang, L. Yang, Stage-structured hematopoiesis model with delays in an almost periodic environment, *Appl. Math. Lett.*, **120** (2021), 107336. https://doi.org/10.1016/j.aml.2021.107336

14. H. S. Ding, Q. J. Liu, J. J. Nieto, Existence of positive almost periodic solutions to a class of hematopoiesis model, *Appl. Math. Model.*, **40** (2016), 3289–3297. https://doi.org/10.1016/j.apm.2015.10.020

15. Z. J. Yao, New results on existence and exponential stability of the unique positive almost periodic solution for hematopoiesis model, *Appl. Math. Model.*, **39** (2015), 7113–7123. https://doi.org/10.1016/j.apm.2015.03.003

16. J. Yuan, L. Liu, Y. Wu, New results on the existence and uniqueness of positive almost periodic solution for the generalized Mackey-Glass hematopoietic model, *Commun. Nonlinear Sci. Numer. Simul.*, **126** (2023) 107487. https://doi.org/10.1016/j.cnsns.2023.107487

17. T. Diagana, H. Zhou, Existence of positive almost periodic solutions to the hematopoiesis model, *Appl. Math. Comput.*, **274** (2016), 644–648. https://doi.org/10.1016/j.amc.2015.10.029

18. P. Amster, R. Balderrama, On the global dynamic behaviour for a generalized haematopoiesis model with almost periodic coefficients and oscillating circulation loss rate, *Math. Meth. Appl. Sci.*, **41** (2018), 3976–3997. https://doi.org/10.1002/mma.4880

19. R. Balderrama, New results on the almost periodic solutions for a model of hematopoiesis with an oscillatory circulation loss rate, *J. Fixed Point Theory Appl.*, **22** (2020), 42. https://doi.org/10.1007/s11784-020-00776-7

20. H. B. Fredj, F. Chérif, Positive pseudo almost periodic solutions to a class of hematopoiesis model: Oscillations and dynamics, *J. Appl. Math. Comput.*, **63** (2020), 479–500. https://doi.org/10.1007/s12190-020-01326-7

21. A. S. Besicovitch, *Almost periodic functions*, New York: Dover Publication INC, 1954.

22. C. Corduneanu, *Almost periodic oscillations and waves*, New York: Springer, 2009. https://doi.org/10.1007/978-0-387-09819-7

23. O. Mellah, P. R. de Fitte, Counterexamples to mean square almost periodicity of the solutions of some SDEs with almost periodic coefficients, *Electron. J. Diff. Equ.*, **2013** (2013), 1–7.

24. Y. Li, X. Wang, Besicovitch almost periodic stochastic processes and almost periodic solutions of Clifford-valued stochastic neural networks, *Discrete Contin. Dyn. Syst. B*, **28** (2023), 2154–2183. https://doi.org/10.3934/dcdsb.2022162

25. A. Klenke, *Probability theory: A comprehensive course*, Cham: Springer, 2020. https://doi.org/10.1007/978-3-030-56402-5

26. N. Huo, Y. Li, $\mathscr{B}$-almost periodic solutions in finite-dimensional distributions for octonion-valued stochastic shunting inhibitory cellular neural networks, *Math. Meth. Appl. Sci.*, **47** (2024), 5136–5160. https://doi.org/10.1002/mma.9858

27. G. D. Prato, J. Zabczyk, *Stochastic equations in infinite dimensions*, Cambridge: Cambridge University Press, 1992. https://doi.org/10.1017/cbo9780511666223

28. M. Kamenskii, O. Mellah, P. R. De Fitte, Weak averaging of semilinear stochastic differential equations with almost periodic coefficients, *J. Math. Anal. Appl.*, **427** (2015), 336–364. https://doi.org/10.1016/j.jmaa.2015.02.036

AIMS Press