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Research article

Pricing geometric average Asian options in the mixed sub-fractional Brownian motion environment with Vasicek interest rate model

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Abstract: Considering the characteristics of long-range correlations in financial markets, the issue of valuing geometric average Asian options is examined, assuming that the variations of the underlying asset follow the mixed sub-fractional Brownian motion, and the dynamics of short-term interest rate satisfies the mixed sub-fractional Vasicek model. Based on the principle of no arbitrage, the definite solution of PDE of a zero-coupon bond for geometric average Asian options under the circumstance of the mixed sub-fractional is given by the delta hedging technique. The derivation of the explicit pricing formula for geometric average Asian options with fixed strike price is achieved through the utilization of multiple variable substitutions. Furthermore, we perform numerical calculations to analyze the performance of the model.

Keywords: option pricing; mixed sub-fractional Brownian motion; geometric average Asian options; Vasicek short rate model
Mathematics Subject Classification: 58J35, 60H10, 91B26

1. Introduction

In 1973, the classic Black-Scholes (BS) model [1] provided an effective method for pricing financial derivatives and greatly simplifying the pricing process. Since then, many scholars have conducted research and promoted the model [2,3]. Asian options, a special type of financial derivative, are appealing due to their unique valuation method, which relies on the mean value of the underlying asset over a set contract period.

However, the BS model may not align with actual financial markets and could fail to provide accurate pricing for complex financial derivatives. In 1968, Mandelbrot and Van Ness [4] proposed a stochastic process: fractional Brownian motion (fBm). If the Hurst parameter $H \neq \frac{1}{2}$, the fractional Brownian motion is not a semi-martingale, which implies that there must be arbitrage opportunities [5,6]. Bojdecki et al. [7] advanced that sub-fractional Brownian motion (sfBm), which not only possesses similar properties to fractional Brownian motion but also features non-stationary increments with weaker correlations on non-overlapping intervals and a faster decay of covariance, aligns better with financial market dynamics. Based on this, Tubor [8] investigated new properties of sfBm. Xu and Li [9] tackled the valuation conundrum linked to compound options. For further applications of sfBm in financial models, please refer to [10–12]. Despite these advancements, the use of sfBm as a stochastic driver may still result in arbitrage opportunities akin to those associated with fBm. Zhang and Xiao [13] showed that the Black-Scholes model driven by fractional Gaussian processes allows for arbitrage opportunities. Since sub-fractional Brownian motion is a more general Gaussian process and is also not a semi-martingale, the application of this model to financial markets necessitates the study of arbitrage possibilities. EI-Nouty and Zili [14] proposed the concept of mixed sub-fractional Brownian motion (msfBm), which lies between Brownian motion and sfBm. The mixed sub-fractional Brownian motion incorporates the semi-martingale condition when $H \in \left(\frac{3}{4}, 1\right)$, making this stochastic process more suitable for inclusion in option pricing models [15]. Furthermore, constructing an appropriate portfolio can enable models with the Hurst parameter $H \in (0,1)$ to avoid arbitrage in financial markets [16]. For example, Guo et al. [17] combined the fractal option pricing model with a new intelligent algorithm to predict the implied volatility in financial assets. Cai et al. [18] found the LSE of the drift parameter of mixed sub-fractional O-U process.

The aforementioned studies collectively presupposed fixed short-term interest rates, which do not reflect the dynamic nature of real interest rates that exhibit mean reversion. The Vasicek model [19], a fundamental interest rate model in finance, describes the evolution of interest rates and has become a cornerstone in the analysis and management of interest rate risks. It provides a theoretical framework that aids in making informed financial decisions and developing sophisticated risk management strategies. Ewald et al. [20] priced options for Asian commodity futures contracts by incorporating stochastic convenience yields, stochastic interest rates, and commodity spot prices, and considered the scenario with jumps. More related studies can be found in [21–25] to further understand the importance of stochastic interest rate models in option pricing. Building on these studies, in this paper, a model for pricing geometric average Asian options is formulated under the msfBm regime, while the short rate follows the Vasicek process.

Then, in Section 2, we state the necessary foundational knowledge. In Section 3, we provide the formula for a zero-coupon bond under the msfBm, based on certain assumptions. In Section 4, we give the solution for valuing geometric average Asian options with fixed strike price. In Section 5, we present relevant numerical calculations and an empirical study to further explore the effects of varying parameters on the model.

2. Preliminaries

The Vasicek model is recognized as a significant model for the short rate that can be combined with other pricing models. The incorporation of mixed sub-fractional Brownian motion can offer more comprehensive risk management solutions. Now, we introduce the relevant knowledge of msfBm, which is covered in [7,14,26].

Definition 2.1. *The Gaussian process* $\xi_H = \{\xi_H(t), t \geq 0\}$ *defined on a probability space* (Ω, F, P) *that satisfies the following conditions*

(1) $\xi_H(0) = 0$,

(2) $E(\xi_H(t) \cdot \xi_H(s)) = s^{2H} + t^{2H} - \frac{1}{2}\left[(s+t)^{2H} + |t-s|^{2H}\right]$,

is called a sub-fractional Brownian motion, where H *is the Hurst parameter with a value range of* $(0,1)$, *and* $B_H(t)$ *is a standard Brownian motion when* $H = \frac{1}{2}$.

Definition 2.2. *Let* (Ω, F, P) *be a probability space. The mixed sub-fractional Brownian motion* $M^{\beta,\gamma,H} = \{M_t^{\beta,\gamma,H}, t \geq 0\}$ *is a stochastic process with* $H \in (0,1)$, *defined by*

$$M^{\beta,\gamma,H} = \beta B(t) + \gamma \xi_H(t), \beta \geq 0, \gamma \geq 0,$$

where $B(t)$ *is a Brownian motion and* $\xi_H(t)$ *is a sub-fractional Brownian motion. We have*

$$E\left(M_t^{\beta,\gamma,H} \cdot M_s^{\beta,\gamma,H}\right) = \beta^2 \min(s,t) + \gamma^2 \left\{s^{2H} + t^{2H} - \frac{1}{2}\left[(s+t)^{2H} + |t-s|^{2H}\right]\right\},$$

where $M_t^{\beta,\gamma,H}$ *is a sfBm when* $\beta = 0$ *and* $\gamma = 1$, $M_t^{\beta,\gamma,H}$ *is a standard Brownian motion when* $\beta = 1$ *and* $\gamma = 0$ *or when* $\beta = 0$, $\gamma = 1$ *and* $H = \frac{1}{2}$.

Definition 2.3. *Asian options are divided into geometric average Asian options and arithmetic average Asian options. Taking fixed strike Asian call options as an example, the payoff is* $(J_T - K)^+$, *where* K *is the strike price,* T *is the expiration date, and* J_t *is the average price of the underlying asset over the predetermined interval. In continuous time, the arithmetic average is represented by*

$$J_t = \frac{1}{t}\int_0^t S_\tau \mathrm{d}\tau,$$

the geometric average is represented by

$$J_t = \exp\left(\frac{1}{t}\int_0^t \ln S_\tau \mathrm{d}\tau\right).$$

Then, we present some basic assumptions of this paper.

(i) Short selling is allowed without penalty; there are no taxes and friction; stocks do not pay dividends; investors borrow and lend at the risk-free rate.

(ii) The stock price $S(t)$ satisfies the mixed sub-fractional Brownian motion of risk neutrality, given by

$$\mathrm{d}S(t) = r(t)S(t)\mathrm{d}t + \sigma_{S_1}S(t)\mathrm{d}B^S(t) + \sigma_{S_2}S(t)\mathrm{d}\xi_H^S(t), \tag{2.1}$$

where $r(t)$ is a short rate and satisfies the following msfBm-Vasicek model

$$\mathrm{d}r(t) = a\big(b - r(t)\big)\mathrm{d}t + \sigma_{r_1}\mathrm{d}B^r(t) + \sigma_{r_2}\mathrm{d}\xi_H^r(t), \tag{2.2}$$

where σ_{S_1}, σ_{S_2}, σ_{r_1}, and σ_{r_2} are constants, $B^S(t)$, $\xi_H^S(t)$, $B^r(t)$, and $\xi_H^r(t)$ are independent of each other.

3. Pricing formula for zero-coupon bond

A zero-coupon bond is issued at a price below its face value and does not pay any interest during the period. Upon maturity at time T, investors are entitled to receive a cash return equivalent to \$1. The price of such a bond is influenced by the passage of time and the variability in interest rates. We denote the price of a zero-coupon bond at time t, maturing at time T, as $P(r,t;T)$.

Theorem 3.1. *In the mixed sub-fractional Vasicek process, the price of a zero-coupon bond with maturity T at time $t \in [0,T]$ is given by*

$$P(r,t;T) = e^{-A(t,T) - rB(t,T)}, \tag{3.1}$$

where

$$\begin{cases} A(t,T) = b(T-t) - bB(t,T) - \dfrac{1}{2}\sigma_{r_1}^2\displaystyle\int_t^T B^2(s,T)\mathrm{d}s - H\left(2 - 2^{2H-1}\right)\sigma_{r_2}^2\displaystyle\int_t^T s^{2H-1}B^2(s,T)\mathrm{d}s, \\ B(t,T) = \dfrac{1 - e^{-a(T-t)}}{a}, \end{cases}$$

Proof. By using the risk hedging formula and Itô formula [19], select two zero-coupon bonds with different maturities, denoted as $P_1 = P_1(r,t;T_1)$ and $P_2 = P_2(r,t;T_2)$, to hedge the risks. Now consider a portfolio Π consisting of one unit of P_1 and Δ units of P_2, we can obtain

$$\Pi = P_1 - \Delta P_2.$$

The change in the portfolio over the time interval $(t, t+\mathrm{d}t)$ is given by

$$\mathrm{d}\Pi = \frac{\partial P_1}{\partial t}\mathrm{d}t + \frac{\partial P_1}{\partial r}\mathrm{d}r + \frac{1}{2}\frac{\partial^2 P_1}{\partial r^2}(\mathrm{d}r)^2 - \Delta\left(\frac{\partial P_2}{\partial t}\mathrm{d}t + \frac{\partial P_2}{\partial r}\mathrm{d}r + \frac{1}{2}\frac{\partial^2 P_2}{\partial r^2}(\mathrm{d}r)^2\right), \tag{3.2}$$

where $(\mathrm{d}r)^2 = \left(\sigma_{r_1}^2 + 2Ht^{2H-1}\left(2-2^{2H-1}\right)\sigma_{r_2}^2\right)\mathrm{d}t + o(\mathrm{d}t)$.

Let $\Delta = \dfrac{\partial P_1/\partial r}{\partial P_2/\partial r}$, Eq (3.2) becomes

$$\begin{aligned}\mathrm{d}\Pi = {} & \frac{\partial P_1}{\partial t}\mathrm{d}t + \frac{1}{2}\left(\sigma_{r_1}^2 + 2Ht^{2H-1}\left(2-2^{2H-1}\right)\sigma_{r_2}^2\right)\frac{\partial^2 P_1}{\partial r^2}\mathrm{d}t \\ & - \frac{\partial P_1/\partial r}{\partial P_2/\partial r}\left(\frac{\partial P_2}{\partial t}\mathrm{d}t + \frac{1}{2}\left(\sigma_{r_1}^2 + 2Ht^{2H-1}\left(2-2^{2H-1}\right)\sigma_{r_2}^2\right)\frac{\partial^2 P_2}{\partial r^2}\mathrm{d}t\right).\end{aligned} \tag{3.3}$$

Furthermore, since the investment portfolio is risk-free, that is, $E(\mathrm{d}\Pi) = r(t)\Pi\mathrm{d}t$, we have

$$\begin{aligned}& \left(\frac{\partial P_1}{\partial t} + \frac{1}{2}\left(\sigma_{r_1}^2 + 2Ht^{2H-1}\left(2-2^{2H-1}\right)\sigma_{r_2}^2\right)\frac{\partial^2 P_1}{\partial r^2} - rP_1\right) / \frac{\partial P_1}{\partial r} \\ & = \left(\frac{\partial P_2}{\partial t} + \frac{1}{2}\left(\sigma_{r_1}^2 + 2Ht^{2H-1}\left(2-2^{2H-1}\right)\sigma_{r_2}^2\right)\frac{\partial^2 P_2}{\partial r^2} - rP_2\right) / \frac{\partial P_2}{\partial r}.\end{aligned} \tag{3.4}$$

Then, we can obtain

$$\left(\frac{\partial P}{\partial t} + \frac{1}{2}\left(\sigma_{r_1}^2 + 2Ht^{2H-1}\left(2-2^{2H-1}\right)\sigma_{r_2}^2\right)\frac{\partial^2 P}{\partial r^2} - rP\right) / \frac{\partial P}{\partial r} = -a\left(b - r(t)\right).$$

Thus, the zero-coupon bond $P(r,t;T)$ satisfies the following partial differential equation given by

$$\begin{cases}\dfrac{\partial P}{\partial t} + a\left(b-r(t)\right)\dfrac{\partial P}{\partial r} + \dfrac{1}{2}\sigma_{r_1}^2\dfrac{\partial^2 P}{\partial r^2} + Ht^{2H-1}\left(2-2^{2H-1}\right)\sigma_{r_2}^2\dfrac{\partial^2 P}{\partial r^2} - rP = 0, \\ P(r,T;T) = 1.\end{cases} \tag{3.5}$$

Given that $A(T,T)=0$ and $B(T,T)=0$, it is not difficult to find a solution for the price of a zero-coupon bond at time t of the following form

$$\begin{cases}\dfrac{\partial P}{\partial t} = P\left[-A'(t,T) - rB'(t,T)\right], \\ \dfrac{\partial P}{\partial r} = -PB(t,T), \\ \dfrac{\partial^2 P}{\partial r^2} = PB^2(t,T).\end{cases} \tag{3.6}$$

Substituting Eq (3.6) into Eq (3.5), we can derive

$$-r\left(B'(t,T) - aB(t,T) + 1\right) - A'(t,T) - abB(t,T) + \frac{1}{2}\sigma_{r_1}^2 B^2(t,T) + Ht^{2H-1}\left(2-2^{2H-1}\right)\sigma_{r_2}^2 B^2(t,T) = 0.$$

After simplification, we have

$$\begin{cases} B'(t,T) - aB(t,T) + 1 = 0, \\ A'(t,T) + abB(t,T) - \frac{1}{2}\sigma_{r_1}^2 B^2(t,T) - Ht^{2H-1}\left(2 - 2^{2H-1}\right)\sigma_{r_2}^2 B^2(t,T) = 0. \end{cases}$$

Then, we can obtain

$$\begin{cases} A(t,T) = b(T-t) - bB(t,T) - \frac{1}{2}\sigma_{r_1}^2 \int_t^T B^2(s,T)\mathrm{d}s - H\left(2 - 2^{2H-1}\right)\sigma_{r_2}^2 \int_t^T s^{2H-1} B^2(s,T)\mathrm{d}s, \\ B(t,T) = \frac{1 - e^{-a(T-t)}}{a}. \end{cases}$$

Proof is completed.

4. Valuation equation for geometric average Asian options

In this section, examine a mixed sub-fractional version of the BS model, i.e., a simple financial market consisting of zero-coupon bonds, underlying assets, and options on underlying assets.

Theorem 4.1. *The value of geometric average Asian call options with fixed strike price is denoted as* $V = V(S,J,r,t)$. *Based on assumptions (2.1) and (2.2), the partial differential equation and boundary conditions are given by*

$$\begin{cases} \frac{\partial V}{\partial t} + \frac{J}{t}\ln\left(\frac{S}{J}\right)\frac{\partial V}{\partial J} + \frac{1}{2}\sigma_{S_1}^2 S^2 \frac{\partial^2 V}{\partial S^2} + Ht^{2H-1}\left(2 - 2^{2H-1}\right)\sigma_{S_2}^2 S^2 \frac{\partial^2 V}{\partial S^2}, \\ + \frac{1}{2}\sigma_{r_1}^2 \frac{\partial^2 V}{\partial r^2} + Ht^{2H-1}\left(2 - 2^{2H-1}\right)\sigma_{r_2}^2 \frac{\partial^2 V}{\partial r^2} + a\left(b - r(t)\right)\frac{\partial V}{\partial r} + rS\frac{\partial V}{\partial S} - rV = 0, \\ V(S,J,r,T) = (J-K)^+. \end{cases} \tag{4.1}$$

Proof. Considering that the portfolio Π consists of one unit option $V(S,J,r,t)$, Δ_{1t} units of underlying assets and Δ_{2t} units of zero-coupon bonds $P(r,t;T)$, the value of the portfolio at time t is given by

$$\Pi_t = V_t - \Delta_{1t} S_t - \Delta_{2t} P_t, \tag{4.2}$$

Choosing the appropriate Δ_{1t} and Δ_{2t} makes the portfolio risk-free in $(t, t+\mathrm{d}t)$, we can obtain

$$
\begin{aligned}
\mathrm{d}\Pi_t &= \mathrm{d}V_t - \Delta_{1t}\mathrm{d}S_t - \Delta_{2t}\mathrm{d}P_t \\
&= \left(\frac{\partial V}{\partial t} + \frac{J}{t}\ln\left(\frac{S}{J}\right)\frac{\partial V}{\partial J} + \frac{1}{2}\sigma_{S_1}^2 S^2 \frac{\partial^2 V}{\partial S^2} + Ht^{2H-1}\left(2-2^{2H-1}\right)\sigma_{S_2}^2 S^2 \frac{\partial^2 V}{\partial S^2} \right)\mathrm{d}t \\
&+ \left(\frac{1}{2}\sigma_{r_1}^2 \frac{\partial^2 V}{\partial r^2} + Ht^{2H-1}\left(2-2^{2H-1}\right)\sigma_{r_2}^2 \frac{\partial^2 V}{\partial r^2} \right)\mathrm{d}t + \left(\frac{\partial V}{\partial S} - \Delta_{1t} \right)\mathrm{d}S + \left(\frac{\partial V}{\partial r} - \Delta_{2t}\frac{\partial P}{\partial r} \right)\mathrm{d}r \\
&- \Delta_{2t}\left(\frac{\partial P}{\partial t} + \frac{1}{2}\sigma_{r_1}^2 \frac{\partial^2 P}{\partial r^2} + Ht^{2H-1}\left(2-2^{2H-1}\right)\sigma_{r_2}^2 \frac{\partial^2 P}{\partial r^2} \right)\mathrm{d}t.
\end{aligned}
$$

Let $\Delta_{1t} = \frac{\partial V}{\partial S}$ and $\Delta_{2t} = \frac{\partial V / \partial r}{\partial P / \partial r}$, using the principle of no arbitrage, we have

$$
E\left(\mathrm{d}\Pi_t\right) = r(t)\Pi \mathrm{d}t = r\left(V_t - \Delta_{1t}S_t - \Delta_{2t}P_t\right)\mathrm{d}t, \tag{4.3}
$$

we can calculate that

$$
\begin{cases}
\frac{\partial V}{\partial t} + \frac{J}{t}\ln\left(\frac{S}{J}\right)\frac{\partial V}{\partial J} + \frac{1}{2}\sigma_{S_1}^2 S^2 \frac{\partial^2 V}{\partial S^2} + Ht^{2H-1}\left(2-2^{2H-1}\right)\sigma_{S_2}^2 S^2 \frac{\partial^2 V}{\partial S^2} \\
+\frac{1}{2}\sigma_{r_1}^2 \frac{\partial^2 V}{\partial r^2} + Ht^{2H-1}\left(2-2^{2H-1}\right)\sigma_{r_2}^2 \frac{\partial^2 V}{\partial r^2} + a\left(b - r(t)\right)\frac{\partial V}{\partial r} + rS\frac{\partial V}{\partial S} - rV = 0, \\
V\left(S, J, r, T\right) = \left(J - K\right)^+.
\end{cases}
$$

Proof is completed.

Theorem 4.2. *Assuming that the stock price satisfies Eq (2.1), and the interest rate satisfies Eq (2.2), the price of the* geometric *average Asian call option at time* $t \in [0, T]$ *with strike price* K *and maturity date* T *is given by*

$$
V\left(S, J, r, t\right) = P\left(r, t; T\right)^{\frac{t}{T}} J^{\frac{t}{T}} S^{\frac{T-t}{T}} e^{L} N\left(d_1\right) - P\left(r, t; T\right) K N\left(d_2\right),
$$

where

$$\begin{cases} d_1 = \dfrac{\dfrac{t\ln J + (T-t)\ln \dfrac{S}{P(r,t;T)}}{T} + \int_t^T \beta_4(s)\mathrm{d}s + 2\int_t^T \beta_3(s)\mathrm{d}s - \ln K}{\sqrt{2\int_t^T \beta_3(s)\mathrm{d}s}}, \\ d_2 = d_1 - \sqrt{2\int_t^T \beta_3(s)\mathrm{d}s}, \\ \beta_1^2(t) = \tilde{\sigma}_S^2 + \tilde{\sigma}_r^2 B^2(t,T), \\ \beta_2(t) = -A(t,T) - rB(t,T), \\ \beta_3(t) = \left(\dfrac{T-t}{T}\right)^2 \beta_1^2(t), \\ \beta_4(t) = \dfrac{1}{T}\beta_2(t) - \dfrac{T-t}{T}\beta_1^2(t), \\ L = \int_t^T \beta_4(s)\mathrm{d}s + \int_t^T \beta_3(s)\mathrm{d}s, \\ \tilde{\sigma}_S^2 = \dfrac{1}{2}\sigma_{S_1}^2 + Ht^{2H-1}\left(2-2^{2H-1}\right)\sigma_{S_2}^2, \\ \tilde{\sigma}_r^2 = \dfrac{1}{2}\sigma_{r_1}^2 + Ht^{2H-1}\left(2-2^{2H-1}\right)\sigma_{r_2}^2, \\ N(x) = \dfrac{1}{\sqrt{2\pi}}\int_{-\infty}^x e^{-\frac{t^2}{2}}\mathrm{d}t. \end{cases}$$

Proof. To simplify the variable-coefficient equation with three variables down to one with two variables, we perform a variable substitution. Thus, let

$$y = \frac{S}{P(r,t;T)}, \quad V_1(y,J,t) = \frac{V(S,J,r,t)}{P(r,t;T)}. \tag{4.4}$$

Sometimes, we denote $P(r,t;T)$ as P to simplify notation. Then, through calculation, we have

$$\begin{cases} \dfrac{\partial V}{\partial J} = P\dfrac{\partial V_1}{\partial J}, \\ \dfrac{\partial V}{\partial t} = V_1\dfrac{\partial P}{\partial t} + P\dfrac{\partial V_1}{\partial t} - y\dfrac{\partial V_1}{\partial y}\dfrac{\partial P}{\partial t}, \\ \dfrac{\partial V}{\partial r} = V_1\dfrac{\partial P}{\partial r} - y\dfrac{\partial V_1}{\partial y}\dfrac{\partial P}{\partial r}, \\ \dfrac{\partial V}{\partial S} = \dfrac{\partial V_1}{\partial y}, \\ \dfrac{\partial^2 V}{\partial r^2} = V_1\dfrac{\partial^2 P}{\partial r^2} - y\dfrac{\partial V_1}{\partial y}\dfrac{\partial^2 P}{\partial r^2} + y^2\dfrac{\partial^2 V_1}{\partial y^2}\dfrac{1}{P}\left(\dfrac{\partial P}{\partial r}\right)^2, \\ \dfrac{\partial^2 V}{\partial r\partial S} = -y\dfrac{\partial^2 V_1}{\partial y^2}\dfrac{1}{P}\dfrac{\partial P}{\partial r}, \\ \dfrac{\partial^2 V}{\partial S^2} = \dfrac{1}{P}\dfrac{\partial^2 V_1}{\partial y^2}. \end{cases} \tag{4.5}$$

Substituting Eq (4.5) into Eq (4.1) and organizing it, we can obtain

$$\begin{aligned} &\frac{\partial V_1}{\partial t} + \left(\frac{1}{2}\frac{1}{P^2}\sigma_{S_1}^2 S^2 + \frac{1}{P^2}Ht^{2H-1}\left(2-2^{2H-1}\right)\sigma_{S_2}^2 S^2 + \frac{1}{2}y^2\frac{1}{P^2}\left(\frac{\partial P}{\partial r}\right)^2\sigma_{r_1}^2\right)\frac{\partial^2 V_1}{\partial y^2} \\ &+y^2\frac{1}{P^2}\left(\frac{\partial P}{\partial r}\right)^2\left(2-2^{2H-1}\right)H\sigma_{r_2}^2 t^{2H-1}\frac{\partial^2 V_1}{\partial y^2} + \frac{J}{t}\ln\left(\frac{S}{J}\right)\frac{\partial V_1}{\partial J} \\ &-\frac{1}{P}y\left(\frac{\partial P}{\partial t} + \frac{1}{2}\frac{\partial^2 P}{\partial r^2}\sigma_{r_1}^2 + Ht^{2H-1}\left(2-2^{2H-1}\right)\sigma_{r_2}^2\frac{\partial^2 P}{\partial r^2} + a\left(b-r(t)\right)\frac{\partial P}{\partial r} - \frac{rS}{y}\right)\frac{\partial V_1}{\partial y} \\ &+\frac{V_1}{P}\left(\frac{\partial P}{\partial t} + \frac{1}{2}\frac{\partial^2 P}{\partial r^2}\sigma_{r_1}^2 + \frac{\partial^2 P}{\partial r^2}Ht^{2H-1}\left(2-2^{2H-1}\right)\sigma_{r_2}^2 + a\left(b-r(t)\right)\frac{\partial P}{\partial r} - rP\right) = 0. \end{aligned}$$

By integrating Eqs (4.4) and (3.5), we can derive

$$\frac{\partial V_1}{\partial t} + \frac{J}{t}\ln\left(\frac{S}{J}\right)\frac{\partial V_1}{\partial J} + y^2\beta_1^2(t)\frac{\partial^2 V_1}{\partial y^2} = 0, \tag{4.6}$$

where

$$\begin{cases} \beta_1^2(t) = \tilde{\sigma}_S^2 + \tilde{\sigma}_r^2 B^2(t,T), \\ \tilde{\sigma}_S^2 = \dfrac{1}{2}\sigma_{S_1}^2 + Ht^{2H-1}\left(2-2^{2H-1}\right)\sigma_{S_2}^2, \\ \tilde{\sigma}_r^2 = \dfrac{1}{2}\sigma_{r_1}^2 + Ht^{2H-1}\left(2-2^{2H-1}\right)\sigma_{r_2}^2. \end{cases}$$

By substituting the deformation of Eq (4.4) into Eq (4.6), the equation can be transformed into

$$\frac{\partial V_1}{\partial t}+\frac{J}{t}\left(\beta_2\left(t\right)+\ln\frac{y}{J}\right)\frac{\partial V_1}{\partial J}+y^2\beta_1^2\left(t\right)\frac{\partial^2 V_1}{\partial y^2}=0, \tag{4.7}$$

where $\beta_2\left(t\right)=-A\left(t,T\right)-rB\left(t,T\right)$.

To simplify Eq (4.7), we can make the following variable substitution

$$x=\frac{t\ln J+\left(T-t\right)\ln y}{T},\quad V_2\left(x,t\right)=V_1\left(y,J,t\right). \tag{4.8}$$

It can be determined that

$$\begin{cases}\dfrac{\partial V_1}{\partial J}=\dfrac{t}{TJ}\dfrac{\partial V_2}{\partial x},\\ \dfrac{\partial V_1}{\partial t}=\dfrac{\partial V_2}{\partial t}+\dfrac{\ln J-\ln y}{T}\dfrac{\partial V_2}{\partial x},\\ \dfrac{\partial V_1}{\partial y}=\dfrac{T-t}{Ty}\dfrac{\partial V_2}{\partial x},\\ \dfrac{\partial^2 V_1}{\partial y^2}=\left(\dfrac{T-t}{Ty}\right)^2\dfrac{\partial^2 V_2}{\partial x^2}-\dfrac{T-t}{Ty^2}\dfrac{\partial V_2}{\partial x}.\end{cases} \tag{4.9}$$

Substituting the above results into Eq (4.7), we find that

$$\begin{cases}\dfrac{\partial V_2}{\partial t}+\beta_3\left(t\right)\dfrac{\partial^2 V_2}{\partial x^2}+\beta_4\left(t\right)\dfrac{\partial V_2}{\partial x}=0,\\ V_2\left(x,T\right)=\left(e^x-K\right)^+,\end{cases} \tag{4.10}$$

where $\beta_3\left(t\right)=\left(\frac{T-t}{T}\right)^2\beta_1^2\left(t\right)$ and $\beta_4\left(t\right)=\frac{1}{T}\beta_2\left(t\right)-\frac{T-t}{T}\beta_1^2\left(t\right)$.

The last time using variable substitution to simplify Eq (4.10) into a heat conduction equation, let

$$V_2\left(x,t\right)=\mu\left(\eta,\theta\right),\quad \eta=x+\int_t^T\beta_4\left(s\right)\mathrm{d}s,\quad \theta=\int_t^T\beta_3\left(s\right)\mathrm{d}s. \tag{4.11}$$

It can be deduced that

$$\begin{cases}\dfrac{\partial V_2}{\partial t}=-\beta_4\left(t\right)\dfrac{\partial\mu}{\partial\eta}-\beta_3\left(t\right)\dfrac{\partial\mu}{\partial\theta},\\ \dfrac{\partial V_2}{\partial x}=\dfrac{\partial\mu}{\partial\eta},\\ \dfrac{\partial^2 V_2}{\partial x^2}=\dfrac{\partial^2\mu}{\partial\eta^2}.\end{cases} \tag{4.12}$$

Then, Eq (4.10) can be converted to

$$\begin{cases} \dfrac{\partial^2 \mu}{\partial \eta^2} = \dfrac{\partial \mu}{\partial \theta}, \\ \mu(\eta, T) = \left(e^{\eta} - K\right)^{+}. \end{cases} \tag{4.13}$$

According to the heat conduction theory, the solution of this equation can be expressed as

$$V_2(x,t) = \mu(\eta,\theta) = e^{\eta+\theta} N(d_5) - KN(d_6) = e^{x+L} N(d_5) - KN(d_6), \tag{4.14}$$

where

$$d_5 = \frac{\eta - \ln K + 2\theta}{\sqrt{2\theta}} = \frac{x + \int_t^T \beta_4(s)\mathrm{d}s + 2\int_t^T \beta_3(s)\mathrm{d}s - \ln K}{\sqrt{2\int_t^T \beta_3(s)\mathrm{d}s}},$$

$$d_6 = d_5 - \sqrt{2\theta} = d_5 - \sqrt{2\int_t^T \beta_3(s)\mathrm{d}s},$$

$$L = \int_t^T \beta_4(s)\mathrm{d}s + \int_t^T \beta_3(s)\mathrm{d}s.$$

Substituting Eq (4.8) into Eq (4.14), we can obtain

$$V_1(y, J, t) = J^{\frac{t}{T}} y^{\frac{T-t}{t}} e^{L} N(d_3) - KN(d_4), \tag{4.15}$$

where

$$d_3 = \frac{\dfrac{t \ln J + (T-t)\ln y}{T} + \int_t^T \beta_4(s)\mathrm{d}s + 2\int_t^T \beta_3(s)\mathrm{d}s - \ln K}{\sqrt{2\int_t^T \beta_3(s)\mathrm{d}s}},$$

$$d_4 = d_3 - \sqrt{2\int_t^T \beta_3(s)\mathrm{d}s}.$$

Combining Eqs (4.4) and (4.15), it follows that

$$V(S,J,r,t) = V_1\left(\frac{S}{P(r,t;T)}, J, t\right) P(r,t;T) = P(r,t;T)^{\frac{t}{T}} J^{\frac{t}{T}} S^{\frac{T-t}{T}} e^{L} N(d_1) - P(r,t;T) KN(d_2),$$

where

$$d_1 = \frac{\dfrac{t \ln J + (T-t)\ln \dfrac{S}{P(r,t;T)}}{T} + \int_t^T \beta_4(s)\mathrm{d}s + 2\int_t^T \beta_3(s)\mathrm{d}s - \ln K}{\sqrt{2\int_t^T \beta_3(s)\mathrm{d}s}},$$

$$d_2 = d_1 - \sqrt{2\int_t^T \beta_3(s)\mathrm{d}s}.$$

Proof is completed.

Theorem 4.3. *The call-put parity relationship for geometric average Asian options with fixed strike price is given by*

$$V(S,J,r,t) - p(S,J,r,t) = P(r,t;T) J^{\frac{t}{T}} \left(\frac{S}{P(r,t;T)} \right)^{\frac{T-t}{T}} e^L - P(r,t;T)K,$$

where $p(S,J,r,t)$ *is the price of geometric average Asian put options and*

$$\begin{cases} \beta_1^2(t) = \tilde{\sigma}_S^2 + \tilde{\sigma}_r^2 B^2(t,T), \\ \beta_2(t) = -A(t,T) - rB(t,T), \\ \beta_3(t) = \left(\frac{T-t}{T} \right)^2 \beta_1^2(t), \\ \beta_4(t) = \frac{1}{T}\beta_2(t) - \frac{T-t}{T}\beta_1^2(t), \\ L = \int_t^T \beta_4(s)\mathrm{d}s + \int_t^T \beta_3(s)\mathrm{d}s, \\ \tilde{\sigma}_S^2 = \frac{1}{2}\sigma_{S_1}^2 + Ht^{2H-1}\left(2-2^{2H-1}\right)\sigma_{S_2}^2, \\ \tilde{\sigma}_r^2 = \frac{1}{2}\sigma_{r_1}^2 + Ht^{2H-1}\left(2-2^{2H-1}\right)\sigma_{r_2}^2. \end{cases}$$

Proof. Let

$$W(S,J,r,t) = V(S,J,r,t) - p(S,J,r,t).$$

According to Theorem 4.1, $W(S,J,r,t)$ satisfies the following definite solution problem

$$\begin{cases} \frac{\partial W}{\partial t} + \frac{J}{t}\ln\left(\frac{S}{J}\right)\frac{\partial W}{\partial J} + \frac{1}{2}\sigma_{S_1}^2 S^2 \frac{\partial^2 W}{\partial S^2} + Ht^{2H-1}\left(2-2^{2H-1}\right)\sigma_{S_2}^2 S^2 \frac{\partial^2 W}{\partial S^2} \\ +\frac{1}{2}\sigma_{r_1}^2 \frac{\partial^2 W}{\partial r^2} + Ht^{2H-1}\left(2-2^{2H-1}\right)\sigma_{r_2}^2 \frac{\partial^2 W}{\partial r^2} + a\left(b-r(t)\right)\frac{\partial W}{\partial r} + rS\frac{\partial W}{\partial S} - rW = 0, \\ W(S,J,r,T) = J - K. \end{cases} \tag{4.16}$$

Making the following variable substitutions

$$y = \frac{S}{P(r,t;T)}, \quad W_1(y,J,t) = \frac{W(S,J,r,t)}{P(r,t;T)}. \tag{4.17}$$

Substituting Eq (4.17) into Eq (4.16), we have

$$\frac{\partial W_1}{\partial t}+\frac{J}{t}\ln\left(\frac{S}{J}\right)\frac{\partial W_1}{\partial J}+y^2\beta_1^2(t)\frac{\partial^2 W_1}{\partial y^2}=0, \tag{4.18}$$

where

$$\begin{cases}\beta_1^2(t)=\tilde{\sigma}_S^2+\tilde{\sigma}_r^2B^2(t,T),\\ \tilde{\sigma}_S^2=\dfrac{1}{2}\sigma_{S_1}^2+Ht^{2H-1}\left(2-2^{2H-1}\right)\sigma_{S_2}^2,\\ \tilde{\sigma}_r^2=\dfrac{1}{2}\sigma_{r_1}^2+Ht^{2H-1}\left(2-2^{2H-1}\right)\sigma_{r_2}^2.\end{cases}$$

Let

$$\xi=\frac{t\ln J+(T-t)\ln y}{T},\quad W_2(\xi,t)=W_1(y,J,t). \tag{4.19}$$

By inserting Eq (4.19) into Eq (4.18), this yields

$$\begin{cases}\dfrac{\partial W_2}{\partial t}+\beta_3(t)\dfrac{\partial^2 W_2}{\partial \xi^2}+\beta_4(t)\dfrac{\partial W_2}{\partial \xi}=0,\\ W_2(\xi,T)=e^{\xi}-K,\end{cases} \tag{4.20}$$

where

$$\beta_2(t)=-A(t,T)-rB(t,T),$$

$$\beta_3(t)=\left(\frac{T-t}{T}\right)^2\beta_1^2(t),$$

$$\beta_4(t)=\frac{1}{T}\beta_2(t)-\frac{T-t}{T}\beta_1^2(t).$$

By letting

$$W_2(\xi,t)=a(t)e^{\xi}-b(t)K, \tag{4.21}$$

then, we can obtain

$$\begin{cases}\dfrac{\partial W_2}{\partial t}=a'(t)e^{\xi}-b'(t)K,\\ \dfrac{\partial W_2}{\partial \xi}=a(t)e^{\xi},\\ \dfrac{\partial^2 W_2}{\partial \xi^2}=a(t)e^{\xi},\end{cases}$$

and

$$\left(a'(t)+\beta_3(t)a(t)+\beta_4(t)a(t)\right)e^{\xi}-b'(t)K=0. \tag{4.22}$$

To solve Eq (4.22), we select appropriate $a(t)$ and $b(t)$ so that

$$\begin{cases} a'(t)+\beta_3(t)a(t)+\beta_4(t)a(t)=0, \\ b'(t)=0, \\ a(T)=1, \\ b(T)=1. \end{cases}$$

Thus, we can derive

$$a(t)=e^{L}, \quad b(t)=1, \tag{4.23}$$

where

$$L=\int_t^T \beta_4(s)\mathrm{d}s+\int_t^T \beta_3(s)\mathrm{d}s.$$

Substituting Eqs (4.21) and (4.23) into Eq (4.17), we have

$$V(S,J,r,t)-p(S,J,r,t)=P(r,t;T)W_1(y,J,t)=P(r,t;T)J^{\frac{t}{T}}\left(\frac{S}{P(r,t;T)}\right)^{\frac{T-t}{T}}e^{L}-P(r,t;T)K.$$

Proof is completed.

Corollary 4.1. *Let the price of the arithmetic average Asian call options with fixed strike price be denoted as* $\hat{V}(S,J,r,t)$*, and let the price of the put options be* $\hat{p}(S,J,r,t)$*. The put-call parity formula for arithmetic average Asian options is*

$$\hat{V}(S,J,r,t)-\hat{p}(S,J,r,t)=P(r,t;T)\left(\frac{tJ}{T}-K\right)+\frac{S}{T}\int_t^T e^{-A(s,T)-rB(s,T)}\mathrm{d}s,$$

where

$$\begin{cases} A(t,T)=b(T-t)-bB(t,T)-\frac{1}{2}\sigma_{r_1}^2\int_t^T B^2(s,T)\mathrm{d}s-H\left(2-2^{2H-1}\right)\sigma_{r_2}^2\int_t^T s^{2H-1}B^2(s,T)\mathrm{d}s, \\ B(t,T)=\dfrac{1-e^{-a(T-t)}}{a}. \end{cases}$$

Proof. Denoting

$$W(S,J,r,t)=\hat{V}(S,J,r,t)-\hat{p}(S,J,r,t).$$

According to Theorem 4.1 and Theorem 4.3, we can similarly derive

$$
\begin{cases}
\dfrac{\partial W}{\partial t}+\dfrac{S-J}{t}\dfrac{\partial W}{\partial J}+\dfrac{1}{2}\sigma_{S_1}^2 S^2\dfrac{\partial^2 W}{\partial S^2}+Ht^{2H-1}\left(2-2^{2H-1}\right)\sigma_{S_2}^2 S^2\dfrac{\partial^2 W}{\partial S^2}\\
+\dfrac{1}{2}\sigma_{r_1}^2\dfrac{\partial^2 W}{\partial r^2}+Ht^{2H-1}\left(2-2^{2H-1}\right)\sigma_{r_2}^2\dfrac{\partial^2 W}{\partial r^2}+a\left(b-r\left(t\right)\right)\dfrac{\partial W}{\partial r}+rS\dfrac{\partial W}{\partial S}-rW=0,\\
W\left(S,J,r,T\right)=J-K.
\end{cases}
\tag{4.24}
$$

By employing a change of variables, we define

$$
y=\frac{S}{P\left(r,t;T\right)},\quad W_1\left(y,J,t\right)=\frac{W\left(S,J,r,t\right)}{P\left(r,t;T\right)}. \tag{4.25}
$$

Substituting Eq (4.25) into Eq (4.24), we can obtain

$$
\frac{\partial W_1}{\partial t}+\frac{yP-J}{t}\frac{\partial W_1}{\partial J}+y^2\beta_1^2\left(t\right)\frac{\partial^2 W_1}{\partial y^2}=0, \tag{4.26}
$$

where

$$
\begin{cases}
\beta_1^2\left(t\right)=\tilde{\sigma}_S^2+\tilde{\sigma}_r^2 B^2\left(t,T\right),\\
\tilde{\sigma}_S^2=\dfrac{1}{2}\sigma_{S_1}^2+Ht^{2H-1}\left(2-2^{2H-1}\right)\sigma_{S_2}^2,\\
\tilde{\sigma}_r^2=\dfrac{1}{2}\sigma_{r_1}^2+Ht^{2H-1}\left(2-2^{2H-1}\right)\sigma_{r_2}^2.
\end{cases}
$$

Let

$$
\xi=\frac{TK-tJ}{y},\quad W_2\left(\xi,t\right)=\frac{T}{y}W_1\left(y,J,t\right). \tag{4.27}
$$

By substituting Eq (4.27) into Eq (4.26), we can deduce

$$
\begin{cases}
\dfrac{\partial W_2}{\partial t}+\beta_1^2\left(t\right)\dfrac{\partial^2 W_2}{\partial \xi^2}\xi^2-P\dfrac{\partial W_2}{\partial \xi}=0,\\
W_2\left(\xi,T\right)=-\xi.
\end{cases}
\tag{4.28}
$$

Let

$$
W_2\left(\xi,t\right)=a\left(t\right)\xi+b\left(t\right), \tag{4.29}
$$

then, we have

$$\begin{cases} \frac{\partial W_2}{\partial t} = a'(t)\xi + b'(t), \\ \frac{\partial W_2}{\partial \xi} = a(t), \\ \frac{\partial^2 W_2}{\partial \xi^2} = 0, \end{cases}$$

and

$$a'(t)\xi + b'(t) - a(t)P = 0. \tag{4.30}$$

By combining Eqs (4.28) and (4.30), we can compare the coefficients to obtain

$$\begin{cases} a'(t) = 0, \\ b'(t) - a(t)P = 0, \\ a(T) = -1, \\ b(T) = 0. \end{cases}$$

Solving the aforementioned system of equations yields

$$a(t) = -1,\ b(t) = \int_t^T e^{-A(s,T)-rB(s,T)} \mathrm{d}s.$$

Finally, we can derive

$$W(S,J,r,t) = \hat{V}(S,J,r,t) - \hat{p}(S,J,r,t) = P(r,t;T)\left(\frac{tJ}{T} - K\right) + \frac{S}{T}\int_t^T e^{-A(s,T)-rB(s,T)} \mathrm{d}s.$$

5. Numerical analysis

In this section, we provide specific values for each parameter to conduct the following numerical calculations.

$$t = 0, S = 30, \sigma_{S_1} = 0.5, \sigma_{S_2} = 0.4, \sigma_{r_1} = 0.3, \sigma_{r_2} = 0.2, a = 2, b = 0.05, r = 0.06, T = 1.$$

From Figure 1, it can be observed that when $H = 0.5$, the msfBm simplifies to a geometric Brownian motion. At this point, the price of geometric average Asian call options peaks, whereas it reaches its minimum at $H = 0.9$. An increment in the Hurst index is indicative of the manifestation of long-range correlations in the asset price, suggesting a persistent price trend. In specific situations, the expected small fluctuations in the asset price in the market will lead to a decrease in the option value. Figure 2 indicates that as the short rate rises, the price of the options gradually increases. This is because an increase in the interest rate could potentially raise the payoff of the options at maturity, thereby enhancing the holding value of the options. Figure 3 shows that the valuation of the options is positively influenced by a rise in the initial stock price. From Figure 4, the price of the options increases with the extension of the expiration date. As the remaining time for the option contract increases, it gives the holders more time to wait for a potential rise in the stock price. Therefore, the possibility of the investors making a profit is greater, and the price of the options will also increase accordingly. Conversely, it diminishes with an upsurge in the strike price. As the strike price increases, buyers have to pay a higher price to exercise the options, which compresses the profit margin and thus reduces the value of the options. Figure 5 illustrates that the pricing of options under the current model closely mirrors that of the sub-fractional Vasicek model. Notably, the pricing discrepancy for options between our model and Vasicek model widens initially and then narrows, and the pricing gap between our model and BS model exhibits a consistent upward trend.

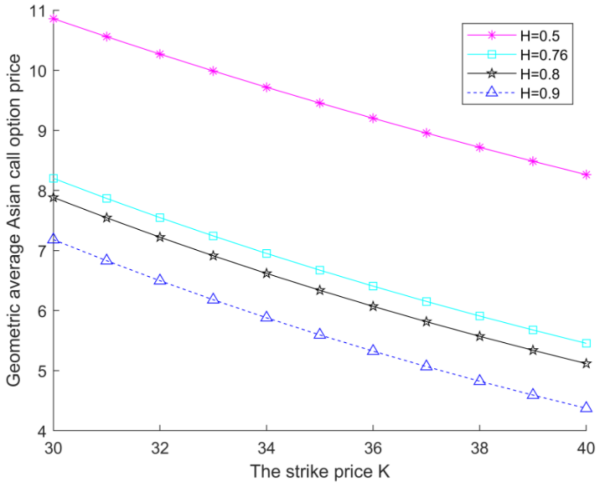

Figure 1. The price of fixed strike geometric average Asian call options under the msfBm-Vasicek model, according to the parameter H.

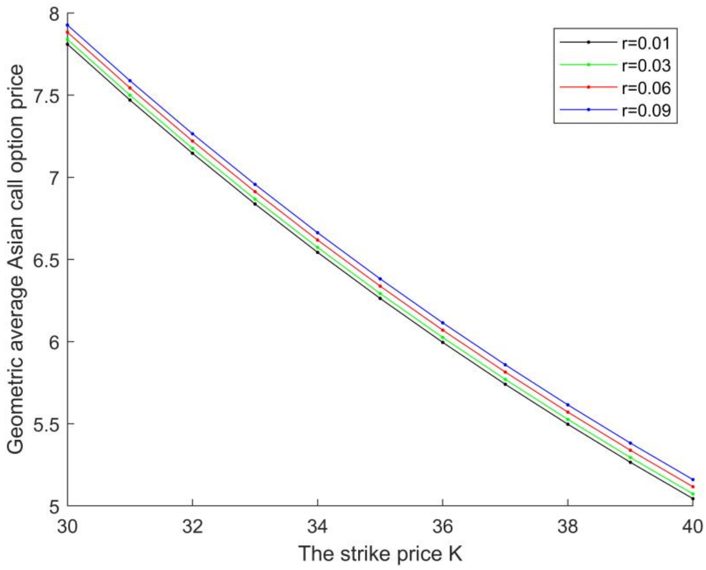

Figure 2. The price of geometric average Asian call options under different r.

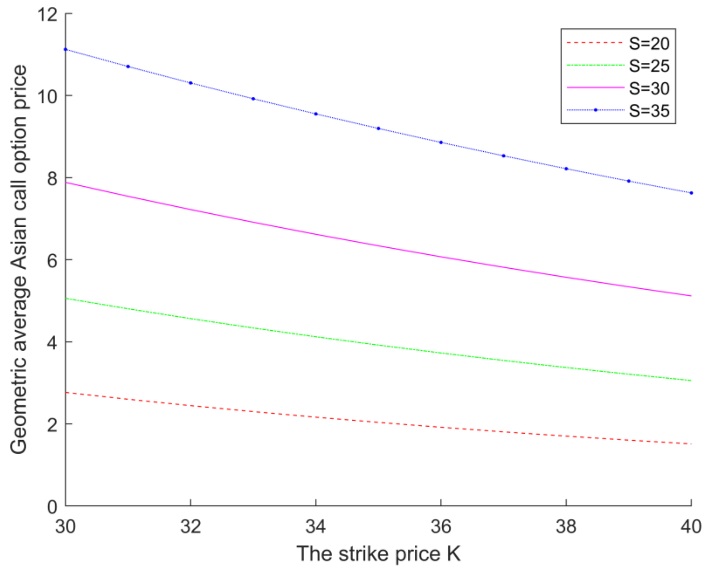

Figure 3. Geometric average Asian call option price across S.

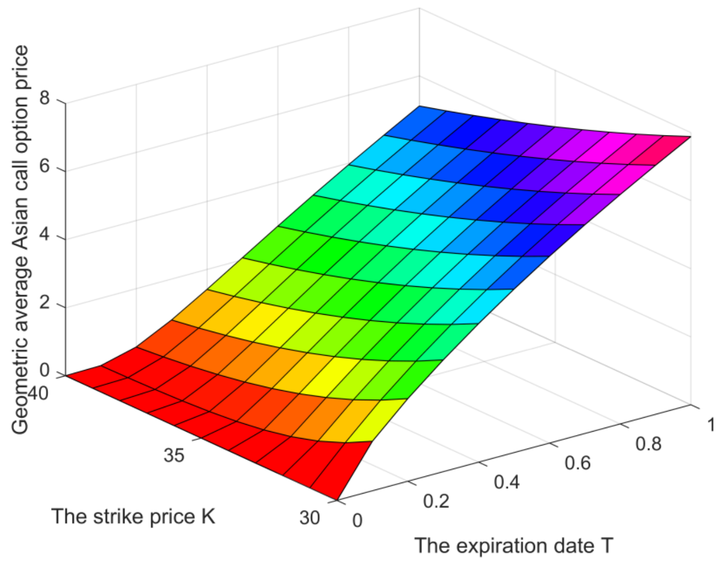

Figure 4. The impact of parameters K and T on the pricing of geometric average Asian call options when $H = 0.8$.

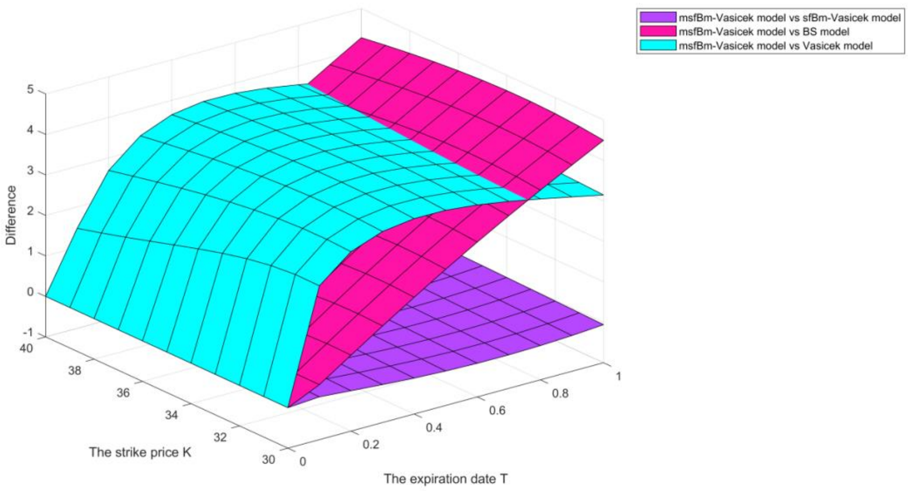

Figure 5. Difference among the msfBm-Vasicek model, sfBm-Vasicek model, Vasicek model, and BS model.

According to Tables 1 and 2, it can be observed that the lowest pricing for geometric average Asian call options is associated with the BS model. In the case of a brief expiration period, the price of the options under the sfBm-Vasicek model marginally surpasses that of our model, especially with an increasing stock price, and there is minimal variation between the two models. For longer expiration dates, the option price under our model exceeds that under the sfBm-Vasicek model, with the price difference becoming more pronounced. Moreover, the price of options under our model and the Vasicek model are getting closer. Overall, our model demonstrates a degree of rationality and is deemed suitable for developing option pricing models in financial markets.

Table 1. The price of the four models under $T = 0.5$ and $K = 40$.

S	Our price	V_{BS}	V_{Vasicek}	$V_{\text{sfBm-Vasicek}}$
20	0.2435	0.0005	1.8389	0.2208
23	0.5498	0.0066	2.8183	0.5202
26	1.0480	0.0426	3.9897	1.0211
29	1.7685	0.1748	5.3360	1.7589
32	2.7255	0.5146	6.8397	2.7506
35	3.9194	1.1863	8.4843	3.9968
38	5.3398	2.2779	10.2543	5.4858
41	6.9698	3.8119	12.1360	7.1977
44	8.7884	5.7485	14.1169	9.1087
47	10.7735	8.0118	16.1861	11.1933
50	12.9033	10.5160	18.3337	13.4270

Table 2. A comparison of the pricing results among different models when $T=2$.

S	Our price	V_{BS}	$V_{Vasicek}$	$V_{sfBm\text{-}Vasicek}$
20	5.1411	0.2085	2.3002	0.9705
23	7.4411	0.4863	4.8981	2.7039
26	9.8921	0.9501	7.6053	4.5926
29	12.4656	1.6332	10.3988	6.6067
32	15.1403	2.5519	13.2620	8.7238
35	17.8998	3.7072	16.1825	10.9266
38	20.7312	5.0881	19.1509	13.2019
41	23.6243	6.6762	22.1600	15.5389
44	26.5707	8.4490	25.2039	17.9291
47	29.5636	10.3825	28.2781	20.3655
50	32.5974	12.4535	31.3787	22.8423

We select copper options for three-month futures as the sample, with a cutoff time of October 2006, and the data are sourced from the London Metal Exchange. Since all trading products on the LME are priced in US dollars, we adopt the interest rate of one-year US Treasury bonds.

We calculate the value of H using the R/S method. Define a return sequence $\left\{R_t, R_t=\frac{\ln P_{t+1}}{\ln P_t}\right\}$ of length N and divide it into A consecutive sub-intervals of length n. Label each sub-interval as $I_a, a=1,\cdots,A.$ Thus, each point in I_a can be represented as

$$R_{k,a}, k=1,\cdots,n; a=1,\cdots,A.$$

For each sub-interval I_a of length n, calculate its mean as

$$e_a=\frac{1}{n}\sum_{k=1}^{n}R_{k,a}.$$

The cumulative mean deviation $X_{k,a}$ for a single sub-interval is calculated as

$$X_{k,a}=\sum_{i=1}^{k}\left(R_{i,a}-e_a\right), k=1,2,\cdots,n.$$

The sum of the cumulative mean deviation sequence $\left\{X_{1,a}, X_{2,a},\cdots X_{n,a}\right\}$ for a single sub-interval is zero. The range of an individual sub-interval is defined as

$$R_{I_a}=\max_k\left(X_{k,a}\right)-\min_k\left(X_{k,a}\right), k=1,2,\cdots,n.$$

Subsequently, the standard deviation S_{I_a} for each sub-interval is given by

$$S_{I_a}=\sqrt{\frac{1}{n}\sum_{k=1}^{n}\left(E_{k,a}-e_a\right)^2}.$$

Therefore, for the partition length n, we can compute the average rescaled range for A sub-intervals as

$$(R/S)_n = \frac{1}{A}\sum_{a=1}^{A}\left(\frac{R_{I_a}}{S_{I_a}}\right).$$

Repeat the above calculation process for different partition lengths (i.e., different time scales) n to obtain multiple average rescaled range values. There is a linear relationship between $\log\left(\frac{R}{S}\right)$ and $\log(n)$ [27]

$$\log\left(\frac{R}{S}\right)_n = a + H\log(n).$$

Finally, a double logarithmic regression is performed on n and R/S, and the slope is the parameter of long-range correlations, that is, the Hurst index. Therefore, we can obtain $H = 0.6$.

From Table 3, it is evident that the RMSE of our model is the smallest, indicating that the option price derived from our model is the closest to the market price. Furthermore, the price from our model is also quite close to that of the sfBm-Vasicek model. This suggests that it is essential to include factors such as the long-range correlations of the underlying asset and stochastic interest rate in the option pricing model, as they significantly influence the option price.

Table 3. Our model compared with other models.

K	Market price	Our price	V_{BS}	V_{Vasicek}	$V_{\mathrm{sfBm\text{-}Vasicek}}$
7800	750	750.8065	747.9587	752.3958	749.0275
7500	694	693.9719	691.9232	696.2100	692.5075
7850	737	737.8847	734.8516	739.3860	736.1443
7900	742	741.1881	739.7236	744.2764	739.8953
7550	670	669.3538	667.6885	672.1255	668.1860
7650	658	658.1121	655.5799	660.1722	656.8193
7700	630	629.6739	627.5145	632.1150	628.5006
7300	584	584.4501	581.6303	586.0389	583.1820
7600	572	572.5130	569.3085	574.0554	571.3721
7450	541	540.6392	538.3216	542.9253	539.7519
7400	508	509.9819	505.4722	510.0512	508.4733
	RMSE	0.8064	2.3756	2.1638	1.2832

6. Conclusions

Geometric average Asian options are important exotic options. The mixed sub-fractional Brownian motion, chosen as a stochastic process, is capable of more accurately depicting the characteristics of long-range correlations. Since the Vasicek model is a very classical model, it is combined with the msfBm to simultaneously address the pricing issue. By using partial differential equations and multiple variable substitution methods, the valuation equations for the model are deduced. Finally, numerical computations are utilized to assess the effects of diverse parameters. The efficacy of our model is demonstrated through a comparative analysis of the pricing discrepancies for options between different models, which provides a reference for option pricing theory and practice.

Author contributions

Xinyi Wang: Conceptualization, Methodology, Writing—original draft, Writing—review & editing, Funding acquisition; Chunyu Wang: Writing—review & editing. All authors have read and approved the final version of the manuscript for publication.

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Conflict of interest

The authors declare no conflicts of interest.

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