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Research article

Synchronization issue of uncertain time-delay systems based on flexible impulsive control

Biwen Li and Qiaoping Huang*

Huangshi Key Laboratory of Metaverse and Virtual Simulation, School of Mathematics and Statistics, Hubei Normal University, Huangshi 435002, Hubei, China

* **Correspondence:** Email: hqp15572910919@163.com; Tel: +8615572910919.

Abstract: This paper discusses a synchronization issue of uncertain time-delay systems via flexible delayed impulsive control. A new Razumikhin-type inequality is presented, considering adjustable parameters the $\varpi(t)$, which relies on flexible impulsive gain. For the uncertain time-delay systems where delay magnitude is not constrained to impulsive intervals, sufficient conditions for global exponential synchronization (GES) are established. Furthermore, based on Lyapunov theory, a new differential inequality and linear matrix inequality design, and a flexible impulsive control method is introduced through using the variable impulsive gain and time-varying delays. It is interesting to find that uncertain time-delay systems can maintain GES by adjusting the impulsive gain and impulsive delay. Finally, two simulations are given to illustrate the effectiveness of the derived results.

Keywords: impulsive control; time delay; synchronization; impulsive gain; parametric uncertainty
Mathematics Subject Classification: 93C30

1. Introduction

Compared to continuous control, impulsive control has received widespread attention in the control field and has been effectively used in real applications such as physics [1–4], cryptography (see [5]), and biological medicine (see [6–8]) because it reduces control cost due to the fact that receiving sampling information only occurs at certain discrete instants. For instance, [1] first combined impulsive control methods with moving vehicles to enable vehicles on the road to travel at the desired safe margin and speed, thereby relieving traffic congestion. [8] set up rational impulsive controllers to explore the issue of optimizing drug to treat influenza, so impulsive control can show some worth for medicine.

Nevertheless, time delays inevitably occur in the sampling, transmission, and processing of impulsive information. Therefore, the time delay problem in impulsive control cannot be ignored. Many researchers have investigated impulsive delay. For example, [2] addresses the problem of the synchronization of time-delay impulsive control in linear dynamic networks with respect to time scales. [9] studied synchronization using distributed delay impulsive control, where the developed Lyapunov function is limited by the size of the impulsive interval. Synchronization of discrete delayed impulsive control with two types of neural networks was analyzed by synchronous impulses, but findings restrict the upper and lower bounds of the impulsive interval [10]. Based on the theory of delayed impulses, the leader-follower synchronization problem for delayed systems was solved in [11]. In particular, the optimal control problem for impulsive time-delay systems has yielded a number of interesting results [12–14].

Synchronization is one of the important dynamics behaviors of impulsive dynamical systems and is very widely used in many different fields [15–17]. The stability and synchronization problems of impulsive dynamical systems with time delay have been a popular topic in the control and analysis of discontinuous dynamical systems, and has attracted the interest of many scholars [18–25]. For example, [18] studied impulsive control of nonlinear delayed systems and applied it to synchronization control of delayed neural networks. An effective impulsive controller for the stabilization of singular delayed systems was proposed in [19]. The class comparison principle (see [21]) and average impulsive interval (AII) method for impulsive delay systems (see [20, 22, 25]) has also been applied to study the stability (or synchronization) of delayed impulsive systems. Furthermore, based on the beneficial impact of impulsive delay on stability, [26] presented an impulsive control scheme with time delay and related criteria for stabilizing the considered system. It is not difficult to find that systems can reach consistent synchronization, asymptotic synchronization, or exponential synchronization by using different impulsive control schemes [27–29]. [30] investigated the GES of the systems using the AII concept and impulsive control with a fixed number of impulses. [31] further derives some innovative and less conservative GES criteria for a class of general delay dynamic networks by employing the idea of AII and comparison principle. It is clear that both [30] and [31], as well as some of the previous literature on delayed impulsive control, focus mainly on the case of fixed impulsive gain. Nevertheless, due to the complexity of practical situations, it is unreasonable to apply the same impulsive gain at each impulse point. In addition, external impulses can desynchronize systems that lack adaptive strategies for restoring synchronization [32, 33].

On the other hand, the parameters of time-delay impulsive dynamical systems can be disturbed by some factors, such as electronic component tolerances, model inaccuracies, and environmental changes. Therefore, the parameter uncertainties, should be taken into account when investigating the stability or synchronization problems of time-delay impulsive dynamical systems, and there have been a number of recent studies in this regard [34–37]. For example, in [36], the synchronization for a kind of switched neural networks involving hybrid delays, parametric uncertainty, and sampling control is discussed.

In summary, this paper focuses on exploring the influence of flexible impulsive gain on synchronization and the potential positive impact of impulses with delays on synchronization by using adjustable impulsive control. The list of contributions of this article is as follows:

1) A new flexible impulsive control scheme for uncertain time-delay systems, relying on the variable gain instead of the common gain commonly of previous studies, is presented to enhance the anti-attack ability of impulsive systems. If systems suffer from external desynchronizing impulses, the novel control method guarantees synchronization of the systems by regulating the impulsive gain so that it satisfies the synchronization criteria. Time-varying impulsive delays are taken into

account equally. When the size of impulsive delay is large enough in the impulsive interval, the unstable impulsive gain can maintain the system synchronization, and the time-delay system can achieve self-synchronization by integrating the acquired impulsive delay and impulsive gain information.

2) The new impulsive delay inequality, which takes into account both AII and average impulsive gain, has been developed. By utilizing such inequality, we derive several sufficient criteria for GES. Time delay limitations of continuous differential equations are relaxed.

The organization of this paper is as follows: Section 2 presents the preliminary knowledge. The major findings are given in Section 3. In Section 4, the results of the simulation are presented, and finally Section 5 draws a conclusion.

2. Preliminaries

The following notations will be used in this article. Let $\Re$ ($\Re_+$, $\Re_+^0$) denote the set of (positive, non-negative) real numbers, and $\mathcal{Z}$ ($\mathcal{Z}_0$) represents the set of positive (non-negative) integer numbers. Denote $\Re^n$ as an n-dimensional real space equipped with Euclidean norm $\|\cdot\|$. $\mathcal{S}(t^+)$ and $\mathcal{S}(t^-)$ stand for the right limit and the left limit of $\mathcal{S}$ at instant t, respectively. For interval $\mathcal{J} \subseteq \Re$, $\mathcal{S} \subseteq \Re^m$ $(1 \le m \le n)$. $\mathcal{PC}(\mathcal{J}, \mathcal{S})$ $=\{\phi \in \mathcal{PC}(\mathcal{J}, \mathcal{S})$: ϕ is continuous everywhere except at a finite number of points t where $\phi(t^+)$ and $\phi(t^-)$ exist, and $\phi(t^+) = \phi(t)\}$. For given $\rho > 0$, $\mathcal{PC}\left([t_0-\rho, t_0], \Re^n\right)$ represents a class of piecewise right continuous functions $x : [t_0-\rho, t_0] \to \Re^n$, in which $\|x\|_\rho \triangleq \sup_{t_0-\rho\le t\le t_0} \|x(t)\|$. Besides, $\mathbb{F} > 0$ ($\mathbb{F} < 0$, $\mathbb{F} \le 0$) indicates $\mathbb{F}$ is a positive (negative, semi) definite symmetric matrix. Let $\lambda_{max}(\mathbb{F})$ and $\lambda_{min}(\mathbb{F})$ denote the maximum and minimum eigenvalue of matrix $\mathbb{F}$, respectively. Let $\mathbb{F}^T$ and $\mathbb{F}^{-1}$ be the transpose and inverse of the matrix $\mathbb{F}$. Let I_n denote an n-dimensional identity matrix. Define the notation $\bullet$ as the symmetric term of a symmetric matrix.

Consider the following class of uncertain time-delay systems:

$$\begin{cases} \dot{s}(t) = (A + \Delta A)s(t) + (B + \Delta B)f(s(t)) + (C + \Delta C)f(s(t-\rho)) + H, & t \ge t_0, \\ s(\hat{t}) = \varrho(\hat{t}), \quad \hat{t} \in [t_0-\rho, t_0], \end{cases} \tag{2.1}$$

where $s(t)$ is the state vector and right continuous, i.e., $s(t) = s(t^+)$, $s(t) \in \Re^n$; H is an external input; A, B, and $C \in \Re^{n\times n}$ stand for the connection weight matrix and the delay connection weight matrix; ΔA, ΔB, and ΔC are the norm-bounded uncertainty terms, which satisfy $\|\Delta A\| \le d_1$, $\|\Delta B\| \le d_2$, and $\|\Delta C\| \le d_3$, and furthermore d_1, d_2, $d_3 > 0$; $f(s(\cdot))$ denotes the activation function; ρ represents the system delay; and $\varrho \in \mathcal{PC}\left([t_0-\rho, t_0], \Re^n\right)$ indicates the initial state.

Refer to system (2.1) as the drive system. The response system is as follows:

$$\begin{cases} \dot{\psi}(t) = (A + \Delta A)\psi(t) + (B + \Delta B)f(\psi(t)) + (C + \Delta C)f(\psi(t-\rho)) + H, & t \ne t_k, \quad t \ge t_0, \\ \psi(\hat{t}) = \iota(\hat{t}), \quad \hat{t} \in [t_0-\rho, t_0], \end{cases} \tag{2.2}$$

where the impulses are driven by

$$\psi(t) = M_k e(t-\eta(t)) + s(t), \quad t = t_k,$$

where $\eta(t)$ is the impulsive delay, and $\{t_k\}$ is the impulse sequence. $\iota \in \mathcal{PC}([t_0-\rho, t_0], \Re^n)$ denotes the initial state. Let the synchronization error be $e(t) = \psi(t) - s(t)$. Thus, the uncertain time-delay error system is as follows:

$$\begin{cases} \dot{e}(t) = (A+\Delta A)e(t) + (B+\Delta B)g(e(t)) + (C+\Delta C)g(e(t-\rho)), & t \neq t_k, \quad t \geq t_0, \\ e(\hat{t}) = \chi(\hat{t}), \quad \hat{t} \in [t_0-\rho, t_0], \end{cases} \tag{2.3}$$

where the impulses are driven by

$$e(t) = M_k e(t-\eta(t)), \quad t = t_k, \tag{2.4}$$

where $g(e(\cdot)) = f(\psi(\cdot)) - f(s(\cdot)), \chi(\hat{t}) = \iota(\hat{t}) - \varrho(\hat{t})$.

Remark 1. The impulsive system discussed here differs from that described using Schwartz-Sobolev theory [38, 39]. While the impulsive control is the solution of several integral equations, the latter is simplified as a the particular type of nonlinear Volterra integral equation. The paper expresses the uncertain time-delay error system formally as a differential equation, whether system delay of the continuous part or impulsive delay of the discrete part is included. Besides, the delay in the discrete portion is a significant factor in synchronization of the overall uncertain time-delay systems in the following analysis.

In the following, we present some assumptions and definitions.

Assumption 1. Suppose there exists a Lipschitz constant θ_i such that $g_i(\cdot) \in \Re$ satisfies

$$|g_i(\bar{u}) - g_i(\bar{v})| \leq \theta_i |\bar{u} - \bar{v}|, \quad \forall \bar{u}, \bar{v} \in \Re,$$

with $i = 1, 2, \cdots, n$ and $\Theta = diag\{\theta_1, \theta_2, \cdots, \theta_n\}$.

Assumption 2. The impulse sequence $\{t_k, k \in \mathcal{Z}_+\}$ satisfies $t_0 < t_1 < \cdots < t_k$, with $t_k \to \infty$ when $k \to \infty$, and such impulse time sequences are defined as $\wp_0$. $\wp$ denotes the set of entire impulse time sequences in $\wp_0$ that satisfy the inequality $\eta(t_k) < t_k - t_{k-1}$. Moreover, when $\eta_k > 0$, for $k \in \mathcal{Z}_0$ and $\zeta_0 = 0$, $\wp_\eta$ indicates the set of all impulse time sequences in $\wp$ that satisfy the inequality $t_k - \eta(t_k) \leq t_{k-1} + \zeta_k$. Every impulse sequence presented in this paper belongs to $\wp$.

Definition 1. ([40]) The response system (2.2) is globally exponentially synchronized with the drive system (2.1) if there exist scalars $D > 0$ and $\gamma > 0$ satisfying

$$\|e(t)\| \leq D \|\psi - \varrho\|_\rho \, exp(-\gamma(t-t_0)), \quad \forall t \geq t_0,$$

where $\psi, \varrho \in \mathcal{PC}([t_0-\rho, t_0], \Re^n)$.

Remark 2. This paper derives sufficient criteria of uncertain time-delay systems synchronization through impulsive controllers $\{t_k, M_k, \eta(t)\}_{k \in \mathcal{Z}_+}$, making the uncertain time-delay systems (2.1) and (2.2) be GES under the flexible impulsive control (2.4). In comparison with impulsive control in [40–42], the design of impulsive gain M_k and $\eta(t)$ are more flexible in this article. By adjusting the two parameters to satisfy synchronization criteria, this paper builds one flexible delayed impulsive control approach.

Definition 2. ([43]) Suppose that there are scalars $\mathcal{N}_0 > 0$ and $T_* > 0$ satisfying

$$\frac{t_\triangleright - t^\triangleright}{T_*} - \mathcal{N}_0 \leq \mathcal{N}(t^\triangleright, t_\triangleright) \leq \frac{t_\triangleright - t^\triangleright}{T_*} + \mathcal{N}_0,$$

where $\mathcal{N}(t^{\triangleright}, t_{\triangleright})$ represent the number of impulses in the interval $(t^{\triangleright}, t_{\triangleright})$. Then, $\mathcal{N}_0$ denotes the elasticity number and T_* is named AII.

Taking into account the impulsive delay $\eta(t)$, there is a piecewise function of the following form:

$$\sigma(t) = \begin{cases} 0, & t \in [t_0, t_1), \\ \sum\limits_{t_i \in \mathfrak{Q}(t_0,t)} \eta(t_i), & t \in [t_k, t_{k+1}), \end{cases}$$

where $\mathfrak{Q}(t_0, t)$ stands for the impulse times $\{t_k, k \in \mathcal{Z}_+\}$ which occur at (t_0, t).

Consider a new Razumikhin-type inequality under above definitions as follows:

$$D^+V(t) \leq \gamma V(t), \quad if \quad V(t-\rho) \leq \Sigma V(t), \quad t \in [t_{k-1}, t_k), \tag{a}$$

$$V(t_k) \leq exp(-\varpi(t_k))V(t_k - \eta(t_k)), \tag{b}$$

where $k \in \mathcal{Z}_+$, $V \in \mathcal{PC}([t_0 - \rho, +\infty), \Re_+)$, $\Sigma = exp\left\{hT_*(\mathcal{N}_0 + 1) + \varpi_*(\frac{\rho}{T_*} + \mathcal{N}_0) + \hat{\varpi}_0\right\} \geq 1$, and γ and h are positive constants with $\gamma < h$.

Definition 3. ([44]) There exist two positive scalars ϖ_* and $\hat{\varpi}_0$ such that

$$\varpi_* \mathcal{N}(t^{\triangleright}, t_{\triangleright}) - \hat{\varpi}_0 \leq \sum_{j=\mathcal{N}(t_0,t^{\triangleright})+1}^{\mathcal{N}(t_0,t_{\triangleright})} \varpi(t_j) \leq \varpi_* \mathcal{N}(t^{\triangleright}, t_{\triangleright}) + \hat{\varpi}_0, \tag{2.5}$$

In the same way, we present a piecewise function related to $\varpi(t_j)$:

$$\xi(t) = \begin{cases} 0, & t \in [t_0, t_1), \\ \sum\limits_{t_i \in \mathfrak{Q}(t_0,t)} \varpi(t_i), & t \in [t_k, t_{k+1}). \end{cases}$$

Remark 3. In order to better handle the influence of flexible impulsive gain, we develop a novel Razumikhin-type inequality in terms of variable parameter $\varpi(t)$ relevant to impulsive gain M_k, see synchronization condition $M_k^T P M_k \leq exp(-\varpi_k)P$. Motivated by average delay impulsive control in [45, 46], we propose a positive scalar ϖ_* in (2.5). Differing from the Razumikhin-type inequality in the previous article, parameter $\varpi(t)$ in the presented inequality does not always need to be positive. It is worth noting that we obtain the lower conservative upper bound of impulsive gain M_k when the flexibility parameter $\varpi(t) < 0$, which was considered to desynchronize systems in existing work, that is have a negative impact on the systems. When the uncertain time-delay systems are driven by a desynchronizing impulsive gain, the synchronization conditions presented are expected to maintain GES.

Lemma 1. ([47]) Given appropriately dimensional real matrices Z, ΔK and appropriately dimensional real vectors r_1, r_2, $\|\Delta K\| \leq z$, there exists a constant $\varepsilon > 0$ that satisfies

$$\pm 2r_1^T Z(\Delta K) r_2 \leq \varepsilon r_1^T Z^T Z r_1 + \frac{z^2}{\varepsilon} r_2^T r_2.$$

Lemma 2. ([48]) Let Λ_1 and Λ_2 be two real matrices. There exists a positive number U and a matrix $E > 0$ such that

$$\Lambda_1^T \Lambda_2 + \Lambda_2^T \Lambda_1 \leq U \Lambda_1^T E \Lambda_1 + \frac{1}{U} \Lambda_2^T E^{-1} \Lambda_2.$$

Lemma 3. ([48]) (Schur Complement) Given

$$Q = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix},$$

where $Q_{11}^T = Q_{11}$, $Q_{12}^T = Q_{21}$, and $Q_{22}^T = Q_{22}$, then if $Q < 0$, we can convert to one of the following conditions:

(1) $Q_{22} < 0$ and $Q_{11} - Q_{12}Q_{22}^{-1}Q_{12}^T < 0$.
(2) $Q_{11} < 0$ and $Q_{22} - Q_{12}^T Q_{11}^{-1} Q_{12} < 0$.

3. Results

Lemma 4. Assume the function $g(t)$ that satisfies inequalities (a) and (b), if there exists a scalar $w \geq 0$ that satisfies

$$h_0 t - h\sigma(t) - \xi(t) \leq w, \quad \forall t \geq t_0, \tag{3.1}$$

then the solution of inequalities (a) and (b) satisfy

$$g(t) \leq exp(h(t - t_0))\hat{g}(t_0)\Gamma_k, \quad \forall t \in [t_{k-1}, t_k), \quad k \in \mathcal{Z}_+, \tag{3.2}$$

over the class $\wp$, where $h_0 > h > \gamma > 0$, $\hat{g}(t_0) = sup\{g(t), t \in [t_0 - \rho, t_0]\}$ and $\Gamma_k = exp(-\xi(t) - h\sigma(t))$. Furthermore, we take the notation D^+ to describe the upper right-hand Dini derivative.

Proof. Let

$$G(t) = \begin{cases} g(t)exp(-h(t - t_{k-1})), & t \in [t_{k-1}, t_k), \quad k \in \mathcal{Z}_+ \\ g(t), & t \in [t_0 - \rho, t_0). \end{cases} \tag{3.3}$$

Subsequently, we shall show that

$$G(t) \leq \Gamma_k \hat{g}(t_0) exp(h(t_{k-1} - t_0)). \tag{3.4}$$

First, when $k = 1$, we will show that (3.4) is true, namely, $G(t) \leq \hat{g}(t_0)$, $t \in [t_0, t_1)$. Apparently, $G(t_0) = g(t_0) \leq \hat{g}(t_0)$. Provided that (3.4) was false for $\bar{t}_0 \in (t_0, t_1)$, there exists $\bar{t}_0 \in (t_0, t_1)$ to make $G(t) > \hat{g}(t_0)$ hold. Let $\bar{t}_0 = inf\{t \in (t_0, t_1) : G(t) > \hat{g}(t_0)\}$, $G^-(\bar{t}_0)$ will be called the left neighborhood of $\bar{t}_0$, $\bar{t}_0^* \in G^-(\bar{t}_0)$, and $G^-(\bar{t}_0^*) = \hat{g}(t_0)$, then we find that $G(\bar{t}_0) > \hat{g}(t_0)$, $G(t) < G(\bar{t}_0)$, $\forall t \in (t_0 - \rho, \bar{t}_0)$, and $D^+G(t)|_{t=\bar{t}_0} \geq 0$.

Case 1. If $t_0 \leq \bar{t}_0 - \rho \leq \bar{t}_0$, then $G(\bar{t}_0 - \rho) < G(\bar{t}_0)$. It follows from (3.3) that $g(\bar{t}_0 - \rho)exp(-h(\bar{t}_0 - \rho - t_0)) < g(\bar{t}_0)exp(-h(\bar{t}_0 - t_0))$, we can get $g(\bar{t}_0 - \rho) < g(\bar{t}_0)exp(-h\rho) < g(\bar{t}_0)$.

Case 2. If $\bar{t}_0 - \rho < t_0$, then $G(\bar{t}_0 - \rho) = g(\bar{t}_0 - \rho) < G(\bar{t}_0) = g(\bar{t}_0)exp(-h(\bar{t}_0 - t_0)) < g(\bar{t}_0)$. Thus, we obtain $g(\bar{t}_0 - \rho) < g(\bar{t}_0) \leq \Sigma g(\bar{t}_0)$. Considering (a) and $\gamma < h$, one can receive

$$\begin{aligned} D^+G(t)|_{t=\bar{t}_0} &= \left[D^+g(t)|_{t=\bar{t}_0} - hg(\bar{t}_0)\right] exp(-h(\bar{t}_0 - t_0)) \\ &\leq (\gamma - h)g(\bar{t}_0)exp(-h(\bar{t}_0 - t_0)) \\ &< 0, \end{aligned}$$

which is a contradiction. Because $\bar{t}_0$ is not an impulsive instant, it follows from the concept of $\bar{t}_0$ that $D^+G(t)|_{t=\bar{t}_0} < 0$.

Afterwards, we suppose that (3.4) is true for $k \le L$, $L \in \mathcal{Z}_+$, that is, $G(t) \le \hat{g}(t_0)\Gamma_k exp(h(t_{k-1} - t_0))$, $t \in [t_{k-1}, t_k)$. Thus, we need to illustrate that $G(t) \le \hat{g}(t_0)\Gamma_{L+1} exp(h(t_L - t_0))$ holds for $t \in [t_L, t_{L+1})$.

When $t = t_L$, one has

$$\begin{aligned} G(t_L) = g(t_L) &\le exp(-\varpi_L)g(t_L - \eta(t_L)) \\ &= exp(-\varpi_L)G(t_L - \eta(t_L))exp(h(t_L - \eta(t_L) - t_{L-1})) \\ &\le exp(-\varpi_L)\hat{g}(t_0)\Gamma_L exp(h(t_{L-1} - t_0))exp(h(t_L - \eta(t_L) - t_{L-1})) \\ &= \hat{g}(t_0)\Gamma_{L+1} exp(h(t_L - t_0)). \end{aligned}$$

Provided that for $\bar{t}_k \in (t_L, t_{L+1})$, $G(t) \le \hat{g}(t_0)\Gamma_{L+1} exp(h(t_L - t_0))$ is wrong, so that there is a constant $\bar{t}_k \in (t_L, t_{L+1})$ that satisfies $G(t) > \hat{g}(t_0)\Gamma_{L+1} exp(h(t_L - t_0))$. Let $\bar{t}_k = inf\{t \in (t_L, t_{L+1}) : G(t) > \hat{g}(t_0)\Gamma_{L+1} exp(h(t_L - t_0))\}$, and $G^-(\bar{t}_k)$ will be called the left neighborhood of $\bar{t}_k$, $\bar{t}_k^* \in G^-(\bar{t}_k)$ and $G^-(\bar{t}_k) = \hat{g}(t_0)\Gamma_{L+1} exp(h(t_L - t_0))$, then we find that $G(\bar{t}_k) > \hat{g}(t_0)\Gamma_{L+1} exp(h(t_L - t_0))$, $G(t) < G(\bar{t}_k)$, $\forall t \in (t_L, \bar{t}_k)$, and $D^+G(t)|_{t=\bar{t}_k} \ge 0$.

Case 1. If $t_L \le \bar{t}_k - \rho \le \bar{t}_k$, then $G(\bar{t}_k - \rho) < G(\bar{t}_k)$, and due to (3.3) we get $g(\bar{t}_k - \rho) < g(\bar{t}_k)exp(-h\rho)$.

Case 2. If $t_{L-1} \le \bar{t}_k - \rho \le t_L$, then $G(\bar{t}_k - \rho) = g(\bar{t}_k - \rho)exp(-h(\bar{t}_k - \rho - t_{L-1})) \le g(\bar{t}_0)\Gamma_L exp(h(t_{L-1} - t_0))$, it leads to

$$\begin{aligned} G(\bar{t}_k - \rho) &= g(\bar{t}_k - \rho)exp(-h(\bar{t}_k - \rho - t_{L-1})) \\ &\le \hat{g}(t_0)\Gamma_L exp(h(t_{L-1} - t_0)) \\ &= \hat{g}(t_0)\Gamma_{L+1} exp(h(t_{L-1} - t_0))exp(\varpi_L + h\eta(t_L)) \\ &< G(\bar{t}_k)exp(h(t_{L-1} - t_L))exp(\varpi_L + h\eta(t_L)) \\ &= g(\bar{t}_k)exp(-h(\bar{t}_k - t_L))exp(h(t_{L-1} - t_L) + \varpi_L + h\eta(t_L)) \\ &= g(\bar{t}_k)exp(-h(\bar{t}_k - t_{L-1}))exp(\varpi_L + h\eta(t_L)), \end{aligned}$$

and we have $g(\bar{t}_k - \rho) < g(\bar{t}_k)exp(-h\rho + \varpi_L + h\eta(t_L))$, which together with $\eta(t_L) \le t_L - t_{L-1}$, yields that $g(\bar{t}_k - \rho) < g(\bar{t}_k)exp(h(t_L - t_{L-1}) + \varpi_L)$.

Case 3. If $t_0 \le \bar{t}_k - \rho \le t_{L-1}$, suppose that $t_0 \le t_{K-1} \le \bar{t}_k - \rho < t_K < \cdots < t_L < \bar{t}_k$, where $K < L$, $K \in \mathcal{Z}_+$. Therefore, $G(\bar{t}_k - \rho) = g(\bar{t}_k - \rho)exp(-h(\bar{t}_k - \rho - t_{K-1})) \le \hat{g}(t_0)\Gamma_K exp(h(t_{K-1} - t_0))$, which leads to

$$\begin{aligned} G(\bar{t}_k - \rho) &= g(\bar{t}_k - \rho)exp(-h(\bar{t}_k - \rho - t_{K-1})) \\ &\le \hat{g}(t_0)\Gamma_K exp(h(t_{K-1} - t_0)) \\ &= \hat{g}(t_0)\Gamma_{L+1} exp(h(t_L - t_0))exp(h(t_{K-1} - t_L))exp\left(\sum_{j=K}^{L} \varpi_j + h\sum_{k=K}^{L} \eta(t_k)\right) \\ &< G(\bar{t}_k)exp(h(t_{K-1} - t_L))exp\left(\sum_{j=K}^{L} \varpi_j + h\sum_{k=K}^{L} \eta(t_k)\right) \\ &= g(\bar{t}_k)exp(-h(\bar{t}_k - t_L))exp(h(t_{K-1} - t_L))exp\left(\sum_{j=K}^{L} \varpi_j + h\sum_{k=K}^{L} \eta(t_k)\right) \\ &= g(\bar{t}_k)exp(-h(\bar{t}_k - t_{K-1}))exp\left(\sum_{j=K}^{L} \varpi_j + h\sum_{k=K}^{L} \eta(t_k)\right), \end{aligned}$$

and we have $g(\bar{t}_k-\rho) < g(\bar{t}_k)exp(-h\rho)exp\left(\sum_{j=K}^{L}\varpi_j + h\sum_{k=K}^{L}\eta(t_k)\right)$. On account of Assumption 2, one can further obtain that

$$\begin{aligned}-h\rho + h\sum_{k=K}^{L}\eta(t_k) &< -h(t_L-t_K) + h(\eta(t_K)+\eta(t_{K+1})+\cdots+\eta(t_{L-1})+\eta(t_L))\\ &\le -h(t_L-t_K) - h(t_K-t_{K-1}+t_{K+1}-t_K+\cdots+t_{L-1}-t_{L-2}+t_L-t_{L-1})\\ &\le h(t_L-t_{L-1}).\end{aligned}$$

Thus, $g(\bar{t}_k-\rho) < g(\bar{t}_k)exp\left(h(t_L-t_{L-1}) + \sum_{j=K}^{L}\varpi_j\right)$.

Case 4. If $\bar{t}_k-\rho < t_0$, it yields that

$$\begin{aligned}G(\bar{t}_k-\rho) = g(\bar{t}_k-\rho) &\le \hat{g}(t_0)\\ &= \hat{g}(t_0)\Gamma_{L+1}exp(h(t_L-t_0))exp(-h(t_L-t_0))exp\left(\sum_{j=1}^{L}\varpi_j + h\sum_{k=1}^{L}\eta(t_k)\right)\\ &< G(\bar{t}_k)exp(-h(t_L-t_0))exp\left(\sum_{j=1}^{L}\varpi_j + h\sum_{k=1}^{L}\eta(t_k)\right)\\ &< g(\bar{t}_k)exp(-h(\bar{t}_k-t_L))exp(-h(t_L-t_0))exp\left(\sum_{j=1}^{L}\varpi_j + h\sum_{k=1}^{L}\eta(t_k)\right)\\ &= g(\bar{t}_k)exp(-h(\bar{t}_k-t_0))exp\left(\sum_{j=1}^{L}\varpi_j + h\sum_{k=1}^{L}\eta(t_k)\right),\end{aligned}$$

hence, $g(\bar{t}_k-\rho) < g(\bar{t}_k)exp(-h(\bar{t}_k-t_0))exp\left(\sum_{j=1}^{L}\varpi_j + h\sum_{k=1}^{L}\eta(t_k)\right)$. Due to $t_L < \bar{t}_k$ and $\eta(t_L) \le t_L - t_{L-1}$, we can further derive that

$$\begin{aligned}-h(\bar{t}_k-t_0) + h\sum_{k=1}^{L}\eta(t_k) &< -h(t_L-t_0) + h(\eta(t_1)+\eta(t_2)+\cdots+\eta(t_{L-1})+\eta(t_L))\\ &\le -h(t_L-t_0) + h(t_1-t_0+t_2-t_1+\cdots+t_{L-1}-t_{L-2}+t_L-t_{L-1})\\ &\le h(t_L-t_{L-1}).\end{aligned}$$

Then, we introduce that $g(\bar{t}_k-\rho) < g(\bar{t}_k)exp\left(h(t_L-t_{L-1}) + \sum_{j=1}^{L}\varpi_j\right)$. Meanwhile, we have

$$g(\bar{t}_k-\rho) < g(\bar{t}_k)exp\left(h(t_L-t_{L-1}) + \xi(t) - \xi(t-\rho)\right), \quad t \in (t_L, t_{L-1}).$$

In the light of Definition 2 and (2.5), one has $exp(h(t_L-t_{L-1}) + \xi(t) - \xi(t-\rho)) \le exp\left(hT_*(\mathcal{N}_0+1) + \varpi^*(\frac{\rho}{T_*}+\mathcal{N}_0) + \hat{\varpi}_0\right)$.

Consequently, all situations lead to

$$\begin{aligned}D^+G(t)|_{t=\bar{t}_k} &= \left[D^+g(t)|_{t=\bar{t}_k} - hg(\bar{t}_k)\right]exp(-h(\bar{t}_k-t_0))\\ &\le (\gamma-h)g(\bar{t}_k)exp(-h(\bar{t}_k-t_0))\\ &< 0,\end{aligned}$$

which is a contradiction. It implies that, $G(t) \leq \hat{g}(t_0)\Gamma_k exp(h(t_{k-1} - t_0))$, $\forall t \in [t_{k-1}, t_k)$, $k \in \mathcal{Z}_+$. This completes the proof.

Remark 4. Note that the conversion from condition (3.1) to (3.5) is a sufficient criterion for GES of uncertain time-delay systems (2.1) and (2.2). Furthermore, Corollary 2 is introduced to satisfy criterion (3.1) (that is condition (3.5)) in practical implementations, which will be discussed later. Subsequently, we derive Theorem 1 for the GES between uncertain time-delay systems (2.1) and (2.2) as follows.

Theorem 1. Under Assumptions 1 and 2, if there are scalars $w \geqslant 0$ and $h_0 > h > 0 > \gamma$, matrix $P > 0$, diagonal matrices $E_1 > 0$ and $E_2 > 0$, and $Q > 0$ satisfies $LQL \leq P$ for every $k \in \mathcal{Z}_+$ with $M_k^T P M_k \leq exp(-\varpi_k)P$ such that

$$h\sigma(t) + \xi(t) \geq h_0 t - w, \tag{3.5}$$

$$\begin{pmatrix} \Pi & PB & PC \\ \bullet & -E_1 & 0 \\ \bullet & \bullet & -E_2 \end{pmatrix} \leq 0, \tag{3.6}$$

where $\Pi = A^T P + PA + (\varepsilon_1 + \varepsilon_2 + \varepsilon_3)P^2 + \frac{d_1^2}{\varepsilon_1} I_n + \frac{d_2^2}{\varepsilon_2} I_n + \Theta E_1 \Theta + (\lambda_1 + \lambda_2)\Sigma P - \gamma P$ with $\Theta = \{\theta_1, \theta_2, \cdots, \theta_n\}$, $\Sigma = exp\left\{hT_*(\mathcal{N}_0 + 1) + \varpi_*(\frac{\rho}{T_*} + \mathcal{N}_0) + \hat{\varpi}_0\right\}$, $\lambda_1 = \lambda_{max}\left(\frac{\Theta E_2 \Theta}{P}\right)$, and $\lambda_2 = \lambda_{max}\left(\frac{d_3 \Theta^2}{\varepsilon_3}\right)$. Then uncertain time-delay systems (2.1) and (2.2) can realize GES over the class $\wp$.

Proof. Let the Lyapunov function $V(t) \triangleq V(e(t)) = e^T(t)Pe(t)$, taking the derivative along the trajectory of error system (2.3), and we have

$$\begin{aligned} D^+V(t) =& 2e^T(t)Pe(t) \\ =& \left[(A + \Delta A)e(t) + (B + \Delta B)g(e(t)) + (C + \Delta C)g(e(t-\rho))\right]^T Pe(t) \\ &+ e^T(t)P\left[(A + \Delta A)e(t) + (B + \Delta B)g(e(t)) + (C + \Delta C)g(e(t-\rho))\right] \\ =& e(t)^T \left[A^T P + PA\right] e(t) + e(t)^T \left[(\Delta A^T)P + P(\Delta A)\right] e(t) + g^T(e(t))B^T Pe(t) + e(t)^T PBg(e(t)) \\ &+ g^T(e(t))(\Delta B)^T Pe(t) + e(t)^T P(\Delta B)g(e(t)) \\ &+ g^T(e(t-\rho))C^T Pe(t) + e(t)^T PCg(e(t-\rho)) \\ &+ g^T(e(t-\rho))(\Delta C)^T Pe(t) + e(t)^T P(\Delta C)g(e(t-\rho)). \end{aligned} \tag{3.7}$$

If $V(t-\rho) \leq \Sigma V(t)$, namely, $e^T(t-\rho)Pe(t-\rho) \leq \Sigma e^T(t)Pe(t)$, then by utilizing Assumption 1 and Lemmas 1 and 2, we have

$$e(t)^T \left[(\Delta A^T)P + P(\Delta A)\right] e(t) \leq \varepsilon_1 e(t)^T P^2 e(t) + \frac{d_1^2}{\varepsilon_1} e(t)^T e(t), \tag{3.8}$$

$$\begin{aligned} g^T(e(t))B^T Pe(t) + e(t)^T PBg(e(t)) &\leq g^T(e(t))E_1 g(e(t)) + e(t)^T PBE_1^{-1}B^T Pe(t) \\ &\leq e(t)^T \left[\Theta E_1 \Theta + PBE_1^{-1}B^T P\right] e(t), \end{aligned} \tag{3.9}$$

$$\begin{aligned} g^T(e(t))(\Delta B)^T Pe(t) + e(t)^T P(\Delta B)g(e(t)) &\leq \varepsilon_2 e(t)^T P^2 e(t) + \frac{1}{\varepsilon_1} g^T(e(t))(\Delta B)^T(\Delta B)g(e(t)) \\ &\leq e(t)^T \left[\varepsilon_2 P^2 + \frac{d_2^2}{\varepsilon_2}\Theta^2\right] e(t), \end{aligned} \tag{3.10}$$

$$
\begin{aligned}
g^T(e(t-\rho))C^TPe(t)+e(t)^TPCg(e(t-\rho)) &\le g^T(e(t-\rho))E_2g(e(t-\rho))+e(t)^TPCE_2^{-1}C^TPe(t)\\
&\le e^T(t-\rho)\Theta E_2\Theta e(t-\rho)+e(t)^TPCE_2^{-1}C^TPe(t)\\
&\le \lambda_{max}\left(\frac{\Theta E_2\Theta}{P}\right)e^T(t-\rho)Pe(t-\rho)+e(t)^TPCE_2^{-1}C^TPe(t)\\
&\le e(t)^T\left[\lambda_1 P\Sigma+PCE_2^{-1}C^TP\right]e(t),
\end{aligned}
\tag{3.11}
$$

$$
\begin{aligned}
g^T(e(t-\rho))(\Delta C)^TPe(t)+e(t)^TP(\Delta C)g(e(t-\rho)) &\le \varepsilon_3 e(t)^TP^2e(t)+\frac{d_3^2}{\varepsilon_1}e^T(t-\rho)\Theta^2e(t-\rho)\\
&\le \varepsilon_3 e(t)^TP^2e(t)+\lambda_2 e^T(t-\rho)Pe(t-\rho)\\
&\le e(t)^T\left[\varepsilon_3P^2+\lambda_2\Sigma P\right]e(t).
\end{aligned}
\tag{3.12}
$$

It follows from Lemma 3, condition (3.6), and inequalities (3.7)–(3.12) that

$$
\begin{aligned}
D^+V(t) \le & e(t)^T\left[A^TP+PA+(\varepsilon_1+\varepsilon_2+\varepsilon_3)P^2+\frac{d_1^2}{\varepsilon_1}I_n+\frac{d_2^2}{\varepsilon_2}I_n+\Theta E_1\Theta+(\lambda_1+\lambda_2)\Sigma P\right]e(t)\\
&+e(t)^T\left[PBE_1^{-1}B^TP+PCE_2^{-1}C^TP\right]e(t)\\
\le & he(t)^TPe(t).
\end{aligned}
$$

For the uncertain time-delay error system (2.3), when $t=t_k$, $k\in\mathcal{Z}_+$, one can get

$$
\begin{aligned}
V(t_k) &\le e^T(t_k-\eta(t_k))M_k^TPM_ke(t_k-\eta(t_k))\\
&\le exp(-\varpi_k)V(t_k-\eta(t_k)).
\end{aligned}
$$

Utilizing Lemma 4 and condition (3.5) leads to

$$
V(t)\le exp(w-h_0t_0)\sup_{s\in[-\rho,0]}V(\chi(s))exp(h-h_0)(t-t_0).
$$

Furthermore, we have

$$
\|e(t)\|\le D\,\|X\|_\rho\, exp(-\gamma(t-t_0)),\quad \forall t\ge 0,
$$

where $D=\sqrt{exp(w-h_0t_0)\lambda_{max}(P)/\lambda_{min}(P)}$, $\gamma=\frac{1}{2}(h_0-h)>0$. Hence, the uncertain time-delay systems (2.1) and (2.2) can reach the GES over the class $\wp$. The proof is completed.

Corollary 1. If there are numbers $h_0>h>0$, matrix $P>0$, diagonal matrices $E_1>0$ and $E_2>0$, and $Q>0$ satisfies $LQL\le P$ for every $k\in\mathcal{Z}_+$ with $M_k^TPM_k\le exp(-(\varpi_*+2\hat{\varpi}_0))$, conditions (3.5) and (3.6) hold. Then, uncertain time-delay systems (2.1) and (2.2) can achieve GES over the class $\wp$.

Proof. According to (2.5), when $t_k\in\wp$, it follows that

$$
(k-1)\varpi_*-\hat{\varpi}_0\le\sum_{j=1}^{k-1}\varpi_j\le(k-1)\varpi_*+\hat{\varpi}_0,
$$

and

$$
k\varpi_*-\hat{\varpi}_0\le\sum_{j=1}^{k}\varpi_j\le k\varpi_*+\hat{\varpi}_0.
$$

Further, we have

$$\begin{aligned}\varpi_k &= \sum_{j=1}^{k} \varpi_j - \sum_{j=1}^{k-1} \varpi_j \\ &\leq (k\varpi_* + \hat{\varpi}_0) - ((k-1)\varpi_* - \hat{\varpi}_0) \\ &\leq \varpi_* + 2\hat{\varpi}_0.\end{aligned}$$

When $t = t_k$, it yields that

$$\begin{aligned}V(t_k) &\leq e^T(t_k - \eta(t_k))M_k^T P M_k e(t_k - \eta(t_k)) \\ &\leq exp(-\varpi_* - 2\hat{\varpi}_0)V(t_k - \eta(t_k)) \\ &\leq exp(-\varpi_k)V(t_k - \eta(t_k)).\end{aligned}$$

Employing Theorem 1, we prove the statement.

Corollary 2. Over the class $\wp_\eta$, the uncertain time-delay systems (2.1) and (2.2) can reach GES, if there exist $t_0 = 0$, $h_0 > h > 0$, $0 < \eta(t_k) \leq \bar{\eta}$, $\zeta_k > 0$, $0 < \mu < 1$,

$$0 \leq \frac{\zeta_k}{t_k - t_{k-1}} \leq \mu, \quad k \in \mathcal{Z}_+, \tag{3.13}$$

and ϖ_* satisfies

$$\varpi_* \geq h_0 T_* - (1-\mu)hT_*. \tag{3.14}$$

Proof. Since $t_k - \eta(t_k) \leq t_{k-1} + \zeta_k$, $k \in \mathcal{Z}_+$, it yields that

$$\begin{aligned}\sigma_k &= \sum_{k=1}^{\mathcal{N}(0,t)} \eta(t_k) \\ &\geq t_k - \sum_{k=1}^{\mathcal{N}(0,t)} \zeta_k \\ &\geq (1-\mu)t_k \\ &\geq (1-\mu)t - \bar{\eta}.\end{aligned} \tag{3.15}$$

It follows from (2.4) and (2.5) that

$$\begin{aligned}\xi_k &= \sum_{k=1}^{\mathcal{N}(0,t)} \varpi(t_k) \\ &\geq \varpi_* \mathcal{N} - \hat{\varpi}_0 \\ &\geq h_0 \mathcal{N} T_* - h(1-\mu)\mathcal{N} T_* - \hat{\varpi}_0.\end{aligned} \tag{3.16}$$

Thereby, we get

$$h\sigma(t) + \xi(t) \geq h(1-\mu)t - h\bar{\eta} + h_0\mathcal{N}T_* - h(1-\mu)\mathcal{N}T_* - \hat{\varpi}_0, \tag{3.17}$$

which, combined with Definition 2 and $(\mathcal{N} - \mathcal{N}_0)T_* \leq t \leq (\mathcal{N} + \mathcal{N}_0)T_*$, can yield

$$\begin{aligned}h\sigma(t) + \xi(t) &\geq h(1-\mu)(\mathcal{N} - \mathcal{N}_0)T_* - h\bar{\eta} + h_0\mathcal{N}T_* - h(1-\mu)\mathcal{N}T_* - \hat{\varpi}_0 \\ &\geq h_0\mathcal{N}T_* - h(1-\mu)\mathcal{N}_0T_* - h\bar{\eta} - \hat{\varpi}_0 \\ &\geq h_0(t - \mathcal{N}_0T_*) - h(1-\mu)\mathcal{N}_0T_* - h\bar{\eta} - \hat{\varpi}_0 \\ &\geq h_0 t - w,\end{aligned} \tag{3.18}$$

where $w = (h_0 + h - \mu h)\mathcal{N}_0 T_* + h\bar{\eta} + \varpi_0$, $\forall \mathcal{N}_0 > 0$. Therefore, $h_0 t - h\sigma(t) - \xi(t) \leq w$, $\forall t \geq 0$, which proves the statement.

Remark 5. It can be found that Theorem 1, criteria (3.6), and $M_k^T P M_k \leq exp(-\varpi_k)P$ are too complex to be tested in practical applications. Hence, we propose Corollaries 1 and 2. We can find that the Constraints of $M_k^T P M_k \leq exp(-\varpi_k)P$ will keep changing as $\varpi(t_k)$ is updated. In order to tackle this issue, a fixed upper bound is proposed to make all variable impulsive gains meet $M_k^T P M_k \leq exp(-\varpi_k)P$ in Corollary 1. Corollary 2 gives an expressive relation between ϖ_* and T_*, which guarantees condition (3.6) completely. We can have a reasonable estimate of ϖ_* and $\varpi(t)$ once the impulsive interval has been identified. According to $M_k^T P M_k \leq exp(-\varpi_k)P$, the uncertain time-delay systems (2.1) and (2.2) can realize GES under the suitable impulsive gain M_k. Meanwhile, there are no restrictions for $\varpi(t)$. If $\varpi(t) < 0$, the discrete or continuous part is not synchronized. Nevertheless, in order to fulfill condition (3.4) for ϖ_*, we only allow limited desynchronizing jumps in impulsive sequences.

Remark 6. Compared with (see [44]), the uncertain time-delay systems we are discussing not only have uncertainties, but also includes both delayed and non-delayed terms at the same time. Therefore, the situation studied in this paper covers the situation of (see [44]), and the results obtained are more comprehensive. Compared with (see [30, 31]), the impulsive gain considered in this paper is more flexible. Even if the uncertain time-delay systems suffers from unstable impulses, the synchronization can be guaranteed by adjusting the impulsive gain, which has not been well reflected in the previous results.

4. Illustrative examples

In this section, two examples are provided to confirm the validity of the theoretical results.

Example 1. Consider the uncertain time-delay error system as follows:

$$\begin{cases} \dot{e}(t) = (A + \Delta A)e(t) + (B + \Delta B)g(e(t)) + (C + \Delta C)g(e(t - \rho)), & t \neq t_k, \quad t \geq t_0, \\ e(t) = M_k e(t - \eta(t)), & t = t_k, \end{cases} \tag{4.1}$$

where $\rho = 0.2$, $f(e(t)) = tanh(e(t))$, the initial value $e(t) = 3$ and $A = 0.4$, $B = 0.2$, $C = 0.2$, $\Delta A = 0.01cos(t)$, $\Delta B = 0.01cos(t)$, and $\Delta C = 0.01cos(t)$, and $\{t_k\} \in \wp$. Suppose $t_k = 0.5k$ and the impulsive delay $\eta(t_k) < 0.5$. We then have following situations: Situation 1: $\varpi(t) = 0.3$, $\eta(t_k) = 0$; Situation 2: $\varpi(t) = 0.3$, $\eta(t_k) = 0.48$; Situation 3: $\varpi(t) = -0.1$, $\eta(t_k) = 0.48$. Then, Figure 1 shows simulation results.

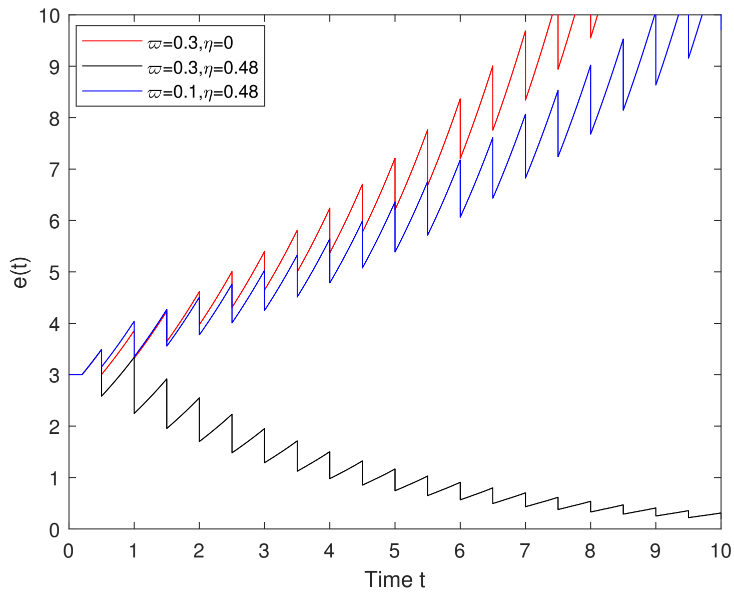

Figure 1. State trajectories of system (4.1) with initial value $e(t)$=3.

Remark 7. It is shown that impulsive delay has a positive effect on synchronization of the uncertain time-delay systems (2.1) and (2.2) from comparison of Situation 1 and Situation 2. At the same time, we can also find that $\varpi(t) = -0.1$ can produce desynchronizing gains when comparing Situation 2 with Situation 3.

Furthermore, assume system (4.1) is regularly disturbed by desynchronizing gain every 0.25s. Thus, when $t_k = 0.25k$, we select

$$\varpi(t) = \begin{cases} -0.1, & t = t_{2k-1}, \\ 0.3, & t = t_{2k}. \end{cases}$$

Recalling the sufficient criteria of Theorem 1, one has $M_k = \sqrt{exp(-\varpi(t))}$ and $\eta(t_k) = 0.21$, and simulation results can be found in Figure 2.

Remark 8. We choose a given value of impulsive interval at the same intervals to more accurately describe the relation between impulsive gain and impulsive delay. The state trajectory of system (4.1) is shown by the blue curve in Figure 2. It is clear that the synchronous result becomes out of synchronization under desynchronizing impulsive gain (yellow curve). However, the uncertain time-delay systems (2.1) and (2.2) return to GES by changing the flexible parameter of impulsive gain and adjusting delay, see Figure 2 (red curve). From Figure 3, it follows that impulsive gain adjustment (blue curve) for synchronization is superior to time delay adjustment (red curve). This means the adjustment of impulsive gain plays an important role in synchronization. Higher robustness of systems synchronization can be achieved by varying impulsive gain in the variable impulsive controller.

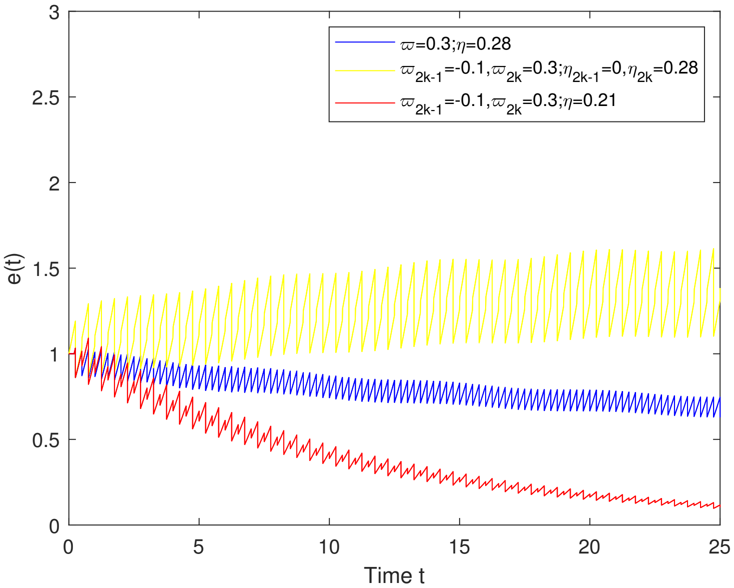

Figure 2. State trajectories under impulsive control with impulsive interference.

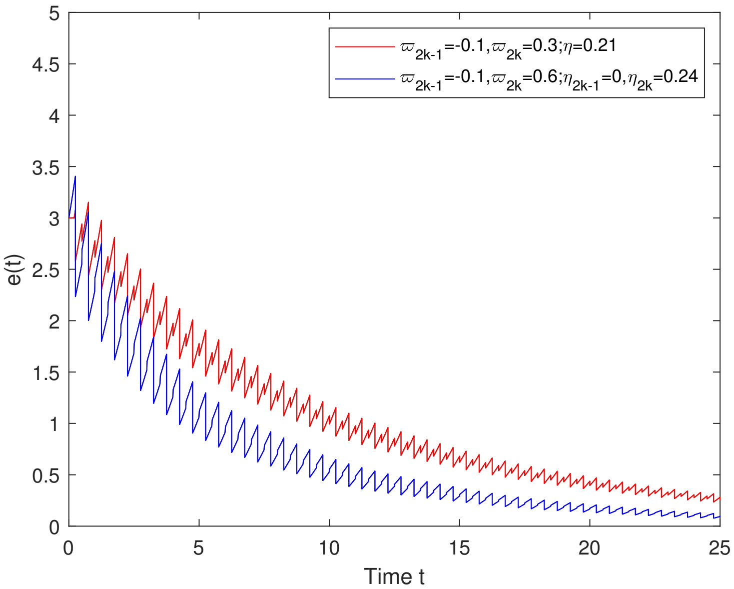

Figure 3. State trajectories under flexible impulsive gain and flexible time-delay.

Example 2. Consider a special case of the same chaotic systems. When transmission delay $\rho = 0$, the value of ΔA, ΔB, and ΔC are 0, respectively, the drive system is as follows:

$$s(t) = As(t) + U(s(t)), \tag{4.2}$$

where $s = (s_1, s_2, s_3)^T \in \Re^3$ and

$$\begin{pmatrix} -\alpha - \alpha m_1 & -\alpha & 0 \\ 1 & -1 & 1 \\ 0 & -\beta & 0 \end{pmatrix}, \quad U(s(t)) = \begin{pmatrix} u_1(s_1(t)) \\ 0 \\ 0 \end{pmatrix},$$

with $U_1(s_1(t)) = 0.5\alpha(m_0 - m_1)(|s_1(t) + 1| - |s_1(t) - 1|)$. The control input of corresponding response system can be described as $K(t) = M_k e(t - \eta(t)) - e(t)$. Hence, the response system model is

$$\begin{cases} \dot{\psi}(t) = A\psi(t), & t \neq t_k, \\ \Delta\psi(\bar{t}) = K(t), & t = t_k. \end{cases} \tag{4.3}$$

The state of synchronization is given by $e(t) = \psi(t) - s(t)$. Then, we give the error system as:

$$\begin{cases} \dot{e}(t) = A\psi(t) + \bar{U}(e(t)), & t \neq t_k, \\ \Delta e(t) = M_k e(t - \eta(t)), & t = t_k, \end{cases} \tag{4.4}$$

where $\bar{U}(e(t)) = U(\psi(t)) - U(s(t)) = (u_1(\psi_1(t)) - u_1(s_1(t)), 0, 0)^T$ and

$$\begin{aligned} |u_1(\psi_1(t)) - u_1(s_1(t))| &= |0.5\alpha(m_0 - m_1)| \cdot |(|\psi_1(t) + 1| - |\psi_1(t) - 1|) - (|s_1(t) + 1| - |s_1(t) - 1|)| \\ &\leq \alpha\, |(m_0 - m_1)| \cdot |\psi_1(t) - s_1(t)|\,, \end{aligned} \tag{4.5}$$

when parameters are setted as $\alpha = 9.2156$, $\beta = 15.9946$, $m_0 = -1.24905$, $m_1 = -0.75735$, $s = (1.2, -0.8, -2.2)^T$, and $\psi = (0.2, 0.2, 0.1)^T$, the error system is illustrated in Figure 4.

Under the situation, impulse sequences satisfy $t_k = 2k$. Let us consider sampling delay as $\eta(t_k) = 1.98$. Based on Corollary 2, $\varpi_* \geq h_0 T_* - (1-\mu)hT_*$. Assume the system experiences a desynchronizing impulse $D_1 = -0.3I$ at time t_1. Recalling Corollary 1 and (3.6), choose $D_k = 0.4I, k \neq 1$.

Figure 4 shows the error variable $|e(t)| = |e_1(t)| + |e_2(t)| + |e_3(t)|$. It is evident that the adjustment of impulsive gain is an effective way for ensuring synchronization.

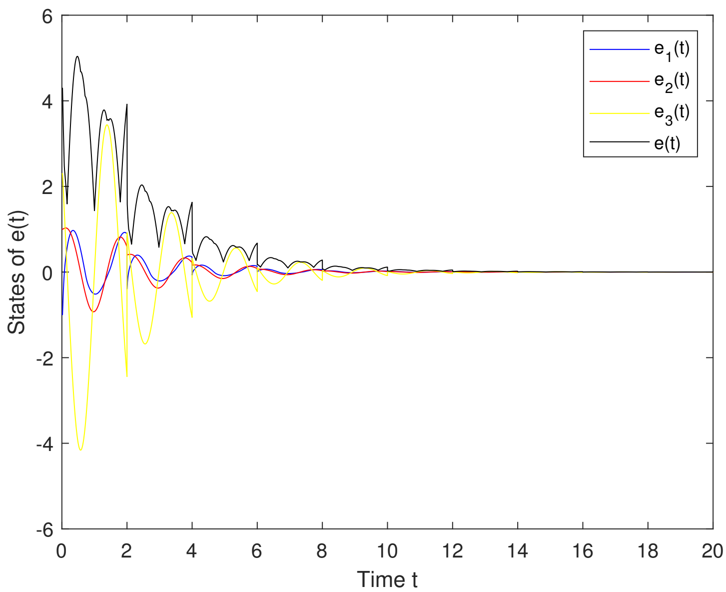

Figure 4. Synchronization error trajectories of Example 2.

5. Conclusions

In this paper, the synchronization problem of uncertain time-delay systems is investigated by delayed impulsive control. Especially, a novel Razumikhin-type inequality was developed. In combination with this inequality, we derive some sufficient conditions for GES. This paper shows that delays in impulsive control are helpful for synchronization. Then, for a desynchronizing impulsive gain, we can also find if the size of the delay in the impulsive interval is adequately large, then the desynchronizing impulsive gain does not break synchronization under the conditions we present. Moreover, uncertain time-delay systems can achieve synchronization through combining impulsive delay and impulsive gain. Especially, there has been a relaxation of $\varpi(t)$. Note that flexible impulsive delays have the upper bound, that is, $\eta(t_k) \leq t_k - t_{k-1}$. Future work will aim at extending the presented results to the impulsive delays over impulsive intervals.

Author contributions

Biwen Li: Software, Validation, Supervision; Qiaoping Huang: Conceptualization, Formal analysis, Methodology, Writing-original draft, Writing-review and editing. All authors have read and approved the final version of the manuscript for publication.

Conflict of interest

The authors declare that there are no conflicts of interest.

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