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Research article

Linear programming-based stochastic stabilization of hidden semi-Markov jump positive systems

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Abstract: This paper focuses on the stochastic stabilization of hidden semi-Markov jump nonlinear positive systems. First, a notion of stochastic stability is introduced for this class of systems. A criterion is addressed to ensure the stochastic stability using a stochastic copositive Lyapunov function. Then, an observed mode is proposed to estimate the emitted value of the hidden semi-Markov process and the mode-dependent controller is designed using an improved matrix decomposition approach. Some auxiliary variables are added to decouple the coupling terms in hidden semi-Markov jump nonlinear positive systems into a tractable condition. All conditions are described in terms of linear programming. Moreover, the proposed design is developed for systems with partially known emission probabilities. Two examples are provided to show the validity of the obtained results.

Keywords: hidden semi-Markov jump positive systems; stochastic stabilization; linear programming
Mathematics Subject Classification: 60K15, 93C10, 93C28, 93E15

1. Introduction

Positive systems have been one of the most active research issues over the last few decades and have extensive applications in the real world [1–3]. As an important kind of positive systems, positive stochastic systems consist of non-negative states, non-negative outputs, and stochastic switching rule [4–6]. Evidence suggests that typical positive stochastic jump systems include Markov jump positive systems (MJ-PSs) and semi-Markov jump positive systems (SMJ-PSs) [7–9]. Compared with MJ-PSs, SMJ-PSs remove the restriction that sojourn time follows the particular distribution in some specific scenarios, i.e., the sojourn time of SMJ-PSs does not follow an exponential distribution [10]. It is worth pointing out that system modes are usually assumed to be exactly known in SMJ-PSs [11]. However, in practice, this idealized assumption is hard to achieve. Moreover, it has been observed that copositive Lyapunov function (CLF) and linear programming (LP) are effective approaches to study positive systems [7, 9, 12, 13]. These novel approaches also bring challenges for research of positive stochastic systems.

As discussed in [11], system modes of SMJ-PSs may be hidden and are hard to be exactly acquired. To overcome this challenge, hidden Markov jump systems were introduced [14–17]. The literature [18] proposed a mixed H_2/H_∞ control framework for hidden Markov jump systems. The work in [19] designed an H_∞ controller to defend multichannel random attacks of nonlinear hidden Markov jump systems. An extended dissipativity-based controller was designed in [20] for singularly perturbed systems with hidden Markov processes. It naturally raises a question whether the sojourn time of the hidden Markov process still needs to follow an exponential distribution or geometric distribution. It is important to note that hidden semi-Markov processes with the two-layer process have been used to address the above issue, i.e., hidden and observed layers [21]. The literature [22] constructed a sojourn time probability density function for non-homogeneous hidden semi-Markov jump systems with limited information. A novel definition of hidden semi-Markov processes was addressed in [23] based on the notion of emission probability. For more details, the reader can refer to [24–26].

Most existing results mainly concentrate on hidden semi-Markov jump systems with a completely known emission probability. However, the emission probability of hidden semi-Markov jump systems is difficult to completely acquire. The work in [27] proposed a hidden semi-Markov process with a partially known emission probability. The work in [28] studied an observed mode-dependent control problem of hidden semi-Markov jump systems with a partially known emission probability. However, the above results are concerned with linear systems. Up to now, to the best of our knowledge, few results are reported for hidden semi-Markov jump nonlinear positive systems (HSMJ-NPSs) with partially known emission probability. This leaves much room to explore the stabilization of HSMJ-PSs with partially known emission probabilities. For this topic, there are several research obstacles. First, how to design the control strategy of HSMJ-NPSs under the hidden Markov process owing to the uncertainties of the process? Unlike the Markov process, it can be directly obtained. It is hard to catch the information of a hidden Markov process, and some suitable estimation approaches need to be presented. This increases the complexity of the control design. Second, how to design the control strategy under the partially known emission probability? It is more difficult to estimate the Markov process when the emission probability is partially known. Thus, the corresponding control strategy is full of challenges. Third, how to establish a unified approach to describe and compute the presented conditions. This paper is to employ CLF and LP to solve the considered design. It is a new topic in the field of positive systems. Therefore, how to design the corresponding gain matrices, how to construct a suitable CLF, and how to address LP-based conditions are key to the design. These inspire the work.

This paper investigates the stabilization of HSMJ-NPSs with partially known emission probabilities. The aim is to understand how these unknown emission probabilities affect HSMJ-NPSs and under which conditions the stabilization can be achieved. By introducing a matrix decomposition technique and stochastic CLF approach, an observed mode-dependent control approach is derived for HSMJ-NPSs. The main contributions are summarized as: (i) Hidden semi-Markov jump systems are developed for positive systems and the corresponding stability analysis is addressed; (ii) A novel matrix decomposition-based control gain design is proposed; and (iii) A unified framework on the stabilization of HSMJ-NPSs is constructed by employing the analysis approach of CLF and the design approach of LP. The rest of this paper is organized as follows: Section II gives the preliminaries. Section III presents the main results. An example is provided in Section IV. Finally, Section V concludes the paper.

Notations: $\mathbb{N}$ and $\mathbb{N}_+$ denote the sets of non-negative and positive integers, respectively. $\mathcal{R}^n$ and $\mathcal{R}^{n\times n}$ are n-dimensional vectors and $n \times n$ matrices. For the probability space $(\Phi, \mathbb{F}, \mathbb{P})$, Φ denotes the sample space, $\mathbb{F}$ denotes the Borel σ-algebra, and $\mathbb{P}$ is the corresponding probability measure on $\mathbb{F}$. $R^\top$ is the transpose of a matrix R. For a matrix $W \in \mathcal{R}^{n\times n}$, w_{ij} is its ith row and jth column element, and $W \succ 0$ $(\succeq 0)$ implies that $w_{ij} > 0$ (≥ 0) for $i, j = 1, 2, \cdots, n$. $\|\cdot\|_1$ is the 1-norm. $\|\cdot\|_2$ is the 2-norm. $\mathbb{E}\{\cdot\}$ is the mathematical expectation. Finally, denote $\mathbf{1}_n^{(i)} = (\underbrace{0, 0, \ldots, 0}_{i-1}, 1, \underbrace{0, \ldots, 0}_{n-i})^\top$.

2. Preliminaries

Consider a semi-Markov jump nonlinear system defined on the probability space $(\Phi, \mathbb{F}, \mathbb{P})$:

$$x(k+1) = A_{O(k)} f(x(k)) + B_{O(k)} u(k), \tag{2.1}$$

where $x(k) \in \mathcal{R}^n$ and $u(k) \in \mathcal{R}^m$ are the state and input, respectively. $f(x(k))$ is the nonlinearities of the system satisfying sector conditions $\underline{\pi} x_i^2(k) \leq f(x_i(k))x_i(k) \leq \overline{\pi} x_i^2(k)$, where $1 \leq i \leq n, 0 < \underline{\pi} \leq \overline{\pi}$ since $f_i(0) = 0$. $O(k), k \in \mathbb{N}$, is a semi-Markov process taking values in the set $\mathcal{P} = \{1, 2, \ldots, P\}, P \in \mathbb{N}_+$, which governs the jumps among system modes. The system matrices of the oth mode are denoted by (A_o, B_o), where $A_o \succeq 0$ and $B_o \succeq 0$ for $O(k) = o \in \mathcal{P}$.

Definition 1. *([1]) A system is called positive when its states are non-negative for any non-negative initial conditions and inputs.*

Lemma 1. *([10]) For a matrix $A \succeq 0$, the following statements are equivalent:*
(i) $A \in \mathcal{R}^{n\times n}$ is a Schur matrix;
(ii) There exists a vector $v \succ 0$ such that $(A^\top - I)v \prec 0$.

Definition 2. *([24]) Suppose that O_p is the index of the system mode at the pth jump, t_p is the time instant at the pth jump with $t_0 = 0$, and κ is the sojourn time. Then the semi-Markov kernel can be represented as:*

$$\varsigma_{oh}(\kappa) = \mathbb{P}(O_{p+1} = h, t_{p+1} - t_p = \kappa | O_p = o) = \eta_{oh}(\kappa)\delta_{oh}, \tag{2.2}$$

where $o, h \in \mathcal{P}, \kappa \in \mathbb{N}$, $\delta_{oh} = \mathbb{P}(O_{p+1} = h | O_p = o)$ is the transition probability with $\delta_{oo} = 0$, and $\eta_{oh}(\kappa) = \mathbb{P}(t_{p+1} - t_p = \kappa | O_p = o, O_{p+1} = h)$ is a probability density function.

Throughout this paper, it is assumed that the semi-Markov processes $O(k), k \in \mathbb{N}$ is hidden, whose values can be estimated by an observed process $O^\star(k)$ with state space $\mathcal{Q} = \{1, 2, \ldots, Q\}, Q \in \mathbb{N}_+$ and the emission probability:

$$\psi_{os} = \mathbb{P}(O^\star(k) = s | O(k) = o), \tag{2.3}$$

where $o \in \mathcal{P}$, $s \in \mathcal{Q}$, $\sum_{s\in\mathcal{Q}} \psi_{os} = 1$, and $\Gamma = [\psi_{os}]_{\mathcal{P}\times\mathcal{Q}}$ is the emission probability matrix.

Definition 3. *System (2.1) is said to be stochastically stable if for any initial condition $x(k_0) = x_0$ and $O(k_0) = O_0$, the following holds:*

$$\lim_{k\to\infty} \mathbb{E}\{\|x(k)\|_1 |_{x_0, O_0}\} = 0. \tag{2.4}$$

Definition 4. *System (2.1) is said to be σ-stochastically stable if for any initial condition $x(k_0) = x_0$ and $\mathcal{O}(k_0) = \mathcal{O}_0$, the following holds:*

$$\lim_{k\to\infty} \mathbb{E}\{\|x(k)\|_1|_{x_0,\mathcal{O}_0,\kappa^o\le\kappa^o_{\max}}\} = 0, \tag{2.5}$$

where $\kappa^o_{\max}$ is the upper bound of sojourn time for the pth subsystem mode, $\sigma_{\text{error}} := |\ln F_o(\kappa^o_{\max})|$, and $F_o(\kappa^o_{\max}) = \mathbb{P}(\kappa_o \le \kappa^o_{\max}) = \sum_{\tau=1}^{\kappa^o_{\max}} \sum_{h\in\mathcal{P}} \varsigma_{oh}(\tau)$.

Remark 1. *Definitions 3 and 4 follow the definitions of non-positive hidden semi-Markov jump systems [24, 26, 28]. For non-positive systems, $\lim_{k\to\infty} \mathbb{E}\{\|x(k)\|_2^2|_{x_0,\mathcal{O}_0}\} = 0$ and $\lim_{k\to\infty} \mathbb{E}\{\|x(k)\|_2^2|_{x_0,\mathcal{O}_0,\kappa^o\le\kappa^o_{\max}}\} = 0$ are always used to describe the stochastic stability conditions of systems. In Definitions 3 and 4, the 2-norm $\|x(k)\|_2$ is replaced by the 1-norm $\|x(k)\|_1$. The main reasons are twofold. First, the 1-norm is more suitable for positive systems. Generally, the 2-norm represents the energy, and thus a quadratic Lyapunov function can be constructed. Positive systems are usually used to model populations, the amount of material, etc. The 1-norm $\|x(k)\|_1$ represents the sum of all components of the state and is suitable to be introduced for positive systems. Under the 1-norm framework, a copositive Lyapunov function can be chosen for a such class of systems. Second, the 1-norm is equivalent to the 2-norm. By the norm equivalence principle, the mentioned two norms are equivalent in a finite space. Thus, it is reasonable to introduce Definitions 3 and 4 for HSMJ-NPSs.*

3. Main results

In this section, the stabilization of HSMJ-NPSs with completely known and partially known emission probabilities are presented. First, an observed mode-dependent controller is designed:

$$u(k) = K_{\mathcal{O}^\star(k)} x(k), \tag{3.1}$$

where $K_{\mathcal{O}^\star(k)}$ is a control gain matrix to be determined. Then,

$$x(k+1) = A_{\mathcal{O}(k)} f(x(k)) + B_{\mathcal{O}(k)} K_{\mathcal{O}^\star(k)} x(k). \tag{3.2}$$

Lemma 2. *Assume that system (3.2) is positive. If there exist a stochastic CLF $V(x(k), \mathcal{O}(k)) = x^\top(k)\lambda_{\mathcal{O}(k)}$ with $\lambda_{\mathcal{O}(k)} \succ 0$ and constants $\rho_{\mathcal{O}(t_p)}, \lambda_1 > 0, \lambda_2 > 0, \lambda_3 > 0$ with $\lambda_2 \ge \lambda_1$ such that*

$$\lambda_1\|x(k)\|_1 \le V(x(k), \mathcal{O}(k)) \le \lambda_2\|x(k)\|_1, \tag{3.3}$$

$$\mathbb{E}\{V(x(t_p + k), \mathcal{O}(t_p + k) = h|x(t_p), \mathcal{O}(t_p) = o\} \le \rho_{\mathcal{O}(t_p)} V(x(t_p), \mathcal{O}(t_p)), \tag{3.4}$$

$$\mathbb{E}\{V(x(t_p + k), \mathcal{O}(t_p + k) = h|x(t_p), \mathcal{O}(t_p) = o\} - V(x(t_p), \mathcal{O}(t_p)) \le -\lambda_3\|x(t_p)\|_1, \tag{3.5}$$

hold, then system (3.2) is stochastically stable for any initial conditions $x(k_0) = x_0$ and $\mathcal{O}(k_0) = \mathcal{O}_0$, where $k \in [1, t_{p+1} - t_p)$ and $\mathcal{O}(k_p) \in \mathcal{P}$.

Proof. By (3.5), it follows that

$$\mathbb{E}\{\lambda_3\|x(t_p)\|_1\} \le \mathbb{E}\{V(x(t_p), \mathcal{O}(t_p))\} - \mathbb{E}\{\mathbb{E}\{V(x(t_p + k), \mathcal{O}(t_p + k) = h|x(t_p), \mathcal{O}(t_p) = o\}\}.$$

Together with $\mathbb{E}\{\mathbb{E}\{V(x(t_p+k), \mathcal{O}(t_p+k) = h|x(t_p), \mathcal{O}(t_p) = o\}\} = \mathbb{E}\{V(x(t_p+k), \mathcal{O}(t_p+k) = h|x(t_p), \mathcal{O}(t_p) = o\}$, we obtain $\mathbb{E}\{\lambda_3\|x(t_p)\|_1\} \leq \mathbb{E}\{V(x(t_p), \mathcal{O}(t_p))\} - \mathbb{E}\{V(x(t_p + k), \mathcal{O}(t_p + k) = h|x(t_p), \mathcal{O}(t_p) = o\}$. Then, $\sum_{n=1}^{\infty} \mathbb{E}\{\lambda_3\|x(t_p)\|_1\} \leq \mathbb{E}\{V(x(t_0), \mathcal{O}(t_0)\} < \infty$, which means that $\lim\limits_{p\to\infty} \mathbb{E}\{\lambda_3\|x(t_p)\|_1\} = 0$. By $\lambda_3 > 0$, we have $\lim\limits_{p\to\infty} \mathbb{E}\{\|x(t_p)\|_1\} = 0$. Since $\lambda_2 \geq \lambda_1 > 0$, then $\mathbb{E}\{\lambda_1\|x(k)\|_1\} \leq \max\limits_{\mathcal{O}(t_p)\in\mathcal{O}}\{\rho_{\mathcal{O}(t_p)}\}\mathbb{E}\{V(x(k), \mathcal{O}(k))\} \leq \max\limits_{\mathcal{O}(t_p)\in\mathcal{O}}\{\rho_{\mathcal{O}(t_p)}\}\mathbb{E}\{\lambda_2\|x(k)\|_1\}$ holds $\forall k \in [t_p, t_{p+1})$. It is not difficult to conclude that $\lim\limits_{p\to\infty} \mathbb{E}\{\|x(k)\|_1\} = 0$. By Definition 3, the system (3.2) is stochastically stable. □

Remark 2. *Unlike the Lyapunov function $V(x(k), \mathcal{O}(k)) = x^\top(k)P_{\mathcal{O}(k)}x(k)$ used in [24, 26], and [28], a stochastic CLF $V(x(k), \mathcal{O}(k)) = x^\top(k)\lambda_{\mathcal{O}(k)}$ is used for HSMJ-NPSs in Lemma 2. Such a Lyapunov function follows the design in [7, 9, 12, 13]. When the state of the system is non-negative, the positivity of CLF is easy to be guaranteed. It is also clear that CLF has a simpler form than the quadratic Lyapunov function.*

Theorem 1. *If there exist constants $\wp_1 > 1, \wp_2 > 1, \rho_o > 0$, $\mathcal{R}^n$ vectors $\lambda_o \succ 0, \mu_s \succ 0$, and $\mathcal{R}^m$ vector $\chi_s^{(i)} \prec 0$ such that the conditions*

$$\overline{\pi} \sum_{i=1}^{\mathfrak{I}} \mathbf{1}_{\mathfrak{I}}^{(i)\top} \underline{B}^\top \sum_{s\in\mathcal{Q}} \tilde{\psi}_s \mu_s A_o + B_o \sum_{i=1}^{\mathfrak{I}} \mathbf{1}_{\mathfrak{I}}^{(i)} \chi_s^{(i)\top} \succ 0, \tag{3.6a}$$

$$\wp_1 \overline{\pi} \sum_{s\in\mathcal{Q}} \tilde{\psi}_s A_o^\top \mu_s - \rho_o \lambda_o + \wp_1 \sum_{i=1}^{\mathfrak{I}} \mathbf{1}_{\mathfrak{I}}^{(i)} \chi_s^{(i)\top} \prec 0, \tag{3.6b}$$

$$\overline{\pi} \sum_{s\in\mathcal{Q}} \tilde{\psi}_s A_o^\top \mu_s - \mu_a + \sum_{i=1}^{\mathfrak{I}} \mathbf{1}_{\mathfrak{I}}^{(i)} \chi_s^{(i)\top} \prec 0, \tag{3.6c}$$

$$\overline{\pi} \sum_{s\in\mathcal{Q}} \tilde{\psi}_s A_o^\top \sum_{\kappa=2}^{\kappa_{\max}^o} \sum_{h\in\mathcal{P}} \tilde{\varsigma}_{oh}(\kappa)\mu_s - \sum_{\kappa=2}^{\kappa_{\max}^o} \sum_{h\in\mathcal{P}} \tilde{\varsigma}_{oh}(\kappa)\mu_a + \sum_{i=1}^{\mathfrak{I}} \mathbf{1}_{\mathfrak{I}}^{(i)} \chi_s^{(i)\top} \prec 0, \tag{3.6d}$$

$$\wp_2 \overline{\pi} \sum_{s\in\mathcal{Q}} \tilde{\psi}_s A_o^\top (\sum_{h\in\mathcal{P}} \tilde{\varsigma}_{oh}(1)\mu_s + \sum_{\kappa=2}^{\kappa_{\max}^o} \sum_{h\in\mathcal{P}} \tilde{\varsigma}_{oh}(\kappa)\mu_s) - \lambda_o + \wp_2 \sum_{i=1}^{\mathfrak{I}} \mathbf{1}_{\mathfrak{I}}^{(i)} \chi_s^{(i)\top} \prec 0, \tag{3.6e}$$

$$\lambda_o \leq \wp_1 \mu_s, \lambda_h \leq \wp_2 \mu_s \tag{3.6f}$$

hold for $(o, h) \in \mathcal{P} \times \mathcal{P}, (s, a) \in \mathcal{Q} \times \mathcal{Q}, \kappa_{\max}^o \in [2, +\infty)$, respectively, then the system (3.2) is positive and σ-stochastically stable under the observed mode-dependent controller (3.1) with

$$K_s = \sum_{i=1}^{\mathfrak{I}} \frac{\mathbf{1}_{\mathfrak{I}}^{(i)} \chi_s^{(i)\top}}{\mathbf{1}_{\mathfrak{I}}^{(i)\top} \underline{B}^\top \sum_{s\in\mathcal{Q}} \tilde{\psi}_s \mu_s}, \tag{3.7}$$

where $i \in \{1, 2, \ldots, \mathfrak{I}\} \in \mathcal{Q}$, $\tilde{\psi}_s = \max\limits_{o\in\mathcal{P}}\{\psi_{os}\}$, $\tilde{\varsigma}_{oh}(\kappa) = \frac{\varsigma_{oh}(\kappa)}{\sum_{o\in\mathcal{P}} \sum_{c=1}^{\kappa_{\max}^o} \varsigma_{ch}(c)}$, $\underline{B} = \min\limits_{o\in\mathcal{P}}\{\underline{B}_o\}$, $\overline{B} = \max\limits_{o\in\mathcal{P}}\{\overline{B}_o\}$, and $\underline{B}_o$ and $\overline{B}_o$ represent the minimal and maximal elements of the matrix B_o.

Proof. Since $\underline{B} \geq 0$, $\overline{\pi} \geq 0$, and $\mu_s \succ 0$, it can follow that $\sum_{i=1}^{\mathfrak{I}} \mathbf{1}_{\mathfrak{I}}^{(i)\top} \underline{B}^\top \sum_{s\in\mathcal{Q}} \tilde{\psi}_s \mu_s \geq 0$. Give a nonnegative initial condition $x(k_0) \geq 0$. Following $A_o \geq 0$ and (3.7) gives $A_o + \frac{B_o \sum_{i=1}^{\mathfrak{I}} \mathbf{1}_{\mathfrak{I}}^{(i)} \chi_s^{(i)\top}}{\sum_{i=1}^{\mathfrak{I}} \mathbf{1}_{\mathfrak{I}}^{(i)\top} \underline{B}^\top \sum_{s\in\mathcal{Q}} \tilde{\psi}_s \mu_s} \geq 0$, which means that $A_o + B_o K_s \geq 0$. By (3.2) and Definition 1, we obtain $x(k_0 + 1) \geq 0$. Thus, one can have $x(k) \geq 0$ using recursive derivation, that is, the system (3.2) is positive. Using the sector condition and (3.2) yields that

$$x(k + 1) \geq \underline{\pi} A_{\mathcal{O}(k)} x(k) + B_{\mathcal{O}(k)} K_{\mathcal{O}^\star(k)} x(k), \tag{3.8}$$

$$x(k + 1) \leq \overline{\pi} A_{\mathcal{O}(k)} x(k) + B_{\mathcal{O}(k)} K_{\mathcal{O}^\star(k)} x(k) = \mathcal{A}_{\mathcal{O}(k)\mathcal{O}^\star(k)} x(k), \tag{3.9}$$

where $\mathcal{A}_{O(k)O^{\star}(k)} = \overline{\pi}A_{O(k)} + B_{O(k)}K_{O^{\star}(k)}$.

Construct a stochastic CLF as: $V(x(k), O(k) = o) = x^{\top}(k)\lambda_o$. Then, $\lambda_{\min}\|x(k)\|_1 \le V(x(k), O(k)) \le \lambda_{\max}\|x(k)\|_1$, where $\lambda_{\min} = \min_{o\in\mathcal{P}}\{\underline{\lambda}_o\}$, $\lambda_{\max} = \max_{o\in\mathcal{P}}\{\overline{\lambda}_o\}$, and $\underline{\lambda}_o$ and $\overline{\lambda}_o$ represent the minimal and maximal elements of the vector λ_o, respectively. Thus, the condition (3.3) is guaranteed. Then,

$$\mathbb{E}\{V(x(t_p + k), O(t_p + k))|_{x_0,O_0}\} = x^{\top}(t_p)\Big(\textstyle\sum_{s_1\in Q} \cdots \sum_{s_k\in Q} \prod_{i=1}^{k} \psi_{os_i}\mathcal{A}^{\top}_{os_i}\lambda_o\Big) \tag{3.10}$$

holds for $\kappa^o_{\max} \in [2, +\infty), k \in [1, t_{p+1} - t_p - 1]$, and $p \in [0, +\infty)$. By (3.6f) and (3.10), it follows that

$$\begin{aligned}
&\textstyle\sum_{s_1\in Q} \cdots \sum_{s_k\in Q} \prod_{i=1}^{k} \psi_{os_i}\mathcal{A}^{\top}_{os_i}\lambda_o - \rho_o\lambda_o\\
&\le \textstyle\sum_{s_1\in Q} \cdots \sum_{s_k\in Q} \prod_{i=1}^{k} \psi_{os_i}\mathcal{A}^{\top}_{os_i}\wp_1\mu_{s_k} - \rho_o\lambda_o\\
&= \wp_1\Big(\textstyle\sum_{s_1\in Q} \cdots \sum_{s_{k-1}\in Q} \prod_{i=1}^{k-1} \psi_{os_i}\mathcal{A}^{\top}_{os_i}(\sum_{s_k\in Q} \psi_{os_k}\mathcal{A}^{\top}_{os_k}\mu_{s_k} - \mu_{s_{k-1}})\\
&\quad + \textstyle\sum_{n=1}^{k-1} \sum_{s_1\in Q} \cdots \sum_{s_{n-1}\in Q} \prod_{i=1}^{n-1} \psi_{os_i}\mathcal{A}^{\top}_{os_i}(\sum_{s_n\in Q} \psi_{os_n}\mathcal{A}^{\top}_{os_n}\mu_{s_n} - \mu_{s_{n-1}})\\
&\quad + \textstyle\sum_{s_1\in Q} \psi_{os_1}\mathcal{A}^{\top}_{os_1}\mu_{s_1}\Big) - \rho_o\lambda_o,
\end{aligned} \tag{3.11}$$

where

$$\begin{aligned}
&\wp_1\Big(\textstyle\sum_{n=1}^{k-1} \sum_{s_1\in Q} \cdots \sum_{s_{n-1}\in Q} \prod_{i=1}^{n-1} \psi_{os_i}\mathcal{A}^{\top}_{os_i}(\sum_{s_n\in Q} \psi_{os_n}\mathcal{A}^{\top}_{os_n}\mu_{s_n} - \mu_{s_{n-1}}) + \sum_{s_1\in Q} \psi_{os_1}\mathcal{A}^{\top}_{os_1}\mu_{s_1}\Big) - \rho_o\lambda_o\\
&= \wp_1\Big(\textstyle\sum_{s_1\in Q} \psi_{os_1}\mathcal{A}^{\top}_{os_1} \cdots \sum_{s_{k-1}\in Q} \psi_{os_{k-2}}\mathcal{A}^{\top}_{os_{k-2}} \sum_{s_{k-1}\in Q} \psi_{os_k}\mathcal{A}^{\top}_{os_{k-1}}\mu_{s_{k-1}}\\
&\quad - \textstyle\sum_{s_1\in Q} \psi_{os_1}\mathcal{A}^{\top}_{os_1}\mu_{s_1} + \sum_{s_1\in Q} \psi_{os_1}\mathcal{A}^{\top}_{os_1}\mu_{s_1}\Big) - \rho_o\lambda_o\\
&= \wp_1\Big(\textstyle\sum_{s_1\in Q} \cdots \sum_{s_k\in Q} \prod_{i=1}^{k-1} \psi_{os_i}\mathcal{A}^{\top}_{os_i}\mu_{s_{k-1}}\Big) - \rho_o\lambda_o.
\end{aligned}$$

Using (3.7) gives $K_s^{\top}\underline{B}^{\top} \sum_{s\in Q} \tilde{\psi}_s\mu_s = \sum_{i=1}^{\mathfrak{I}} \mathbf{1}^{(i)}_{\mathfrak{I}}\chi^{(i)\top}_s$. It is also clear that $K_s \prec 0$. Together with $\underline{B} \ge 0$, $\tilde{\psi}_s = \max_{o\in\mathcal{P}}\{\psi_{os}\}$, and (3.6b) yields that

$$\begin{aligned}
\wp_1 \textstyle\sum_{s_1\in Q} \psi_{os_1}\mathcal{A}^{\top}_{os_1}\mu_{s_1} - \rho_o\lambda_o &\le \wp_1 \textstyle\sum_{s_1\in Q} \tilde{\psi}_{s_1}\mathcal{A}^{\top}_{os_1}\mu_{s_1} - \rho_o\lambda_o\\
&\le \wp_1\overline{\pi} \textstyle\sum_{s_1\in Q} \tilde{\psi}_{s_1}A^{\top}_o\mu_{s_1} + \wp_1 \sum_{i=1}^{\mathfrak{I}} \mathbf{1}^{(i)}_{\mathfrak{I}}\chi^{(i)\top}_{s_1} - \rho_o\lambda_o \prec 0.
\end{aligned} \tag{3.12}$$

By (3.6c), we have

$$\begin{aligned}
\textstyle\sum_{s_n\in Q} \psi_{os_n}\mathcal{A}^{\top}_{os_n}\mu_{s_n} - \mu_{s_{n-1}} &\le \textstyle\sum_{s_k\in Q} \psi_{os_k}\mathcal{A}^{\top}_{os_k}\mu_{s_k} - \mu_{s_{k-1}}\\
&\le \overline{\pi} \textstyle\sum_{s_k\in Q} \tilde{\psi}_{s_k}A^{\top}_o\mu_{s_k} + K^{\top}_{s_k}B^{\top}_o \sum_{s_k\in Q} \tilde{\psi}_{s_k}\mu_{s_k} - \mu_{s_{k-1}}\\
&\le \overline{\pi} \textstyle\sum_{s_k\in Q} \tilde{\psi}_{s_k}A^{\top}_o\mu_{s_k} + \sum_{i=1}^{\mathfrak{I}} \mathbf{1}^{(i)}_{\mathfrak{I}}\chi^{(i)\top}_{s_k} - \mu_{s_{k-1}} \prec 0.
\end{aligned} \tag{3.13}$$

Substituting (3.12) and (3.13) into (3.11) gives $\wp_1(\sum_{s_1\in Q} \cdots \sum_{s_k\in Q} \prod_{i=1}^{k} \psi_{os_i}\mathcal{A}^{\top}_{os_i}\mu_{s_k}) - \rho_o\lambda_o \prec 0$. Thus, it is not difficult to conclude that condition (3.4) is true.

Furthermore, we consider $O(t_p) = o, O(t_{p+1}) = h, \forall o, h \in \mathcal{P}, o \ne h$, and $\kappa = t_{p+1} - t_p$. Then,

$$\mathbb{E}\{V(x(t_{p+1}), O(t_{p+1}))|_{x_0,O_0,\kappa\le\kappa^o_{\max}}\} = x^{\top}(t_p)(\textstyle\sum_{\kappa=1}^{\kappa^o_{\max}} \sum_{s_1\in Q} \cdots \sum_{s_\kappa\in Q} \prod_{i=1}^{\kappa} \psi_{os_i}\mathcal{A}^{\top}_{os_i} \sum_{h\in\mathcal{P}} \varsigma_{oh}(\kappa)\lambda_h) \tag{3.14}$$

holds for $p \in [0, +\infty)$. Together with (3.6f) yields that

$$\begin{aligned}
&\textstyle\sum_{\kappa=1}^{\kappa^o_{\max}} \sum_{s_1\in Q} \cdots \sum_{s_\kappa\in Q} \prod_{i=1}^{\kappa} \psi_{os_i}\mathcal{A}^{\top}_{os_i} \sum_{h\in\mathcal{P}} \varsigma_{oh}(\kappa)\lambda_h - \lambda_o\\
&\le \textstyle\sum_{\kappa=1}^{\kappa^o_{\max}} \sum_{s_1\in Q} \cdots \sum_{s_\kappa\in Q} \prod_{i=1}^{\kappa} \psi_{os_i}\mathcal{A}^{\top}_{os_i} \sum_{h\in\mathcal{P}} \varsigma_{oh}(\kappa)\wp_2\mu_{s_i} - \lambda_o\\
&= \wp_2\Big(\textstyle\sum_{\kappa=2}^{\kappa^o_{\max}} \Big(\sum_{\varrho=1}^{\kappa-1} \sum_{s_1\in Q} \cdots \sum_{s_\varrho\in Q} \prod_{i=1}^{\varrho} \psi_{os_i}\mathcal{A}^{\top}_{os_i}\\
&\quad \times (\textstyle\sum_{s_{\varrho+1}\in Q} \psi_{os_{\varrho+1}}\mathcal{A}^{\top}_{os_{\varrho+1}} \sum_{h\in\mathcal{P}} \varsigma_{oh}(\kappa)\mu_{s_{\varrho+1}} - \sum_{h\in\mathcal{P}} \varsigma_{oh}(\kappa)\mu_{s_\varrho})\Big)\\
&\quad + \textstyle\sum_{\kappa=2}^{\kappa^o_{\max}} \sum_{s_1\in Q} \psi_{os_1}\mathcal{A}^{\top}_{os_1} \sum_{h\in\mathcal{P}} \varsigma_{oh}(\kappa)\mu_{s_1} + \sum_{s_1\in Q} \psi_{os_1}\mathcal{A}^{\top}_{os_1} \sum_{h\in\mathcal{P}} \varsigma_{oh}(1)\mu_{s_1}\Big) - \lambda_o,
\end{aligned} \tag{3.15}$$

where $\varrho \in [1, \kappa-1]$. Note the facts $K_s^\top \underline{B}^\top \sum_{s\in Q} \tilde{\psi}_s \mu_s = \sum_{i=1}^{\mathfrak{I}} \mathbf{1}_{\mathfrak{I}}^{(i)} \chi_s^{(i)\top}$, $K_s \prec 0$, $\tilde{\psi}_s = \max_{o\in\mathcal{P}}\{\psi_{os}\}$, and $\underline{B} \succeq 0$. Together with (3.6d) gives

$$\begin{aligned}
&\textstyle\sum_{s_{\varrho+1}\in Q} \psi_{os_{\varrho+1}} \mathcal{A}_{os_{\varrho+1}}^\top \sum_{\kappa=2}^{\kappa_{\max}^o} \sum_{h\in\mathcal{P}} \tilde{\varsigma}_{oh}(\kappa)\mu_{s_{\varrho+1}} - \sum_{\kappa=2}^{\kappa_{\max}^o} \sum_{h\in\mathcal{P}} \tilde{\varsigma}_{oh}(\kappa)\mu_{s_\varrho} \\
&\le \bar{\pi} \textstyle\sum_{s_{\varrho+1}\in Q} \tilde{\psi}_{s_{\varrho+1}} A_o^\top \sum_{\kappa=2}^{\kappa_{\max}^o} \sum_{h\in\mathcal{P}} \tilde{\varsigma}_{oh}(\kappa)\mu_{s_{\varrho+1}} + K_{s_{\varrho+1}}^\top B_o^\top \sum_{s_{\varrho+1}\in Q} \tilde{\psi}_{s_{\varrho+1}} \sum_{\kappa=2}^{\kappa_{\max}^o} \\
&\quad \times \textstyle\sum_{h\in\mathcal{P}} \tilde{\varsigma}_{oh}(\kappa)\mu_{s_{\varrho+1}} - \sum_{\kappa=2}^{\kappa_{\max}^o} \sum_{h\in\mathcal{P}} \tilde{\varsigma}_{oh}(\kappa)\mu_{s_\varrho} \\
&\le \bar{\pi} \textstyle\sum_{s_{\varrho+1}\in Q} \tilde{\psi}_{s_{\varrho+1}} A_o^\top \sum_{\kappa=2}^{\kappa_{\max}^o} \sum_{h\in\mathcal{P}} \tilde{\varsigma}_{oh}(\kappa)\mu_{s_{\varrho+1}} - \sum_{\kappa=2}^{\kappa_{\max}^o} \sum_{h\in\mathcal{P}} \tilde{\varsigma}_{oh}(\kappa)\mu_{s_\varrho} + \sum_{i=1}^{\mathfrak{I}} \mathbf{1}_{\mathfrak{I}}^{(i)} \chi_{s_{\varrho+1}}^{(i)\top} \prec 0.
\end{aligned} \tag{3.16}$$

By (3.6e), we obtain

$$\begin{aligned}
&\wp_2 \textstyle\sum_{s_1\in Q} \psi_{os_1} \mathcal{A}_{os_1}^\top (\sum_{h\in\mathcal{P}} \tilde{\varsigma}_{oh}(1)\mu_{s_1} + \sum_{\kappa=2}^{\kappa_{\max}^o} \sum_{h\in\mathcal{P}} \tilde{\varsigma}_{oh}(\kappa)\mu_{s_1}) - \lambda_o \\
&\le \wp_2 \bar{\pi} \textstyle\sum_{s_1\in Q} \tilde{\psi}_{s_1} A_o^\top (\sum_{h\in\mathcal{P}} \tilde{\varsigma}_{oh}(1)\mu_{s_1} + \sum_{\kappa=2}^{\kappa_{\max}^o} \sum_{h\in\mathcal{P}} \tilde{\varsigma}_{oh}(\kappa)\mu_{s_1}) + \wp_2 K_{s_1}^\top B_o^\top \sum_{s_1\in Q} \tilde{\psi}_{s_1} \\
&\quad \times (\textstyle\sum_{h\in\mathcal{P}} \tilde{\varsigma}_{oh}(1)\mu_{s_1} + \sum_{\kappa=2}^{\kappa_{\max}^o} \sum_{h\in\mathcal{P}} \tilde{\varsigma}_{oh}(\kappa)\mu_{s_1}) - \lambda_o \\
&\le \wp_2 \bar{\pi} \textstyle\sum_{s_1\in Q} \tilde{\psi}_{s_1} \mathcal{A}_{os_1}^\top (\sum_{h\in\mathcal{P}} \tilde{\varsigma}_{oh}(1)\mu_{s_1} + \sum_{\kappa=2}^{\kappa_{\max}^o} \sum_{h\in\mathcal{P}} \tilde{\varsigma}_{oh}(\kappa)\mu_{s_1}) - \lambda_o + \wp_2 \sum_{i=1}^{\mathfrak{I}} \mathbf{1}_{\mathfrak{I}}^{(i)} \chi_{s_1}^{(i)\top} \prec 0.
\end{aligned} \tag{3.17}$$

Substituting (3.16) and (3.17) into (3.15) yields $\wp_2(\sum_{\kappa=1}^{\kappa_{\max}^o} \sum_{s_1\in Q} \cdots \sum_{s_\kappa\in Q} \prod_{i=1}^{\kappa} \psi_{os_i} \mathcal{A}_{os_i}^\top \sum_{h\in\mathcal{P}} \tilde{\varsigma}_{oh}(\kappa)\mu_{s_i}) - \lambda_o \prec 0$. It is not difficult to conclude that the condition (3.5) under $\kappa_{\max}^o \in [2, +\infty)$ is true. Therefore, the system (3.2) is positive and σ-stochastically stable. This completes the proof. □

Remark 3. *Theorem 1 designs the controller gain matrix (3.7) using a matrix decomposition technique. Under the matrix decomposition technique, the controller gain matrix K_s can be transformed into a set of vector variables, i.e.,*

$$\begin{aligned}
K_s &= \begin{pmatrix} k_{s11} & k_{s12} & \cdots & k_{s1\mathfrak{I}} \\ k_{s21} & k_{s22} & \cdots & k_{s2\mathfrak{I}} \\ \vdots & \vdots & \ddots & \vdots \\ k_{s\mathfrak{I}1} & k_{s\mathfrak{I}2} & \cdots & k_{s\mathfrak{I}\mathfrak{I}} \end{pmatrix} = \begin{pmatrix} k_{s11} & k_{s12} & \cdots & k_{s1\mathfrak{I}} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} + \cdots + \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ k_{s\mathfrak{I}1} & k_{s\mathfrak{I}2} & \cdots & k_{s\mathfrak{I}\mathfrak{I}} \end{pmatrix} \\
&= \mathbf{1}_{\mathfrak{I}}^{(1)} \times (k_{s11}k_{s12}\cdots k_{s1\mathfrak{I}}) + \cdots + \mathbf{1}_{\mathfrak{I}}^{(\mathfrak{I})} \times (k_{s\mathfrak{I}1}k_{s\mathfrak{I}2}\cdots k_{s\mathfrak{I}\mathfrak{I}}) = \frac{\mathbf{1}_{\mathfrak{I}}^{(1)}\chi_s^{(1)\top}}{\mathbf{1}_{\mathfrak{I}}^{(1)\top}\underline{B}^\top \sum_{s\in Q}\tilde{\psi}_s\mu_s} + \cdots + \frac{\mathbf{1}_{\mathfrak{I}}^{(\mathfrak{I})}\chi_s^{(\mathfrak{I})\top}}{\mathbf{1}_{\mathfrak{I}}^{(\mathfrak{I})\top}\underline{B}^\top \sum_{s\in Q}\tilde{\psi}_s\mu_s} \\
&= \sum_{i=1}^{\mathfrak{I}} \frac{\mathbf{1}_{\mathfrak{I}}^{(i)}\chi_s^{(i)\top}}{\mathbf{1}_{\mathfrak{I}}^{(i)\top}\underline{B}^\top \sum_{s\in Q}\tilde{\psi}_s\mu_s}.
\end{aligned}$$

This means that these vector variables can be computed by using the LP toolbox in MATLAB. In fact, the term $\mathbf{1}_{\mathfrak{I}}^{(i)\top}\underline{B}^\top \sum_{s\in Q}\tilde{\psi}_s\mu_s$ is introduced to transform (3.7) into an LP form as shown in the conditions (3.12), (3.13), (3.16), and (3.17).

Remark 4. *From the derivation in Theorem 1, it is easy to achieve the positivity and stochastic stability of the system (3.2) for $\kappa_{\max}^o = 2$. For the special case $\kappa_{\max}^o = 1$, two points are stated. On one hand, it can follow from (3.6e) that $\wp_2\bar{\pi}\sum_{s\in Q}\tilde{\psi}_s A_o^\top \sum_{h\in\mathcal{P}}\tilde{\varsigma}_{oh}(1)\mu_s - \lambda_o + \wp_2\sum_{i=1}^{\mathfrak{I}}\mathbf{1}_{\mathfrak{I}}^{(i)}\chi_s^{(i)\top} \prec 0$. Together with the remaining conditions in (3.6), the positivity and stochastic stability of the system (3.2) can be verified for $\kappa_{\max}^o = 1$. On the other hand, it is assumed that the term $\sum_{\kappa=2}^{\kappa_{\max}^o}\sum_{h\in\mathcal{P}}\tilde{\varsigma}_{oh}(\kappa)\mu_s$ equals to zero when $\kappa_{\max}^o = 1$. This further enhances the validity of the condition (3.6e) for $\kappa_{\max}^o = 1$.*

Remark 5. *As discussed in the Introduction, there are two widely used switching rules in positive stochastic systems, i.e., Markov and the semi-Markov processes [9, 10, 29, 30]. Compared with the Markov process, the semi-Markov process removes the restriction that the sojourn time follows an* *exponential distribution. In [9], the obtained results were concerned with MJ-PSs. In [10, 29, 30], the stochastic stability of SMJ-PSs was addressed. In fact, system modes of SMJ-PSs may be hidden [14–17]. However, so far, no efforts have been devoted to the hidden mode issues of MJ-PSs, not mentioning HSMJ-NPSs. Theorem 1 attempts to construct a novel σ-stochastic stabilization framework for HSMJ-NPSs in a linear approach. The obtained framework develops the semi-Markov process of positive systems in [10, 29, 30] to the hidden semi-Markov process. The difficulty lies in how to design a control strategy for HSMJ-NPSs under the hidden Markov process owing to the uncertainties of the process. The results in [10, 29, 30] can be regarded as some special cases of the results in Theorem 1.*

Theorem 1 gives sufficient conditions for the σ-stochastic stabilization of HSMJ-NPSs in observable emission probabilities. In practice, it is hard to obtain full emission probabilities. Thus, we consider the system (3.2) with partially known emission probabilities. Inspired by the results in [27] and [28], we introduce the following formal definition of partially known emission probabilities. Denote $\mathcal{Q} = \mathcal{Q}_{o,Z} \cup \mathcal{Q}_{o,U}$ with $\mathcal{Q}_{o,Z} = \{s \in \mathcal{Q} : \psi_{os} \text{ is known}\}$ and $\mathcal{Q}_{o,U} = \{s \in \mathcal{Q} : \psi_{os} \text{ is unknown}\}$. On the other hand, if $\mathcal{Q}_o^Z \neq \emptyset$, then denote $\mathcal{Q}_{o,Z} = \{Z_o^1, Z_o^2, \ldots, Z_o^{q_o}\}, 1 \leq q_o \leq \mathcal{Q}$, and $\psi_o^Z = \sum_{s\in\mathcal{Q}_{o,Z}} \psi_{os}$, where $Z_o^{q_o}$ is the column number of the qth known element in the oth row in the emission probability matrix.

Theorem 2. *If there exist constants $\wp_1 > 1, \wp_2 > 1, \rho_o > 0$, $\mathcal{R}^n$ vectors $\lambda_o \succ 0, \mu_s \succ 0$, and $\mathcal{R}^m$ vector $\chi_s^{(i)} \prec 0$ such that the conditions*

$$\bar{\pi} \sum_{i=1}^{\mathfrak{J}} \mathbf{1}_{\mathfrak{J}}^{(i)\top} \underline{B}^\top \sum_{s\in\mathcal{Q}} \tilde{\psi}_s \mu_s A_o + B_o \sum_{i=1}^{\mathfrak{J}} \mathbf{1}_{\mathfrak{J}}^{(i)} \chi_s^{(i)\top} \succ 0, \tag{3.18a}$$

$$\wp_1 \bar{\pi} \sum_{s\in\mathcal{Q}_{o,Z}} \tilde{\psi}_s A_o^\top \mu_s - \psi_o^Z \rho_o \lambda_o + \wp_1 \sum_{i=1}^{\mathfrak{J}} \mathbf{1}_{\mathfrak{J}}^{(i)} \chi_s^{(i)\top} \prec 0, \tag{3.18b}$$

$$\bar{\pi} \sum_{s\in\mathcal{Q}_{o,Z}} \tilde{\psi}_s A_o^\top \mu_s - \psi_o^Z \mu_a + \sum_{i=1}^{\mathfrak{J}} \mathbf{1}_{\mathfrak{J}}^{(i)} \chi_s^{(i)\top} \prec 0, \tag{3.18c}$$

$$\bar{\pi} \sum_{s\in\mathcal{Q}_{o,Z}} \tilde{\psi}_s A_o^\top \sum_{\kappa=2}^{\kappa_{\max}^o} \sum_{h\in\mathcal{P}} \tilde{\varsigma}_{oh}(\kappa) \mu_s - \psi_o^Z \sum_{\kappa=2}^{\kappa_{\max}^o} \sum_{h\in\mathcal{P}} \tilde{\varsigma}_{oh}(\kappa) \mu_a + \sum_{i=1}^{\mathfrak{J}} \mathbf{1}_{\mathfrak{J}}^{(i)} \chi_s^{(i)\top} \prec 0, \tag{3.18d}$$

$$\wp_2 \bar{\pi} \sum_{s\in\mathcal{Q}_{o,Z}} \tilde{\psi}_s A_o^\top \Big(\sum_{h\in\mathcal{P}} \tilde{\varsigma}_{oh}(1) \mu_s + \sum_{\kappa=2}^{\kappa_{\max}^o} \sum_{h\in\mathcal{P}} \tilde{\varsigma}_{oh}(\kappa) \mu_s\Big) - \psi_o^Z \lambda_o + \wp_2 \sum_{i=1}^{\mathfrak{J}} \mathbf{1}_{\mathfrak{J}}^{(i)} \chi_s^{(i)\top} \prec 0, \tag{3.18e}$$

$$\lambda_o \leq \wp_1 \mu_s, \lambda_h \leq \wp_2 \mu_s, \tag{3.18f}$$

hold for $s \in \mathcal{Q}_{o,Z}$ and $\kappa_{\max}^o \in [2, +\infty)$ and the conditions

$$\bar{\pi} \sum_{i=1}^{\mathfrak{J}} \mathbf{1}_{\mathfrak{J}}^{(i)\top} \underline{B}^\top \mu_s A_o + B_o \sum_{i=1}^{\mathfrak{J}} \mathbf{1}_{\mathfrak{J}}^{(i)} \chi_s^{(i)\top} \succ 0, \tag{3.19a}$$

$$\wp_1 \bar{\pi} A_o^\top \mu_s - \rho_o \lambda_o + \wp_1 \sum_{i=1}^{\mathfrak{J}} \mathbf{1}_{\mathfrak{J}}^{(i)} \chi_s^{(i)\top} \prec 0, \tag{3.19b}$$

$$\bar{\pi} A_o^\top \mu_s - \mu_a + \sum_{i=1}^{\mathfrak{J}} \mathbf{1}_{\mathfrak{J}}^{(i)} \chi_s^{(i)\top} \prec 0, \tag{3.19c}$$

$$\bar{\pi} A_o^\top \sum_{\kappa=2}^{\kappa_{\max}^o} \sum_{h\in\mathcal{P}} \tilde{\varsigma}_{oh}(\kappa) \mu_s - \sum_{\kappa=2}^{\kappa_{\max}^o} \sum_{h\in\mathcal{P}} \tilde{\varsigma}_{oh}(\kappa) \mu_a + \sum_{i=1}^{\mathfrak{J}} \mathbf{1}_{\mathfrak{J}}^{(i)} \chi_s^{(i)\top} \prec 0, \tag{3.19d}$$

$$\wp_2 \bar{\pi} A_o^\top \Big(\sum_{h\in\mathcal{P}} \tilde{\varsigma}_{oh}(1) \mu_s + \sum_{\kappa=2}^{\kappa_{\max}^o} \sum_{h\in\mathcal{P}} \tilde{\varsigma}_{oh}(\kappa) \mu_s\Big) - \lambda_o + \wp_2 \sum_{i=1}^{\mathfrak{J}} \mathbf{1}_{\mathfrak{J}}^{(i)} \chi_s^{(i)\top} \prec 0, \tag{3.19e}$$

$$\lambda_o \leq \wp_1 \mu_s, \lambda_h \leq \wp_2 \mu_s, \tag{3.19f}$$

hold for $s \in \mathcal{Q}_{o,U}$ and $\kappa^o_{\max} \in [2, +\infty)$, respectively, then the system (3.2) is positive and σ-stochastically stable under the observed mode-dependent controller (3.1) with

$$K_s = \sum_{i=1}^{\mathfrak{I}} \frac{\mathbf{1}_{\mathfrak{I}}^{(i)} \chi_s^{(i)\top}}{\mathbf{1}_{\mathfrak{I}}^{(i)\top} \underline{B}^\top \sum_{s\in\mathcal{Q}} \tilde{\psi}_s \mu_s}, \quad s \in \mathcal{Q}_{o,Z}, \tag{3.20a}$$

$$K_s = \sum_{i=1}^{\mathfrak{I}} \frac{\mathbf{1}_{\mathfrak{I}}^{(i)} \chi_s^{(i)\top}}{\mathbf{1}_{\mathfrak{I}}^{(i)\top} \underline{B}^\top \mu_s}, \quad s \in \mathcal{Q}_{o,U}. \tag{3.20b}$$

Proof. Recalling the fact $\mathcal{Q} = \mathcal{Q}_{o,Z} \cup \mathcal{Q}_{o,U}$, the conditions (3.18)–(3.20) can guarantee the validity of the conditions (3.6) and (3.7). This completes the proof. □

Remark 6. *In [9] and [10], the stochastic stabilization of Markov jump positive systems $x(k+1) = A_{O(k)}x(k) + B_{O(k)}u(k)$ was designed as: $u(k) = K_{O(k)}x(k) = \frac{\sum_{i=1}^{\mathfrak{I}} \mathbf{1}_{\mathfrak{I}}^{(i)} \chi_{O(k)}^{(i)\top}}{\mathbf{1}_{\mathfrak{I}}^\top \underline{B}^\top \sum_{O(k)\in\mathcal{Q}} \tilde{\psi}_{O(k)} \mu_{O(k)}} x(k)$. It should be pointed out that an additional condition $\chi_{O(k)}^{(i)} \preceq \chi_{O(k)}$ is introduced in these literature. This increases the conservatism of the design. In Theorems 1 and 2, an improved control design is proposed in (3.7) and (3.20). Under the novel design approach, the additional restriction condition $\chi_s^{(i)} \preceq \chi_s$ is removed.*

Remark 7. *For non-positive Markov jump systems [21–24, 30], quadratic Lyapunov functions were always used. Owing to the positivity property, quadratic Lyapunov functions are not very effective for positive systems and it has been observed that CLF and LP are more suitable [7, 9, 10, 12, 13, 29, 31]. How to design the control gain matrices, how to construct a suitable CLF for HSMJ-NPSs, and how to address LP-based conditions are key to handling the issues of HSMJ-NPSs. A novel stochastic stabilization framework is constructed in Theorems 1 and 2 for HSMJ-NPSs. It should be noted that such a design framework is easily developed for other issues of positive stochastic jump systems.*

4. Illustrative examples

In [10] and [31], a communication network model with three nodes was established via SMJ-PSs in the following form:

$$x(k+1) = A_{O(k)}x(k) + B_{O(k)}u_{O(k)}(k), \tag{4.1}$$

where $O(k)$ denotes busy- and idle-time models of communication networks by the semi-Markov process and $x(k) = (x_1(k), x_2(k), x_3(k))^\top$ is the number of data transmitted between the three nodes.

Note the fact that switching between busy- and idle-time cases is usually uncertain and random in communication network operation. Moreover, many uncertainties, such as environment and temperature, exist in communication network operations. This is a typical nonlinearity phenomenon. In [26–28], switching rules with uncertain and random are described via a hidden semi-Markov process. Inspired by [26–28], we further assume switching rules between busy- and idle-time cases of communication networks under uncertainties are hidden. Thus, the communication network model (4.1) is changed as:

$$x(k+1) = A_{O(k)}f(x(k)) + B_{O(k)}u_{O^\star(k)}(k), \tag{4.2}$$

where $O^\star(k)$ is an observed process for $O(k)$.

By the analysis above, it is reasonable to use the system (2.1) with hidden semi-Markov processes to improve the communication network model, where

$$A_1 = \begin{pmatrix} 0.42 & 0.37 & 0.29 \\ 0.29 & 0.38 & 0.33 \\ 0.30 & 0.45 & 0.49 \end{pmatrix}, B_1 = \begin{pmatrix} 0.25 & 0.29 \\ 0.26 & 0.28 \\ 0.29 & 0.27 \end{pmatrix}, A_2 = \begin{pmatrix} 0.52 & 0.36 & 0.28 \\ 0.30 & 0.35 & 0.37 \\ 0.30 & 0.24 & 0.45 \end{pmatrix},$$
$$B_2 = \begin{pmatrix} 0.29 & 0.31 \\ 0.28 & 0.28 \\ 0.26 & 0.28 \end{pmatrix}, A_3 = \begin{pmatrix} 0.62 & 0.29 & 0.26 \\ 0.30 & 0.39 & 0.37 \\ 0.31 & 0.29 & 0.49 \end{pmatrix}, B_3 = \begin{pmatrix} 0.30 & 0.32 \\ 0.26 & 0.29 \\ 0.29 & 0.29 \end{pmatrix},$$

where $f(x(k)) = 0.3x_i + \frac{x_i}{x_i^2+1}$ and $o, s = \{1, 2, 3\}$. The upper bounds of the sojourn time are $\kappa^1_{\max} = \kappa^2_{\max} = \kappa^3_{\max} = 5$. The emission probability matrix is assumed to have four cases as listed in Table 1, where ? is an unknown element of the emission probability matrix. The system modes are governed by the semi-Markov process $[\varsigma_{oh}(\kappa)] = [\eta_{oh}(\kappa)\delta_{oh}]$, which is borrowed from [27]:

$$[\delta_{oh}] = \begin{pmatrix} 0 & 0.7 & 0.3 \\ 0.4 & 0 & 0.6 \\ 0.5 & 0.5 & 0 \end{pmatrix}, [\eta_{oh}(\kappa)] = \begin{pmatrix} 0 & \eta^{\kappa}_{12} & \eta^{\kappa}_{13} \\ \eta^{\kappa}_{21} & 0 & \eta^{\kappa}_{23} \\ \eta^{\kappa}_{31} & \eta^{\kappa}_{32} & 0 \end{pmatrix},$$

where

$$\eta^{\kappa}_{12} = \frac{0.6^{\kappa} \cdot 0.4^{10-\kappa} \cdot 10!}{(10-\kappa)! \cdot \kappa!}, \eta^{\kappa}_{13} = \frac{0.4^{\kappa} \cdot 0.6^{10-\kappa} \cdot 10!}{(10-\kappa)! \cdot \kappa!},$$
$$\eta^{\kappa}_{21} = 0.9^{(\kappa-1)^2} - 0.9^{\kappa^2}, \eta^{\kappa}_{23} = \frac{0.5^{10} \cdot 10!}{(10-\kappa)! \cdot \kappa!}, \eta^{\kappa}_{31} = 0.4^{(\kappa-1)^{1.3}} - 0.4^{\kappa^{1.3}}, \eta^{\kappa}_{33} = 0.3^{(\kappa-1)^{0.8}} - 0.3^{\kappa^{1.3}}.$$

Table 1. Four cases of the emission probability matrix Γ.

Case 1	Case 2	Case 3	Case 4
$\begin{pmatrix} 0.70 & 0.20 & 0.10 \\ 0.10 & 0.60 & 0.30 \\ 0.40 & 0.50 & 0.10 \end{pmatrix}$	$\begin{pmatrix} 0.70 & 0.20 & 0.10 \\ 0.10 & ? & ? \\ 0.40 & 0.50 & 0.10 \end{pmatrix}$	$\begin{pmatrix} ? & 0.20 & ? \\ 0.10 & 0.60 & 0.30 \\ 0.40 & 0.50 & 0.10 \end{pmatrix}$	$\begin{pmatrix} ? & ? & 0.10 \\ 0.10 & 0.60 & 0.30 \\ ? & ? & 0.10 \end{pmatrix}$

Choose $\rho_1 = 1.1, \rho_2 = 1.2, \rho_3 = 1.3, \underline{\pi} = 0.2, \overline{\pi} = 0.4$, and $\wp_1 = \wp_2 = 1.1$. By Theorem 1, one can obtain for Case 1 that

$$K_1^{\text{Case1}} = \begin{pmatrix} -0.1363 & -0.1236 & -0.1377 \\ -0.1156 & -0.1000 & -0.1105 \end{pmatrix},$$
$$K_2^{\text{Case1}} = \begin{pmatrix} -0.1296 & -0.1175 & -0.1309 \\ -0.1099 & -0.0950 & -0.1051 \end{pmatrix}, K_3^{\text{Case1}} = \begin{pmatrix} -0.1266 & -0.1148 & -0.1279 \\ -0.1074 & -0.0928 & -0.1026 \end{pmatrix}.$$

The initial condition of the system is set as $x_0 = \begin{pmatrix} 25.5 & 20.5 & 17 \end{pmatrix}^{\top}$. In this condition, Figures 1 and 2 show the state of open- and closed-loop systems in Case 1. From Figure 1, the states under completely known emission probabilities are positive and stochastically stable. It means that the proposed observed mode-dependent controller is effective. By Theorem 2, one can obtain for Cases 2–4 that

$$K_1^{\text{Case2}} = \begin{pmatrix} -0.1889 & -0.1727 & -0.1896 \\ -0.1691 & -0.1488 & -0.1558 \end{pmatrix},$$
$$K_2^{\text{Case2}} = \begin{pmatrix} -0.1756 & -0.1605 & -0.1762 \\ -0.1572 & -0.1383 & -0.1448 \end{pmatrix}, K_3^{\text{Case2}} = \begin{pmatrix} -0.1774 & -0.1621 & -0.1779 \\ -0.1588 & -0.1397 & -0.1462 \end{pmatrix},$$
$$K_1^{\text{Case3}} = \begin{pmatrix} -0.2096 & -0.2079 & -0.2236 \\ -0.1294 & -0.1033 & -0.0704 \end{pmatrix},$$
$$K_2^{\text{Case3}} = \begin{pmatrix} -0.2209 & -0.2190 & -0.2356 \\ -0.1363 & -0.1089 & -0.0742 \end{pmatrix}, K_3^{\text{Case3}} = \begin{pmatrix} -0.2403 & -0.2383 & -0.2564 \\ -0.1483 & -0.1184 & -0.0807 \end{pmatrix},$$

and

$$K_1^{\text{Case4}} = \begin{pmatrix} -0.2010 & -0.1872 & -0.2072 \\ -0.1494 & -0.1072 & -0.0934 \end{pmatrix},$$
$$K_2^{\text{Case4}} = \begin{pmatrix} -0.2418 & -0.2252 & -0.2492 \\ -0.1797 & -0.1290 & -0.1123 \end{pmatrix}, K_3^{\text{Case4}} = \begin{pmatrix} -0.1888 & -0.1759 & -0.1946 \\ -0.1403 & -0.1007 & -0.0877 \end{pmatrix}.$$

Figures 3 and 4 show the states of open- and closed-loop systems in Case 2, Figures 5 and 6 show the states of open- and closed-loop systems in Case 3, and Figures 7 and 8 show the states of open- and closed-loop systems in Case 4. Simulation results show that the designed controller is still effective under partially known emission probabilities.

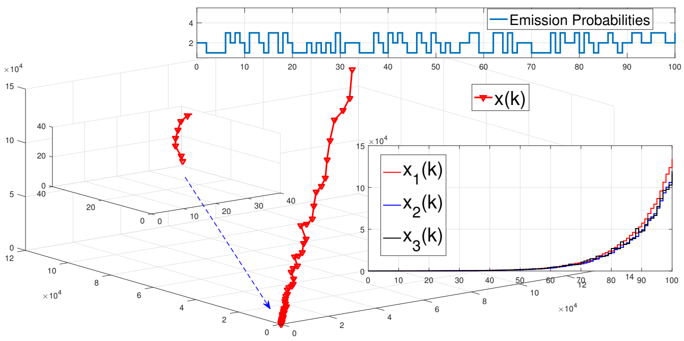

Figure 1. States of the open-loop system in Case 1.

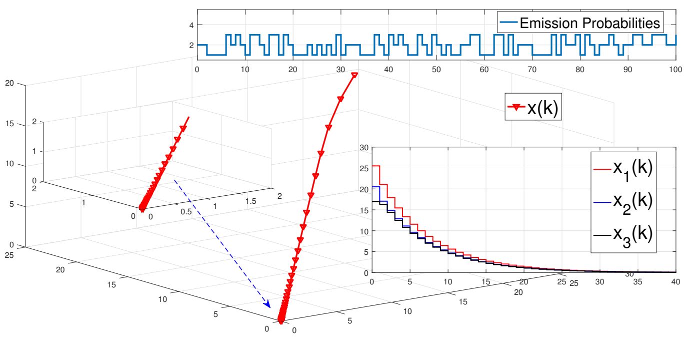

Figure 2. States of the closed-loop system in Case 1.

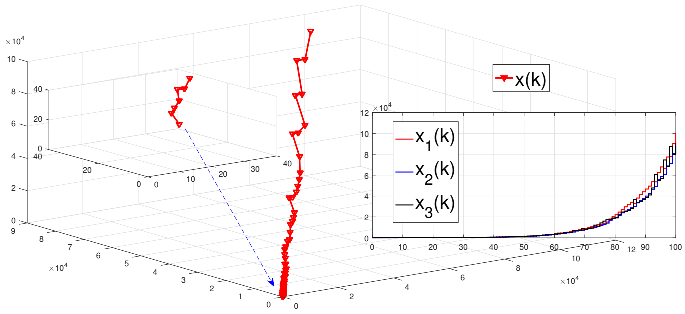

Figure 3. States of the open-loop system in Case 2.

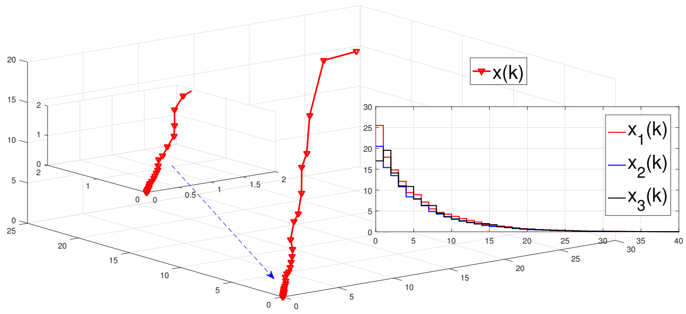

Figure 4. States of the closed-loop system in Case 2.

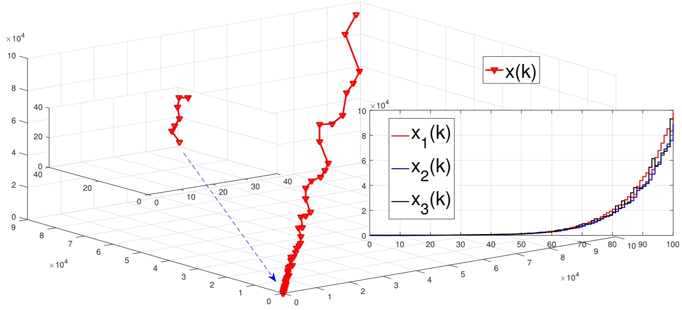

Figure 5. States of the open-loop system in Case 3.

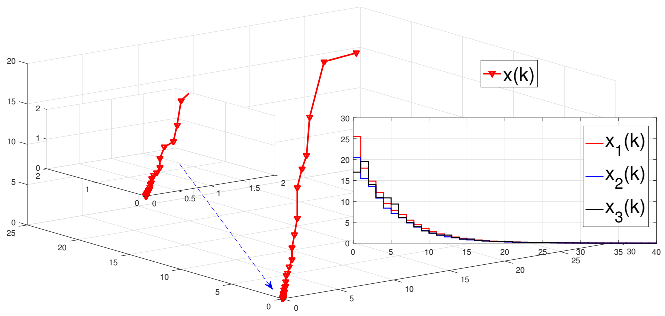

Figure 6. States of the closed-loop system in Case 3.

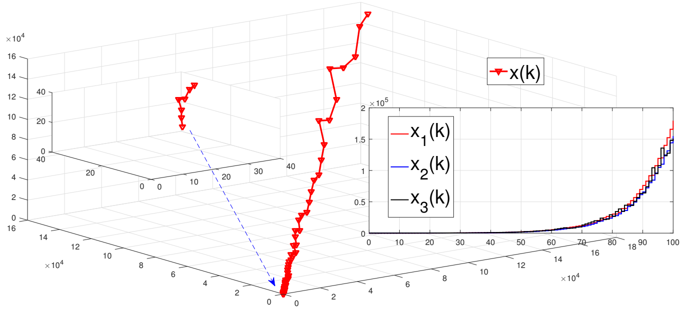

Figure 7. States of the open-loop system in Case 4.

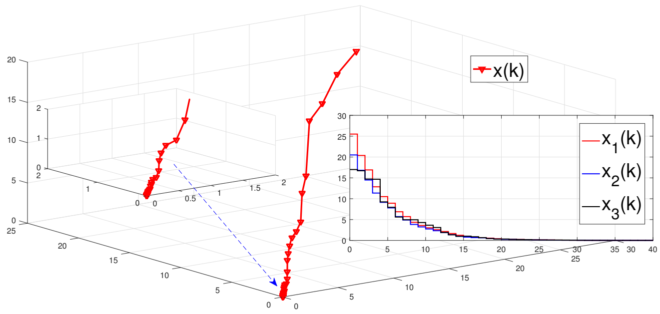

Figure 8. State for the closed-loop system under Case 4.

5. Conclusions

This paper has studied the stabilization of HSMJ-NPSs with completely known- and partially known emission probabilities. Hidden semi-Markov jump processes are employed to govern the switching of systems. Using a CLF and LP, the observed mode-dependent controller is constructed. The designed stabilization framework will be further developed for other issues of positive systems such as stability analysis, nonlinear observer design, event-triggered controller design, etc. The results obtained in this paper are based on an idealized assumption that the considered systems do not contain external perturbations. Thus, how to construct a stochastic stabilization framework on HSMJ-NPSs with completely unknown emission probabilities in the presence of disturbance is an interesting topic in future.

Author contributions

Xuan Jia: Investigation, Writing-original draft, Validation, Writing-review & editing; Junfeng Zhang: Conceptualization, Methodology, Investigation, Supervision; Tarek Raïssi: Writing-original draft, Software. All authors have read and agreed to the published version of the manuscript.

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Conflict of interest

The authors declare no conflicts of interest.

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