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Research article

Commuting Toeplitz operators and H-Toeplitz operators on Bergman space

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Abstract: In this paper, we consider the H-Toeplitz and Toeplitz operators acting on the Bergman space. First, we describe the characterizations of commutativity of two H-Toeplitz operators with certain harmonic symbols. For the general case, it seems very hard. As an extension to the study of Toeplitz operators on the Bergman space, we present the necessary and sufficient conditions of the commutativity of the H-Toeplitz operator and the Toeplitz operator with non-harmonic symbols.

Keywords: H-Toeplitz operators; Toeplitz operators; Bergman space
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1. Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$. As usual, $L^2(\mathbb{D})$ denotes the Hilbert space of all Lebesgue square integrable functions on $\mathbb{D}$ with the inner product

$$\langle f, g\rangle = \int_{\mathbb{D}} f(z)\overline{g(z)}dA(z),$$

for $f, g \in L^2(\mathbb{D})$, where dA is the normalized area measure on $\mathbb{D}$. The Bergman space $L_a^2(\mathbb{D})$ is the subset of $L^2(\mathbb{D})$, consisting of all analytic functions on $\mathbb{D}$. Let P be the orthogonal projection from $L^2(\mathbb{D})$ onto $L_a^2(\mathbb{D})$, then

$$Pf(w) = \int_{\mathbb{D}} \frac{f(z)}{(1-\bar{z}w)^2}dA(z),$$

for $w \in \mathbb{D}$ and $f \in L^2(\mathbb{D})$. For each $z \in \mathbb{D}$, the reproducing kernel function in the Bergman space is denoted by K_z which is given by

$$K_z(w) = \frac{1}{(1-\bar{z}w)^2},\ w \in \mathbb{D}.$$

It is clear that $\{e_n\}_{n=0}^{+\infty}$ forms an orthonormal basis for $L_a^2(\mathbb{D})$, where $e_n(w) = \sqrt{n+1}w^n$, see [1].

Let $L^{\infty}(\mathbb{D})$ be the set of all bounded measurable functions on $\mathbb{D}$. Fix $f \in L^{\infty}(\mathbb{D})$, the Toeplitz operator T_f on the Bergman space is defined by

$$T_f g = P(fg), \quad g \in L_a^2(\mathbb{D}).$$

Toeplitz operators and matrices have applications in control and signal-processing, see [2]. The harmonic Bergman space $L_h^2(\mathbb{D})$ is the collection of all harmonic functions in $L^2(\mathbb{D})$. Define a unitary operator $K : L_a^2 \to L_h^2$ by $K(e_{2n}) = e_n$ and $K(e_{2n+1}) = \overline{e_{n+1}}$ for non-negative integers n. The H-Toeplitz operator B_f with symbol f on the Bergman space is defined by

$$B_f g = P(fK(g)), \quad g \in L_a^2(\mathbb{D}).$$

One may check that B_1 is not the identity operator.

Commuting Toeplitz operators has been studied on various function spaces in recent years. In [3], Brown and Halmos obtained the necessary and sufficient conditions for commuting Toeplitz operators on the Hardy space. In [4], Axler and Cuckovic proved a similar result for Toeplitz operators with bounded harmonic symbols on the Bergman space. In [5], Louhichi and Zakariasy studied the same problem with the quasihomogeneous symbols. All known results have shown that the characterization of commuting Toeplitz operators is quite hard.

In 2007, Arora and Paliwal [6] studied the H-Toeplitz operator which have clubbed the notion of Toeplitz and Hankel operators together on the Hardy space. But this operator is neither Toeplitz nor Hankel operator. They also described the partial isometry, compact and hyponormal properties of H-Toeplitz operators on the Hardy space in their paper. In 2021, Gupta and Singh first studied the H-Toeplitz operator on the Bergman space [7]. They described the commutativity of H-Toeplitz operators with analytic symbols which have nonzero real coefficients $\{a_n\}_{n=0}^{+\infty}$ and $\{b_n\}_{n=0}^{+\infty}$ satisfying $\frac{b_{n+k}}{a_{n+k}} \geq \frac{b_{2n+1}}{a_{2n+1}}$ for any n and k. In 2022, Liang et al. characterized the commuting H-Toeplitz operators with quasihomogeneous symbols on the Bergman space [8]. The obtained results in [7, 8] show that the commutativity of H-Toeplitz operators is quite different from the case of Toeplitz operators.

Some other properties of the H-Toeplitz operator have been studied. In 2022, Kim and Lee give the necessary and sufficient conditions for contractive and expansive H-Toeplitz operators, see [9]. In 2023, Ding and Chen characterize the case when the product of two H-Toeplitz operators is another H-Toeplitz operator with one general and another quasihomogeneous symbols, and also describe the product of the H-Toeplitz operator and the Toeplitz operator to be another H-Toeplitz operator with certain harmonic symbols, see [10].

Motivated by the above, in this paper we will characterize the commuting two H-Toeplitz operators and the commuting Toeplitz and H-Toeplitz operators on the Bergman space as an extension to the study of Toeplitz operators on the Bergman space. We will obtain several results as follows.

Theorem 1.1. *Suppose p_1, p_2, q_1, q_2 are positive integers, $f = a_1 w^{p_1} + \overline{a_2 w^{p_2}}$, $g = b_1 w^{q_1} + \overline{b_2 w^{q_2}}$. Then, $B_f B_g = B_g B_f$ if and only if there exist $\alpha, \beta \in \mathbb{C}$, $|\alpha| + |\beta| \neq 0$ such that $\alpha f + \beta g = 0$.*

Theorem 1.2. *Suppose f is a bounded harmonic function on $\mathbb{D}$. Let p be a non-negative integer, then $B_{w^p} B_f = B_f B_{w^p}$ if and only if there exist $c \in \mathbb{C}$ such that $f = cw^p$.*

The above two results show that two H-Toeplitz operators with certain harmonic symbols commute when two symbols are linearly dependent. The following result shows that, if a H-Toeplitz operator and a Toeplitz operator with non-harmonic symbols commute, the symbols are either constant or zero.

Theorem 1.3. *Let ϕ be a bounded radial function and p be a non-negative integer. Suppose*

$$f = \sum_{k\geq 0} e^{ik\theta}\varphi_k(r) \in L^\infty(\mathbb{D}),$$

where each φ_k is a bounded radial function. Then, $T_f B_{e^{ip\theta}\phi} = B_{e^{ip\theta}\phi} T_f$ holds if and only if $\phi = 0$ or f is a constant.

The contents of the paper are organized as follows. In Section 2, we shall collect some results as preliminaries which will be used frequently in this paper. In Section 3, we study the commuting H-Toeplitz operators with certain harmonic symbols and prove Theorems 1.1 and 1.2. In Section 4, we focus on the commutativity of a H-Toeplitz operator and a Toeplitz operator with non-harmonic symbols, which proves Theorem 1.3.

2. Preliminaries

In this section, we present some useful lemmas which come from [8].

Lemma 2.1. *Let p be a non-negative integer. For any non-negative n,*

$$B_{w^p}(w^{2n}) = \sqrt{\frac{n+1}{2n+1}}\, w^{n+p},$$

$$B_{w^p}(w^{2n+1}) = \begin{cases} \sqrt{\dfrac{n+2}{2n+2}}\dfrac{p-n}{p+1} w^{p-n-1}, & n \leq p-1, \\ 0 & , \ n > p-1, \end{cases}$$

$$B_{\overline{w}^p}(w^{2n}) = \begin{cases} \sqrt{\dfrac{n+1}{2n+1}}\dfrac{n-p+1}{n+1} w^{n-p}, & n \geq p, \\ 0 & , \ n < p, \end{cases}$$

$$B_{\overline{w}^p}(w^{2n+1}) = 0.$$

Lemma 2.2. *Let p and q be non-negative integers, then the following statements are equivalent:*

(1) $B_{w^p}B_{w^q} = B_{w^q}B_{w^p}$,
(2) $B_{\overline{w}^p}B_{\overline{w}^q} = B_{\overline{w}^q}B_{\overline{w}^p}$,
(3) $p = q$.

Lemma 2.3. *Let p be a non-negative integer, then $B_{w^p}B_{\overline{w}^p} = B_{\overline{w}^p}B_{w^p}$ if and only if $p = 0$.*

Lemma 2.4. *Let p and q be different non-negative integers, $a, b \in \mathbb{C}$, then $B_{a\overline{w}^p}B_{bw^q} = B_{bw^q}B_{a\overline{w}^p}$ if and only if $ab = 0$.*

3. The proof of Theorems 1.1 and 1.2

In this section, we prove the necessary and sufficient conditions for the commutativity of two H-Toeplitz operators with certain harmonic symbols. First, we consider harmonic monomial symbols.

Proof of Theorem 1.1. The sufficiency is obvious, so now we prove the necessity. Suppose $B_fB_g = B_gB_f$. It is noted that

$$B_fB_g = a_1b_1B_{w^{p_1}}B_{w^{q_1}} + a_1\overline{b_2}B_{w^{p_1}}B_{\overline{w}^{q_2}} + \overline{a_2}b_1B_{\overline{w}^{p_2}}B_{w^{q_1}} + \overline{a_2b_2}B_{\overline{w}^{p_2}}B_{\overline{w}^{q_2}}, \tag{3.1}$$

$$B_gB_f = b_1a_1B_{w^{q_1}}B_{w^{p_1}} + \overline{b_2}a_1B_{w^{q_1}}B_{\overline{w}^{p_2}} + b_1\overline{a_2}B_{\overline{w}^{q_2}}B_{w^{p_1}} + \overline{b_2a_2}B_{\overline{w}^{q_2}}B_{\overline{w}^{p_2}}. \tag{3.2}$$

By Lemma 2.1, for any non-negative integers n, it follows from the above that

$$B_fB_g(w^{2n+1}) = (a_1b_1B_{w^{p_1}}B_{w^{q_1}} + \overline{a_2}b_1B_{\overline{w}^{p_2}}B_{w^{q_1}})(w^{2n+1}),$$

$$B_gB_f(w^{2n+1}) = (b_1a_1B_{w^{q_1}}B_{w^{p_1}} + b_1\overline{a_2}B_{\overline{w}^{q_2}}B_{w^{p_1}})(w^{2n+1}).$$

Case 1. We consider the case of $p_1 \neq q_1$.

For this case, we will first obtain $a_1b_1 = 0$. Without loss of generality, we assume $p_1<q_1$. It is divided into the following two cases.

Case 1.1. Let $p_1 = q_1 - 1$, then $p_1 - 1<q_1 - 1$. Let $n = q_1 - 1$,

$$\begin{aligned} B_fB_g(w^{2q_1-1}) &= (a_1b_1B_{w^{p_1}}B_{w^{q_1}} + \overline{a_2}b_1B_{\overline{w}^{p_2}}B_{w^{q_1}})(w^{2q_1-1}) \\ &= a_1b_1\sqrt{\frac{q_1+1}{2q_1}}\frac{1}{p_1+1}w^{p_1}, \end{aligned}$$

but

$$B_gB_f(w^{2q_1-1}) = (b_1a_1B_{w^{q_1}}B_{w^{p_1}} + b_1\overline{a_2}B_{\overline{w}^{q_2}}B_{w^{p_1}})(w^{2q_1-1}) = 0,$$

therefore we have $a_1b_1 = 0$.

Case 1.2. Let $p_1<q_1 - 1$, then $p_1 - 1<q_1 - 2$. Let $n = q_1 - 2$,

$$\begin{aligned} B_fB_g(w^{2q_1-3}) &= (a_1b_1B_{w^{p_1}}B_{w^{q_1}} + \overline{a_2}b_1B_{\overline{w}^{p_2}}B_{w^{q_1}})(w^{2q_1-3}) \\ &= a_1b_1\sqrt{\frac{q_1}{2q_1-2}}\frac{2}{q_1+1}\frac{p_1}{p_1+1}w^{p_1-1}, \end{aligned}$$

but

$$B_gB_f(w^{2q_1-3}) = (b_1a_1B_{w^{q_1}}B_{w^{p_1}} + b_1\overline{a_2}B_{\overline{w}^{q_2}}B_{w^{p_1}})(w^{2q_1-3}) = 0,$$

we also get $a_1b_1 = 0$.

The above two cases show that $a_1 = 0$ or $b_1 = 0$ when $p_1 \neq q_1$. Based on this, we will get $\overline{a_2}b_1 = 0$. For this, the argument is also divided into the following two cases.

Case 1.3. Let $a_1 = 0$. Then $f = \overline{a_2 w^{p_2}}$, $g = b_1 w^{q_1} + \overline{b_2 w^{q_2}}$. Choose $n > \max\{q_2, p_2\}$, then

$$\begin{aligned} B_f B_g(w^{2n}) &= B_{\overline{a_2 w^{p_2}}} B_{b_1 w^{q_1} + \overline{b_2 w^{q_2}}}(w^{2n}) \\ &= \sqrt{\frac{n+1}{2n+1}} \overline{a_2} B_{\overline{w^{p_2}}}(b_1 w^{q_1+n} + \overline{b_2} \frac{n-q_2+1}{n+1} w^{n-q_2}), \end{aligned} \tag{3.3}$$

$$\begin{aligned} B_g B_f(w^{2n}) &= B_{b_1 w^{q_1} + \overline{b_2 w^{q_2}}} B_{\overline{a_2 w^{p_2}}}(w^{2n}) \\ &= \sqrt{\frac{n+1}{2n+1}} \overline{a_2} B_{b_1 w^{q_1} + \overline{b_2 w^{q_2}}}(\frac{n-p_2+1}{n+1} w^{n-p_2}). \end{aligned} \tag{3.4}$$

Case 1.3.1. If both q_1 and q_2 are even, choose $n \geq 2(q_1 + p_2 + q_2)$ where n is even, then (3.3) becomes

$$\begin{aligned} &\overline{a_2} \sqrt{\frac{n+1}{2n+1}} \Big(b_1 \sqrt{\frac{\frac{q_1+n}{2}+1}{q_1+n+1}} \frac{\frac{q_1+n}{2} - p_2 + 1}{\frac{q_1+n}{2}+1} w^{\frac{q_1+n}{2} - p_2} \\ &\qquad + \overline{b_2} \frac{n-q_2+1}{n+1} \sqrt{\frac{\frac{n-q_2}{2}+1}{n-q_2+1}} \frac{\frac{n-q_2}{2} - p_2 + 1}{\frac{n-q_2}{2}+1} w^{\frac{n-q_2}{2} - p_2} \Big). \end{aligned} \tag{3.5}$$

Note that, $\frac{q_1+n}{2} - p_2 > \frac{n-q_2}{2} - p_2$.

Now, if p_2 is also even, then for even $n \geq 2(q_1 + p_2 + q_2)$, (3.4) becomes

$$\sqrt{\frac{n+1}{2n+1}} \overline{a_2} \frac{n-p_2+1}{n+1} \sqrt{\frac{\frac{n-p_2}{2}+1}{n-p_2+1}} \Big(b_1 w^{q_1 + \frac{n-p_2}{2}} + \overline{b_2} \frac{\frac{n-p_2}{2} - q_2 + 1}{\frac{n-p_2}{2}+1} w^{\frac{n-p_2}{2} - q_2} \Big). \tag{3.6}$$

Note that, $q_1 + \frac{n-p_2}{2} > \frac{n-p_2}{2} - q_2$.

Comparing with (3.5) and (3.6) and observing the degree of w, we have that: If $\frac{q_1+n}{2} - p_2 = q_1 + \frac{n-p_2}{2}$ and $\frac{n-q_2}{2} - p_2 = \frac{n-p_2}{2} - q_2$, then $p_2 = -q_1$, $p_2 = q_2$. Under this condition, $p_2 = -q_1$, which is a contradiction since p_2, q_2, q_1 are positive integers. Hence, $\overline{a_2} b_1 = 0$, $p_2 = q_2$. If $\frac{q_1+n}{2} - p_2 = \frac{n-p_2}{2} - q_2$ or $q_1 + \frac{n-p_2}{2} = \frac{n-q_2}{2} - p_2$, it means that the coefficient of the biggest degree of w in either (3.6) or (3.5) is zero. Hence, $\overline{a_2} b_1 = 0$.

If p_2 is odd, for even $n \geq 2(q_1 + p_2 + q_2)$, (3.4) equals to zero. Immediately, it follows from (3.5) that $\overline{a_2} b_1 = 0$.

Case 1.3.2. If both q_1 and q_2 are odd, let $n \geq 2(q_1 + p_2 + q_2)$ where n is odd, meaning that both $q_1 - n$ and $q_2 - n$ are even, then (3.5) still holds. If p_2 is odd then $p_2 - n$ is even, (3.6) is still true, with the similar argument as done before, we have $\overline{a_2} b_1 = 0$, $p_2 = q_2$. If p_2 is even, (3.4) becomes zero, then $\overline{a_2} b_1 = 0$ is similarly as above.

Case 1.3.3. If one of q_1 and q_2 is even and another is odd: Assume that q_1 is even, we can choose the even n satisfying $n \geq 2(q_1 + p_2 + q_2)$. For (3.3) equals (3.4),

$$\overline{a_2} b_1 \sqrt{\frac{n+1}{2n+1}} \sqrt{\frac{\frac{q_1+n}{2}+1}{q_1+n+1}} \frac{\frac{q_1+n}{2} - p_2 + 1}{\frac{q_1+n}{2}+1} w^{\frac{q_1+n}{2} - p_2} = \begin{cases} 0, & p_2 \textit{ is odd}, \\ (3.6), & p_2 \textit{ is even}. \end{cases}$$

We see the degree of w in above equation satisfies either $\frac{q_1+n}{2} - p_2 = q_1 + \frac{n-p_2}{2}$ or $\frac{q_1+n}{2} - p_2 = \frac{n-p_2}{2} - q_2$ or $\frac{q_1+n}{2} - p_2 = 0$. Each of these three conditions implies $\overline{a_2} b_1 = 0$. If q_1 is odd, we can choose the odd n satisfying $n \geq 2(q_1 + p_2 + q_2)$. With the similar argument, we also have $\overline{a_2} b_1 = 0$.

In summary, when $a_1 = 0$, we get $\overline{a_2}b_1 = 0$. If $a_1 = \overline{a_2} = 0$, then $f = 0$. If $a_1 = b_1 = 0$, then $B_fB_g = B_gB_f$ becomes $\overline{a_2b_2}B_{\overline{w}^{p_2}}B_{\overline{w}^{q_2}} = \overline{a_2b_2}B_{\overline{w}^{q_2}}B_{\overline{w}^{p_2}}$, by Lemma 2.2, then $\overline{a_2b_2} = 0$ or $p_2 = q_2$.

Case 1.4. Let $b_1 = 0$. By (3.3) and (3.4), using the same arguments as done in Case 1.3, it follows that $g = 0$ or $a_1 = b_1 = 0$, then $\overline{a_2b_2} = 0$ or $p_2 = q_2$.

Case 2. We now turn to consider the case of $p_1 = q_1$.

If $B_fB_g = B_gB_f$, by (3.1) and (3.2), it follows that

$$\begin{aligned}&(a_1\overline{b_2}B_{w^{p_1}}B_{\overline{w}^{q_2}} + \overline{a_2}b_1B_{\overline{w}^{p_2}}B_{w^{p_1}} + \overline{a_2b_2}B_{\overline{w}^{p_2}}B_{\overline{w}^{q_2}})(w^{2n})\\ &\quad= (b_1\overline{a_2}B_{w^{p_1}}B_{\overline{w}^{p_2}} + \overline{b_2}a_1B_{\overline{w}^{q_2}}B_{w^{p_1}} + \overline{b_2a_2}B_{\overline{w}^{p_2}}B_{\overline{w}^{q_2}})(w^{2n}),\end{aligned} \tag{3.7}$$

for any non-negative integers n.

Case 2.1. We now consider the case of $p_2 \neq q_2$. Choose $n = \min\{p_2, q_2\}$. Assume that $p_2<q_2$, then $n = p_2$ and (3.7) becomes

$$\sqrt{\frac{p_2+1}{2p_2+1}}\overline{a_2}b_1B_{\overline{w}^{p_2}}(w^{p_1+p_2}) = \sqrt{\frac{p_2+1}{2p_2+1}}\Big(\overline{a_2}b_1\frac{1}{p_2+1}w^{p_1} + \overline{b_2}a_1B_{\overline{w}^{q_2}}(w^{p_1+p_2})\Big). \tag{3.8}$$

For the left side of (3.8), the degree of w is either $\frac{p_1-p_2}{2}$ or zero. For the right side of (3.8), the degree of w of $B_{\overline{w}^{q_2}}(w^{p_1+p_2})$ is either $\frac{p_1+p_2}{2} - q_2$ or zero. As we assume, $p_2 < q_2$ and p_1, p_2, q_2 are positive integers, thus $p_1 \neq \frac{p_1-p_2}{2}$ and $p_1 \neq \frac{p_1+p_2}{2} - q_2$. Hence, the coefficient of w^{p_1} is zero, that is, $\overline{a_2}b_1 = 0$, which yields $a_2 = 0$ or $b_1 = 0$.

Case 2.1.1. If $a_2 = 0$, then $f = a_1w^{p_1}$, $g = b_1w^{q_1} + \overline{b_2w^{q_2}}$. (3.7) then becomes

$$a_1\overline{b_2}B_{w^{p_1}}B_{\overline{w}^{q_2}}(w^{2n}) = \overline{b_2}a_1B_{\overline{w}^{q_2}}B_{w^{p_1}}(w^{2n}).$$

By Lemma 2.4, $a_1\overline{b_2} = 0$. It follows that $f = 0$ or $a_2 = b_2 = 0$, which is $f = a_1w^{p_1}$, $g = b_1w^{p_1}$.

Case 2.1.2. If $b_1 = 0$, then $f = a_1w^{p_1} + \overline{a_2w^{p_2}}$, $g = \overline{b_2w^{q_2}}$. (3.7) then becomes

$$a_1\overline{b_2}B_{w^{p_1}}B_{\overline{w}^{q_2}}(w^{2n}) + \overline{a_2b_2}B_{\overline{w}^{p_2}}B_{\overline{w}^{q_2}}(w^{2n}) = \overline{b_2}a_1B_{\overline{w}^{q_2}}B_{w^{p_1}}(w^{2n}) + \overline{b_2a_2}B_{\overline{w}^{p_2}}B_{\overline{w}^{q_2}}(w^{2n}).$$

For $p_2 < q_2$, set $n > 2q_2 + p_2 + 1$. From the above equation, it follows that

$$\begin{aligned}&\frac{n-q_2+1}{n+1}(a_1\overline{b_2}B_{w^{p_1}}(w^{n-q_2}) + \overline{a_2b_2}B_{\overline{w}^{p_2}}(w^{n-q_2}))\\ &\quad= \Big(\overline{b_2}a_1B_{\overline{w}^{q_2}}(w^{p_1+n}) + \overline{b_2a_2}\frac{n-p_2+1}{n+1}B_{\overline{w}^{q_2}}(w^{n-p_2})\Big).\end{aligned} \tag{3.9}$$

Now we choose n such that $n - q_2$ is even.Then the left side of (3.9) becomes

$$\frac{n-q_2+1}{n+1}\sqrt{\frac{\frac{n-q_2}{2}+1}{n-q_2+1}}\Big(a_1\overline{b_2}w^{p_1+\frac{n-q_2}{2}} + \overline{a_2b_2}\frac{\frac{n-q_2}{2}-p_2+1}{\frac{n-q_2}{2}+1}w^{\frac{n-q_2}{2}-p_2}\Big). \tag{3.10}$$

If $p_1 + n$ and $n - p_2$ are even, the right of (3.9) becomes

$$\begin{aligned}&a_1\overline{b_2}\sqrt{\frac{\frac{n+p_1}{2}+1}{n+p_1+1}}\frac{\frac{n+p_1}{2}-q_2+1}{\frac{n+p_1}{2}+1}w^{\frac{n+p_1}{2}-q_2}\\ &\quad+\overline{a_2b_2}\frac{n-p_2+1}{n+1}\sqrt{\frac{\frac{n-p_2}{2}+1}{n-p_2+1}}\frac{\frac{n-p_2}{2}-q_2+1}{\frac{n-p_2}{2}+1}w^{\frac{n-p_2}{2}-q_2}.\end{aligned} \tag{3.11}$$

Looking the degree of w, it follows that $p_1 + \frac{n-q_2}{2} > \frac{n-q_2}{2} - p_2$ in (3.10) and $\frac{n+p_1}{2} - q_2 > \frac{n-p_2}{2} - q_2$ in (3.11).

Comparing (3.10) with (3.11), if $p_1 + \frac{n-q_2}{2} = \frac{n+p_1}{2} - q_2$ and $\frac{n-q_2}{2} - p_2 = \frac{n-p_2}{2} - q_2$, then $p_1 = -q_2$, $q_2 = p_2$. These both contradict that p_2, q_2, p_1 are positive integers and $p_2 \neq q_2$. So, we have $a_1\overline{b_2} = 0$. If $p_1 + \frac{n-q_2}{2} = \frac{n-p_2}{2} - q_2$ or $\frac{n-q_2}{2} - p_2 = \frac{n+p_1}{2} - q_2$, then either the last term of (3.11) or the last term of (3.10) is zero. This also gives $a_1\overline{b_2} = 0$.

If at least one of $p_1 + n$ and $n - p_2$ is odd, then (3.11) becomes

$$\overline{a_2 b_2}\frac{n-p_2+1}{n+1}\sqrt{\frac{\frac{n-p_2}{2}+1}{n-p_2+1}}\frac{\frac{n-p_2}{2}-q_2+1}{\frac{n-p_2}{2}+1}w^{\frac{n-p_2}{2}-q_2}$$

or

$$a_1\overline{b_2}\sqrt{\frac{\frac{n+p_1}{2}+1}{n+p_1+1}}\frac{\frac{n+p_1}{2}-q_2+1}{\frac{n+p_1}{2}+1}w^{\frac{n+p_1}{2}-q_2}.$$

Applying similar and easier arguments as done above, we also get that $a_1\overline{b_2} = 0$.

To sum up, when $b_1 = 0$, we get $a_1\overline{b_2} = 0$. If $b_1 = \overline{b_2} = 0$, then $g = 0$. If $a_1 = b_1 = 0$, then by Lemma 2.2, $\overline{a_2 b_2} = 0$ or $p_2 = q_2$.

Case 2.2. We consider the case when $p_1 = q_1$, $p_2 = q_2$, that is $f = a_1 w^{p_1} + \overline{a_2 w^{p_2}}$, $g = b_1 w^{p_1} + \overline{b_2 w^{p_2}}$. By (3.1) and (3.2), this implies that

$$(a_1\overline{b_2} - b_1\overline{a_2})(B_{w^{p_1}}B_{\overline{w}^{p_2}} - B_{\overline{w}^{p_2}}B_{w^{p_1}}) = 0.$$

By Lemmas 2.3 and 2.4, $B_{w^{p_1}}B_{\overline{w}^{p_2}} - B_{\overline{w}^{p_2}}B_{w^{p_1}} \neq 0$. We then obtain $a_1\overline{b_2} - b_1\overline{a_2} = 0$. The proof is complete. □

Now we turn to more general cases. Before we consider the case where one symbol is an analytic monomial and another is a harmonic function, we need the following two lemmas.

Lemma 3.1. *Suppose f is a bounded harmonic function on $\mathbb{D}$. Then, $B_f B_1 = B_1 B_f$ if and only if f is a constant.*

Proof. Write $f = f_+ + \overline{f_-}$ where $f_+ = \sum_{i=1}^{\infty} a_i w^i$ and $f_- = \sum_{j=0}^{\infty} b_j w^j$ are analytic functions. By Lemma 2.1, $B_1(w) = 0$, implying that $B_1 B_f(w) = 0$. Again, by Lemma 2.1

$$B_1 B_f(w) = \sum_{i=0}^{\infty} a_{2i+1}\frac{2i+1}{2i+2}\sqrt{\frac{i+1}{2i+1}}w^i = 0.$$

So $a_{2i+1} = 0$, $i \geq 0$. By using $B_f B_1(1) = B_1 B_f(1)$, we get the following equation

$$\sum_{i=1}^{\infty} a_{2i} w^{2i} = \sum_{i=1}^{\infty} a_{2i}\sqrt{\frac{i+1}{2i+1}}w^i.$$

Comparing the lowest degree of w, it is easy to see that $a_{2i} = 0$ for all $i \geq 1$. Hence, $f_+ = 0$.

Now we turn to the co-analytic part of f. By

$$B_{\overline{f_-}}B_1(w^{4n+2}) = \sqrt{\frac{2n+2}{4n+3}}B_{\overline{f_-}}(w^{2n+1}) = 0,$$

we have

$$B_1 B_{\overline{f_-}}(w^{4n+2}) = \sum_{j=0}^{n} \overline{b_{2j+1}}\frac{2n-j+1}{2n+2}\sqrt{\frac{n-j+1}{2n-2j+1}}w^{n-j} = 0.$$

Hence, $b_{2j+1} = 0$, $j = 0, 1, 2\dots, n$. Because n is any non-negative integer, then $b_{2j+1} = 0$ for all $j \geq 0$. Also, for any positive integer n, by $B_{\overline{f_-}}B_1(w^{4n}) = B_1 B_{\overline{f_-}}(w^{4n})$, we get the following equation

$$\sqrt{\frac{n+1}{2n+1}}\sum_{j=1}^{n}\overline{b_{2j}}\frac{2n-j+1}{2n+1}w^{n-2j} = \sum_{j=1}^{n}\overline{b_{2j}}\frac{2n-2j+1}{2n+1}\sqrt{\frac{n-j+1}{2n-2j+1}}w^{n-j}.$$

Comparing the highest degree of w, we easily obtain that $b_{2j} = 0$, $j = 1, 2, 3, \dots, n$. Because n is any positive integer, then $b_{2j} = 0$, $j > 0$. Hence, $f_- = b_0$ is a constant. Therefore, f is a constant. □

Lemma 3.2. *Suppose g is a bounded analytic function on $\mathbb{D}$ with $g(0) = 0$. Let p be a non-negative integer, then $B_{w^p}B_{\overline{g}} = B_{\overline{g}}B_{w^p}$ if and only if $g = 0$.*

Proof. The sufficiency is obvious, so now we prove the necessity. If $p = 0$, by Lemma 3.1, $g = 0$. Now we consider the case when $p \geq 1$. Suppose that $B_{w^p}B_{\overline{g}} = B_{\overline{g}}B_{w^p}$, and write g as $\sum_{j=1}^{\infty} b_j w^j$.
Case 1. p is even. Then, for any non-negative integer n, $p + 2n + 1$ is odd. By Lemma 2.1, $B_{\overline{g}}B_{w^p}(w^{4n+2}) = \sqrt{\frac{2n+2}{4n+3}}B_{\overline{g}}(w^{p+2n+1}) = 0$. Since

$$B_{w^p}B_{\overline{g}}(w^{4n+2}) = \sqrt{\frac{2n+2}{4n+3}}B_{w^p}\Big(\sum_{j=0}^{n}\overline{b_{2j+1}}\frac{2n-2j+1}{2n+1}w^{2n-2j} + \sum_{j=0}^{n}\overline{b_{2j}}\frac{n-j+1}{n+1}w^{2n-2j+1}\Big),$$

hence

$$\begin{aligned} B_{w^p}B_{\overline{g}}(w^{4n+2}) = \sqrt{\frac{2n+2}{4n+3}}\Big[&\sum_{j=0}^{n}\overline{b_{2j+1}}\frac{2n-2j+1}{2n+1}\sqrt{\frac{n-j+1}{2n-2j+1}}w^{n-j+p} \\ &+ P\Big(w^p\sum_{j=0}^{n}\overline{b_{2j}}\frac{n-j+1}{n+1}\sqrt{\frac{n-j+2}{2n-2j+2}}\overline{w}^{n-j+1}\Big)\Big] = 0. \end{aligned} \tag{3.12}$$

The degree of w in the first term of above equation is from p to $n + p$, while in the second term, it is not greater than $p - 1$. So immediately we get $b_{2j+1} = 0$, $j = 0, 1, 2, ..., n$.

Because n is any non-negative integer, $b_{2j+1}=0$ for any integer $j\geq 0$. Now again, by (3.12),

$$
\begin{aligned}
0 &= P\Big(w^p\sum_{j=0}^{n}\overline{b_{2j}}\frac{n-j+1}{n+1}\sqrt{\frac{n-j+2}{2n-2j+2}}\overline{w}^{n-j+1}\Big)\\
&= \begin{cases} \displaystyle\sum_{j=0}^{n-1}\overline{b_{2j}}\frac{n-j+1}{n+1}\sqrt{\frac{n-j+2}{2n-2j+2}}\frac{p-n+j}{p+1}w^{p-n+j-1}, & n\leq p-1,\\ \displaystyle\sum_{j=n-p}^{n-1}\overline{b_{2j}}\frac{n-j+1}{n+1}\sqrt{\frac{n-j+2}{2n-2j+2}}\frac{p-n+j}{p+1}w^{p-n+j-1}, & n>p-1. \end{cases}
\end{aligned}
$$

Since n is any non-negative integer, by the above, it is easy to see that $b_{2j}=0$ for $j\geq 0$. Hence, $b_j=0$ for any $j\geq 0$, we get that $g=0$.

Case 2. p is odd. Then, for any non-negative integer n, we see that $p+2n$ is odd. Thus, with similar arguments as Case 1 when applying $B_{w^p}B_{\overline{g}}(w^{4n})=B_{\overline{g}}B_{w^p}(w^{4n})$, one may get the desired conclusion. □

We are now ready to prove the commutativity of two H-Toeplitz operators with one symbol being an analytic monomial and another being a harmonic one. For the general case, it seems very hard.

Proof of Theorem 1.2. The sufficiency is obvious, so now we prove the necessity. If $p=0$, by Lemma 3.1, f is constant. We only need to prove the case when p is a positive integer. Write $f=f_+ + \overline{f_-}$ and assume that $f_+=\sum_{i=0}^{\infty}a_iw^i$. Suppose $B_{w^p}B_f=B_fB_{w^p}$, then

$$B_{w^p}B_{f_+}+B_{w^p}B_{\overline{f_-}}=B_{f_+}B_{w^p}+B_{\overline{f_-}}B_{w^p}. \tag{3.13}$$

For non-negative integer n, by Lemma 2.1 we see that $B_{w^p}B_{\overline{f_-}}(w^{2n+1})=0$. When $n\geq p$, we have $B_{w^p}(w^{2n+1})=0$, thus (3.13) implies that $B_{w^p}B_{f_+}(w^{2n+1})=0$ when $n\geq p$.

Case 1. p is even. Let $n=p$, then a direct computation gives that

$$
\begin{aligned}
0 &= B_{w^p}B_{f_+}(w^{2p+1})\\
&= \sqrt{\frac{p+2}{2p+2}}B_{w^p}\Big(\sum_{i=\frac{p}{2}}^{\infty}a_{2i+1}\frac{2i-p+1}{2i+2}w^{2i-p}+\sum_{i=\frac{p}{2}+1}^{\infty}a_{2i}\frac{2i-p}{2i+1}w^{2i-p-1}\Big)\\
&= \sqrt{\frac{p+2}{2p+2}}\Big(\sum_{i=\frac{p}{2}}^{\infty}a_{2i+1}\frac{2i-p+1}{2i+2}\sqrt{\frac{i-\frac{p}{2}+1}{2i-p+1}}w^{i+\frac{p}{2}}\\
&\quad+\sum_{i=\frac{p}{2}+1}^{\frac{3p}{2}}a_{2i}\frac{2i-p}{2i+1}\sqrt{\frac{i-\frac{p}{2}+1}{2i-p}}\frac{\frac{3p}{2}-i+1}{p+1}w^{\frac{3p}{2}-i}\Big).
\end{aligned}
\tag{3.14}
$$

By (3.14), we get $a_{2i+1}=0$ for any $i\geq\frac{p}{2}$, which implies $a_{p+1}=a_{p+3}=\cdots=0$. Let $n=p+1$, then a direct computation gives that

$$
\begin{aligned}
0 &= B_{w^p}B_{f_+}(w^{2p+3})\\
&= \sqrt{\frac{p+3}{2p+4}}B_{w^p}\Big(\sum_{i=\frac{p}{2}+1}^{\infty} a_{2i}\frac{2i-p-1}{2i+1}w^{2i-p-2} + \sum_{i=\frac{p}{2}+1}^{\infty} a_{2i+1}\frac{2i-p}{2i+2}w^{2i-p-1}\Big)\\
&= \sqrt{\frac{p+3}{2p+4}}\Big(\sum_{i=\frac{p}{2}+1}^{\infty} a_{2i}\frac{2i-p-1}{2i+1}\sqrt{\frac{i-\frac{p}{2}}{2i-p-1}}w^{\frac{p}{2}+i-1}\\
&\quad + \sum_{i=\frac{p}{2}+1}^{\frac{3p}{2}} a_{2i+1}\frac{i-\frac{p}{2}}{i+1}\sqrt{\frac{i-\frac{p}{2}+1}{2i-p}}\frac{\frac{3p}{2}-i+1}{p+1}w^{\frac{3p}{2}-i}\Big).
\end{aligned}
\tag{3.15}
$$

By (3.15), we get $a_{2i} = 0$ for any $i \geq \frac{p}{2}+1$, which implies $a_{p+2} = a_{p+4} = \cdots = 0$. Thus, $f_+ = \sum_{i=0}^{p} a_i w^i$. Next, we will show $a_0, ..., a_{p-1}$ are zero.

Again by Lemma 2.1, $B_{w^p}B_{\overline{f_-}}(w) = B_{\overline{f_-}}B_{w^p}(w) = 0$, and by (3.13) we obtain the following two equations which are equal:

$$
\begin{aligned}
B_{w^p}B_{f_+}(w) &= B_{w^p}\Big(\sum_{i=1}^{\frac{p}{2}} a_{2i}\frac{2i}{2i+1}w^{2i-1} + \sum_{i=0}^{\frac{p}{2}-1} a_{2i+1}\frac{2i+1}{2i+2}w^{2i}\Big)\\
&= \sum_{i=1}^{\frac{p}{2}} a_{2i}\frac{2i}{2i+1}\sqrt{\frac{i+1}{2i}}\frac{p-i+1}{p+1}w^{p-i} + \sum_{i=0}^{\frac{p}{2}-1} a_{2i+1}\frac{2i+1}{2i+2}\sqrt{\frac{i+1}{2i+1}}w^{i+p},
\end{aligned}
\tag{3.16}
$$

$$
B_{f_+}B_{w^p}(w) = \frac{p}{p+1}\sqrt{\frac{\frac{p}{2}+1}{p}}\sum_{i=\frac{p}{2}}^{p} a_i\frac{i-\frac{p}{2}+1}{i+1}w^{i-\frac{p}{2}}. \tag{3.17}
$$

Note that the degree of w of the first term in (3.16) is from $\frac{p}{2}$ to $p-1$, and the second term is from p to $\frac{3p}{2}-1$. But in (3.17), the degree of w is from 0 to $\frac{p}{2}$. This means that in (3.16), the second term is zero, and in the first term, the coefficients of w^{p-i} are zero except $w^{\frac{p}{2}}$. This implies that $a_0 = a_1 = \cdots = a_{p-1} = 0$. Hence, $a_i = 0$ except a_p. At this time, both (3.16) and (3.17) are equal to $\sqrt{\frac{\frac{p}{2}+1}{p}}\frac{p}{p+1}a_p w^{\frac{p}{2}}$. Hence, $f_+ = cw^p$ with $c = a_p$.

Case 2. p is odd. Let $n = p$, then a direct computation gives that

$$
\begin{aligned}
0 &= B_{w^p}B_{f_+}(w^{2p+1})\\
&= \sqrt{\frac{p+2}{2p+2}}B_{w^p}\Big(\sum_{i=\frac{p+1}{2}}^{\infty} a_{2i+1}\frac{2i-p+1}{2i+2}w^{2i-p} + \sum_{i=\frac{p+1}{2}}^{\infty} a_{2i}\frac{2i-p}{2i+1}w^{2i-p-1}\Big)\\
&= \sqrt{\frac{p+2}{2p+2}}\Big(\sum_{i=\frac{p+1}{2}}^{\frac{3p-1}{2}} a_{2i+1}\frac{2i-p+1}{2i+2}\sqrt{\frac{i-\frac{p-1}{2}+1}{2i-p+1}}\frac{\frac{3p-1}{2}-i+1}{p+1}w^{\frac{3p-1}{2}-i}\\
&\quad + \sum_{i=\frac{p+1}{2}}^{\infty} a_{2i}\frac{2i-p}{2i+1}\sqrt{\frac{i-\frac{p+1}{2}+1}{2i-p}}w^{i+\frac{p-1}{2}}\Big).
\end{aligned}
\tag{3.18}
$$

Let $n = p + 1$, then we obtain

$$
\begin{aligned}
0 &= B_{w^p} B_{f_+}(w^{2p+3}) \\
&= \sqrt{\frac{p+3}{2p+4}} B_{w^p}\Big(\sum_{i=\frac{p+1}{2}}^{\infty} a_{2i+1} \frac{2i-p}{2i+2} w^{2i-p-1} + \sum_{i=\frac{p+1}{2}+1}^{\infty} a_{2i} \frac{2i-p-1}{2i+1} w^{2i-p-2} \Big) \\
&= \sqrt{\frac{p+3}{2p+4}} \Big(\sum_{i=\frac{p+1}{2}}^{\infty} a_{2i+1} \frac{2i-p}{2i+2} \sqrt{\frac{i-\frac{p+1}{2}+1}{2i-p}} w^{i+\frac{p-1}{2}} \\
&+ \sum_{i=\frac{p+1}{2}+1}^{\frac{3p+1}{2}} a_{2i} \frac{2i-p-1}{2i+1} \sqrt{\frac{i-\frac{p+1}{2}+1}{2i-p-1}} \frac{\frac{3p+1}{2}-i+1}{p+1} w^{\frac{3p+1}{2}-i} \Big).
\end{aligned} \tag{3.19}
$$

By (3.18) and (3.19), with similar arguments as done in Case 1, we get $f_+ = \sum_{i=0}^{p} a_i w^i$. On the other hand, by Lemma 2.1, $B_{w^p} B_{\overline{f_-}}(1) = B_{\overline{f_-}} B_{w^p}(1) = 0$, and by (3.13), we see that $B_{w^p} B_{f_+}(1) = B_{f_+} B_{w^p}(1)$. A direct computation gives that

$$
\begin{aligned}
B_{w^p} B_{f_+}(1) &= B_{w^p}\Big(\sum_{i=0}^{\frac{p-1}{2}} a_{2i} w^{2i} + \sum_{i=0}^{\frac{p-1}{2}} a_{2i+1} w^{2i+1} \Big) \\
&= \sum_{i=0}^{\frac{p-1}{2}} a_{2i} \sqrt{\frac{i+1}{2i+1}} w^{i+p} + \sum_{i=0}^{\frac{p-1}{2}} a_{2i+1} \sqrt{\frac{i+2}{2i+2}} \frac{p-i}{p+1} w^{p-1-i},
\end{aligned} \tag{3.20}
$$

$$
B_{f_+} B_{w^p}(1) = \sqrt{\frac{\frac{p+1}{2}+1}{p+1}} \sum_{i=\frac{p+1}{2}}^{p} a_i \frac{i-\frac{p+1}{2}+1}{i+1} w^{i-\frac{p+1}{2}}. \tag{3.21}
$$

By (3.20) and (3.21) and applying similar arguments as done in Case 1, we obtain $f_+ = cw^p$, where $c = a_p$.

In summary, since $f_+ = cw^p$, $B_f B_{w^p} = B_{w^p} B_f$ gives that $B_{w^p} B_{\overline{f_-}} = B_{\overline{f_-}} B_{w^p}$, which means $f_- = 0$ by Lemma 3.2. The proof is complete. □

Now we shall prove the result about commuting H-Toeplitz operators where one symbol is a co-analytic monomial and another is an analytic function.

Theorem 3.3. *Suppose f is a bounded analytic function on $\mathbb{D}$. Let q be a positive integer, then $B_{\overline{w}^q} B_f = B_f B_{\overline{w}^q}$ if and only if $f = 0$.*

Proof. The sufficiency is obvious, so now we only need to prove the necessity. Suppose $B_{\overline{w}^q} B_f = B_f B_{\overline{w}^q}$ and write f as $\sum_{i=0}^{\infty} a_i w^i$. By Lemma 2.1, $B_f B_{\overline{w}^q}(1) = B_f B_{\overline{w}^q}(w) = 0$, so we have

$$
B_{\overline{w}^q} B_f(1) = \sum_{i=q}^{\infty} a_{2i} \sqrt{\frac{i+1}{2i+1}} \frac{i-q+1}{i+1} w^{i-q} = 0,
$$

$$B_{\overline{w}^q}B_f(w)=\sum_{i=q}^{\infty}a_{2i+1}\frac{2i+1}{2i+2}\sqrt{\frac{i+1}{2i+1}}\frac{i-q+1}{i+1}w^{i-q}=0.$$

It follows that $a_i=0$ for any $i\geq 2q$. It remains to be shown that $a_i=0$, $i=1,2,\cdots,2q-1$. Again by Lemma 2.1 ,

$$\begin{aligned}B_fB_{\overline{w}^q}(w^{6q})&=\sqrt{\frac{3q+1}{6q+1}}\frac{2q+1}{3q+1}B_f(w^{2q})\\&=\sqrt{\frac{3q+1}{6q+1}}\frac{2q+1}{3q+1}\sqrt{\frac{q+1}{2q+1}}\sum_{i=0}^{2q-1}a_iw^{i+q},\end{aligned}\tag{3.22}$$

$$\begin{aligned}B_{\overline{w}^q}B_f(w^{6q})&=\sqrt{\frac{3q+1}{6q+1}}B_{\overline{w}^q}(\sum_{i=0}^{q-1}a_{2i+1}w^{2i+1+3q}+\sum_{i=0}^{q-1}a_{2i}w^{2i+3q})\\&=\begin{cases}\sqrt{\frac{3q+1}{6q+1}}\sum_{i=0}^{p-1}a_{2i}\sqrt{\frac{i+\frac{3q}{2}+1}{2i+3q+1}}\frac{i+\frac{q}{2}+1}{i+\frac{3q}{2}+1}w^{i+\frac{q}{2}}, & q\ is\ even,\\ \sqrt{\frac{3q+1}{6q+1}}\sum_{i=0}^{q-1}a_{2i+1}\sqrt{\frac{i+\frac{3q+1}{2}+1}{2i+3q+2}}\frac{i+\frac{q+1}{2}+1}{i+\frac{3q+1}{2}+1}w^{i+\frac{q+1}{2}}, & q\ is\ odd.\end{cases}\end{aligned}\tag{3.23}$$

Case 1. q is even, so $p\geq 2$. Note that (3.22) equals (3.23), so the following equation holds

$$\frac{2q+1}{3q+1}\sqrt{\frac{q+1}{2q+1}}\sum_{i=0}^{2q-1}a_iw^{i+q}=\sum_{i=0}^{q-1}a_{2i}\sqrt{\frac{i+\frac{3q}{2}+1}{2i+3q+1}}\frac{i+\frac{q}{2}+1}{i+\frac{3q}{2}+1}w^{i+\frac{q}{2}}.\tag{3.24}$$

The degree of w in the left side is from q to $3q-1$ and the right is from $\frac{q}{2}$ to $\frac{3q}{2}-1$, so we get $a_0=0$ immediately. Since $3q-1>\frac{3q}{2}-1$, on the left side of (3.24), the coefficients of w^t from $t=\frac{3q}{2}$ to $3q-1$ are zero, meaning $a_{\frac{q}{2}}=\cdots=a_{2q-1}=0$. If $q=2$, the desired conclusion holds. If $q\geq 4$, substitute $a_0=0$ and $a_{\frac{q}{2}}=\cdots=a_{2q-1}=0$ into (3.24), notice that $a_q=0$, then we get

$$\frac{2q+1}{3q+1}\sqrt{\frac{q+1}{2q+1}}\sum_{i=1}^{\frac{q}{2}-1}a_iw^{i+q}=\sum_{i=1}^{\frac{q}{2}-1}a_{2i}\sqrt{\frac{i+\frac{3q}{2}+1}{2i+3q+1}}\frac{i+\frac{q}{2}+1}{i+\frac{3q}{2}+1}w^{i+\frac{q}{2}}.\tag{3.25}$$

The degree of w on the left side is from $q+1$ to $\frac{3q}{2}-1$, and the right is from $\frac{q}{2}+1$ to $q-1$, but $\frac{q}{2}+1<q-1<q+1\leq\frac{3q}{2}-1$, thus on the left side of (3.25), $a_1=\cdots=a_{\frac{q}{2}-1}=0$, which is the desired conclusion.

Case 2. p is odd. Because (3.22) equals (3.23), the following equation holds:

$$\frac{2q+1}{3q+1}\sqrt{\frac{q+1}{2q+1}}\sum_{i=0}^{2q-1}a_iw^{i+q}=\sum_{i=0}^{q-1}a_{2i+1}\sqrt{\frac{i+\frac{3q+1}{2}+1}{2i+3q+2}}\frac{i+\frac{q+1}{2}+1}{i+\frac{3q+1}{2}+1}w^{i+\frac{q+1}{2}}.$$

If $q=1$, the above equation becomes $\frac{3}{4}\sqrt{\frac{2}{3}}(a_0w+a_1w^2)=\frac{2}{3}\sqrt{\frac{3}{5}}a_1w$, then $a_0=a_1=0$. If $q\geq 3$, from the above equation we immediately get $a_0=0$ and applying the same arguments as done when q is even, we get the desired result.

Therefore, $a_i=0$ for any $i\geq 0$, that is, $f=0$. The proof is complete. □

4. The proof of Theorem 1.3

The aim of this section is to find the necessary and sufficient conditions of the commutativity of H-Toeplitz and Toeplitz operators with non-harmonic symbols.

To discuss one of our main results, we will use the Mellin transform $\widehat{\varphi}$ of the function $\varphi \in L^1([0,1], rdr)$, which is defined by

$$\widehat{\varphi}(w) = \int_0^1 \varphi(r) r^{w-1} dr.$$

It is known that φ is analytic on $\{w : \operatorname{Re} w > 2\}$. The following lemmas has been proved in [8], which will be used in the following.

Lemma 4.1. *Let $\varphi \in L^1([0,1], rdr)$. If there exist a sequence of positive integers n_k such that $\sum\limits_{k=1}^{\infty} \frac{1}{n_k} = \infty$ and $\widehat{\varphi}(n_k) = 0$ for all k, then $\varphi = 0$.*

Let $\mathcal{R}$ be the space of square integrable functions on $[0,1]$ with respect to the measure rdr. It is clear that the functions in $\mathcal{R}$ are radial functions on $\mathbb{D}$. Since trigonometric polynomials are dense in L^2 and $e^{ik_1\theta}\mathcal{R}$ is orthogonal to $e^{ik_2\theta}\mathcal{R}$ for $k_1 \neq k_2$, one can see

$$L^2 = \bigoplus_{k\in\mathbb{Z}} e^{ik\theta}\mathcal{R}.$$

So, for each $f \in L^2(\mathbb{D})$, it can be written as (see [11])

$$f(re^{i\theta}) = \sum_{k\in\mathbb{Z}} e^{ik\theta}\varphi_k(r),$$

where each $\varphi_k \in \mathcal{R}$ is a bounded radial function when $f \in L^\infty(\mathbb{D})$. Each function in $e^{ik\theta}\mathcal{R}$ is called a quasihomogeneous function with degree k.

Lemma 4.2. *Let ϕ be a bounded radial function and p an integer. For any non-negative integer n,*

$$B_{e^{ip\theta}\phi}(w^{2n}) = \begin{cases} 2\sqrt{\dfrac{n+1}{2n+1}}(n+p+1)\widehat{\phi}(2n+p+2)w^{n+p}, & n+p \geq 0, \\ 0 & , \ n+p < 0, \end{cases}$$

$$B_{e^{ip\theta}\phi}(w^{2n+1}) = \begin{cases} 2\sqrt{\dfrac{n+2}{2n+2}}(p-n)\widehat{\phi}(p+2)w^{p-n-1}, & n+1 \leq p, \\ 0 & , \ n+1 > p, \end{cases}$$

By Lemma 4.2, we obtain the following lemmas immediately.

Lemma 4.3. *Suppose $f = \sum_{k=0}^{\infty} e^{is\theta}\varphi_k(r) \in L^\infty(\mathbb{D})$ where each φ_k is a radial function. For any non-negative integer n,*

$$B_f(w^{2n}) = 2\sum_{k=0}^{\infty} \sqrt{\frac{n+1}{2n+1}}(n+k+1)\widehat{\varphi}_k(k+2n+2)w^{n+k}.$$

The following lemma is proved in [5] which will be used later.

Lemma 4.4. *Let p be an integer and ψ be a bounded radial function on $\mathbb{D}$. Then, for any non-negative integer n,*

$$T_{e^{ip\theta}\psi}(w^n) = \begin{cases} 2(n+p+1)\widehat{\psi}(2n+p+2)w^{n+p}, & n+p \geq 0, \\ 0, & n+p < 0. \end{cases}$$

In order to prove Theorem 1.3, we first give the following lemma.

Lemma 4.5. *Let $\phi \neq 0$ be a bounded radial function and p an integer. Write*

$$f = \sum_{k\in\mathbb{Z}} e^{ik\theta}\varphi_k(r) \in L^\infty(\mathbb{D}),$$

where each φ_k is a bounded radial function. If $T_f B_{e^{ip\theta}\phi} = B_{e^{ip\theta}\phi} T_f$, then $\varphi_{2k+1} = 0$ for each k.

Proof. Choose n satisfying $n > |p|$, which means $n+1 > p$. By Lemma 4.2, $B_{e^{ip\theta}\phi}(w^{2n+1}) = 0$, and by Lemma 4.4,

$$\begin{aligned} 0 = B_{e^{ip\theta}\phi}T_f(w^{2n+1}) &= 2B_{e^{ip\theta}\phi}\Big(\sum_{k=-2n-1}^{\infty}(2n+2+k)\widehat{\varphi}_k(4n+4+k)w^{k+2n+1}\Big) \\ &= 2B_{e^{ip\theta}\phi}\Big(\sum_{k=-n-1}^{\infty}(2n+3+2k)\widehat{\varphi}_{2k+1}(4n+5+k)w^{2k+2n+2} \\ &\quad + \sum_{k=-n}^{\infty}(2n+2+2k)\widehat{\varphi}_{2k}(4n+4+2k)w^{2k+2n+1}\Big). \end{aligned} \tag{4.1}$$

We will show $\varphi_{2k+1} = 0$ for each integer k in the following two cases.

Case 1. $p \leq 0$. In this case, (4.1) becomes

$$\begin{aligned} 0 = \sum_{k=-n-1-p}^{\infty}&(2n+3+2k)\widehat{\varphi}_{2k+1}(4n+5+k)\sqrt{\frac{k+n+2}{2k+2n+3}} \\ &\times (k+n+2+p)\widehat{\phi}(2k+2n+3+p)w^{k+n+1+p}, \end{aligned}$$

then

$$\widehat{\varphi}_{2k+1}(4n+5+k)\widehat{\phi}(2k+2n+3+p) = 0$$

holds, where $k > -n-1-p$. Set

$$E_k = \Big\{n \geq |p| : \quad \widehat{\phi}(2k+2n+3+p) \neq 0\Big\},$$

As we assume $\phi \neq 0$, by Lemma 4.1, $\sum_{n\in E_k}\frac{1}{n} = \infty$. For each k, choose $n \in E_k$ then $\widehat{\varphi}_{2k+1}(4n+5+k) = 0$ with $\sum_{n\in E_k}\frac{1}{4n+5+k} = \infty$. By Lemma 4.1, we get $\varphi_{2k+1} = 0$, where $k > -n-1-p$. For any integer n with $n \geq |p|$, we get $\varphi_{2k+1} = 0$ for each integer k.

Case 2. $p > 0$. So (4.1) becomes

$$0 = \sum_{k=-n-1}^{\infty}(2n+3+2k)\widehat{\varphi}_{2k+1}(4n+5+k)\sqrt{\frac{k+n+2}{2k+2n+3}}$$

$$\times (k+n+2+p)\widehat{\phi}(2k+2n+3+p)w^{k+n+1+p},$$

and then

$$\widehat{\varphi}_{2k+1}(4n+5+k)\widehat{\phi}(2k+2n+3+p) = 0$$

holds, where $k > -n-1$. With the similar arguments as done in Case 1, we can obtain $\varphi_{2k+1} = 0$ for each integer k. □

Lemma 4.6. *Let ϕ and φ be bounded radial functions and p be a non-negative integer. $T_\varphi B_{e^{ip\theta}\phi} = B_{e^{ip\theta}\phi}T_\varphi$ holds if and only if $\phi = 0$ or φ is a constant.*

Proof. For any non-negative integer n, by Lemmas 4.2 and 4.4,

$$\begin{aligned} T_\varphi B_{e^{ip\theta}\phi}(w^{2n}) &= 2\sqrt{\frac{n+1}{2n+1}}(n+p+1)\widehat{\phi}(2n+p+2)T_\varphi(w^{n+p}) \\ &= 4\sqrt{\frac{n+1}{2n+1}}(n+p+1)\widehat{\phi}(2n+p+2)(n+p+1)\widehat{\varphi}(2n+2p+2)w^{n+p}, \end{aligned}$$

$$\begin{aligned} B_{e^{ip\theta}\phi}T_\varphi(w^{2n}) &= 2(2n+1)\widehat{\varphi}(4n+2)B_{e^{ip\theta}\phi}(w^{2n}) \\ &= 4\sqrt{\frac{n+1}{2n+1}}(2n+1)\widehat{\varphi}(4n+2)(n+p+1)\widehat{\phi}(2n+p+2)w^{n+p}. \end{aligned}$$

Hence, we get that, for any non-negative integer n,

$$[(2n+1)\widehat{\varphi}(4n+2) - (n+p+1)\widehat{\varphi}(2n+2p+2)]\widehat{\phi}(2n+p+2) = 0. \tag{4.2}$$

Set

$$E = \left\{n \geq |p| : \quad \widehat{\phi}(2n+p+2) = 0\right\}.$$

By Lemma 4.1, if $\sum_{n\in E}\frac{1}{n} = \infty$, then $\phi = 0$. Otherwise, $\sum_{n\in E^c}\frac{1}{n} = \infty$, where E^c is the complement of E in the set of non-negative integers. Then,

$$(2n+1)\widehat{\varphi}[2(2n+1)] = (n+p+1)\widehat{\varphi}[2(n+1)+2p], \quad n \in E^c.$$

This implies that

$$\frac{1}{2}z\widehat{\varphi}(z) = [\frac{1}{4}(z+2)+p]\widehat{\varphi}[\frac{1}{2}(z+2)+2p], \ \forall\, \mathrm{Re} z \geq 2.$$

So,

$$\widehat{1}(\frac{1}{2}z+2p+1)\widehat{\varphi}(z) = \widehat{1}(z)\widehat{\varphi}(\frac{1}{2}z+2p+1).$$

Denote that

$$G(z) = \frac{\widehat{\varphi}(z)}{\widehat{1}(z)}.$$

Notice that $G(z) = G(\frac{1}{2}z+2p+1)$, and it can be written as $G(2w) = G(w+2p+1)$ where $z = 2w$. With similar proof of Theorem 3.6 in [8], it follows that $G(z)$ is a constant, namely λ. Then,

$$\widehat{\varphi}(z) = \lambda\widehat{1}(z) = \widehat{\lambda}(z).$$

Therefore, φ is constant. □

Proof of Theorem 1.3. If $\phi = 0$, the conclusion is obvious. In the following we assume $\phi \neq 0$. By Lemma 4.5, $\varphi_{2k+1} = 0$ for each integer k. Now, f can be written as

$$f = \sum_{k\geq 0} e^{i(2k)\theta}\varphi_{2k}(r).$$

We now show that $\varphi_{2k} = 0$ for $k > 0$. For any non-negative integer n, by Lemmas 4.3 and 4.4, then

$$\begin{aligned} T_f B_{e^{ip\theta}\phi}(w^{2n}) &= 2\sqrt{\frac{n+1}{2n+1}}(n+p+1)\widehat{\phi}(2n+p+2)T_f(w^{n+p}) \\ &= 4\sqrt{\frac{n+1}{2n+1}}(n+p+1)\widehat{\phi}(2n+p+2) \\ &\quad \times \sum_{k=0}^{+\infty}(n+p+2k+1)\widehat{\varphi_{2k}}(2n+2p+2k+2)w^{n+p+2k}, \end{aligned}$$

$$\begin{aligned} B_{e^{ip\theta}\phi}T_f(w^{2n}) &= 2\sum_{k=0}^{+\infty}(2n+2k+1)\widehat{\varphi_{2k}}(4n+2k+2)B_{e^{ip\theta}\phi}(w^{2n+2k}) \\ &= 4\sum_{k=0}^{+\infty}(2n+2k+1)\widehat{\varphi_{2k}}(4n+2k+2) \\ &\quad \times \sqrt{\frac{n+k+1}{2n+2k+1}}(n+k+p+1)\widehat{\phi}(2n+2k+p+2)w^{n+k+p}. \end{aligned}$$

We have that

$$\begin{aligned} &\sqrt{\frac{n+1}{2n+1}}(n+p+1)\widehat{\phi}(2n+p+2) \\ &\times \sum_{k=0}^{+\infty}(n+p+2k+1)\widehat{\varphi_{2k}}(2n+2p+2k+2)w^{n+p+2k} \\ &= \sum_{k=0}^{+\infty}(2n+2k+1)\widehat{\varphi_{2k}}(4n+2k+2) \\ &\times \sqrt{\frac{n+k+1}{2n+2k+1}}(n+k+p+1)\widehat{\phi}(2n+2k+p+2)w^{n+k+p}. \end{aligned} \tag{4.3}$$

Comparing two sides of (4.3), the degrees of w are the same when $k = 0$. Using the same argument as done in Lemma 4.6, we get that φ_0 is constant. So, (4.3) becomes

$$\begin{aligned} &\sqrt{\frac{n+1}{2n+1}}(n+p+1)\widehat{\phi}(2n+p+2) \\ &\times \sum_{k=1}^{+\infty}(n+p+2k+1)\widehat{\varphi_{2k}}(2n+2p+2k+2)w^{n+p+2k} \\ &= \sum_{k=1}^{+\infty}(2n+2k+1)\widehat{\varphi_{2k}}(4n+2k+2) \\ &\times \sqrt{\frac{n+k+1}{2n+2k+1}}(n+k+p+1)\widehat{\phi}(2n+2k+p+2)w^{n+k+p}. \end{aligned} \tag{4.4}$$

By (4.4), the lowest of the degree of w on the left is $n+p+2$, but on the right it is $n+p+1$. It is clear that the coefficient of w^{n+p+1} is zero, that is

$$\widehat{\varphi}_2(4n+4)\widehat{\phi}(2n+p+4)=0.$$

Set

$$E_2=\Big\{n\geq|p|:\quad \widehat{\phi}(2n+p+4))\neq 0\Big\}.$$

As we assume $\phi\neq 0$, by Lemma 4.1, $\sum_{n\in E_2}\frac{1}{n}=\infty$. Choosing $n\in E_2$, then $\widehat{\varphi}_2(4n+4)=0$ with $\sum_{n\in E_2}\frac{1}{4n+4}=\infty$. By Lemma 4.1 we get $\varphi_2=0$. Then, (4.3) becomes

$$\begin{aligned}&\sqrt{\frac{n+1}{2n+1}}(n+p+1)\widehat{\phi}(2n+p+2)\\&\times\sum_{k=2}^{+\infty}(n+p+2k+1)\widehat{\varphi}_{2k}(2n+2p+2k+2)w^{n+p+2k}\\&=\sum_{k=2}^{+\infty}(2n+2k+1)\widehat{\varphi}_{2k}(4n+2k+2)\\&\times\sqrt{\frac{n+k+1}{2n+2k+1}}(n+k+p+1)\widehat{\phi}(2n+2k+p+2)w^{n+k+p}.\end{aligned}\tag{4.5}$$

Again, by (4.5), the lowest of the degree of w on the left is $n+p+4$, but on the right it is $n+p+2$. The coefficient of w^{n+p+2} on the right is zero, that is

$$\widehat{\varphi}_4(4n+6)\widehat{\phi}(2n+p+6)=0.$$

Set

$$E_4=\Big\{n\geq|p|:\quad \widehat{\phi}(2n+p+6))\neq 0\Big\}.$$

By using similar argument as done above, we get $\varphi_4=0$. One can see that if the lowest degrees of w on both sides of (4.5) are different, then the coefficients should be zero. Thus, we have $\varphi_{2k}=0$ for $k>0$. Further, $\varphi_k=0$ for $k>0$. Therefore, f is a constant. □

As the special case of Theorem 1.3, we obtain two corollaries in the following.

Corollary 4.7. *Let ϕ and φ be bounded radial functions. $T_\phi B_\varphi=B_\varphi T_\phi$ holds if and only if $\phi=0$ or f is constant.*

Corollary 4.8. *Let ϕ be a bounded radial function and p a non-negative integer. Suppose f is analytic in $L^\infty(\mathbb{D})$. Then, $T_fB_{e^{ip\theta}\phi}=B_{e^{ip\theta}\phi}T_f$ holds if and only if $\phi=0$ or f is constant.*

5. Conclusions

In this research, we obtain the following characterizations for the commuting Toeplitz operators and H-Toeplitz operators on the Bergman space.

(1) Suppose p_1, p_2, q_1, q_2 are positive integers, $f=a_1w^{p_1}+\overline{a_2w^{p_2}}$, $g=b_1w^{q_1}+\overline{b_2w^{q_2}}$. Then, $B_fB_g=B_gB_f$ if and only if there exist $\alpha,\beta\in\mathbb{C}$, $|\alpha|+|\beta|\neq 0$ such that $\alpha f+\beta g=0$.

(2) Suppose f is a bounded harmonic function on $\mathbb{D}$. Let p be a non-negative integer. Then $B_{w^p}B_f = B_f B_{w^p}$ if and only if there exist $c \in \mathbb{C}$, such that $f = cw^p$.

(3) Let ϕ be a bounded radial function and p be a non-negative integer. Suppose

$$f = \sum_{k\geq 0} e^{ik\theta}\varphi_k(r) \in L^\infty(\mathbb{D}),$$

where each φ_k is a bounded radial function. Then, $T_f B_{e^{ip\theta}\phi} = B_{e^{ip\theta}\phi} T_f$ holds if and only if $\phi = 0$ or f is a constant.

Use of AI tools declaration

The author declares she has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares that she has no competing interests.

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