Research article

Using computational techniques of fixed point theory for studying the stationary infinite horizon problem from the financial field

Abdelkader Belhenniche[1], Amelia Bucur[2,*], Liliana Guran[3] and Adrian Nicolae Branga[2]

[1] SYSTEC, Faculty of Engineering, Porto University, Institute for Systems and Robotics Rua Dr. Roberto Frias s/n, 4200–465 Porto, Portugal

[2] Department of Mathematics and Informatics, Faculty of Sciences, Lucian Blaga University of Sibiu, I.Raţiu, Street, No.5–7, 550012, Sibiu, Romania

[3] Department of Hospitality Services, Babeş-Bolyai University, Horea street, no 7, 400174, Cluj-Napoca, Romania

* **Correspondence:** Email: amelia.bucur@ulbsibiu.ro; Tel: +40740381318; Fax: +40269216617.

Abstract: In this article, we will solve optimization problems from the financial and economic field with constants, infinite-horizon iterative techniques and elements from fixed point theory. We will resort to Ćirić contractions in Banach space and the main result consists of the existence of a fixed point to solve an important class of infinite-horizon iterative schemes for optimization problems with state constraints for which the value function is merely lower semi-continuous. The developed tools allowed us to solve the stationary infinite-horizon optimization problems, especially for the maximization of the utility of households. We present some fixed point results that are fundamental for the development of our contributions: Notably, existence, monotonicity, attainability and results in the Ćirić contribution. We show the convergence in norm with probability for an iterative procedure defined for our problem under the stated assumptions. By using the Ćirić operator and the Reich-Rus type ψF-contraction, we prove the existence of the results of the optional cost function of an infinite horizon problem in a complete metric space. For a particular case, we realize a numerical simulation in C++. The conclusions are that the convergence, the existence and the uniqueness results of an optimal cost function of an infinite horizon problem in a Banach space can be treated by resorting to the Ćirić operator.

Keywords: infinite horizon problem; fixed point; Blackwell condition; Bellman equation; sustainable growth

Mathematics Subject Classification: 47H10, 90C39, 91B06

1. Introduction

The applicability of dynamic programming results for optimal control and for optimal growth problems [1–4], in which fixed-point theory-based methods were used, is well-known. In this article we investigate the application of the Ćirić fixed-point theorem to prove the existence and uniqueness results of the solution of the dynamic programming Bellman's equation under assumptions that are significantly weaker than the ones generally considered in the specialty literature [5].

Our work afforded us to further extend the topic of an already relevant class of methods to solve optimization problems (notably optimal control, respectively dynamic programming), that have a long study in the history of mathematical techniques.

Discrete Markov decision models for the financial field may be studied using the dynamic programming principles developed by Richard Bellman (1956). Dynamic programming is based on the Principle of Optimality, which was explained by Bellman in the following text: "An optimal policy has the property that, whatever the initial state and decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision" [6].

For the simulation of the mathematical model, Ricard Torres published in 2014 a MATLAB script implementing policy iteration [7]. Torres' result provides insight to how we can programmatically find the optimal policies, by developing a way to programmatically organize the Dynamic Programming problem into a cost network.

Dynamic programming and fixed point theory are important mathematical tools in the financial and economic fields (for modern financial, respectively econometric modulization). The book by Stokey and Lucas [8] published in 1989 is an example of an important book in this direction. Numerical methods to solve dynamic programming (DP) models can be DP models with sequential decision making, the optimal inventory model (Arrow et al. [9]), the optimal investment model (Lucas and Prescott [10]), the optimal growth model under uncertainty (Brock and Mirman [11]), the asset pricing models (Lucas [12] and Brock [13]), the business cycle model (Kydland and Prescott [14]) or DP formulated as a Markov decision process (MDP) (Androulakis [15]).

Blackwell (1965) and Denardo (1967) showed that the Bellman operator is a contraction mapping. Kamihigashi [16] showed that the Bellman operator has a unique fixed point, and this fixed point is a value function. The consistent Bellman operator (Bellemare et al. [17]) is a modified operator that addresses the problem of inconsistency of the optimal action-value functions for suboptimal actions. The distributional Bellman operator (Bellemare et al. [18]) enables operation of the whole return distribution, instead of its expectation, i.e., the value function (Bellemare et al. [19], Bellman [20]). The logistic Bellman operator uses a logistic loss to solve a convex linear programming problem to find optimal value functions (Bas-Serrano et al. [21]).

The Markov decision process (MDP) is a fundamental framework for stochastic games, control design in stochastic environments, reinforcement learning, etc. (Filar and Vrieze [22]). Relying on the fact that the optimal value function is the fixed point of the Bellman operator, dynamic programming methods iteratively use different types of the Bellman operator to converge to the optimal value function (Puterman [23], Alden and Smith [24]).

The literature on infinite horizon optimization and its applications is rich. For example, in 1989, Schochetman and Smith [25] considered the general problem of choosing a discounted cost. They solved examples for equipment replacement and production planning, by applying the study of the

minimizing infinite sequence of decisions from a closed subset of the product space formed by a sequence of arbitrary compact metric spaces.

Modeling for optimizing production was analyzed in 1955 by Chanes et al. [26] and by Modigliani and Hohn [27], and sequential production planning was described in an article from the year 1957 (Johnson [28]).

The existence of forecast horizons in undiscounted discrete-time lot size models was investigated in 1990 by Chand et al. [29].

The literature on infinite horizon optimization is vast and encompasses diverse fields, including finance, electrical engineering (especially control systems), economics, operation research, management science, statistics and mathematics (Sethi and Thompson [30], Denardo [31], Cheevaprawatdomrong [32], Bes and Sethi [33]).

The planning of the horizon model of cash management was investigated by Sethi in 1971 (Sethi [34]), and optimal backlogging over an infinite horizon under time-varying convex production and inventory costs was studied by Ghate and Smith in an article from the year 2009 (Ghate and Smith [35]).

The use of mathematics within the field of finance has been increasing, for analyzing and solving problems such as development [36], control theory, differential game theory, capital asset pricing model from stochastic optimum [37], risk management, derivative security pricing and valuation, portofolio creation and structuring [38], efficiency quantification of capital markets [39], quantitative investing strategies, models of growth, exchange economy and effects of mergers on consumers on both sides of the market [40], etc.

Fixed point theory plays an important role in topology, nonlinear analysis, dynamic optimization and obtaining results in the theory of differential and integral equations, notably in the existence of differential and integral equations or inclusions. These results are essential in many branches of science, economics, management and finance; thus, its increasing use in control and optimization is not surprising [1, 41, 42].

In [43], Richard Bellman introduced dynamic programming, which is typically useful to investigate problems that involve choices to be made over an infinite number of periods, such as the problem that considers a periodic wage offer for a worker. The acceptance implies receiving this wage in all future periods while the rejection implies receiving a new wage offer in the next period. Another example that is increasingly important concerns choosing how to allocate output between consumption and investment, that is, the consumption yields utility in the current period, while investment increases future production.

Based on the work of Torres, in this article, we will develop tools that allow us to solve the stationary, infinite-horizon optimization problems, that is, precisely, the maximization of the utility of households under weaker assumptions than those considered so far.

Let us formulate our problem as the following:

$$\max_{k_i \in K_t} \sum_{t=0}^{\infty} \beta^t U[f(k_t) - k_{t+1}], \tag{1.1}$$

$$s.t. \quad 0 \le k_{t+1} \le f(k_t), \forall t, \text{ and given } k_0 > 0.$$

We can solve (1.1) by defining and solving the following associated functional equation:

$$Tv(k) = \max_{0 \leq y \leq f(k)} U[f(k) - y] + \beta v(y), \tag{1.2}$$

where $K_t = \{k_t\}_0^\infty$, k_t is Capital at start of period t, and $f(k_t)$ is the production. c_t is the consumption, and this means

$$k_{t+1} = f(k_t) - c_t.$$

$U(.)$ is the current period of utility function, and $\beta \in (0, 1)$.

The well-known Banach contraction principle states that, if (X, d) is a complete metric space, and $T : X \to X$ is a mapping satisfying

$$d(T(x), T(y)) \leq \sigma d(x, y), \tag{1.3}$$

for some $\sigma \in (0, 1)$ and for all $x, y \in X$, then T has a unique fixed point x^*, and the sequence $\{x_n\}$ generated by the iterative process $x_{n+1} = Tx_n$ converges to x^* for some $x \in X$.

The Banach contraction has been widely generalized in several settings. In [44], Ćirić introduced quasi contraction map and showed that the condition of quasi contraction implies all the conditions of Banach's contraction principle.

In this paper we will consider Ćirić contractions in Banach space, and the main result consists of the existence of a fixed point. The practical relevance of our result is that it can be applied even to discontinuous operators. An important class of problems consists of, for example, infinite-horizon iterative schemes for optimization problems with state constraints for which the value function is merely lower semi-continuous.

In this paper we investigate the Ćirić operator to prove the properties of infinite-horizon problems. A self map $T : X \to X$ on a metric space (X, d) is said to be a Ćirić mapping if, for all x and y in X,

$$d(T(x), T(y)) \leq \sigma \max \left\{ d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2} \left[d(x, Ty) + d(y, Tx) \right] \right\}. \tag{1.4}$$

We can write

$$M(x, y) = \max \left\{ d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2} \left[d(x, Ty) + d(y, Tx) \right] \right\}.$$

To see the relevance of this extension, just consider the following very simple example Ćirić contractive mapping that is not a contraction. Let $T : X \to X$ be defined by

$$Tx = \begin{cases} \dfrac{x}{4} & for\ x \in [0, 1], \\[2mm] \dfrac{x}{5} & for\ x \in (1, 2]. \end{cases} \tag{1.5}$$

The mapping T is Ćirić type with $\sigma = \dfrac{3}{5}$. Indeed, if both x and y are in X_1 or in X_2, then

$$d(Tx, Ty) = \frac{1}{4}|x - \frac{4}{5}y| \leq \frac{1}{4}|x - \frac{1}{5}y| = \frac{1}{4}d(x, Ty);$$

$$d(Tx, Ty) = \frac{1}{5}|\frac{5}{4}x - y| \leq \frac{1}{5}|y - Tx| \leq \frac{1}{4}d(y, Tx).$$

Therefore, T satisfies the condition

$$d(Tx, Ty) \leq \frac{1}{4} \max\{d(x, y), d(x, Ty), d(y, Tx)\}$$

and hence the inequality (1.4).

To show that T is not Banach contraction on X, let $x = \dfrac{999}{1000}$ and $y = \dfrac{1001}{1000}$.

Then, we have $d(x, y) = |\dfrac{999}{1000} - \dfrac{1001}{1000}| = \dfrac{40}{20000}$

and, on other side, $d(Tx, Ty) = \dfrac{991}{20000}$.

Then, we get, $d(Tx, Ty) = \dfrac{991}{20000} > \sigma\dfrac{40}{80} = \sigma d(x, y)$, as $\sigma \in (0, 1)$. Therefore, the Banach contraction condition is not satisfied.

The Ćirić contractive map does not have to be continuous in general, but it is always convergent to a fixed point.

This article is organized as follows. In the next section, we formulate the infinite horizon problem that we are going to investigate in two cases: Implicit and fully explicit. Then, we provide the basic definitions, as well as the assumptions to be satisfied by its data. In the ensuing section, Section 3, we present several fixed point results that are fundamental for the development of our contributions: Notably, existence, monotonicity, and attainability. Also pertinent to our results is the Ćirić contraction [44], which will also be introduced in this section. In Section 5, we show that the convergence in norm with probability is one of the iterative procedures defined for our problem under the stated assumptions and application to dynamical programming. Section 6, numerical simulation, contains an algorithm in C++, which was applied for a particular case. Finally, some conclusions, and prospective future work are briefly addressed in Section 7.

2. Problem formulation

This section is organized into two parts. In the first part, we introduce the operator that is defined via an explicit function in a complete metric space, while in the second part, we study the one that defines an implicit function.

2.1. Explicit case

The optimal value function is a unique solution of the Bellman equation given by

$$Tv^*(x) = \max_{0 \leq y \leq f(x)} U[f(x) - y] + \beta v(y) \text{ for all } x \in X.$$

Given some positive function $v : X \Rightarrow R$, we denote by $C(X)$ the set of functions v such that $\|v\| < \infty$, where the norm $\| \cdot \|$ on $C(X)$ is defined by $\| \cdot \|$:

$$\|v\| = \sup_{x \in X} |v(x)|.$$

Hence, $C(X)$ is complete with metric induced by norm. To proceed further and in order to get the optimal cost function, we need to consider for each policy $y \in \mathcal{M}$ the mapping $T : C(X) \rightarrow C(X)$

defined by

$$Tv(x) = \max_{0 \leq y \leq f(x)} U[f(x) - y] + \beta v(y) \text{ for all } x \in X.$$

Thus, in order to solve Bellman's equation, we only need to introduce the material that leads us to prove the existence and uniqueness of the fixed point, that is, $Tv^* = v^*$.

Example 2.1. *One sector sustainable growth*

Further, we need the following notations related to the economy concepts.

- Capital at the start of period t is denoted by k_t.

- Production (including depreciated capital) is $f(k_t)$ and consumption is c_t, so $k_{t+1} = f(k_t) - c_t$.

Given $k_0 \geq 0$ and $0 \leq k_{t+1} \leq f(k_t)$, the planner maximizes

$$\sum_{t=0}^{\infty} \delta^t U(c_t) = \sum_{t=0}^{\infty} \delta^t U(f(k_t) - k_{t+1}).$$

Let $U : \mathbb{R}_+ \to \mathbb{R}$ be an increasing and bounded map and let $\delta \in (0, 1)$. Suppose the problem has a solution for all $k_0 \geq 0$. Let us define the value function $v : \mathbb{R}_+ \to \mathbb{R}$, where $v(k_0)$ is value of the maximized objective function given k_0.

Then, the planner's problem in period 0 is $\max_{c_0, k_1} (U(c_0) + \delta V(k_1)$, subject to $c_0 + k_1 = f(k_0)$, $c_0, k_1 \geq 0$ and $k_0 \geq 0$ given.

The value function must satisfy the following functional equation:

$$Tv(k) = \max_{0 \leq y \leq f(k)} (U(f(k) - y) + \delta v(y)).$$

Note $v \in B(X)$, where $X = \mathbb{R}_+$.

We define $T : C(X) \to C(X)$ as follows:

$$Tv(k) = \max_{0 \leq y \leq f(k)} (U(f(k) - y) + \delta v(y)).$$

A solution is a function v^* satisfying $v^* = Tv^*$.

2.2. Implicit case

Note, however, that evaluating an optimal policy requires not only availability of the optimal value function v^* but also the dynamics function f and stage cost function U. If f and U are unknown, then knowing v^* is not sufficient for evaluating an optimal policy. To overcome this issue, we will use the optimal H-function to compute an optimal policy without knowing f and U explicitly.

We consider a set X of states, a set N of controls (the current period of utility) and, for each $x \in X$, a nonempty control constraint $N(x) \subset N$. We denote by $\mathcal{M}$ the set of all functions $\mu : X \to N$ with $\mu(x) \in N(x)$ for all $x \in X$, which will be referred to as current period choices.

We denote by $\mathcal{V}(X)$ the set of functions $v : X \to \mathbb{R}$ and by $\bar{\mathcal{V}}(X)$ the set of functions $v : X \to \overline{\mathbb{R}}$ where $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$. We study the operator of the form

$$H : X \times N \times \mathcal{V}(X) \to \mathbb{R},$$

and, for each policy $\mu \in \mathcal{M}$, we consider the mapping $T : \mathcal{V}(X) \to \bar{\mathcal{V}}(X)$ defined by

$$Tv(x) = \max_{\mu \in \mathcal{M}} H(x, \mu, v) \ \forall x \in X.$$

In view of the definition of $\mathcal{M}$, T and T_μ, we have the following relations:

$$Tv(x) = \max_{\mu \in \mathcal{M}} \{H(x, \mu(x), v)\} = \max_{u \in U} \{H(x, u, v)\}.$$

Example 2.2. *Let $T : C(X) \to C(X)$ be defined as follows:*

$$Tv(x) = \begin{cases} \dfrac{v(x)}{4} & \text{for } x \in S_1, \\[2mm] \dfrac{v(x)}{5} & \text{for } x \in S_2, \end{cases} \tag{2.1}$$

where

$$S_1 = \{v \in C(X) : 0 \le v(x) \le 1\},$$

and

$$S_2 = \{v \in C(X) : 1 < v(x) \le 2\}.$$

Then, it is clear that T is a Ćirić contractive map, but it does not satisfy the Banach contraction condition.

3. Auxiliary results

In this section, many results that will be instrumental in the proof of the main result of this article are presented.

Definition 3.1. *(Ćirić map) A self map T of $B(X)$ is called a Ćirić contractive map if*

$$\|Tv - Tv'\| \le \sigma \max\{\|v - v'\|, \|v - Tv\|, \|v' - Tv'\|, \frac{1}{2}[\|v - Tv'\| + \|v' - Tv\|]\},$$

for all v, $v' \in C(X)$, where $\sigma \in (0, 1)$.

Assumption 3.1. *The self map T_μ is a Ćirić contractive map.*

Theorem 3.1. *(Existence) Let the operators $T, T_\mu : C(X) \to C(X)$ be Ćirić contractive. Then, T and T_μ have, respectively, v^* and v_μ as fixed points.*

Lemma 3.1. *The following hold:*

   i) *For an arbitrary $v_0 \in C(X)$, the sequence $\{v_k\}$ defined by $v_{k+1} = T_\mu v_k$ converges in norm to v_μ.*
   ii) *For an arbitrary $v_0 \in C(X)$, the sequence $\{v_k\}$ defined by $v_{k+1} = Tv_k$ converges in norm to v^*.*

Lemma 3.2. *$C(X)$ is complete with respect to the topology induced by $\|\cdot\|$.*

It is not difficult to observe that $C(X)$ is closed and convex. Thus, given $\{v_k\}_{k=1}^{\infty} \subset C(X)$ and $v \in C(X)$, if $v_k \to V$ in the sense that $\lim_{k \to \infty} \|v_k - v\| = 0$, then $\lim_{k \to \infty} v_k(x) = v(x)$ for all $x \in X$.

Now, we introduce the following standard assumptions:

Assumption 3.2. *(Well-posedness) For all $v \in C(X)$ and for all $\mu \in \mathcal{M}$, we have that $T_\mu v \in C(X)$ and $Tv \in C(X)$.*

From Definition 3.1, we conclude that every contraction T is also a Ćirić contractive map. However, the inverse is not always true.

We will require the following properties to hold.

Assumption 3.3. *(Monotonicity) For all $v, v' \in C(X)$, we have that $v \leq v'$ implies*

$$H(x, u, v) \leq H(x, u, v'), \quad \forall x \in X, \ u \in U(x),$$

where "$\leq$" is defined in a pointwise sense in X.

Assumption 3.4. *(Attainability) For all $v \in C(X)$, there exists $\mu \in \mathcal{M}$ such that $T_\mu v = Tv$.*

4. Main results

Blackwell's sufficient condition for Ćirić contractive map

Theorem 4.1. *(Blackwell [7]) Let T be an operator defined on $C(X)$ satisfying the following properties:*
(a) [monotonicity] For all $x \in X$ and for all $v, v' \in C(X)$, $v(x) \leq v'(x)$, implies that for all $x \in X, Tv(x) \leq Tv'(x)$.
(b) [discounting] There exists some $\beta \in (0, 1)$ such that
for all $v \in C(X), a \geq 0, x \in X, [T(v + a)](x) \leq (Tv)(x) + \beta a$.
Then, T is a Ćirić map with modulus β.

Proof. For any v and $v' \in C(X)$ and any $x \in X$,

$$v(x) - v'(x) \leq \|v - v'\| \leq M(v, v').$$

Hence, $v(x) \leq M(v, v') + v'(x)$ for all $x \in X$, so

$$v \leq M(v, v') + v'.$$

Suppose $T : C(X) \to C(X)$ satisfies Blackwell's condition. Then, there exists $\beta \in (0, 1)$ such that

$$T(v) \leq T(v' + M(v, v')) \leq Tv' + \beta M(v, v').$$

By going through the exact same reasoning, by reversing the role of v and v' we will end up with

$$T(v') \leq T(v + M(v, v')) \leq Tv + \beta M(v, v').$$

Hence, for all $x \in X$, with $|Tv(x) - Tv'(x)| \leq \beta M(v, v')$, then

$$\|Tv - Tv'\|_\infty \leq \beta M(v, v').$$

Thus, T is a Ćirić contractive map.

5. Convergence result

Theorem 5.1. *Let $T : C(X) \to C(X)$ be a Ćirić contractive map. Then, we have the following.*
1) There exists a unique solution to the Bellman equation $Tv = v$.
2) The solution can be found by iterating the relation $v_{k+1} = Tv_k$.

Remark 5.1. *The unique solution of the Bellman equation here is the optimal cost function, which is the maximum of the utility.*

Remark 5.2. *Unlike the Banach contraction operator, the Ćirić operator is not necessarily continuous.*

5.1. Example and implementation

Stochastic shortest path problem

Next, we present an example illustrating our results.
Let

$$Tv(k) = \begin{cases} \max_{0 \le y \le f(k)} (U(f(k) - y) + \frac{1}{4}v(y)) & for \ v \in S_1, \\[2ex] \max_{0 \le y \le f(k)} (U(f(k) - y) + \frac{1}{5}v(y)) & for \ v \in S_2, \end{cases} \tag{5.1}$$

where

$$S_1 = \{v \in C(X) : v(x) \in [0, 1]\},$$
$$S_2 = \{v \in C(X) : v(x) \in (1, 2]\}.$$

$g : X \times U \to \mathbb{R}$ is incurred when control $u \in U(x)$ is selected at state x.

Let $v(x) = 1$, for all $x \in X$. Given an arbitrary $v_0 \in B(X)$, since $H(\cdot, \cdot, \cdot)$ is a Ćirić contractive map, it is clear that the Bellman equation defined by

$$Tv(k) = \begin{cases} \max_{0 \le y \le f(k)} (U(f(k) - y) + \frac{1}{4}v(y)) & for \ v \in S_1, \\[3ex] \max_{0 \le y \le f(k)} (U(f(k) - y) + \frac{1}{5}v(y)) & for \ v \in S_2 \end{cases} \tag{5.2}$$

is not a Banach contraction, but it is a Ćirić contractive map.

5.2. Another application to dynamical programming

If in [37] we have

$$Tv(k) = \max_{0 \le y \le f(x)} \{U[f(k) - y] + \beta(y)\}, \tag{5.3}$$

where $f \in B_d(K_t)$ is the set of all real-valued bounded functions on K_t, we define a norm on $B_d(K_t)$ by $\|f\| = \sup_{t \in [0, \infty)} |f(k_t)|$ for all $f \in B_d(K_t)$.

Then, $(B_d(K_t), \| \cdot \|)$ forms a Banach space equipped with the metric defined by $d(f_1, f_2) := \sup_{t \in [0, \infty)} |f_1(k_t) - f_2(k_t)|$ for all $f_1, f_2 \in B_d(K_t)$.

We are going to analyze the existence of the solution for the functional equation [46] by using the concept of the extended interpolative Reich-Rus type ψF-contraction. For this, we use an operator $T : B_d(K_t) \to B_d(K_t)$ defined by

$$Tv(k) = \max_{0 \leq y \leq f(k)} \{U[f(x) - y] + \beta v(y)\}, \tag{5.4}$$

for all $f \in B_d(K_t)$.

Obviously, for the cases in which U and v are bounded, T becomes well-defined.

Theorem 5.2. *Let $T : B_d(K_t) \to B_d(K_t)$ be an operator defined by (5.4) and the following properties:*

1) U and v are continuous and bounded.
2) For all $f_1, f_2 \in B_d(K_t) \backslash Fix(T)$, satisfying

$$|U[f_1(k) - y] - U[f_2(k) - y]| \leq \frac{1}{e} \sqrt{d(f_1, f_2) \cdot d(f_1, Tf_1) \cdot d(f_2, Tf_2)}, \tag{5.5}$$

the Eq (5.3) has a bounded solution.

Proof. Let $\epsilon > 0$ and $f_1, f_2 \in B_d(K_t) \backslash Fix(T)$. Then, there exist k_1, k_2 such that

$$U[f_1(k_1) - y] + \beta v(y) > Tf_1(k) - \epsilon, \tag{5.6}$$

$$U[f_2(k_2) - y] + \beta v(y) > Tf_2(k) - \epsilon, \tag{5.7}$$

$$Tf_1(k) \geq U[f_2(k_2) - y] + \beta v(y), \tag{5.8}$$

$$Tf_2(k) \geq U[f_1(k_1) - y] + \beta v(y). \tag{5.9}$$

From (5.6) and (5.7) results

$$Tf_1(k) - Tf_2(k) < U[f_1(k_1) - y] + \beta v(y) - U[f_2(k_2) - y] + \beta v(y) + \varepsilon \leq$$

$$\leq |U[f_1(k_1) - y] - U[f_2(k_2) - y]| + \varepsilon \leq \tag{5.10}$$

$$\leq \frac{1}{e} \sqrt[3]{d(f_1, f_2) \cdot d(f_1, Tf_1) \cdot d(f_2, Tf_2)}, \text{ for all } k.$$

Analogously, (5.8) and (5.9) imply

$$Tf_2(k) - Tf_1(k) < \frac{1}{e} \sqrt[3]{d(f_1, f_2) \cdot d(f_1, Tf_1) \cdot d(f_2, Tf_2)} + \varepsilon, \text{ for all } k. \tag{5.11}$$

From (5.10) and (5.11) results

$$|Tf_1(k) - Tf_2(k)| < \frac{1}{e} \sqrt[3]{d(f_1, f_2) \cdot d(f_1, Tf_1) \cdot d(f_2, Tf_2)} + \varepsilon, \text{ for all } k. \tag{5.12}$$

Since $\varepsilon > 0$ is arbitrary, we have

$$|Tf_1(k) - Tf_2(k)| \leq \frac{1}{e} \cdot \sqrt[3]{d(f_1, f_2) \cdot d(f_1, Tf_1) \cdot d(f_2, Tf_2)},$$

for all k, and thus,

$$d(Tf_1(k), Tf_2(k)) \leq \frac{1}{e} \cdot \sqrt[3]{d(f_1, f_2) \cdot d(f_1, Tf_1) \cdot d(f_2, Tf_2)}.$$

We apply the logarithm function, and we obtain

$$\begin{aligned}
lnd(Tf_1, Tf_2) &\leq -1 + \frac{1}{3}lnd(f_1, f_2) + \frac{1}{3}lnd(f_1, Tf_1) + \frac{1}{3}lnd(f_2, Tf_2) = \\
&\frac{1}{3}[lnd(f_1, f_2) - 1] + \frac{1}{3}[lnd(f_1, Tf_1) - 1] + \frac{1}{3}[lnd(f_2, Tf_2) - 1].
\end{aligned} \tag{5.13}$$

The relation (5.13) is nothing but extended interpolative Reich-Rus type ψF-contractions for the operator T, with $F(x) = lnx$, $\psi(x) = x - 1$, $a = b = c = \frac{1}{3}$ and $s = 1$, because: A mapping $T : B_d(k_t) \to B_d(k_t)$ ($B_d(k_t)$ being a b - metric space) is an extended interpolative Reich-Rus type ψT-contraction if there exist $F \in \mathcal{F}$ and $\psi \in \Psi$ such that for all $f_1, f_2 \in B_d(k_t) \backslash Fix(T)$ with $Tf_1 \neq Tf_2$,

$$F(d(Tf_1, Tf_2)) \leq a\psi(F(d(f_1, f_2))) + b\psi(F(d(f_1, Tf_1))) + c\psi(F(d(f_2, Tf_2))),$$

for some constants $a, b, c \in [0, 1]$ with $0 < a + b + c \leq 1$,

where $\mathcal{F}$ denotes the class of all functions $F : (0, \infty) \to \mathbb{R}$ for which the following hold:

(F_1) $\lim_{n \to \infty} x_n = 0 \Leftrightarrow \lim_{n \to \infty} F(x_n) = -\infty$, for all sequences $\{x_n\} \in (0, \infty)$.

(F_2) There exists $k \in (0, 1)$, and $x^k F(x) \to 0$ for $x \to 0^+$.

$\Psi = \{\psi : \mathbb{R} \to \mathbb{R} | \psi$ is monotone increasing and $\psi^n(t) \to -\infty$ for $n \to \infty$ for all $t \in \mathbb{R}\}$; ψ^n is the $n - t$ composition of ψ [45].

We have $\psi^n(x) = x - n$, and for $f^* \in B_d(K_t) \backslash Fix(T)$, $\beta^* = d(f^*, Tf^*) = \sup_{t \in [0, \infty)} |f^*(t) - Tf^*(t)| \in \mathbb{R}^*$. Then, for all $p \in (0, 1)$, the series $\sum_n |\psi^n F(f^*)|^{-1/p}$ is convergent. Hence, by Theorem 3.1 from [46], T has a fixed point in $B_d(K_t)$, and that implies that Eq (5.3) has a bounded solution.

6. Numerical simulation

Example 6.1. *We consider the functions $f : \mathbb{R} \to \mathbb{R}_+$, $U : \mathbb{R}_+ \to \mathbb{R}$, $U(t) = \frac{t+1}{t+2}$, $v : \mathbb{R} \to \mathbb{R}$, $v(y) = \frac{2y+1}{y+1}$, the parameter $\beta \in (0, 1)$ and a point $x \in \mathbb{R}$. Define the function $h_{\beta,x} : [0, f(x)] \to \mathbb{R}$,*

$$h_{\beta,x}(y) = U[f(x) - y] + \beta v(y) = \frac{f(x) - y + 1}{f(x) - y + 2} + \beta \frac{2y + 1}{y + 1} =$$

$$= 1 - \frac{1}{f(x) - y + 2} + \beta(2 - \frac{1}{y + 1}).$$

After a simple calculation we find

$$h'_{\beta,x}(y) = -\frac{1}{(f(x) - y + 2)^2} + \frac{\beta}{(y + 1)^2}.$$

It follows that $h'_{\beta,x}(y) = 0$ if and only if $y = y_0 = \frac{f(x) + 2 - \frac{1}{\sqrt{\beta}}}{\frac{1}{\sqrt{\beta}} + 1}$. If $y_0 \in [0, f(x)]$, then $h'_{\beta,x}(y) \geq 0$ for $y \in [0, y_0]$, and $h'_{\beta,x}(y) \leq 0$ for $y \in [y_0, f(x))$. Consequently, $y_0 = \frac{f(x) + 2 - \frac{1}{\sqrt{\beta}}}{\frac{1}{\sqrt{\beta}} + 1}$ is the unique maximum point

of the function $h_{\beta,x}(y)$ on the interval $[0, f(x)]$. Moreover, the maximum value of the function $h_{\beta,x}(y)$ on the interval $[0, f(x)]$ is given by

$$h_0 = h_{\beta,x}(y_0) = 1 - \frac{1}{f(x) - y_0 + 2} + \beta(2 - \frac{1}{y_0 + 1}) =$$

$$= 1 - \frac{1}{f(x) - \frac{f(x)+2-\frac{1}{\sqrt{\beta}}}{\frac{1}{\sqrt{\beta}}+1} + 2} + \beta(2 - \frac{1}{\frac{f(x)+2-\frac{1}{\sqrt{\beta}}}{\frac{1}{\sqrt{\beta}}+1} + 1}) = 1 + 2\beta - \frac{(1 + \sqrt{\beta})^2}{f(x) + 3}.$$

Considering the particular case for $\beta = \frac{1}{4}$, we find that $y_0 = \frac{f(x)}{3} \in [0, f(x)]$ is the unique maximum point of the function $h_{\frac{1}{4},x}(y)$ on the interval $[0, f(x)]$. The maximum value of the function $h_{\frac{1}{4},x}(y)$ on the interval $[0, f(x)]$ is

$$h_0 = h_{\frac{1}{4},x}(y_0) = \frac{3}{4}(1 + \frac{f(x)}{f(x) + 3}).$$

ALGORITHM

Input parameters
  - The analytical expression of the function $f : \mathbb{R} \to \mathbb{R}_+$;
  - The analytical expression of the function $U : \mathbb{R}_+ \to \mathbb{R}$;
  - The analytical expression of the function $v : \mathbb{R} \to \mathbb{R}$;
  - The parameter $\beta \in (0, 1)$;
  - The point $x \in \mathbb{R}$ for which we calculate the solution of the optimization problem
    $\max_{y \in [0, f(x)]}\{U[f(x) - y] + \beta v(y)\} = -\min_{y \in [0, f(x)]}\{-U[f(x) - y] - \beta v(y)\}$;
  - The error for the computation of the maximum point of the optimization problem, $\varepsilon > 0$.

Output parameters
  x_{max} - the maximum point of the optimization problem, calculated with the error ε;
  g_{max} - the maximum value of the function to be optimized;
  n - the number of iterations.

Pseudocode
Define the function $g_{\beta,x} : [0, f(x)] \to \mathbb{R}$,
  $g_{\beta,x}(y) = -U[f(x) - y] - \beta v(y)$
$A = 0$
$B = f(x)$
$F_0 = 1$
$F_1 = 1$
$n = 1$
While $F_n \le \frac{B-A}{\varepsilon}$ do
    $n = n + 1$
    $F_n = F_{n-1} + F_{n-2}$
End While
$a_1 = A$
$b_1 = B$
$\lambda_1 = a_1 + \frac{F_{n-2}}{F_n} * (b_1 - a_1)$
$\mu_1 = a_1 + \frac{F_{n-1}}{F_n} * (b_1 - a_1)$
For $k = 1, 2, ..., n - 1$ do

If $g_{\beta,x}(\lambda_k) < g_{\beta,x}(\mu_k)$ then

$\qquad a_{k+1} = a_k$

$\qquad b_{k+1} = \mu_k$

$\qquad \lambda_{k+1} = a_{k+1} + \frac{F_{n-k-2}}{F_{n-k}} * (b_{k+1} - a_{k+1})$

$\qquad \mu_{k+1} = \lambda_k$

else

$\qquad a_{k+1} = \lambda_k$

$\qquad b_{k+1} = b_k$

$\qquad \lambda_{k+1} = \mu_k$

$\qquad \mu_{k+1} = a_{k+1} + \frac{F_{n-k-1}}{F_{n-k}} * (b_{k+1} - a_{k+1})$

End If

End For

$x_{max} = \frac{a_n + b_n}{2}$

$g_{max} = -g_{\beta,x}(x_{max})$

PROCEDURE IN C++

```cpp
#include <iostream>
#include <math.h>
using namespace std;
double f(double x)
{
    return pow(2,-fabs(x));
}
double U(double t)
{
    return (t+1)/(t+2);
}
double v(double y)
{
    return (2*y+1)/(y+1);
}
double g(double beta, double x, double y)
{
    return -U(f(x)-y)-beta*v(y);
}
void InputParameters(double& beta, double& x, double& eps)
{
    cout << "beta="; cin >> beta;
    cout << "x="; cin >> x;
    cout << "eps="; cin >> eps;
}
void OptimizationMethod(double beta, double x, double eps, int& n, double F[100], double a[100],
double b[100], double lambda[100], double miu[100], double& ymax, double& gmax)
```

```
{
    double A, B;
    int k;
    A=0;
    B=f(x);
    F[0]=1;
    F[1]=1;
    n=1;
    while (F[n]<=(B-A)/eps)
    {
        n++;
        F[n]=F[n-1]+F[n-2];
    }
    a[1]=A;
    b[1]=B;
    lambda[1]=a[1]+F[n-2]*(b[1]-a[1])/F[n];
    miu[1]=a[1]+F[n-1]*(b[1]-a[1])/F[n];
    for(k=1;k<=n-1;k++)
        if (g(beta, x, lambda[k])<g(beta, x, miu[k]))
        {
            a[k+1]=a[k];
            b[k+1]=miu[k];
            lambda[k+1]=a[k+1]+F[n-k-2]*(b[k+1]-a[k+1])/F[n-k];
            miu[k+1]=lambda[k];
        }
        else
        {
            a[k+1]=lambda[k];
            b[k+1]=b[k];
            lambda[k+1]=miu[k];
            miu[k+1]=a[k+1]+F[n-k-1]*(b[k+1]-a[k+1])/F[n-k];
        }
    ymax=(a[n]+b[n])/2;;
    gmax=-g(beta, x, ymax);
}
void OutputParameters(double beta, double x, int n, double F[100], double a[100], double b[100],
double lambda[100], double miu[100], double ymax, double gmax)
{
    int k;
    cout<<"The number of iterations is n="<<n<<endl;
    cout<<"k   g(beta, x, lambda[k])   g(beta, x, miu[k])   a[k]   b[k]   lambda[k]   miu[k]"<<endl;
    cout.precision(7);
    for(k=1;k<= n;k++)
```

```
        cout<<k<<"  "<<g(beta, x, lambda[k])<<"   "<<g(beta, x, miu[k])<<"   "<<a[k]<<"   "
            <<b[k]<<"   "<<lambda[k]<<"   "<<miu[k]<<endl;
    cout<<"The maximum point of the function is ymax="<<ymax<<endl;
    cout<<"The maximum value of the function is gmax="<<gmax<<endl;
}
int main()
{
    int n;
    double beta, x, eps, F[100], a[100], b[100], lambda[100], miu[100], ymax, gmax;
    InputParameters(beta, x, eps);
    OptimizationMethod(beta, x, eps, n, F, a, b, lambda, miu, ymax, gmax);
    OutputParameters(beta, x, n, F, a, b, lambda, miu, ymax, gmax);
    return 0;
}
```

TEST DATA

$f : \mathbb{R} \to \mathbb{R}_+$, $f(x) = 2^{-|x|}$

$U : \mathbb{R}_+ \to \mathbb{R}$, $U(t) = \frac{t+1}{t+2}$

$v : \mathbb{R} \to \mathbb{R}$, $v(y) = \frac{2y+1}{y+1}$

beta=0.25

x=-0.25

eps=0.00001

RESULTS

The number of iterations is n=25

The maximum point of the function is ymax=0.2803

The maximum value of the function is gmax=0.9141993

The outputs and outcomes are presented in Table 1 and also in a graphical representation in Figure 1:

Table 1. Outputs and outcomes.

k	$g(beta, x, lambda[k])$	$g(beta, x, miu[k])$	$a[k]$	$b[k]$	$lambda[k]$	$miu[k]$
1	-1.249433	-1.249433	-0.6	-0.4	-0.5238095	-0.4761905
2	-0.9129618	-0.913905	0	0.5197026	0.1985087	0.3211938
3	-0.913905	-0.9118618	0.1985087	0.5197026	0.3211938	0.3970174
4	0.9141929	-0.913905	0.1985087	0.3970174	0.2743323	0.3211938
5	-0.9139782	-0.9141929	0.1985087	0.3211938	0.2453703	0.2743323
6	-0.9141929	-0.9141739	0.2453703	0.3211938	0.2743323	0.2922318
7	-0.9141471	-0.9141929	0.2453703	0.2922318	0.2632698	0.2743323
8	-0.9141929	-0.9141991	0.2632698	0.2922318	0.2743323	0.2811693
9	-0.9141991	-0.9141946	0.2743323	0.2922318	0.2811693	0.2853948
10	-0.9141987	-0.9141991	0.2743323	0.2853948	0.2785578	0.2811693
11	-0.9141991	-0.9141981	0.2785578	0.2853948	0.2811693	0.2827833
12	-0.9141992	-0.9141991	0.2785578	0.2827833	0.2801718	0.2811693
13	-0.9141992	-0.9141992	0.2785578	0.2811693	0.2795553	0.2801718
14	-0.9141992	-0.9141992	0.2795553	0.2811693	0.2801718	0.2805528
15	-0.9141992	-0.9141992	0.2795553	0.2805528	0.2799363	0.2801718
16	-0.9141992	-0.9141993	0.2799363	0.2805528	0.2801718	0.2803173
17	-0.9141993	-0.9141992	0.2801718	0.2805528	0.2803173	0.2804073
18	-0.9141992	-0.9141993	0.2801718	0.2804073	0.2802619	0.2803173
19	-0.9141993	-0.9141992	0.2802619	0.2804073	0.2803173	0.2803519
20	-0.9141993	-0.9141993	0.2802619	0.2803519	0.2802965	0.2803173
21	-0.9141993	-0.9141993	0.2802619	0.2803173	0.2802826	0.2802965
22	-0.9141993	-0.9141993	0.2802826	0.2803173	0.2802965	0.2803034
23	-0.9141993	-0.9141993	0.2802826	0.2803034	0.2802896	0.2802965
24	-0.9141993	-0.9141993	0.2802896	0.2803034	0.2802965	0.2802965
25	-0.9141993	-0.9141993	0.2802965	0.2803034	0.2802965	0.2803034
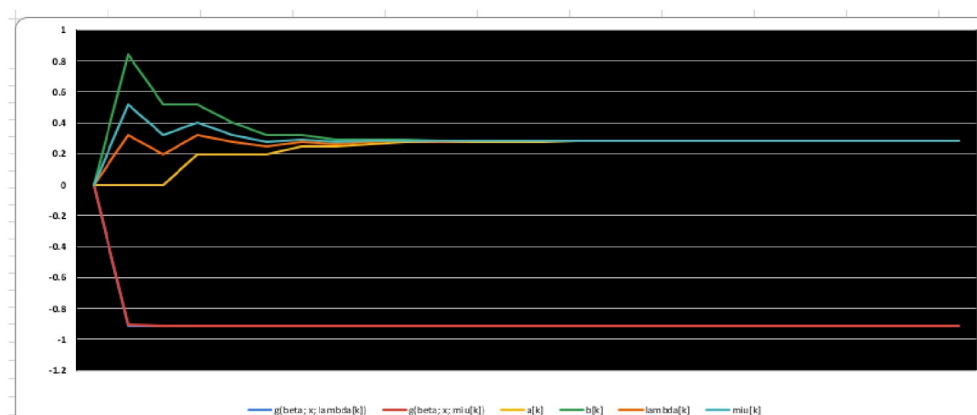
Figure 1. A graphical representation in Microsoft Excel.

7. Conclusions

In this article, we approached an economic/financial problem, the stationary infinite horizon problem, under a much wider class of contractive operators by applying methods of fixed point theory. By resorting to the Ćirić operator and the Reich-Rus type ψF-contraction, we proved the convergence, the existence and the uniqueness of the results of the optimal cost function of the infinite horizon problem in Banach space. The numerical simulation showed that a stationary infinite horizon problem can be solved for particular cases with applications in the financial/economic field.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare that they have no conflict of interest. The authors contributed equally to the writing of this manuscript. All authors have approved the final version of the manuscript.

References

1. D. P. Bertsekas, *Dynamic programming and optimal control*, Belmont: Athena Scientific, 1995.

2. D. Acemoglu, *Introduction to modern economic growth*, 2007. Available from: `https://www.theigc.org/sites/default/files/2016/06/acemoglu-2007.pdf`.

3. Y. Guo, X. B. Shu, F. Xu, C. Yang, HJB equation for optimal control system with random impulses, *Optimization*, 2022. https://10.1080/02331934.2022.2154607

4. F. Xu, Y. Lai, X. B. Shu, Chaos in integer order and fractional order financial systems and their synchronization, *Chaos Soliton. Fract.*, **117** (2018), 125–136. https://doi.org/10.1016/j.chaos.2018.10.005

5. D. P. Bertsekas, *Dynamic programming and optimal control*, Belmont: Athena Scientific, 2005. Available from: `http://athenasc.com/dpbook.html`.

6. M. J. Miranda, P. L. Fackler, *Applied computational economics and finance*, The MIT Press, 2004. Available from: `https://mitpress.mit.edu/9780262633093/applied-computational-economics-and-finance/`.

7. R. Torres, *Dynamic programming (ECO 10401 - 001)*, 2014. Available from: `http://ciep.itam.mx/~rtorres/progdin/`.

8. N. L. Stokey, R. E. Lucas Jr., E. C. Prescott, *Recursive methods in economic dynamics*, Cambridge: Harvard University Press, 1989. https://doi.org/10.2307/j.ctvjnrt76

9. K. J. Arrow, T. Harris, J. Marschak, Optimal inventory policy, *Econometrica*, **19** (1951), 250–272. Available from: `https://www.or.mist.i.u-tokyo.ac.jp/takeda/FreshmanCourse/ArrowHarrisMarschak.pdf`.

10. R. E. Lucas Jr., E. C. Prescott, Investment under uncertainty, *Econometrica*, **39** (1971), 659–681. https://doi.org/10.2307/1909571

11. W. A. Brock, L. J. Mirman, Optimal economic growth and uncertainty: The discounted case, *J. Econ. Theory*, **4** (1972), 479–513. https://doi.org/10.1016/0022-0531(72)90135-4

12. R. E. Lucas, Asset prices in an exchange economy, *Econometrica*, **46** (1978), 1429–1445. Available from: `http://www.jstor.org/stable/1913837`.

13. W. A. Brock, *The economics of information and uncertainty*, Chapter title: Asset prices in a production economy, Chicago: University of Chicago Press, 1982. Available from: `https://www.nber.org/system/files/chapters/c4431/c4431.pdf`.

14. F. E. Kydland, E. C. Prescott, Time to build and aggregate fluctuations, *Econometrica*, **50** (1982), 1345–1370. Available from: `http://www.jstor.org/stable/1913386?origin=JSTOR-pdf`.

15. I. P. Androulakis, *Dynamic programming: Infinite horizon problems overview*, In: C. Floudas, P. Pardalos (eds), Encyclopedia of optimization, Boston: Springer, 2008. https://doi.org/10.1007/978-0-387-74759-0-148

16. T. Kamihigashi, Existence and uniqueness of a fixed point for the Bellman operator in deterministic dynamic programming, *REB Disscution*, 2011.

17. M. G. Bellemare, G. Ostrovski, A. Guez, P. Thomas, R. Munos, Increasing the action gap: New operators for reinforcement learning, *Pro. AAAI Conf. Artif. Intell.*, **30** (2016). https://doi.org/10.48550/arXiv.1512.04860

18. M. G. Bellemare, W. Dabney, R. Munos, A distributional perspective on reinforcement learning, *Int. Conf. Machine Learn.*, 2017, 449–458. https://doi.org/10.48550/arXiv.1707.06887

19. M. G. Bellemare, W. Dabney, M. Rowland, Distributional reinforcement learning, MIT Press, 2023. Available from: `http://www.distributional-rl.org`.

20. R. Bellman, Dynamic programming, *Science*, **153** (1966), 34–37.

21. J. B. Serrano, S. Curi, A. Krause, G. Neu, Logistic q-learning, *Int. Conf. Artif. Intell. Statis.*, 2021, 3610–3618. Available from: `https://proceedings.mlr.press/v130/bas-serrano21a.html`.

22. J. Filar, K. Vrieze, *Competitive Markov decision processes*, Springer Science & Business Media, 2012. Available from: `https://link.springer.com/book/10.1007/978-1-4612-4054-9`.

23. M. L. Puterman, *Markov decision processes: Discrete stochastic dynamic programming*, John Wiley & Sons, 2014.

24. J. M. Alden, R. L. Smith, Rolling horizon procedures in nonhomogeneous Markov decision processes, *Oper. Res.*, **40** (1984), 183–194. http://dx.doi.org/10.1287/opre.40.3.S183

25. I. E. Schochetman, R. L. Smith, Infinite horizon optimization, *Math. Oper. Res.*, **14** (1989), 559–574. http://dx.doi.org/10.1287/moor.14.3.559

26. A. Charnes, W. W. Cooper, B. Mellon, A model for optimizing production by reference to cost surrogates, *Econometrica*, **23** (1955), 307–323. http://dx.doi.org/10.2307/1910387

27. F. Modigliani, F. E. Hohn, Production planning over time and the nature of the expectation and planning horizon, *Econometrica*, **23** (1955), 46–66. http://dx.doi.org/10.2307/1905580

28. S. M. Johnson, Sequential production planning over time at minimum cost, *Manag. Sci.*, **3** (1957), 435–437. http://dx.doi.org/10.1287/mnsc.3.4.435

29. S. Chand, S. P. Sethi, J. M. Proth, Existence of forecast horizons in undiscounted discrete-time lot size models, *Oper. Res.*, **38** (1990), 884–892. http://dx.doi.org/10.1287/opre.38.5.884

30. S. P. Sethi, G. L. Thompson, *Optimal control theory*, New York: Springer, 2005. http://dx.doi.org/10.1007/978-1-4757-6097-2

31. E. V. Denardo, *Dynamic programming: Models and applications*, Mineola: Dover, 2003.

32. T. Cheevaprawatdomrong, R. L. Smith, Infinite horizon production scheduling in time varying systems under stochastic demand, *Oper. Res.*, **52** (2004), 105–115. http://dx.doi.org/10.1287/opre.1030.0080

33. C. Bes, S. P. Sethi, Concepts of forecast and decision horizons: Applications to dynamic stochastic optimization problems, *Math. Oper. Res.*, **13** (1988), 295–310. http://dx.doi.org/10.1287/moor.13.2.295

34. S. P. Sethi, A note on a planning horizon model of cash management, *J. Financ. Quant. Anal.*, **6** (1971), 659–665. http://dx.doi.org/10.2307/2330136

35. A. Ghate, R. L. Smith, Optimal backlogging over an infinite horizon under time-varying convex production and inventory costs, *Manuf. Serv. Op. Manag.*, **11** (2009), 362–368. http://dx.doi.org/10.1287/msom.1080.0218

36. N. L. Stokey, *Recursive methods in economic dynamics*, Harvard University Press, 1969. Available from: `https://www.hup.harvard.edu/catalog.php?isbn=9780674750968`.

37. X. Yang, There important applications of mathematics in financial mathematics, *Am. J. Indust. Bus. Manage.*, **7** (2017), 1096–1100. https://doi.org/10.4236/ajibm.2017.79077

38. V. Brătian, A. Bucur, C. Oprean, *Finanţe cantitative: Evaluarea valorilor mobiliare şi gestiunea portofoliului*, Sibiu: Lucian Blaga Publishing House, 2016. Available from: `http://biblioteca.ulbsibiu.ro`.

39. C. O. Stan, C. Tănăsescu, A. Bucur, A new proposal for efficiency quantification of capital markets in the context of complex nonlinear dynamics and chaos, *Econ. Res.-Ekon. Istraž.*, **30** (2017), 1669–1693. http://dx.doi.org/10.1080/1331677X.2017.1383172

40. V. Brătian, A. Bucur, C. Oprean, C. Tănăsescu, Discretionary vs nondiscretionary in fiscal mechanism-nonautomatic fiscal stabilities, *Econ. Res.-Ekon. Istraž.*, **29** (2017), 1–17. http://dx.doi.org/10.1080/1331677X.2015.1106330

41. D. E. Kirk, *Optimal control theory: An introduction*, New Jersey: Prentice-Hall, Englewood Cliffs, 1998. Available from: `https://books.google.ro`.

42. M. Sniedovich *Dynamic programming: Foundations and principles*, CRC Press, 2010. https://doi.org/10.1201/EBK0824740993

43. R. Bellman *The theory of dynamic programming*, Santa Monica: Rand Corporation Report, 1954. Available from: `https://www.rand.org/content/dam/rand/pubs/papers/2008/P550.pdf`.

44. L. B. Ćirić, A generalization of Banach's contraction principle, *P. Am. Math. Soc.*, **45** (1974), 267–273. Available from: `https://www.ams.org/journals/proc/1974-045-02/S0002-9939-1974-0356011-2/S0002-9939-1974-0356011-2.pdf`.

45. N. A. Secelean, D. Wardowski, ψF-contraction: Not necessarily nonexpansive Picard operators, *Results Math.*, **70** (2016), 41–431. http://dx.doi.org/10.1007/s00025-016-0570-7

46. S. Panya, K. Roy, M. Soha, Fixed point for a class of extended interpolative ψF-contraction maps over *a b*-metric space and its applications to dynamical programming, *U. P. B. SCI Bull Series A,* **83** (2021), 59–70. Available from: `https://www.scientificbulletin.upb.ro`.