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Review

Spectrum and analytic functional calculus in real and quaternionic frameworks: An overview

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Abstract: An approach to the elementary spectral theory for quaternionic linear operators was presented by the author in a recent paper, quoted and discussed in the Introduction, where, unlike in works by other authors, the construction of the analytic functional calculus used a Riesz-Dunford-Gelfand type kernel, and the spectra were defined in the complex plane. In fact, the present author regards the quaternionic linear operators as a special class of real linear operators, a point of view leading to a simpler and a more natural approach to them. The author's main results in this framework are summarized in the following, and other pertinent comments and remarks are also included in this text. In addition, a quaternionic joint spectrum for pairs of operators is discussed, and an analytic functional calculus which uses a Martinelli type kernel in two variables is recalled.

Keywords: spectrum in real algebras; conjugation; real operators; quaternionic operators; analytic functional calculus
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1. Introduction

The aim of the present work is to exhibit an overview of the main results from the author's work [1], and partially from [2], adding some relevant comments and new remarks. In the article [1], in order to construct an analytic functional calculus for quaternionic linear operators, the class of the *quaternionic slice regular functions* (or *slice holomorphic functions*; see [3]), defined on subsets of the quaternionic algebra, is replaced by a class of vector-valued holomorphic functions, called *stem functions* (see Remark 2), defined on subsets of the complex plane. In fact, these two classes are isomorphic via a Cauchy type transform (see Theorem 4 below, or Theorem 6 from [2]), and we use the latter to construct an analytic functional calculus for what are called quaternionic linear operators. Nevertheless, the images of the analytic functional calculi obtained by these two methods are the same, as one might expect (see Remark 8).

Our motivation comes from the following remark. Regarding the quaternions as elements of the real Hamilton algebra $\mathbb{H}$, and imbedding $\mathbb{H}$ into its complexification $\mathbb{M}$, the quaternions have a spectrum in the complex plane, and the quaternionic slice regular functions can be obtained via an analytic functional calculus with $\mathbb{M}$-valued stem functions, which are holomorphic functions, symmetric with respect to the real axis, used in [2] (see also [13] for a more general definition). It is this idea which led us to shortcat the construction of the analytic functional for quaternionic operators, using directly $\mathbb{M}$-valued analytic functions instead of quaternionic slice regular ones. At the same time, we have noticed that quaternionic linear operators are particular cases of real linear operators, allowing us to refine and adapt some techniques used in the study of the latter to get significant results valid for the former.

The spectral theory for quaternionic linear operators has already been discussed in numerous works, in particular, in the monographs [3, 4], where the construction of an analytic functional calculus (called *S-analytic functional calculus*) mounts to associating a quaternionic linear operator and a given function from the class of slice regular functions with another quaternionic linear operator, via an integral formula, using a certain non-commutative kernel. Unlike in these works, in the paper [1] one first considers the case of linear operators on real Banach spaces, whose spectrum is in the complex plane, and one sketches the construction of an analytic functional calculus for them, using some classical ideas (see Theorem 2 below). Then, regarding the quaternionic operators as particular cases of the real ones, this framework is extended to the quaternionic case, showing that the approach from the real case can be adapted to that more intricate situation. Unlike in [3] or [4], the functional calculus is obtained via a Riesz-Dunford-Gelfand formula, defined in a partially commutatative framework, rather than the non-commutative Cauchy type formula used by some previous authors. This is possible because the S-spectrum, introduced by F. Colombo and I. Sabadini (see [3]), can be replaced by a spectrum in the complex plane. As already mentioned above, one can show that the analytic functional calculus obtained with $\mathbb{M}$-valued stem functions is equivalent to the analytic functional calculus obtained with slice holomorphic functions in [3] or [4], in the sense that the images of these functional calculi coincide, as shown in Remark 8. Unlike in the original paper [1], answering a question of the referee, we have added a complete proof of equality (4.9).

The analytic functional calculus can actually be defined for a class of analytic operator valued functions, whose definition extends that of stem functions, and it applies, in particular, to a large family of quaternionic linear operators.

In Subsection 4.2, we recall the construction of the *quaternionic Cauchy transform* from [2], showing later that the quaternionic Cauchy kernel from [3] can be obtained from the Riesz-Dunford kernel in $\mathbb{M}$, via the Cauchy transform, which is a new result (see Example 3).

The case of pairs of commuting real operators is also exhibited, following [1], and some connections with the quaternionic case are indicated. In fact, we define a quaternionic spectrum for them and construct an analytic functional calculus using a Martinelli type formula (see [5] for the original approach).

2. Spectrum in real algebras and conjugation

If $\mathcal{A}$ is an arbitrary unital real Banach algebra (see [6] for details), the (complex) spectrum of an element $a \in \mathcal{A}$ may be defined by the equality

$$\sigma_{\mathbb{C}}(a) = \{u + iv; (u - a)^2 + v^2 \text{ is not invertible}, u, v \in \mathbb{R}\}, \tag{2.1}$$

where i is the imaginary unit.

This definition goes back to Kaplansky (see [7]), and it applies in particular, to linear operators, acting on real vector spaces.

The set $\sigma_{\mathbb{C}}(a)$ is *conjugate symmetric*, meaning that $u + iv \in \sigma_{\mathbb{C}}(a)$ if and only if $u - iv \in \sigma_{\mathbb{C}}(a)$.

Fixing a unital real Banach algebra $\mathcal{A}$, we denote by $\mathcal{A}_{\mathbb{C}}$ the complexification of $\mathcal{A}$, that is, $\mathcal{A}_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{R}} \mathcal{A}$, written simply as $\mathcal{A} + i\mathcal{A}$, via the identification of the element $1 \otimes a + i \otimes b$ with the element $a + ib$, for all $a, b \in \mathcal{A}$.

This tensor product $\mathcal{A}_{\mathbb{C}}$ is a unital complex Banach algebra, with the product of two elements given by $(a + ib)(c + id) = ac - bd + i(ad + bc)$ for all $a, b, c, d \in \mathcal{A}$, and with a (not necessarily unique) norm, which we fix to be given by $\|a + ib\| = \|a\| + \|b\|$, where $\| * \|$ is the norm of $\mathcal{A}$. It is important to emphasize that in the algebra $\mathcal{A}_{\mathbb{C}}$, the complex numbers commute with all elements of $\mathcal{A}$.

Another useful property of the algebra $\mathcal{A}_{\mathbb{C}}$ is a the existence of a *conjugation*, given by

$$\mathcal{A}_{\mathbb{C}} \ni a + ib \mapsto a - ib \in \mathcal{A}_{\mathbb{C}},\ a, b \in \mathcal{A},$$

which is a unital conjugate-linear automorphism, whose square is the identity. Note also that an arbitrary element $a + ib$ is invertible if and only if $a - ib$ is invertible.

The ordinary spectrum, defined for every element $a \in \mathcal{A}_{\mathbb{C}}$, will be denoted by $\sigma(a)$. Identifying the algebra $\mathcal{A}$ with a real subalgebra of $\mathcal{A}_{\mathbb{C}}$ via the map $a \mapsto 1 \otimes a$, one gets the following.

Lemma 1. *For every $a \in \mathcal{A}$ we have the equality $\sigma_{\mathbb{C}}(a) = \sigma(a)$.*

Indeed, as we have the obvious identity

$$(u - a)^2 + v^2 = (u + iv - a)(u - iv - a)$$

for an arbitrary complex number $u + iv$ with $u, v \in \mathbb{R}$, the assertion follows easily.

We will apply the discussion from above to the context of linear operators. In what follows, we consider real, complex and quaternionic linear operators, that is, $\mathbb{R}$-, $\mathbb{C}$- and $\mathbb{H}$-linear operators, respectively.

The spectral theory for real linear operators is not a new subject. A pertinent discussion exists actually in the framework of linear relations (see [8]). Moreover, the slightly different approach to real linear operators developed in [1], can be applied, with minor changes, to the case of some quaternionic operators.

For a real or complex Banach space $\mathcal{V}$, we denote by $\mathcal{B}(\mathcal{V})$ the algebra of all bounded $\mathbb{R}$- (respectively $\mathbb{C}$-)linear operators on $\mathcal{V}$. As before, the multiples of the identity will be identified with the corresponding scalars.

Let $\mathcal{V}$ be a real Banach space, and let $\mathcal{V}_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{R}} \mathcal{V}$ be its complexification, which is identified with the direct sum $\mathcal{V} + i\mathcal{V}$. Every operator $T \in \mathcal{B}(\mathcal{V})$ has a unique extension to an operator $T_{\mathbb{C}} \in \mathcal{B}(\mathcal{V}_{\mathbb{C}})$, given by $T_{\mathbb{C}}(x + iy) = Tx + iTy,\ x, y \in \mathcal{V}$ and the map $\mathcal{B}(\mathcal{V}) \ni T \mapsto T_{\mathbb{C}} \in \mathcal{B}(\mathcal{V}_{\mathbb{C}})$ is unital, $\mathbb{R}$-linear and multiplicative. Particularly, $T \in \mathcal{B}(\mathcal{V})$ is invertible if and only if $T_{\mathbb{C}} \in \mathcal{B}(\mathcal{V}_{\mathbb{C}})$ is invertible.

Given an operator $S \in \mathcal{B}(\mathcal{V}_{\mathbb{C}})$, we define the operator $S^{\flat} \in \mathcal{B}(\mathcal{V}_{\mathbb{C}})$ to be equal to CSC, where $C : \mathcal{V}_{\mathbb{C}} \mapsto \mathcal{V}_{\mathbb{C}}$ is the (natural) conjugation of $\mathcal{V}_{\mathbb{C}}$ given by $x + iy \mapsto x - iy,\ x, y \in \mathcal{V}$. The map

$\mathcal{B}(\mathcal{V}_{\mathbb{C}}) \ni S \mapsto S^\flat \in \mathcal{B}(\mathcal{V}_{\mathbb{C}})$ is a unital conjugate-linear automorphism, whose square is the identity on $\mathcal{B}(\mathcal{V}_{\mathbb{C}})$. We have $S^\flat = S$ if and only if $S(\mathcal{V}) \subset \mathcal{V}$, and so $T_{\mathbb{C}}^\flat = T_{\mathbb{C}}$.

Noting that $(S + S^\flat)(\mathcal{V}) \subset \mathcal{V}, i(S - S^\flat)(\mathcal{V}) \subset \mathcal{V}$, we deduce that the algebras $\mathcal{B}(\mathcal{V}_{\mathbb{C}})$ and $\mathcal{B}(\mathcal{V})_{\mathbb{C}}$ are isomorphic and they will often be identified. So $\mathcal{B}(\mathcal{V})$ may be regarded as a (real) subalgebra of $\mathcal{B}(\mathcal{V})_{\mathbb{C}}$. In particular, if $S = U + iV$, with $U, V \in \mathcal{B}(\mathcal{V})$, we have $S^\flat = U - iV$, implying that the map $S \mapsto S^\flat$ is the conjugation of the complex algebra $\mathcal{B}(\mathcal{V})_{\mathbb{C}}$ induced by the conjugation C of $\mathcal{V}_{\mathbb{C}}$.

For every operator $S \in \mathcal{B}(\mathcal{V}_{\mathbb{C}})$, we denote by $\sigma(S)$ its usual spectrum. As $\mathcal{B}(\mathcal{V})$ is a real algebra, the (complex) spectrum of an operator $T \in \mathcal{B}(\mathcal{V})$ is given by the equality (2.1):

$$\sigma_{\mathbb{C}}(T) = \{u + iv; (u - T)^2 + v^2 \text{ is not invertible}, u, v \in \mathbb{R}\}.$$

Corollary 1. *For every $T \in \mathcal{B}(\mathcal{V})$ we have the equality $\sigma_{\mathbb{C}}(T) = \sigma(T_{\mathbb{C}})$.*

3. Analytic functional calculus for real operators

Using the concept of spectrum for real operators, an important step for further development is the construction of an analytic functional calculus (see [8]). As in [2], in what follows, we shall present a similar construction for real linear operators. Unlike in [8], we perform our construction using a class of operator valued analytic functions instead of scalar valued analytic functions. Moreover, this approach looks simpler for this case, and it is a model to get an analytic functional calculus for quaternionic linear operators.

If $\mathcal{V}$ is a real Banach space, and so each operator $T \in \mathcal{B}(\mathcal{V})$ has a complex spectrum $\sigma_{\mathbb{C}}(T)$, which is compact and nonempty, one can use the classical Riesz-Dunford functional calculus, in a slightly generalized form (that is, replacing the scalar-valued analytic functions by operator-valued analytic ones, which is a well known idea difficult to track).

To use various versions of the Cauchy formula, we adopt the following definition. Let $U \subset \mathbb{C}$ be open. An open subset $\Delta \subset U$ will be called a *Cauchy domain* (in U) if the boundary of Δ is in U and it consists of a finite family of closed curves, piecewise smooth, positively oriented. A Cauchy domain is bounded but not necessarily connected.

Remark 1. If $\mathcal{V}$ is a real Banach space, and $T \in \mathcal{B}(\mathcal{V})$, we have the usual analytic functional calculus for the operator $T_{\mathbb{C}} \in \mathcal{B}(\mathcal{V}_{\mathbb{C}})$ (see [9]). That is, in a slightly generalized form, and for later use, if $U \supset \sigma(T_{\mathbb{C}})$ is an open set in $\mathbb{C}$ and $F : U \mapsto \mathcal{B}(\mathcal{V}_{\mathbb{C}})$ is analytic, we set

$$F(T_{\mathbb{C}}) = \frac{1}{2\pi i} \int_{\Gamma} F(\zeta)(\zeta - T_{\mathbb{C}})^{-1} d\zeta,$$

where Γ is the boundary of a Cauchy domain Δ containing $\sigma(T_{\mathbb{C}})$ in U. In fact, because $\sigma(T_{\mathbb{C}})$ is conjugate symmetric, we may and shall assume that both U and Γ are conjugate symmetric. Because the function $\zeta \mapsto F(\zeta)(\zeta - T_{\mathbb{C}})^{-1}$ is analytic in $U \setminus \sigma(T_{\mathbb{C}})$, the integral does not depend on the particular choice of the Cauchy domain Δ containing $\sigma(T_C)$.

A natural question is to find an appropriate condition to infer that $F(T_{\mathbb{C}})^\flat = F(T_{\mathbb{C}})$, which would imply the invariance of $\mathcal{V}$ under $F(T_{\mathbb{C}})$.

With the notation of Remark 1, we have the following.

Theorem 1. *Let $U \subset \mathbb{C}$ be open and conjugate symmetric. If $F : U \mapsto \mathcal{B}(\mathcal{V}_{\mathbb{C}})$ is analytic and $F(\zeta)^\flat = F(\bar{\zeta})$ for all $\zeta \in U$, then $F(T_{\mathbb{C}})^\flat = F(T_{\mathbb{C}})$ for all $T \in \mathcal{B}(\mathcal{V})$ with $\sigma_{\mathbb{C}}(T) \subset U$.*

This statement, as well as its proof, can be found in [1], as Theorem 1.

Remark 2. If $\mathcal{A}$ is a unital real Banach algebra, $\mathcal{A}_{\mathbb{C}}$ its complexification, and $U \subset \mathbb{C}$ is open, we denote by $\mathcal{O}(U, \mathcal{A}_{\mathbb{C}})$ the algebra of all analytic $\mathcal{A}_{\mathbb{C}}$-valued functions. If U is conjugate symmetric, and $\mathcal{A}_{\mathbb{C}} \ni a \mapsto \bar{a} \in \mathcal{A}_{\mathbb{C}}$ is its natural conjugation, we denote by $\mathcal{O}_s(U, \mathcal{A}_{\mathbb{C}})$ the real subalgebra of $\mathcal{O}(U, \mathcal{A}_{\mathbb{C}})$ consisting of those functions F with the property $F(\bar{\zeta}) = \overline{F(\zeta)}$ for all $\zeta \in U$. Adapting a known terminology, such functions will be called ($\mathcal{A}_{\mathbb{C}}$-*valued*) *stem functions* (see also [10] or [13] for a more general definition).

When $\mathcal{A} = \mathbb{R}$, so $\mathcal{A}_{\mathbb{C}} = \mathbb{C}$, the space $\mathcal{O}_s(U, \mathbb{C})$ will be denoted by $\mathcal{O}_s(U)$, which is a real algebra. Note that $\mathcal{O}_s(U, \mathcal{A}_{\mathbb{C}})$ is also a bilateral $\mathcal{O}_s(U)$-module.

Definition 1. *For an operator $T \in \mathcal{B}(\mathcal{V})$ and for a given function $F \in \mathcal{O}_s(U, \mathcal{B}(\mathcal{V})_{\mathbb{C}})$, we shall write*

$$F(T) = \frac{1}{2\pi i}\left(\int_\Gamma F(\zeta)(\zeta - T_{\mathbb{C}})^{-1} d\zeta\right)_{|\mathcal{V}},$$

and the map $F \mapsto F(T)$, will be called the (left) analytic functional calculus of T.

We note that the right hand side of the integral from above belongs to $\mathcal{B}(\mathcal{V})$, by the previous theorem. The analytic functional calculus given by Definition 1 has the following properties.

Theorem 2. *Let $\mathcal{V}$ be a real Banach space, let $U \subset \mathbb{C}$ be a conjugate symmetric open set, and let $T \in \mathcal{B}(\mathcal{V})$, with $\sigma_{\mathbb{C}}(T) \subset U$. Then the assignment*

$$\mathcal{O}_s(U, \mathcal{B}(\mathcal{V})_{\mathbb{C}}) \ni F \mapsto F(T) \in \mathcal{B}(\mathcal{V})$$

is an $\mathbb{R}$-linear map, and the assignment

$$\mathcal{O}_s(U) \ni f \mapsto f(T) \in \mathcal{B}(\mathcal{V})$$

is a unital real algebra morphism.

Moreover, the following properties are true:

(1) For all $F \in \mathcal{O}_s(U, \mathcal{B}(\mathcal{V})_{\mathbb{C}})$, $f \in \mathcal{O}_s(U)$, we have $(Ff)(T) = F(T)f(T)$, where $(Ff)(\zeta) = F(\zeta)f(\zeta)$ for all $\zeta \in U$.

(2) For every polynomial $P(\zeta) = \sum_{n=0}^m A_n \zeta^n$, $\zeta \in \mathbb{C}$, with $A_n \in \mathcal{B}(\mathcal{V})$ for all $n = 0, 1, \ldots, m$, we have $P(T) = \sum_{n=0}^m A_n T^n \in \mathcal{B}(\mathcal{V})$.

This result, and its proof as well, based on arguments that are more or less standard (see [9]), can be found in [1] as Theorem 2.

4. Analytic functional calculus for quaternionic operators

4.1. Spectrum of a quaternion

Some known definitions and elementary facts about quaternions can be found in [3], Section 4.6.

The abstract algebra of quaternions $\mathbb{H}$ is the four-dimensional $\mathbb{R}$-algebra generated by the unit 1, and by the "imaginary units" $\{\mathbf{j},\mathbf{k},\mathbf{l}\}$, satisfying

$$\mathbf{jk}=-\mathbf{kj}=\mathbf{l},\ \mathbf{kl}=-\mathbf{lk}=\mathbf{j},\ \mathbf{lj}=-\mathbf{jl}=\mathbf{k},\ \mathbf{jj}=\mathbf{kk}=\mathbf{ll}=-1.$$

We have $\mathbb{H}\supset\mathbb{R}$ by identifying every number $x\in\mathbb{R}$ with the element $x1\in\mathbb{H}$.

The natural (multiplicative) norm on $\mathbb{H}$ is given by

$$\|\mathbf{x}\|=\sqrt{x_0^2+x_1^2+x_2^2+x_0^2},\ \ \mathbf{x}=x_0+x_1\mathbf{j}+x_2\mathbf{k}+x_3\mathbf{l},\ \ x_0,x_1,x_2,x_3\in\mathbb{R},$$

while the map

$$\mathbb{H}\ni\mathbf{x}=x_0+x_1\mathbf{j}+x_2\mathbf{k}+x_3\mathbf{l}\mapsto\mathbf{x}^*=x_0-x_1\mathbf{j}-x_2\mathbf{k}-x_3\mathbf{l}\in\mathbb{H}$$

is a natural involution.

As we have $\mathbf{xx}^*=\mathbf{x}^*\mathbf{x}=\|\mathbf{x}\|^2$, it follows, in particular, that every element $\mathbf{x}\in\mathbb{H}\setminus\{0\}$ is invertible, and $\mathbf{x}^{-1}=\|\mathbf{x}\|^{-2}\mathbf{x}^*$.

For an arbitrary quaternion $\mathbf{x}=x_0+x_1\mathbf{j}+x_2\mathbf{k}+x_3\mathbf{l}$, $x_0,x_1,x_2,x_3\in\mathbb{R}$, we set $\Re\mathbf{x}=x_0=(\mathbf{x}+\mathbf{x}^*)/2$, and $\Im\mathbf{x}=x_1\mathbf{j}+x_2\mathbf{k}+x_3\mathbf{l}=(\mathbf{x}-\mathbf{x}^*)/2$, that is, the *real* and *imaginary part* of $\mathbf{x}$, respectively.

The complexification $\mathbb{C}\otimes_{\mathbb{R}}\mathbb{H}$ of the $\mathbb{R}$-algebra $\mathbb{H}$ (appearing also in [10]) will be, as above, identified with the direct sum $\mathbb{M}=\mathbb{H}+i\mathbb{H}$, which is an algebra containing the complex field $\mathbb{C}$. Clearly, in the algebra $\mathbb{M}$, the elements of $\mathbb{H}$, in particular the "imaginary units" $\mathbf{j},\mathbf{k},\mathbf{l}$, commute with all complex numbers.

The natural conjugation in the algebra $\mathbb{M}$ is given by $\bar{\mathbf{a}}=\mathbf{b}-i\mathbf{c}$, where $\mathbf{a}=\mathbf{b}+i\mathbf{c}$ is arbitrary in $\mathbb{M}$, with $\mathbf{b},\mathbf{c}\in\mathbb{H}$ (see also [10]). Note that $\overline{\mathbf{a}+\mathbf{b}}=\bar{\mathbf{a}}+\bar{\mathbf{b}}$, and $\overline{\mathbf{ab}}=\bar{\mathbf{a}}\bar{\mathbf{b}}$, $\overline{r\mathbf{a}}=r\bar{\mathbf{a}}$ in particular, for all $\mathbf{a},\mathbf{b}\in\mathbb{M}$ and $r\in\mathbb{R}$. In addition, $\bar{\mathbf{a}}=\mathbf{a}$ if and only if $\mathbf{a}\in\mathbb{H}$, which characterizes the elements of $\mathbb{H}$ among those of $\mathbb{M}$.

Remark 3. In the algebra $\mathbb{M}$ we have the identities

$$(\lambda-\mathbf{x}^*)(\lambda-\mathbf{x})=(\lambda-\mathbf{x})(\lambda-\mathbf{x}^*)=\lambda^2-\lambda(\mathbf{x}+\mathbf{x}^*)+\|\mathbf{x}\|^2\in\mathbb{C},$$

for all $\lambda\in\mathbb{C}$ and $\mathbf{x}\in\mathbb{H}$. If the complex number $\lambda^2-2\lambda\Re\mathbf{x}+\|\mathbf{x}\|^2$ is nonnull, then both elements $\lambda-\mathbf{x}^*$, $\lambda-\mathbf{x}$ are invertible. Conversely, if $\lambda-\mathbf{x}$ is invertible, we must have $\lambda^2-2\lambda\Re\mathbf{x}+\|\mathbf{x}\|^2$ nonnull. In this way, the *spectrum* of a quaternion $\mathbf{x}\in\mathbb{H}$ is given by the equality $\sigma(\mathbf{x})=\{s_\pm(\mathbf{x})\}$, where $s_\pm(\mathbf{x})=\Re\mathbf{x}\pm i\|\Im\mathbf{x}\|$ are the *eigenvalues* of $\mathbf{x}$ (see also [1,2]).

We also need the concept of *resolvent set* of a quaternion $\mathbf{x}\in\mathbb{H}$, denoted by $\rho(\mathbf{x})$, and given by $\mathbb{C}\setminus\sigma(\mathbf{x})$.

The polynomial $P_{\mathbf{x}}(\lambda)=\lambda^2-2\lambda\Re\mathbf{x}+\|\mathbf{x}\|^2$ is the *minimal polynomial* of $\mathbf{x}$. In fact, the equality $\sigma(\mathbf{y})=\sigma(\mathbf{x})$ for some $\mathbf{x},\mathbf{y}\in\mathbb{H}$ is an equivalence relation in the algebra $\mathbb{H}$, which holds if and only if $P_{\mathbf{x}}=P_{\mathbf{y}}$.

Let $\mathbb{S}=\{\mathfrak{s}=x_1\mathbf{j}+x_2\mathbf{k}+x_3\mathbf{l};x_1,x_2,x_3\in\mathbb{R},x_1^2+x_2^2+x_3^2=1\}$, that is, the unit sphere of "purely imaginary" quaternions. It is clear that $\mathfrak{s}^*=-\mathfrak{s}$, and so $\mathfrak{s}^2=-1$, $\mathfrak{s}^{-1}=-\mathfrak{s}$, and $\|\mathfrak{s}\|=1$ for all $\mathfrak{s}\in\mathbb{S}$.

Representing an arbitrary quaternion $\mathbf{x}$ under the form $x_0+y_0\kappa_0$, with $x_0,y_0\in\mathbb{R}$ and $\kappa_0\in\mathbb{S}$, a quaternion $\mathbf{y}$ is equivalent to $\mathbf{x}$ if and only if it is of the form $x_0+y_0\kappa$ for some $\kappa\in\mathbb{S}$ (see [2] for some details).

Remark 4. As in [2], a subset $\Omega \in \mathbb{H}$ is said to be *spectrally saturated* (called *axially symmetric* in [3]) if for every $\mathbf{x} \in \mathbb{H}$ with $\sigma(\mathbf{x}) = \sigma(\mathbf{q})$ for some $\mathbf{q} \in \Omega$, we also have $\mathbf{x} \in \Omega$. For a conjugate symmetric set $U \subset \mathbb{C}$ we put $U_{\mathbb{H}} = \{\mathbf{q} \in \mathbb{H}; \sigma(\mathbf{q}) \subset U\}$, which is a spectrally saturated set. Conversely, if $\Omega \subset \mathbb{H}$ is a spectrally saturated set, we put $\mathfrak{S}(\Omega) = \{\zeta \in \mathbb{C}; \exists \mathbf{q} \in \Omega, \zeta \in \sigma(\mathbf{q})\}$, which is a conjugate symmetric set in $\mathbb{C}$. If $U = \mathfrak{S}(\Omega)$, we have $U_{\mathbb{H}} = \Omega$ and $\mathfrak{S}(\Omega) = U$. Moreover, U is open in $\mathbb{C}$ if and only if $U_{\mathbb{H}}$ is open in $\mathbb{H}$.

4.2. Slice holomorphic functions and the Cauchy transform

For $\mathbb{M}$-valued functions defined on subsets of $\mathbb{H}$, the concept of slice regularity is defined as follows.

Let $\Omega \subset \mathbb{H}$ be an open set, and let $F : \Omega \mapsto \mathbb{M}$ be a differentiable function. Following [3], Definition 4.1.1 (see also [13] for a more general context), we say that F is *slice right regular* on Ω if for all $\mathfrak{s} \in \mathbb{S}$,

$$\bar{\partial}_{\mathfrak{s}} F(x + y\mathfrak{s}) := \frac{1}{2}\left(\frac{\partial}{\partial x} + R_{\mathfrak{s}}\frac{\partial}{\partial y}\right) F(x + y\mathfrak{s}) = 0,$$

on the set $\Omega \cap (\mathbb{R} + \mathbb{R}\mathfrak{s})$, where $R_{\mathfrak{s}}$ is the right multiplication of the elements of $\mathbb{M}$ by $\mathfrak{s}$.

A slice left regularity can also be defined via the left multiplication of the elements of $\mathbb{M}$ by elements from $\mathbb{S}$. In what follows, we use only the slice right regularity, which will be simply called *slice regularity*. See also [4, 10–12] for other connections in the "slice" context.

Let us note that the convergent series of the form $\sum_{k\geq 0} a_k \mathbf{q}^k$, on quaternionic balls $\{\mathbf{q} \in \mathbb{H}; \|\mathbf{q}\| < r\}$, with $r > 0$ and $a_k \in \mathbb{H}$ for all $k \geq 0$, are $\mathbb{H}$-valued slice regular on their domain of definition. In fact, if actually $a_k \in \mathbb{M}$, such functions are ($\mathbb{M}$-valued) slice regular on their domain of definition.

Following [2], Definition 1, we consider the following concept.

Definition 2. The $\mathbb{M}$-*valued Cauchy kernel* on the open set $\Omega \subset \mathbb{H}$ is given by

$$\rho(\mathbf{q}) \times \Omega \ni (\zeta, \mathbf{q}) \mapsto (\zeta - \mathbf{q})^{-1} \in \mathbb{M}. \tag{4.1}$$

As shown in [2], Example 2, the $\mathbb{M}$-*valued Cauchy kernel* on the open set $\Omega \subset \mathbb{H}$ is slice regular.

Using the $\mathbb{M}$-valued Cauchy kernel, we may define a concept of *quaternionic Cauchy transform* (see [2], Section 5).

As before, for a given open set $U \subset \mathbb{C}$, we denote by $\mathcal{O}(U, \mathbb{M})$ the complex algebra of all $\mathbb{M}$-valued analytic functions on U. If $U \subset \mathbb{C}$ is open and conjugate symmetric, let $\mathcal{O}_s(U, \mathbb{M})$ be the real subalgebra of $\mathcal{O}(U, \mathbb{M})$ consisting of all $\mathbb{M}$-valued stem functions from $\mathcal{O}(U, \mathbb{M})$.

Because $\mathbb{C} \subset \mathbb{M}$, we have $\mathcal{O}(U) \subset \mathcal{O}(U, \mathbb{M})$, where $\mathcal{O}(U)$ is the complex algebra of all complex-valued analytic functions on the open set U. Similarly, when $U \subset \mathbb{C}$ is open and conjugate symmetric, $\mathcal{O}_s(U) \subset \mathcal{O}_s(U, \mathbb{M})$, where $\mathcal{O}_s(U)$ is the real subalgebra consisting of all functions f from $\mathcal{O}(U)$ which are complex stem functions.

For example, if $\Delta \subset \mathbb{C}$ is an open disk centered at 0, each function $F \in \mathcal{O}_s(\Delta, \mathbb{M})$ can be represented as a convergent series $F(\zeta) = \sum_{k\geq 0} a_k \zeta^k$, $\zeta \in \Delta$, with $a_k \in \mathbb{H}$ for all $k \geq 0$.

Definition 3. Let $U \subset \mathbb{C}$ be a conjugate symmetric open set, and let $F \in \mathcal{O}(U, \mathbb{M})$. For every $\mathbf{q} \in U_{\mathbb{H}}$ we set

$$C[F](\mathbf{q}) = \frac{1}{2\pi i}\int_{\Gamma} F(\zeta)(\zeta - \mathbf{q})^{-1} d\zeta, \tag{4.2}$$

where Γ is the boundary of a Cauchy domain in U containing the spectrum $\sigma(\mathbf{q})$. The function $C[F] : U_{\mathbb{H}} \mapsto \mathbb{M}$ is called the *(quaternionic) Cauchy transform* of the function $F \in \mathcal{O}(U, \mathbb{M})$. Clearly, the function $C[F]$ does not depend on the choice of Γ because the function

$$U \setminus \sigma(\mathbf{q}) \ni \zeta \mapsto F(\zeta)(\zeta - \mathbf{q})^{-1} \in \mathbb{M}$$

is analytic.

We put

$$\mathcal{R}(U_{\mathbb{H}}, \mathbb{M}) = \{C[F]; F \in \mathcal{O}(U, \mathbb{M})\}. \tag{4.3}$$

Proposition 1. *Let $U \subset \mathbb{C}$ be open and conjugate symmetric, and let $F \in \mathcal{O}(U, \mathbb{M})$. Then function $C[F] \in \mathcal{R}(U_{\mathbb{H}}, \mathbb{M})$ is slice regular on $U_{\mathbb{H}}$.*

The proof of this assertion can be found in [2], as Proposition 1.

A condition insuring that the Cauchy transform is actually $\mathbb{H}$-valued is obtained from the following result, proved as Theorem 4 in [2].

Theorem 3. *Let $U \subset \mathbb{C}$ be open and conjugate symmetric, and let F be in $\mathcal{O}(U, \mathbb{M})$. The Cauchy transform $C[F]$ is $\mathbb{H}$-valued if and only if F belongs to $\mathcal{O}_s(U, \mathbb{M})$.*

If $\Omega \subset \mathbb{H}$ be a spectrally saturated open set, we put

$$\mathcal{R}_s(\Omega, \mathbb{H}) = \{C[F]; F \in \mathcal{O}_s(U, \mathbb{M})\}, \tag{4.4}$$

where $U = \mathfrak{S}(\Omega)$.

Theorem 4. *Let $\Omega \subset \mathbb{H}$ be a spectrally saturated open set, and let $\Phi : \Omega \mapsto \mathbb{H}$. The function Φ is slice regular if and only if we have $\Phi \in \mathcal{R}_s(\Omega, \mathbb{H})$, with $U = \mathfrak{S}(\Omega)$. Moreover, the assignment*

$$\mathcal{O}_s(U, \mathbb{M}) \ni F \mapsto C[F] \in \mathcal{R}_s(\Omega, \mathbb{H})$$

is a linear isomorphism.

This statement follows from Theorems 5 and 6 from [2].

4.3. Quaternionic and complex spectra

Remark 5. Following [3], a *right $\mathbb{H}$-vector space $\mathcal{V}$* is a real vector space having a right multiplication with the elements of $\mathbb{H}$, such that $(x + y)\mathbf{q} = x\mathbf{q} + y\mathbf{q}$, $x(\mathbf{q} + \mathbf{s}) = x\mathbf{q} + x\mathbf{s}$, $x(\mathbf{q}\mathbf{s}) = (x\mathbf{q})\mathbf{s}$ for all $x, y \in \mathcal{V}$ and $\mathbf{q}, \mathbf{s} \in \mathbb{H}$.

If $\mathcal{V}$ is also a real Banach space, the operator $T \in \mathcal{B}(\mathcal{V})$ is *right $\mathbb{H}$-linear* if $T(x\mathbf{q}) = T(x)\mathbf{q}$ for all $x \in \mathcal{V}$ and $\mathbf{q} \in \mathbb{H}$. The set of right $\mathbb{H}$ linear operators will be denoted by $\mathcal{B}^{\mathrm{r}}(\mathcal{V})$, which is, in particular, a unital real algebra.

In a similar way, one defines the concept of a *left $\mathbb{H}$-vector space*. A real vector space $\mathcal{V}$ will be said to be an $\mathbb{H}$-*vector space* if it is simultaneously a right $\mathbb{H}$- and a left $\mathbb{H}$-vector space. As noticed in [3], the framework of $\mathbb{H}$-vector spaces is an appropriate one for the study of right $\mathbb{H}$-linear operators.

If $\mathcal{V}$ is an $\mathbb{H}$-vector space which is also a real Banach space, then $\mathcal{V}$ is said to be a *Banach $\mathbb{H}$-space*. In this case, we also assume that $R_{\mathbf{q}} \in \mathcal{B}(\mathcal{V})$, and the map $\mathbb{H} \ni \mathbf{q} \mapsto R_{\mathbf{q}} \in \mathcal{B}(\mathcal{V})$ is norm continuous, where $R_{\mathbf{q}}$ is the right multiplication of the elements of $\mathcal{V}$ by a given quaternion $\mathbf{q} \in \mathbb{H}$. Similarly, if $L_{\mathbf{q}}$ is the left multiplication of the elements of $\mathcal{V}$ by the quaternion $\mathbf{q} \in \mathbb{H}$, we assume that $L_{\mathbf{q}} \in \mathcal{B}(\mathcal{V})$ for all $\mathbf{q} \in \mathbb{H}$, and that the map $\mathbb{H} \ni \mathbf{q} \mapsto L_{\mathbf{q}} \in \mathcal{B}(\mathcal{V})$ is norm continuous. Note also that

$$\mathcal{B}^{\mathrm{r}}(\mathcal{V}) = \{T \in \mathcal{B}(\mathcal{V}); TR_{\mathbf{q}} = R_{\mathbf{q}}T,\ \mathbf{q} \in \mathbb{H}\}.$$

To adapt the discussion regarding the real algebras to this case, we first consider the complexification $\mathcal{V}_{\mathbb{C}}$ of $\mathcal{V}$. Because $\mathcal{V}$ is an $\mathbb{H}$-bimodule, the space $\mathcal{V}_{\mathbb{C}}$ is actually an $\mathbb{M}$-bimodule, via the multiplications

$$(\mathbf{q} + i\mathbf{s})(x + iy) = \mathbf{q}x - \mathbf{s}y + i(\mathbf{q}y + \mathbf{s}x), (x + iy)(\mathbf{q} + i\mathbf{s}) = x\mathbf{q} - y\mathbf{s} + i(y\mathbf{q} + x\mathbf{s}),$$

for all $\mathbf{q} + i\mathbf{s} \in \mathbb{M}$, $\mathbf{q}, \mathbf{s} \in \mathbb{H}$, $x + iy \in \mathcal{V}_{\mathbb{C}}$, $x, y \in \mathcal{V}$. Moreover, the operator $T_{\mathbb{C}}$ is right $\mathbb{M}$-linear, that is $T_{\mathbb{C}}((x + iy)(\mathbf{q} + i\mathbf{s})) = T_{\mathbb{C}}(x + iy)(\mathbf{q} + i\mathbf{s})$ for all $\mathbf{q} + i\mathbf{s} \in \mathbb{M}$, $x + iy \in \mathcal{V}_{\mathbb{C}}$, via a direct computation.

Let C be the natural conjugation of $\mathcal{V}_{\mathbb{C}}$. As in the real case, for every $S \in \mathcal{B}(\mathcal{V}_{\mathbb{C}})$, we put $S^{\flat} = CSC$. The left and right multiplication with the quaternion $\mathbf{q}$ on $\mathcal{V}_{\mathbb{C}}$ will also be denoted by $L_{\mathbf{q}}, R_{\mathbf{q}}$, respectively, as elements of $\mathcal{B}(\mathcal{V}_{\mathbb{C}})$. We set

$$\mathcal{B}^{\mathrm{r}}(\mathcal{V}_{\mathbb{C}}) = \{S \in \mathcal{B}(\mathcal{V}_{\mathbb{C}}); SR_{\mathbf{q}} = R_{\mathbf{q}}S,\ \mathbf{q} \in \mathbb{H}\},$$

which is a unital complex algebra containing all operators $L_{\mathbf{q}}, \mathbf{q} \in \mathbb{H}$. Note that if $S \in \mathcal{B}^{\mathrm{r}}(\mathcal{V}_{\mathbb{C}})$, then $S^{\flat} \in \mathcal{B}^{\mathrm{r}}(\mathcal{V}_{\mathbb{C}})$. Indeed, because $CR_{\mathbf{q}} = R_{\mathbf{q}}C$, we also have $S^{\flat}R_{\mathbf{q}} = R_{\mathbf{q}}S^{\flat}$. In fact, as we have $(S + S^{\flat})(\mathcal{V}) \subset \mathcal{V}$ and $i(S - S^{\flat})(\mathcal{V}) \subset \mathcal{V}$, it folows that the algebras $\mathcal{B}^{\mathrm{r}}(\mathcal{V}_{\mathbb{C}})$, $\mathcal{B}^{\mathrm{r}}(\mathcal{V})_{\mathbb{C}}$ are isomorphic, and they will often be identified, where $\mathcal{B}^{\mathrm{r}}(\mathcal{V})_{\mathbb{C}} = \mathcal{B}^{\mathrm{r}}(\mathcal{V}) + i\mathcal{B}^{\mathrm{r}}(\mathcal{V})$ is the complexification of $\mathcal{B}^{\mathrm{r}}(\mathcal{V})$, which is also a unital complex Banach algebra.

Inspired by the Definition 4.8.1 from [3] (see also [4]), we used in [1] the following.

Definition 4. For a given operator $T \in \mathcal{B}^{\mathrm{r}}(\mathcal{V})$, the set

$$\sigma_{\mathbb{H}}(T) := \{\mathbf{q} \in \mathbb{H}; T^2 - 2(\Re\mathbf{q})T + \|\mathbf{q}\|^2) \text{ not invertible}\}$$

is called the *quaternionic spectrum* (or simply the *Q-spectrum*) of T.

The complement $\rho_{\mathbb{H}}(T) = \mathbb{H} \setminus \sigma_{\mathbb{H}}(T)$ is called the *quaternionic resolvent* (or simply the *Q-resolvent*) of T.

We decided to call *Q-spectrum* the concept given by previous definition rather than *S-spectrum* (as in [3], and earlier introduced by F. Colombo and I. Sabadini), to stress its connection with the quaternionic algebra. In fact, in the construction of the analytic functional calculus for quaternionic linear operators, and unlike in [3, 4], this spectrum will be replaced by a strongly related one (see Lemma 2), nevertheless defined in the complex plane (see Theorem 7).

Note that, if $\mathbf{q} \in \sigma_{\mathbb{H}}(T))$, then $\{\mathbf{s} \in \mathbb{H}; \sigma(\mathbf{s}) = \sigma(\mathbf{q})\} \subset \sigma_{\mathbb{H}}(T)$. In other words, the spectrum $\sigma_{\mathbb{H}}(T)$ is a spectrally saturated set.

Assuming that $\mathcal{V}$ is a Banach $\mathbb{H}$-space, then $\mathcal{B}^{\mathrm{r}}(\mathcal{V})$ is a unital real Banach $\mathbb{H}$-algebra (that is, a Banach algebra which also a Banach $\mathbb{H}$-space), via the algebraic operations $(\mathbf{q}T)(x) = \mathbf{q}T(x)$, and $(T\mathbf{q})(x) = T(\mathbf{q}x)$ for all $\mathbf{q} \in \mathbb{H}$ and $x \in \mathcal{V}$. Hence the complexification $\mathcal{B}^{\mathrm{r}}(\mathcal{V})_{\mathbb{C}}$ is, in particular, a unital complex Banach algebra. Also note that the complex numbers, regarded as elements of $\mathcal{B}^{\mathrm{r}}(\mathcal{V})_{\mathbb{C}}$, commute with the elements of $\mathcal{B}^{\mathrm{r}}(\mathcal{V})$. For this reason, for each $T \in \mathcal{B}^{\mathrm{r}}(\mathcal{V})$ we have the resolvent set

$$\rho_{\mathbb{C}}(T) = \{\lambda \in \mathbb{C}; (T^2 - 2(\Re\lambda)T + |\lambda|^2)^{-1} \in \mathcal{B}^{\mathrm{r}}(\mathcal{V})\} =$$

$$\{\lambda \in \mathbb{C}; (\lambda - T_{\mathbb{C}})^{-1} \in \mathcal{B}^{\mathrm{r}}(\mathcal{V}_{\mathbb{C}})\} = \rho(T_{\mathbb{C}}),$$

and the associated spectrum $\sigma_{\mathbb{C}}(T) = \sigma(T_{\mathbb{C}})$.

One can see that there exists a strong connection between $\sigma_{\mathbb{H}}(T)$ and $\sigma_{\mathbb{C}}(T)$. Specifically, one can prove the following.

Lemma 2. *For every $T \in \mathcal{B}^{\mathrm{r}}(\mathcal{V})$ we have the equalities*

$$\sigma_{\mathbb{H}}(T) = \{\mathbf{q} \in \mathbb{H}; \sigma_{\mathbb{C}}(T) \cap \sigma(\mathbf{q}) \neq \emptyset\}, \tag{4.5}$$

and

$$\sigma_{\mathbb{C}}(T) = \{\lambda \in \sigma(\mathbf{q}); \mathbf{q} \in \sigma_{\mathbb{H}}(T)\}. \tag{4.6}$$

This statement, and its proof as well, can be found in [1] as Lemma 2. Moreover, as in [14], Remark 14, we can also prove that

$$\sigma_{\mathbb{H}}(T) = \{\Re(\lambda) + |\Im(\lambda)|; \lambda \in \sigma_{\mathbb{C}}(T), \mathbf{s} \in \mathbb{S}\}.$$

Remark. As expected, the set $\sigma_{\mathbb{H}}(T)$ is nonempty and bounded, which follows easily from Lemma 2. It is also compact, as a consequence of Definition 4, because the set of invertible elements in $\mathcal{B}^{\mathrm{r}}(\mathcal{V})$ is open.

4.4. Analytic functional calculus

If $\mathcal{V}$ is a real Banach $\mathbb{H}$-space, because $\mathcal{B}^{\mathrm{r}}(\mathcal{V})$ is also a real Banach space, each operator $T \in \mathcal{B}^{\mathrm{r}}(\mathcal{V})$ has a complex spectrum $\sigma_{\mathbb{C}}(T)$. Therefore, applying the corresponding result for real operators, we may construct an analytic functional calculus using the classical Riesz-Dunford functional calculus, in a slightly generalized form, as done in Theorem 2. In this new case, our basic complex algebra is $\mathcal{B}^{\mathrm{r}}(\mathcal{V})_{\mathbb{C}}$, endowed with the conjugation $\mathcal{B}^{\mathrm{r}}(\mathcal{V})_{\mathbb{C}} \ni S \mapsto S^{\flat} \in \mathcal{B}^{\mathrm{r}}(\mathcal{V})_{\mathbb{C}}$.

Theorem 5. *Let $U \subset \mathbb{C}$ be open and conjugate symmetric. If $F : U \mapsto \mathcal{B}^{\mathrm{r}}(\mathcal{V}_{\mathbb{C}})$ is analytic and $F(\zeta)^{\flat} = F(\bar{\zeta})$ for all $\zeta \in U$, then $F(T_{\mathbb{C}})^{\flat} = F(T_{\mathbb{C}})$ for all $T \in \mathcal{B}^{\mathrm{r}}(\mathcal{V})$ with $\sigma_{\mathbb{C}}(T) \subset U$.*

This assertion coincides with that of Theorem 3 from [1].

As in the real case, we may identify the algebra $\mathcal{B}^{\mathrm{r}}(\mathcal{V})$ with a subalgebra of $\mathcal{B}^{\mathrm{r}}(\mathcal{V})_{\mathbb{C}}$. So, when $F \in \mathcal{O}_s(U, \mathcal{B}^{\mathrm{r}}(\mathcal{V})_{\mathbb{C}}) = \{F \in \mathcal{O}(U, \mathcal{B}^{\mathrm{r}}(\mathcal{V})_{\mathbb{C}}); F(\bar{\zeta}) = F(\zeta)^{\flat}\ \forall \zeta \in U\}$ (see also Remark 2), we can write, via Theorem 5, that

$$F(T) = \frac{1}{2\pi i}\int_{\Gamma} F(\zeta)(\zeta - T)^{-1} d\zeta \in \mathcal{B}^{\mathrm{r}}(\mathcal{V}),$$

for a suitable choice of Γ.

The next result provides an *analytic functional calculus* for operators from the real algebra $\mathcal{B}^{\mathrm{r}}(\mathcal{V})$.

Theorem 6. *Let $\mathcal{V}$ be a real Banach $\mathbb{H}$-space, let $U \subset \mathbb{C}$ be a conjugate symmetric open set, and let $T \in \mathcal{B}^{\mathrm{r}}(\mathcal{V})$, with $\sigma_{\mathbb{C}}(T) \subset U$. Then, the map*

$$\mathcal{O}_s(U, \mathcal{B}^{\mathrm{r}}(\mathcal{V})_{\mathbb{C}}) \ni F \mapsto F(T) \in \mathcal{B}^{\mathrm{r}}(\mathcal{V})$$

is $\mathbb{R}$-linear, and the map

$$\mathcal{O}_s(U) \ni f \mapsto f(T) \in \mathcal{B}^{\mathrm{r}}(\mathcal{V})$$

is a unital real algebra morphism.

Moreover, the following properties are true:

(1) *For all $F \in \mathcal{O}_s(U, \mathcal{B}^{\mathrm{r}}(\mathcal{V})_{\mathbb{C}})$, $f \in \mathcal{O}_s(U)$, we have $(Ff)(T) = F(T)f(T)$, where $(Ff)(\zeta) = F(\zeta)f(\zeta)$ for all $\zeta \in U$.*

(2) *For every polynomial $P(\zeta) = \sum_{n=0}^{m} A_n\zeta^n$, $\zeta \in \mathbb{C}$, with $A_n \in \mathcal{B}^{\mathrm{r}}(\mathcal{V})$ for all $n = 0, 1, \ldots, m$, we have $P(T) = \sum_{n=0}^{m} A_n T^n \in \mathcal{B}^{\mathrm{r}}(\mathcal{V})$.*

This statement can be found in [1] as Theorem 4.

Remark 6. That Theorems 5 and 6 have practically the same proofs as Theorems 1 and 2 (respectively) is due to the fact that all of them can be obtained as particular cases of more general results. Indeed, considering a unital real Banach algebra $\mathcal{A}$, and its complexification $\mathcal{A}_{\mathbb{C}}$, identifying $\mathcal{A}$ with a real subalgebra of $\mathcal{A}_{\mathbb{C}}$, for a function $F \in \mathcal{O}_s(U, \mathcal{A}_{\mathbb{C}})$, where $U \subset \mathbb{C}$ is open and conjugate symmetric, the element $F(b) \in \mathcal{A}$ for each $b \in \mathcal{A}$ with $\sigma_{\mathbb{C}}(b) \subset U$. The assertion follows as in the proof of Theorem 1. The other results also have their counterparts. We omit the details.

Remark 7. The algebra $\mathbb{H}$ is clearly a Banach $\mathbb{H}$-space. Moreover, the left multiplications $L_{\mathbf{q}}$, $\mathbf{q} \in \mathbb{H}$, are elements of $\mathcal{B}^{\mathrm{r}}(\mathbb{H})$, and the map $\mathbb{H} \ni \mathbf{q} \mapsto L_{\mathbf{q}} \in \mathcal{B}^{\mathrm{r}}(\mathbb{H})$ is an injective morphism of real algebras allowing the identification of $\mathbb{H}$ with a subalgebra of $\mathcal{B}^{\mathrm{r}}(\mathbb{H})$.

If $\Omega \subset \mathbb{H}$ is a spectrally saturated open set, let $U = \mathfrak{S}(\Omega)$. Because $\mathcal{O}_s(U) \subset \mathcal{O}_s(U, \mathbb{M})$, we must have

$$\mathcal{R}_s(\Omega) := \{C[f]; f \in \mathcal{O}_s(U)\} \subset \mathcal{R}_s(\Omega, \mathbb{H}).$$

The next theorem, which is an analytic functional calculus for quaternions (see [2], Theorem 5), can be obtained as a particular case of Theorem 6.

Theorem 7. *Let $\Omega \subset \mathbb{H}$ be a spectrally saturated open set, and let $U = \mathfrak{S}(\Omega)$. The space $\mathcal{R}_s(\Omega)$ is a unital commutative real algebra, the space $\mathcal{R}_s(\Omega, \mathbb{H})$ is a right $\mathcal{R}_s(\Omega)$-module, the map*

$$\mathcal{O}_s(U, \mathbb{M}) \ni F \mapsto C[F] \in \mathcal{R}_s(\Omega, \mathbb{H})$$

is a right module isomorphism, and its restriction

$$\mathcal{O}_s(U) \ni f \mapsto C[f] \in \mathcal{R}_s(\Omega)$$

is a real algebra isomorphism.

Moreover, for every polynomial $P(\zeta) = \sum_{n=0}^{m} a_n\zeta^n$, $\zeta \in \mathbb{C}$, with $a_n \in \mathbb{H}$ for all $n = 0, 1, \ldots, m$, we have $C[P](\mathbf{q}) = \sum_{n=0}^{m} a_n q^n \in \mathbb{H}$ for all $q \in \mathbb{H}$.

For other details concerning this assertion see Theorem 5 from [1].

Remark 8. It is shown in [2], Theorem 6, that the space $\mathcal{R}_s(\Omega,\mathbb{H})$ coincides with the space of all $\mathbb{H}$-valued functions which are *slice regular*, independently defined in [3], Definition 4.1.1. Such functions are used in [3] to define a quaternionic functional calculus for quaternionic linear operators (see also [4]). This construction is largely explained in the fourth chapter of [3].

Our Theorem 6 constructs an analytic functional calculus with functions from $\mathcal{O}_s(U,\mathcal{B}^{\mathrm{r}}(\mathcal{V})_{\mathbb{C}})$, where U is a a neighborhood of the complex spectrum of a given quaternionic linear operator, leading to another quaternionic linear operator, replacing formally the complex variable with that operator.

Identifying the algebra $\mathbb{H}$ with a subalgebra of $\mathcal{B}^{\mathrm{r}}(\mathcal{V})_{\mathbb{C}}$ by regarding the elements of $\mathbb{H}$ as left multiplication operators, we can show that the functional calculus from [3] is equivalent to our functional calculus with functions from $\mathcal{O}_s(U,\mathbb{M})$ in the sense that they have the same images. This is a consequence of the fact that the class of regular quaternionic-valued functions $\mathcal{R}_s(\Omega,\mathbb{H})$ used by the construction in [3] is isomorphic to the class of analytic functions $\mathcal{O}_s(U,\mathbb{M})$, via Theorem 7. Our approach looks simpler and there is a stronger connection with the classical approach, because we use spectra defined in the complex plane, and our Cauchy type kernels are partially commutative.

As in [1], Remark 8, a direct argument concerning the equivalence of those analytic functional calculi can be given. For an operator $T\in\mathcal{B}^{\mathrm{r}}(\mathcal{V})$, the *right S-resolvent* is defined via the formula

$$S_R^{-1}(\mathbf{s},T)=-(T-\mathbf{s}^*)(T^2-2\Re(\mathbf{s})T+\|\mathbf{s}\|)^{-1},\ \mathbf{s}\in\rho_{\mathbb{H}}(T) \tag{4.7}$$

(see [3], formula (4.47)). Fixing an element $\kappa\in\mathbb{S}$, and a spectrally saturated open set $\Omega\subset\mathbb{H}$, for $\Phi\in\mathcal{R}_s(\Omega,\mathbb{H})$ one puts

$$\Phi(T)=\frac{1}{2\pi}\int_{\partial(\Sigma_\kappa)}\Phi(\mathbf{s})d\mathbf{s}_\kappa S_R^{-1}(\mathbf{s},T), \tag{4.8}$$

where $\Sigma\subset\Omega$ is a spectrally saturated open set containing $\sigma_{\mathbb{H}}(T)$, such that $\Sigma_\kappa=\{u+v\kappa\in\Sigma;u,v\in\mathbb{R}\}$ is a subset whose boundary $\partial(\Sigma_\kappa)$ consists of a finite family of closed curves, piecewise smooth, positively oriented, and $d\mathbf{s}_\kappa=-\kappa du\wedge dv$. Formula (4.8) is a (right) quaternionic functional calculus, as defined in [3], Section 4.10.

Because the space $\mathcal{V}_{\mathbb{C}}$ is also an $\mathbb{H}$-space, we may extend these formulas to the operator $T_{\mathbb{C}}\in\mathcal{B}^{\mathrm{r}}(\mathcal{V}_{\mathbb{C}})$, replacing the operator T by $T_{\mathbb{C}}$ in formulas (4.7) and (4.8). For the function $\Phi\in\mathcal{R}_s(\Omega,\mathbb{H})$ there exists a function $F\in\mathcal{O}_s(U,\mathbb{M})$ such that $C[F]=\Phi$. Denoting by Γ_κ the boundary of a Cauchy domain in $U\subset\mathbb{C}$ containing the compact set $\cup\{\sigma(\mathbf{s});\mathbf{s}\in\overline{\Sigma_\kappa}\}$, we have

$$\Phi(T_{\mathbb{C}})=\frac{1}{2\pi}\int_{\partial(\Sigma_\kappa)}\left(\frac{1}{2\pi i}\int_{\Gamma_\kappa}F(\zeta)(\zeta-\mathbf{s})^{-1}d\zeta\right)d\mathbf{s}_\kappa S_R^{-1}(\mathbf{s},T_{\mathbb{C}})=$$
$$\frac{1}{2\pi i}\int_{\Gamma_\kappa}F(\zeta)\left(\frac{1}{2\pi}\int_{\partial(\Sigma_\kappa)}(\zeta-\mathbf{s})^{-1}d\mathbf{s}_\kappa S_R^{-1}(\mathbf{s},T_{\mathbb{C}})\right)d\zeta. \tag{4.9}$$

The intertwining of the integral in formula (4.9) can be explained in the following way. We consider the parametrizations $\phi:[0,1]\mapsto\mathbb{C}$ and $\psi:[0,1]\mapsto\mathbb{C}_\kappa:=\{(a+b\kappa;a,b\in\mathbb{R}\}$ of the curves Γ_κ and $\partial(\Sigma_\kappa)$ respectively, having continuous derivatives except a finite number of points. We also put $\psi_\kappa=-\kappa\psi$. Note that these functions commute, because, in our framework, complex numbers and quaternions commute. As $d\zeta=\phi'(u)du, d\mathbf{s}_\kappa=\psi'_\kappa(v)dv, u,v\in[0,1]$, we have

$$\frac{1}{2\pi}\int_{\partial(\Sigma_\kappa)}\left(\frac{1}{2\pi i}\int_{\Gamma_\kappa}F(\zeta)(\zeta-\mathbf{s})^{-1}d\zeta\right)d\mathbf{s}_\kappa S_R^{-1}(\mathbf{s},T_{\mathbb{C}}) =$$

$$\frac{1}{2\pi}\int_0^1\left(\frac{1}{2\pi i}\int_0^1 F(\phi(u)))(\phi(u)-\psi(v))^{-1}\phi'(u)du\right)\psi_\kappa'(v)S_R^{-1}(\psi(v),T_{\mathbb{C}})dv =$$

$$\frac{1}{2\pi}\int_0^1\left(\frac{1}{2\pi i}\int_0^1 F(\phi(u)))(\phi(u)-\psi(v))^{-1}\phi'(u)\psi_\kappa'(v)S_R^{-1}(\psi(v),T_{\mathbb{C}})du\right)dv =$$

$$\frac{1}{2\pi}\int_0^1\left(\frac{1}{2\pi i}\int_0^1 F(\phi(u)))(\phi(u)-\psi(v))^{-1}\psi_\kappa'(v)S_R^{-1}(\psi(v),T_{\mathbb{C}})\phi'(u)du\right)dv =$$

$$\frac{1}{2\pi i}\int_0^1\left(\frac{1}{2\pi}\int_0^1 F(\phi(u)))(\phi(u)-\psi(v))^{-1}\psi_\kappa'(v)S_R^{-1}(\psi(v),T_{\mathbb{C}})dv\right)\phi'(u)du =$$

$$\frac{1}{2\pi i}\int_{\Gamma_\kappa}F(\zeta)\left(\frac{1}{2\pi}\int_{\partial(\Sigma_\kappa)}(\zeta-\mathbf{s})^{-1}d\mathbf{s}_\kappa S_R^{-1}(\mathbf{s},T_{\mathbb{C}})\right)d\zeta,$$

via Fubini's theorem. Then we have

$$\frac{1}{2\pi i}\int_{\Gamma_\kappa}F(\zeta)\left(\frac{1}{2\pi}\int_{\partial(\Sigma_\kappa)}(\zeta-\mathbf{s})^{-1}d\mathbf{s}_\kappa S_R^{-1}(\mathbf{s},T_{\mathbb{C}})\right)d\zeta \tag{4.10}$$

$$\frac{1}{2\pi i}\int_{\Gamma_\kappa}F(\zeta)(\zeta-T_{\mathbb{C}})^{-1}d\zeta.$$

To obtain formula (4.10), we use an argument from [1], Remark 9. Specifically, it follows from the complex linearity of $S_R^{-1}(\mathbf{s},T_{\mathbb{C}})$, and from formula (4.49) in [3], that

$$(\zeta-\mathbf{s})S_R^{-1}(\mathbf{s},T_{\mathbb{C}}) = S_R^{-1}(\mathbf{s},T_{\mathbb{C}})(\zeta-T_{\mathbb{C}})-1,$$

whence

$$(\zeta-\mathbf{s})^{-1}S_R^{-1}(\mathbf{s},T_{\mathbb{C}}) = S_R^{-1}(\mathbf{s},T_{\mathbb{C}})(\zeta-T_{\mathbb{C}})^{-1}+(\zeta-\mathbf{s})^{-1}(\zeta-T_{\mathbb{C}})^{-1},$$

and therefore,

$$\frac{1}{2\pi}\int_{\partial(\Sigma_\kappa)}(\zeta-\mathbf{s})^{-1}d\mathbf{s}_\kappa S_R^{-1}(\mathbf{s},T_{\mathbb{C}}) = \frac{1}{2\pi}\int_{\partial(\Sigma_\kappa)}d\mathbf{s}_\kappa S_R^{-1}(\mathbf{s},T_{\mathbb{C}})(\zeta-T_{\mathbb{C}})^{-1}+$$

$$\frac{1}{2\pi}\int_{\partial(\Sigma_\kappa)}(\zeta-\mathbf{s})^{-1}d\mathbf{s}_\kappa(\zeta-T_{\mathbb{C}})^{-1} = (\zeta-T_{\mathbb{C}})^{-1},$$

because

$$\frac{1}{2\pi}\int_{\partial(\Sigma_\kappa)}d\mathbf{s}_\kappa S_R^{-1}(\mathbf{s},T_{\mathbb{C}}) = 1 \text{ and } \frac{1}{2\pi}\int_{\partial(\Sigma_\kappa)}(\zeta-\mathbf{s})^{-1}d\mathbf{s}_\kappa = 0,$$

as in Theorem 4.8.11 from [3], since the $\mathbb{M}$-valued function $\mathbf{s}\mapsto(\zeta-\mathbf{s})^{-1}$ is analytic in a neighborhood of the set $\overline{\Sigma_\kappa}\subset\mathbb{C}_\kappa$ for each $\zeta\in\Gamma_\kappa$, respectively.

Consequently, $\Phi(T)=\Phi(T_{\mathbb{C}})_{|\mathcal{V}}=F(T_{\mathbb{C}})_{|\mathcal{V}}=F(T)$

In particular, regarding a quaternion $\mathbf{q}\in\mathbb{H}$ as a left multiplication operator, formula (4.8), written as

$$\Phi(\mathbf{q}) = \frac{1}{2\pi}\int_{\partial(\Sigma_\kappa)}\Phi(\mathbf{s})d\mathbf{s}_\kappa S_R^{-1}(\mathbf{s},\mathbf{q}), \tag{4.11}$$

is formula (2.39) from [3] in the quaternionic case, via Proposition 2.7.20, with a more direct proof.

Remark 9. Our approach permits to prove a version of the *spectral mapping theorem* in a classical style, via direct arguments (see also [3, 4] for a different context). For every operator $T \in \mathcal{B}^{\mathrm{r}}(\mathcal{V})$ and each function $\Phi \in \mathcal{R}_s(\Omega)$ one has $\sigma_{\mathbb{H}}(\Phi(T)) = \Phi(\sigma_{\mathbb{H}}(T))$, via Theorem 3.5.9 from [3]. Using our approach, for every function $f \in \mathcal{O}_s(U)$, one has $f(\sigma_{\mathbb{C}}(T)) = \sigma_{\mathbb{C}}(f(T))$, directly from the corresponding (classical) spectral mapping theorem in [9]. This result is parallel to that from [3] mentioned above, also giving an explanation for the former, via the isomorphism of the spaces $\mathcal{O}_s(U)$ and $\mathcal{R}_s(\Omega)$ by Theorem 7.

5. Some examples

Example 1. This is Example 2 from [1]. The space $\mathcal{V} = \mathbb{H}$ is itself a (simple) Banach $\mathbb{H}$-space. Because $\mathcal{V}_{\mathbb{C}} = \mathbb{M}$, for a fixed element $\mathbf{q} \in \mathbb{H}$, we may consider the operator $L_{\mathbf{q}} \in \mathcal{B}^{\mathrm{r}}(\mathbb{H})$, whose complex spectrum is given by $\sigma_{\mathbb{C}}(L_{\mathbf{q}}) = \sigma(\mathbf{q}) = \{\Re\mathbf{q} \pm i\|\Im\mathbf{q}\|\}$. If $U \subset \mathbb{C}$ is a conjugate symmetric open set containing $\sigma_{\mathbb{C}}(L_{\mathbf{q}})$, and $F \in \mathcal{O}_s(U, \mathbb{M})$, then we have

$$F(L_{\mathbf{q}}) = F(s_+(\mathbf{q}))\iota_+(\mathfrak{s}_{\tilde{\mathbf{q}}}) + F(s_-(\mathbf{q}))\iota_-(\mathfrak{s}_{\tilde{\mathbf{q}}}) \in \mathbb{M}, \tag{5.1}$$

where we have $s_\pm(\mathbf{q}) = \Re\mathbf{q} \pm i\|\Im\mathbf{q}\|$, $\tilde{\mathbf{q}} = \Im\mathbf{q}$, $\mathfrak{s}_{\tilde{\mathbf{q}}} = \tilde{\mathbf{q}}\|\tilde{\mathbf{q}}\|^{-1}$, and $\iota_\pm(\mathfrak{s}_{\tilde{\mathbf{q}}}) = 2^{-1}(1 \mp i\mathfrak{s}_{\tilde{\mathbf{q}}})$, provided $\tilde{\mathbf{q}} \neq 0$, via [2], Remark 3. The case $\tilde{\mathbf{q}} = 0$ is trivial.

Of course, this formula can be extended to a larger class of functions.

Example 2. This is Example 3 from [1]. Let $C(\mathfrak{X}, \mathbb{M})$ be the space of $\mathbb{M}$-valued continuous functions on the compact space $\mathfrak{X}$. Then the space $C(\mathfrak{X}, \mathbb{H})$, consisting of $\mathbb{H}$-valued functions, is the real subspace of $C(\mathfrak{X}, \mathbb{M})$, which is also a Banach $\mathbb{H}$-space with respect to the operations $(\mathbf{q}F)(x) = \mathbf{q}F(x)$ and $(F\mathbf{q})(x) = F(x)\mathbf{q}$ for all $F \in C(\mathfrak{X}, \mathbb{H})$ and $x \in \mathfrak{X}$. Moreover, $C(\mathfrak{X}, \mathbb{H})_{\mathbb{C}} = C(\mathfrak{X}, \mathbb{H}_{\mathbb{C}}) = C(\mathfrak{X}, \mathbb{M})$.

Fixing a function $\Theta \in C(\mathfrak{X}, \mathbb{H})$, we define the operator $T \in \mathcal{B}(C(\mathfrak{X}, \mathbb{H}))$ by the equality $(TF)(x) = \Theta(x)F(x)$ for all $F \in C(\mathfrak{X}, \mathbb{H})$ and $x \in \mathfrak{X}$. Note that $(T(F\mathbf{q}))(x) = \Theta(x)F(x)\mathbf{q} = ((TF)\mathbf{q})(x)$ for all $F \in C(\mathfrak{X}, \mathbb{H}), \mathbf{q} \in \mathbb{H}$, and $x \in \mathfrak{X}$. In other words, $T \in \mathcal{B}^{\mathrm{r}}(C(\mathfrak{X}, \mathbb{H}))$. Note also that the operator T is invertible if and only if the function Θ has no zero in $\mathfrak{X}$.

According to Definition 4, we have

$$\rho_{\mathbb{H}}(T) = \{\mathbf{q} \in \mathbb{H}; (T^2 - 2\Re\mathbf{q}\, T + \|\mathbf{q}\|^2)^{-1} \in \mathcal{B}^{\mathrm{r}}(C(\mathfrak{X}, \mathbb{H}))\}.$$

Consequently, $\mathbf{q} \in \sigma_{\mathbb{H}}(T)$ if and only if zero is in the range of the function

$$\tau(\mathbf{q}, x) := \Theta(x)^2 - 2\Re\mathbf{q}\, \Theta(x) + \|\mathbf{q}\|^2,\ x \in \mathfrak{X}.$$

Similarly,

$$\rho_{\mathbb{C}}(T) = \{\lambda \in \mathbb{C}; (T^2 - 2\Re\lambda\, T + \|\lambda\|^2)^{-1} \in \mathcal{B}^{\mathrm{r}}(C(\mathfrak{X}, \mathbb{H}))\},$$

and so $\lambda \in \sigma_{\mathbb{C}}(T)$ if and only if zero is in the range of the function

$$\tau(\lambda, x) := \Theta(x)^2 - 2\Re\lambda\, \Theta(x) + |\lambda|^2,\ x \in \mathfrak{X}.$$

Looking for solutions $u + iv, u, v \in \mathbb{R}$, of the equation $(u - \Theta(x))^2 + v^2 = 0$, a direct calculation shows that $u = \Re\Theta(x)$ and $v = \pm\|\Im\Theta(x)\|$. Hence,

$$\sigma_{\mathbb{C}}(T) = \{\Re\Theta(x) \pm i\|\Im\Theta(x)\|; x \in \mathfrak{X}\} = \cup_{x\in\mathfrak{X}}\sigma(\Theta(x)).$$

Moreover, for every open conjugate symmetric subset $U \subset \mathbb{C}$ containing $\sigma_{\mathbb{C}}(T)$, and for every function $\Phi \in \mathcal{O}_c(U, \mathscr{B}^{\mathrm{r}}(C(\mathfrak{X}, \mathbb{M})))$, we may construct the operator $\Phi(T) \in \mathscr{B}^{\mathrm{r}}(C(\mathfrak{X}, \mathbb{H}))$, via Theorem 6.

Example 3. Now, we show that the non-commutative Cauchy kernel from [3] is the Cauchy transform of the complex Cauchy kernel associated to a quaternion.

Let $\mathbf{s}, \mathbf{q} \in \mathbb{H}$ with $\sigma(\mathbf{s}) \cap \sigma(\mathbf{q}) = \emptyset$, and so $\mathbf{s} \neq \mathbf{q}$. In particular, the quaternion $\mathbf{s}^2 - 2\Re(\mathbf{q})\mathbf{s} + \|\mathbf{q}\|^2$ is invertible. Indeed, if $\zeta = \Re\mathbf{q} + i\|\Im\mathbf{q}\| \in \sigma(\mathbf{q})$, we have in $\mathbb{M}$

$$\mathbf{s}^2 - 2\Re(\mathbf{q})\mathbf{s} + \|\mathbf{q}\|^2 = (\mathbf{s} - \zeta)(\mathbf{s} - \bar{\zeta}),$$

and both $\mathbf{s} - \zeta, \mathbf{s} - \bar{\zeta}$ are invertible because $\zeta \notin \sigma(\mathbf{s})$.

Let us consider the function

$$S_R^{-1}(\mathbf{q}, \mathbf{s}) = -(\mathbf{s} - \mathbf{q}^*)(\mathbf{s}^2 - 2\Re(\mathbf{q})\mathbf{s} + \|\mathbf{q}\|^2)^{-1},$$

which is the *right noncommutative Cauchy kernel*, as defined in [3], as formula (2.33).

Note also that the function $\rho(\mathbf{q}) \ni \zeta \mapsto (\zeta - \mathbf{q})^{-1} \in \mathbb{M}$ is in the space $\mathcal{O}_s(\rho(\mathbf{q}), \mathbb{M})$.

We can show the equality

$$S_R^{-1}(\mathbf{q}, \mathbf{s}) = -\frac{1}{2\pi i}\int_{\Gamma_{\mathbf{s}}} (\zeta - \mathbf{q})^{-1}(\zeta - \mathbf{s})^{-1} d\zeta,$$

where $\Gamma_{\mathbf{s}}$ surrounds a Cauchy domain containing $\sigma(\mathbf{s})$, whose closure is disjoint of $\sigma(\mathbf{q})$. Indeed,

$$\frac{1}{2\pi i}\int_{\Gamma_{\mathbf{s}}} (\zeta - \mathbf{q})^{-1}(\zeta - \mathbf{s})^{-1} d\zeta =$$
$$\frac{1}{2\pi i}\int_{\Gamma_{\mathbf{s}}} [(\zeta - \mathbf{q})(\zeta - \mathbf{q}^*)]^{-1}(\zeta - \mathbf{q}^*)(\zeta - \mathbf{s})^{-1} d\zeta =$$
$$\frac{1}{2\pi i}\int_{\Gamma_{\mathbf{s}}} (\zeta^2 - 2\zeta\Re(\mathbf{q}) + \|\mathbf{q}\|^2)^{-1}(\zeta - \mathbf{q}^*)(\zeta - \mathbf{s})^{-1} d\zeta =$$
$$(\mathbf{s} - \mathbf{q}^*)(\mathbf{s}^2 - 2\Re(\mathbf{q})\mathbf{s} + \|\mathbf{q}\|^2)^{-1},$$

showing that the kernel $S_R^{-1}(\mathbf{s}, \mathbf{q})$ is the Cauchy transform of the function $\rho(\mathbf{q}) \ni \zeta \mapsto -(\zeta - \mathbf{q})^{-1} \in \mathbb{M}$.

6. Quaternionic joint spectrum of pairs

Sometimes, especially in applications, it is more convenient to work with matrix quaternions rather than with abstract quaternions. Specifically, one considers the injective unital algebra morphism

$$\mathbb{H} \ni x_1 + y_1\mathbf{j} + x_2\mathbf{k} + y_2\mathbf{l} \mapsto \begin{pmatrix} x_1 + iy_1 & x_2 + iy_2 \\ -x_2 + iy_2 & x_1 - iy_1 \end{pmatrix} \in \mathbb{M}_2,$$

with $x_1, y_1, x_2, y_2 \in \mathbb{R}$, where $\mathbb{M}_2$ is the complex algebra of 2×2 matrices, whose image, denoted by $\mathbb{H}_2$ is the real algebra of matrix quaternions. The elements of $\mathbb{H}_2$ can be also written as matrices of the form

$$Q(\mathbf{z}) = \begin{pmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{pmatrix}, \ \mathbf{z} = (z_1, z_2) \in \mathbb{C}^2.$$

A natural connection between the spectral theory of pairs of commuting operators in a complex Hilbert space and the algebra of quaternions has firstly been noticed in [15]. Another connection will be presented in the following.

If $\mathcal{V}$ is an arbitrary vector space, we denote by $\mathcal{V}^2$ the Cartesian product $\mathcal{V} \times \mathcal{V}$.

Let $\mathcal{V}$ be a real Banach space, and let $\mathbf{T} = (T_1, T_2) \in \mathcal{B}(\mathcal{V})^2$ be a pair of commuting operators. The extended pair $\mathbf{T}_{\mathbb{C}} = (T_{1\mathbb{C}}, T_{2\mathbb{C}}) \in \mathcal{B}(\mathcal{V}_{\mathbb{C}})^2$ also consists of commuting operators. For simplicity, we set

$$Q(\mathbf{T}_{\mathbb{C}}) := \begin{pmatrix} T_{1\mathbb{C}} & T_{2\mathbb{C}} \\ -T_{2\mathbb{C}} & T_{1\mathbb{C}} \end{pmatrix}$$

which acts on the complex Banach space $\mathcal{V}_{\mathbb{C}}^2$.

One can define the quaternionic resolvent set and spectrum for the case of a pair of operators (see [1], Definition 2), inspired by the case of a single operator (see [15]).

Definition 5. Let $\mathcal{V}$ be a real Banach space. For a given pair $\mathbf{T} = (T_1, T_2) \in \mathcal{B}(\mathcal{V})^2$ of commuting operators, the set of those $Q(\mathbf{z}) \in \mathbb{H}_2$, $\mathbf{z} = (z_1, z_2) \in \mathbb{C}^2$, such that the operator

$$T_1^2 + T_2^2 - 2\Re(z_1)T_1 - 2\Re(z_2)T_2 + |z_1|^2 + |z_2|^2$$

is invertible in $\mathcal{B}(\mathcal{V})$ is said to be the *quaternionic joint resolvent* (or simply the *Q-joint resolvent*) of $\mathbf{T}$, and it is denoted by $\rho_{\mathbb{H}}(\mathbf{T})$.

The complement $\sigma_{\mathbb{H}}(\mathbf{T}) = \mathbb{H}_2 \setminus \rho_{\mathbb{H}}(\mathbf{T})$ is called the *quaternionic joint spectrum* (or simply the *Q-joint spectrum*) of $\mathbf{T}$.

For every pair $\mathbf{T}_{\mathbb{C}} = (T_{1\mathbb{C}}, T_{2\mathbb{C}}) \in \mathcal{B}(\mathcal{V}_{\mathbb{C}})^2$ we put $\mathbf{T}_{\mathbb{C}}^c = (T_{1\mathbb{C}}, -T_{2\mathbb{C}}) \in \mathcal{B}(\mathcal{V}_{\mathbb{C}})^2$, and for every pair $\mathbf{z} = (z_1, z_2) \in \mathbb{C}^2$ we put $\mathbf{z}^c = (\bar{z}_1, -z_2) \in \mathbb{C}^2$.

Lemma 3. *A matrix quaternion $Q(\mathbf{z})\,(\mathbf{z} \in \mathbb{C}^2)$ is in the set $\rho_{\mathbb{H}}(\mathbf{T})$ if and only if the operators $Q(\mathbf{T}_{\mathbb{C}}) - Q(\mathbf{z})$, $Q(\mathbf{T}_{\mathbb{C}}^c) - Q(\mathbf{z}^c)$ are invertible in $\mathcal{B}(\mathcal{V}_{\mathbb{C}}^2)$.*

The complete proof of this assertion can be found in [1], as Lemma 3, and it is based on the equalities

$$\begin{pmatrix} T_{1\mathbb{C}} - z_1 & T_{2\mathbb{C}} - z_2 \\ -T_{2\mathbb{C}} + \bar{z}_2 & T_{1\mathbb{C}} - \bar{z}_1 \end{pmatrix} \begin{pmatrix} T_{1\mathbb{C}} - \bar{z}_1 & -T_{2\mathbb{C}} + z_2 \\ T_{2\mathbb{C}} - \bar{z}_2 & T_{1\mathbb{C}} - z_1 \end{pmatrix} =$$

$$\begin{pmatrix} T_{1\mathbb{C}} - \bar{z}_1 & -T_{2\mathbb{C}} + z_2 \\ T_{2\mathbb{C}} - \bar{z}_2 & T_{1\mathbb{C}} - z_1 \end{pmatrix} \begin{pmatrix} T_{1\mathbb{C}} - z_1 & T_{2\mathbb{C}} - z_2 \\ -T_{2\mathbb{C}} + \bar{z}_2 & T_{1\mathbb{C}} - \bar{z}_1 \end{pmatrix} =$$

$$[(T_{1\mathbb{C}} - z_1)(T_{1\mathbb{C}} - \bar{z}_1) + (T_{2\mathbb{C}} - z_2)(T_{2\mathbb{C}} - \bar{z}_2)]\mathbf{I},$$

for all $\mathbf{z} = (z_1, z_2) \in \mathbb{C}^2$, where $\mathbf{I}$ is the identity of $\mathcal{B}(\mathcal{V}_{\mathbb{C}})^2$.

Lemma 3 shows that we have the property $Q(\mathbf{z}) \in \sigma_{\mathbb{H}}(\mathbf{T})$ if and only if $Q(\mathbf{z}^c) \in \sigma_{\mathbb{H}}(\mathbf{T}^c)$. Putting

$$\sigma_{\mathbb{C}^2}(\mathbf{T}) := \{\mathbf{z} \in \mathbb{C}^2; Q(\mathbf{z}) \in \sigma_{\mathbb{H}}(\mathbf{T})\},$$

the set $\sigma_{\mathbb{C}^2}(\mathbf{T})$ has a similar property, specifically, $\mathbf{z} \in \sigma_{\mathbb{C}^2}(\mathbf{T})$ if and only if $\mathbf{z}^{\mathbf{c}} \in \sigma_{\mathbb{C}^2}(\mathbf{T}^{\mathbf{c}})$.

Remark 10. The extended pair $\mathbf{T}_{\mathbb{C}} = (T_{1\mathbb{C}}, T_{2\mathbb{C}}) \in B(V_{\mathbb{C}})^2$ of the commuting pair $\mathbf{T} = (T_1, T_2) \in \mathcal{B}(\mathcal{V})$ can be discussed in connection with the *joint spectral theory* of J. L. Taylor (see [16,17]; see also [19]). Namely, if the operator $T_{1\mathbb{C}}^2 + T_{2\mathbb{C}}^2 - 2\Re(z_1)T_{1\mathbb{C}} - 2\Re(z_2)T_{2\mathbb{C}} + |z_1|^2 + |z_2|^2$ is invertible, then the point $\mathbf{z} = (z_1, z_2)$ belongs to the joint resolvent of $\mathbf{T}_{\mathbb{C}}$ (see Remark 9 from [1]). We also have that the point $\mathbf{z}^c$ belongs to the joint resolvent of $\mathbf{T}_{\mathbb{C}}^c$. In addition, if $\sigma(\mathbf{T}_{\mathbb{C}})$ designates the Taylor spectrum of $\mathbf{T}_{\mathbb{C}}$, we have the inclusion $\sigma(\mathbf{T}_{\mathbb{C}}) \subset \sigma_{\mathbb{C}^2}(\mathbf{T})$. In particular, for every complex-valued function f analytic in a neighborhood of $\sigma_{\mathbb{C}^2}(\mathbf{T})$, the operator $f(\mathbf{T}_{\mathbb{C}})$ can be computed via Taylor's analytic functional calculus. In fact, we have a Martinelli type formula for the analytic functional calculus:

Theorem 8. *Let $\mathcal{V}$ be a real Banach space, let $\mathbf{T} = (T_1, T_2) \in \mathcal{B}(\mathcal{V})^2$ be a pair of commuting operators, let $U \subset \mathbb{C}^2$ be an open set, let $D \subset U$ be a bounded domain containing $\sigma_{\mathbb{C}^2}(\mathbf{T})$, with piecewise-smooth boundary Σ, and let $f \in \mathcal{O}(U)$. Then, we have*

$$f(\mathbf{T}_{\mathbb{C}}) = \frac{1}{(2\pi i)^2} \int_{\Sigma} f(\mathbf{z}) L(\mathbf{z}, \mathbf{T}_{\mathbb{C}})^{-2} [(\bar{z}_1 - T_{1\mathbb{C}}) d\bar{z}_2 - (\bar{z}_2 - T_{2\mathbb{C}}) d\bar{z}_1] dz_1 dz_2$$

where

$$L(\mathbf{z}, \mathbf{T}_{\mathbb{C}}) = T_{1\mathbb{C}}^2 + T_{2\mathbb{C}}^2 - 2\Re(z_1)T_{1\mathbb{C}} - 2\Re(z_2)T_{2\mathbb{C}} + |z_1|^2 + |z_2|^2.$$

The proof of this result can be found in [1], Theorem 6.

Remark 11. (1) The previous functional calculus can be extended to $\mathcal{B}(\mathcal{V}_{\mathbb{C}})$-valued analytic functions, setting, for such a function F, and with the notation from above,

$$F(\mathbf{T}_{\mathbb{C}}) = \frac{1}{(2\pi i)^2} \int_{\Sigma} F(\mathbf{z}) L(\mathbf{z}, \mathbf{T}_{\mathbb{C}})^{-2} [(\bar{z}_1 - T_{1\mathbb{C}}) d\bar{z}_2 - (\bar{z}_2 - T_{2\mathbb{C}}) d\bar{z}_1] dz_1 dz_2$$

(see [1] Remark 10(1)).

In particular, if $F(\mathbf{z}) = \sum_{j,k \geq 0} A_{jk\mathbb{C}} z_1^j z_2^k$, with $A_{j,k} \in \mathcal{B}(\mathcal{V})$, where the series is convergent in a neighbourhood of $\sigma_{\mathbb{C}^2}(\mathbf{T})$, we obtain

$$F(\mathbf{T}) := F(\mathbf{T}_{\mathbb{C}})|\mathcal{V} = \sum_{j,k \geq 0} A_{jk} T_1^j T_2^k \in \mathcal{B}(\mathcal{V}).$$

(2) In the case of Hilbert spaces, there is a stronger connection between the spectral theory of pairs and the algebra of quaternions (see [1], Remark 10(2)). Specifically, if $\mathcal{H}$ is a complex Hilbert space and $\mathbf{V} = (V_1, V_2)$ is a commuting pair of bounded linear operators on $\mathcal{H}$, a point $\mathbf{z} = (z_1, z_2) \in \mathbb{C}^2$ is in the joint resolvent of $\mathbf{V}$ if and only if the operator $Q(\mathbf{V}) - Q(\mathbf{z})$ is invertible in $\mathcal{H}^2$, where

$$Q(\mathbf{V}) = \begin{pmatrix} V_1 & V_2 \\ -V_2^* & V_1^* \end{pmatrix},$$

(see also [15] for other details). In this case, there is also a Martinelli type formula which can be used to construct the associated analytic functional calculus (see [18,19]). An approach to such a construction in Banach spaces, by using a so-called splitting joint spectrum, can be found in [20].

7. Discussion

A parallel theory valid for operators acting in a Cliffordian context is presented in the article [14] (see also [21]). Inspired by this general approach, another contribution of the present author, developing the spectral decompositions of normal operators in a quaternionic setting by extending results from the real context appears in a work published in arXiv: 2103.16266v2.

8. Conclusions

The approach to the elementary spectral theory for the class of quaternionic linear operators regarded as a special class of real linear ones leads to a simpler and more natural construction of the analytic functional calculus, avoiding many of the difficulties which appear when working with the spectrum defined in the quaternionic algebra. This analytic functional calculus with stem functions defined in the complex plane replaces the analytic functional calculus with slice holomorphicc functions, which are defined in the quaternionic algebra.

Use of AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares no conflict of interest in this paper

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