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General decay of solutions for a von Karman plate system with general type of relaxation functions on the boundary

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Abstract: In this paper, we investigate a von Karman plate system with general type of relaxation functions on the boundary. We derive the general decay rate result without requiring the assumption that the initial value $w_0 \equiv 0$ on the boundary, using the multiplier method and some properties of the convex functions. Here we consider the resolvent kernels $k_i (i = 1, 2)$, namely $k_i''(t) \geq -\xi_i(t)G_i(-k_i'(t))$, where G_i are convex and increasing functions near the origin and ξ_i are positive nonincreasing functions. Moreover, the energy decay rates depend on the functions ξ_i and G_i. These general decay estimates allow for certain relaxation functions which are not necessarily of exponential or polynomial decay and therefore improve earlier results in the literature.

Keywords: von Karman plate; general decay; memory term; relaxation function; convexity; boundary condition
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1. Introduction

The purpose of this work is to investigate the general decay of the solutions to the von Karman plate system with a memory condition on the boundary:

$$w_{tt} + \Delta^2 w = [w, v] \quad \text{in } \Omega \times (0, \infty), \tag{1.1}$$

$$\Delta^2 v = -[w, w] \quad \text{in } \Omega \times (0, \infty), \tag{1.2}$$

$$v = \frac{\partial v}{\partial \nu} = 0 \quad \text{on } \Gamma \times (0, \infty), \tag{1.3}$$

$$\frac{\partial w}{\partial \nu} + \int_0^t h_1(t-s)\Big(\mathcal{A}_1 w(s) + \alpha_1 \frac{\partial w(s)}{\partial \nu}\Big)ds = 0 \quad \text{on } \Gamma \times (0, \infty), \tag{1.4}$$

$$w - \int_0^t h_2(t-s)(\mathcal{A}_2 w(s) - \alpha_2 w(s))ds = 0 \quad \text{on } \Gamma \times (0, \infty), \tag{1.5}$$

$$w(x,0) = w_0(x), \quad w_t(x,0) = w_1(x) \ \text{ in } \Omega, \tag{1.6}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain with smooth boundary Γ and $x = (x_1, x_2)$. The constants α_1 and α_2 are positive. The von Karman bracket $[w, u]$ denotes the bilinear expression

$$[w, u] = w_{x_1x_1}u_{x_2x_2} - 2w_{x_1x_2}u_{x_1x_2} + w_{x_2x_2}u_{x_1x_1}.$$

Let us denote by $\nu = (\nu_1, \nu_2)$ the external unit normal vector on Γ and by $\tau = (-\nu_2, \nu_1)$ the corresponding unit tangent vector. Denoting by the differential operators $\mathcal{A}_1$ and $\mathcal{A}_2$

$$\mathcal{A}_1 w = \Delta w + (1-\lambda)A_1 w, \quad \mathcal{A}_2 w = \frac{\partial \Delta w}{\partial \nu} + (1-\lambda)\frac{\partial A_2 w}{\partial \tau},$$

where

$$A_1 w = 2\nu_1\nu_2 w_{x_1x_2} - \nu_1^2 w_{x_2x_2} - \nu_2^2 w_{x_1x_1},$$
$$A_2 w = (\nu_1^2 - \nu_2^2)w_{x_1x_2} + \nu_1\nu_2(w_{x_2x_2} - w_{x_1x_1}),$$

and the constant $\lambda \in (0, \frac{1}{2})$ represents Poisson's ratio.

From the physical point of view, w represents the transversal displacement, and v is the Airy-stress function of the vibrating plate subjected to boundary viscoelastic damping. We see that the memory effect described in integral Eqs (1.4) and (1.5) can be caused by the interaction with another viscoelastic element. The problems (1.1)–(1.6) are interesting not only from the point of view of PDE general theory but also due to its applications in mechanics. This equation is used to predict the shape or deformation of the plate, for example, to understand and design the behavior of the plate in aircraft wings, the support structure of buildings or various mechanical systems. Furthermore, the von Karman plates with memory on the boundary is also widely used in communication and signal processing. In particular, understanding the energy decay that occurs when a signal passes through a boundary can contribute to improving the performance of the communication system.

Recently, some authors have applied the diffusion PDE model to study practical problems such as viral infection, cancer prevention and treatment and online game addiction, and they have achieved good results, see [1–4]. Fractional order differential equations (FODEs) have attracted considerable attention from researchers due to their applications in various scientific and engineering fields. Since most physical, technical and dynamical problems are influenced by certain boundary conditions, the study of boundary value problems is important. Shah et al. [5] established some sufficient conditions for the existence and uniqueness of solutions to impulsive FODEs with integral boundary condition. Recently, Shah et al. [6] investigated the existence and uniqueness of solutions to nonlinear impulsive FODEs under multipoint boundary conditions. Furthermore, fractional order delay differential equations (FODDEs) play crucial roles in modeling various physical and biological processes and phenomena. FODDEs have a wide range of applications in various fields, including electrodynamics, growth cells, quantum mechanics and astrophysics. Shah et al. [7] considered the existence and uniqueness of solutions to the boundary value problem of variable FODDEs.

In recent decades, many authors [8–13] have considered the mathematical analysis of Kirchhoff plates, investigating aspects such as global existence, uniqueness and stability under various boundary feedback conditions. Kang [14] studied the general decay of solutions to the Kirchhoff plates with a memory condition at the boundary. Mustafa and Abusharkh [15] established the general decay rate result for the plate equations with viscoelastic boundary damping. Recently, Al-Mahdi [16] proved the general and optimal decay rate result for the Kirchhoff plate equations with nonlinear damping. This result is a generalization of the work in [14, 15, 17].

On the other hand, the energy decay of the solutions for the von Karman system has been studied by many authors. In [18, 19], the authors proved the stability of the solutions to a von Karman plate with nonlinear boundary feedback. Rivera and Menzala [20] showed the asymptotic behavior of the solution for the following von Karman plates with memory

$$\begin{aligned} &w_{tt} - h\Delta w_{tt} + \Delta^2 w - \int_0^t g(t-s)\Delta^2 w(s)ds = [w, v] \ \text{ in } \Omega \times (0, \infty), \\ &\Delta^2 v = -[w, w] \ \text{ in } \Omega \times (0, \infty), \end{aligned} \tag{1.7}$$

where h is a constant representing the thickness. Recently, Kang [21] investigated the general decay rates for the von Karman plate model (1.7) under the more general conditions

$$g'(t) \leq -\zeta(t)G(g(t)), \ t \geq 0, \tag{1.8}$$

where ζ is a positive nonincreasing differentiable function and G satisfies the suitable conditions. This result improved earlier results in [20,22,23]. Recently, Balegh et al. [24] established the general energy decay result for system (1.7) with nonlinear boundary delay term when g satisfies condition (1.8). The general stability result of the viscoelastic equation, for relaxation function g satisfying condition (1.8), has been investigated in [25–27].

For the case $\alpha_1 = \alpha_2 = 0$ in (1.1)–(1.6), Park and Park [28] studied the asymptotic behavior of the solutions, provided the resolvent kernels satisfy

$$k_i(0) > 0, \ k_i'(t) \leq -C_1 k_i(t), \ k_i''(t) \geq -C_2 k_i'(t), \ \forall t \geq 0, \ (i = 1, 2), \tag{1.9}$$

for some positive constants C_1 and C_2. Kang [29] considered the following generalized condition

$$k_i(0) > 0, \ \ \lim_{t\to\infty} k_i(t) = 0, \ \ k_i'(t) \leq 0, \ \ k_i''(t) \geq H(-k_i'(t)), \ \ (i = 1, 2), \tag{1.10}$$

where H is a positive function, with $H(0) = H'(0) = 0$, and H is linear or it is strictly increasing and strictly convex on $(0, r]$, for some $0 < r < 1$. The inequality in (1.10) has been introduced for the first time in [30]. These are weaker conditions on H than those introduced in [30]. Thus, Kang [29] extended the decay result of [28]. Park [31], using the same assumption on the kernel in [29], obtained the general decay result of energy under $w_0 \neq 0$ on one part of the boundary. Recently, Feng and Soufyane [32] showed the general decay of the solution when the initial condition $w_0 = 0$ on one part of boundary and the resolvent kernels k_i satisfy

$$k_i(0) > 0, \ \ \lim_{t\to\infty} k_i(t) = 0, \ \ k_i'(t) \leq 0, \ \ k_i''(t) \geq \xi_i(t)G_i(-k_i'(t)), \ \ (i = 1, 2), \tag{1.11}$$

where $\xi_i(t)$ are nonincreasing continuous functions and G_i are positive functions, with $G_i(0) = G_i'(0) = 0$, and the G_i are linear or strictly increasing and strictly convex on $(0, r]$, $r \geq -k_i'(0)$.

For problems (1.1)–(1.6), Rivera et al. [33] proved that the solution decays exponentially, provided the resolvent kernels satisfy condition (1.9). Moreover, when the relaxation functions decay polynomially, they showed that the solution decays polynomially. Santos and Soufyane [34] improved the decay result of [33]. They assumed that the resolvent kernels satisfy

$$k_i(0) > 0,\ k_i(t) \geq 0,\ k_i'(t) \leq 0,\ k_i''(t) \geq \eta_i(t)(-k_i'(t)),\ \forall t \geq 0,\ (i = 1, 2), \tag{1.12}$$

where $\eta_i : \mathbb{R}^+ \to \mathbb{R}^+$ are functions satisfying conditions

$$\eta_i(t) > 0,\ \eta_i'(t) \leq 0 \ \text{ and } \int_0^{+\infty} \eta_i(t) dt = +\infty.$$

Motivated by their results, we prove the asymptotic behavior of the solution for system (1.1)–(1.6) when the initial condition $w_0 \neq 0$ on Γ and the resolvent kernels k_i satisfy condition (1.11). This condition is more general compared to the previous conditions (1.10) and (1.12). Therefore, these general decay estimates improve the earlier results of [29, 31, 33, 34]. Moreover, using the multiplier method and some properties of convex functions, we obtain the general decay of solution for system (1.1)–(1.6) that depends on the functions ξ_i and G_i.

The paper is organized as follows. In Section 2, we present some notations and assumptions needed for our work. In Section 3, we prove the general decay of the solutions for the von Karman plate system with memory condition on the boundary.

2. Preliminaries

In this section, we present some material needed in the proof of our main result. Throughout this paper we denote $\|\cdot\|_{L^2(\Omega)}$ and $\|\cdot\|_{L^2(\Gamma)}$ by $\|\cdot\|$ and $\|\cdot\|_\Gamma$, respectively. Let us define the bilinear form

$$a(w, u) = \int_\Omega \{w_{x_1x_1}u_{x_1x_1} + w_{x_2x_2}u_{x_2x_2} + \lambda(w_{x_1x_1}u_{x_2x_2} + w_{x_2x_2}u_{x_1x_1}) + 2(1-\lambda)w_{x_1x_2}u_{x_1x_2}\}dx.$$

We assume that there exists $x_0 \in \mathbb{R}^2$, such that

$$\Gamma = \{x \in \Gamma : m(x) \cdot \nu(x) > 0\},$$

where $m(x) = x - x_0$. The compactness of Γ implies that there exists $\delta > 0$, such that

$$m(x) \cdot \nu(x) \geq \delta > 0, \quad \forall x \in \Gamma. \tag{2.1}$$

As shown in [33, 34], we use the boundary conditions (1.4) and (1.5) to estimate the terms $\mathcal{A}_1 w$ and $\mathcal{A}_2 w$. Differentiating (1.4) and (1.5) and applying Volterra's inverse operator, we have

$$\mathcal{A}_1 w = -\alpha_1 \frac{\partial w}{\partial \nu} - \gamma_1 \Big\{ \frac{\partial w_t}{\partial \nu} - k_1(t) \frac{\partial w_0}{\partial \nu} + k_1(0) \frac{\partial w}{\partial \nu} + k_1' * \frac{\partial w}{\partial \nu} \Big\}, \tag{2.2}$$

$$\mathcal{A}_2 w = \alpha_2 w + \gamma_2 \{w_t - k_2(t) w_0 + k_2(0) w + k_2' * w\}, \tag{2.3}$$

where $\gamma_i = \frac{1}{h_i(0)}, (i = 1, 2)$ and the resolvent kernels $k_i, (i = 1, 2)$ satisfy

$$k_i + \frac{1}{h_i(0)} h_i' * k_i = -\frac{1}{h_i(0)} h_i',$$

where $*$ is the convolution product. Thus, we use boundary conditions (2.2) and (2.3) instead of (1.4) and (1.5).

The following identity will be used later.

Lemma 2.1. *([35]) For any $w \in H^4(\Omega)$ and $u \in H^2(\Omega)$, we have*

$$\int_\Omega (\Delta^2 w) u dx = a(w,u) + \int_\Gamma (\mathcal{A}_2 w)u - (\mathcal{A}_1 w)\frac{\partial u}{\partial \nu} d\Gamma, \tag{2.4}$$

$$\begin{aligned}\int_\Omega (m\cdot\nabla w)\Delta^2 w dx = a(w,w) + \int_\Gamma \Big[(\mathcal{A}_2 w)(m\cdot\nabla w) - (\mathcal{A}_1 w)\frac{\partial(m\cdot\nabla w)}{\partial\nu}\Big] d\Gamma \\ + \frac{1}{2}\int_\Gamma (m\cdot\nu)[w_{x_1x_1}^2 + w_{x_2x_2}^2 + 2\lambda w_{x_1x_1}w_{x_2x_2} + 2(1-\lambda)w_{x_1x_2}^2] d\Gamma.\end{aligned} \tag{2.5}$$

We state the relative results of the Airy stress function and von Karman bracket $[\cdot,\cdot]$.

Lemma 2.2. *([8]) Let w, u be functions in $H^2(\Omega)$ and v in $H_0^2(\Omega)$, where Ω is an open, bounded and connected set of $\mathbb{R}^2$ with regular boundary. Then,*

$$\int_\Omega [w,v]u dx = \int_\Omega [w,u] v dx. \tag{2.6}$$

By differentiating the term $h\Box w$, we obtain the following lemma.

Lemma 2.3. *For $h, w \in C^1([0,\infty):\mathbb{R})$, we have*

$$(h * w)w_t = -\frac{1}{2}h(t)|w(t)|^2 + \frac{1}{2}h'\Box w - \frac{1}{2}\frac{d}{dt}\Big[h\Box w - \Big(\int_0^t h(s)ds\Big)|w|^2\Big], \tag{2.7}$$

where $(h\Box w)(t) := \int_0^t h(t-s)|w(t)-w(s)|^2 ds$.

As in [32, 36], we consider the following assumptions on k_i $(i = 1,2)$.
(A) The resolvent kernels $k_i : \mathbb{R}_+ \to \mathbb{R}_+$ are twice differentiable functions, such that

$$k_i(0) > 0, \;\; k_i'(t) \le 0, \;\; \lim_{t\to\infty} k_i(t) = 0, \;\; \int_0^\infty k_i(s)ds < \infty, \tag{2.8}$$

and there exist positive functions $G_i \in C^1(\mathbb{R}_+)$ and the G_i are linear or strictly increasing and strictly convex C^2 functions on $(0,r]$, $r < 1$, with $G_i(0) = G_i'(0) = 0$, such that

$$k_i''(t) \ge \xi_i(t) G_i(-k_i'(t)), \;\; \forall t > 0, \tag{2.9}$$

where $\xi_i : \mathbb{R}^+ \to \mathbb{R}^+$ are nonincreasing differentiable functions.

From (A), we easily see that there exists $t_0 > 0$ large enough such that

$$0 < -k_i'(t_0) \le -k_i'(t) \le -k_i'(0), \;\text{ for } t \in [0,t_0], \tag{2.10}$$

and

$$\max\{k_i(t), -k_i'(t), k_i''(t)\} < \min\{r, G(r)\}, \;\text{ for } t \ge t_0, \tag{2.11}$$

where $G = \min\{G_1, G_2\}$.

As $\xi_i(t)$ and $-k_i'(t)$ are positive nonincreasing continuous functions and $G_i(t)$ is a positive continuous function, there exist positive constants a_i and b_i such that

$$a_i \le \xi_i(t)G_i(-k_i'(t)) \le b_i, \quad \text{for } t \in [0, t_0].$$

Therefore, for all $t \in [0, t_0]$, we obtain

$$k_i''(t) \ge \xi_i(t)G_i(-k_i'(t)) \ge -c_i k_i'(t), \tag{2.12}$$

where $c_i = -\frac{a_i}{k_i'(0)}$.

The well-posedness of von Karman system plates with boundary conditions of memory type is given by the following theorem.

Theorem 2.1. *([33]) Let $k_i(i = 1, 2) \in C^2(\mathbb{R}_+)$ be such that $k_i,\ -k_i',\ k_i'' \ge 0$. If the initial conditions $(w_0, w_1) \in (H^4(\Omega) \cap H^2(\Omega)) \times H^2(\Omega)$ satisfy the compatibility conditions*

$$\mathcal{A}_1 w_0 + \alpha_1 \frac{\partial w_0}{\partial \nu} + \gamma_1 \frac{\partial w_1}{\partial \nu} = 0, \quad \mathcal{A}_2 w_0 - \alpha_2 w_0 - \gamma_2 w_1 = 0 \ \ on\ \Gamma,$$

then the solution of (1.1)–(1.6) has the following regularity

$$w \in C^1([0, T] : H^2(\Omega)) \cap C^0([0, T] : H^4(\Omega)).$$

The energy function of system (1.1)–(1.6) is given by

$$\begin{aligned} E(t) = &\frac{1}{2}\|w_t\|^2 + \frac{1}{2}a(w, w) + \frac{1}{4}\|\Delta v\|^2 + \frac{\alpha_1}{2}\Big\|\frac{\partial w}{\partial \nu}\Big\|_\Gamma^2 + \frac{\gamma_1}{2}k_1(t)\Big\|\frac{\partial w}{\partial \nu}\Big\|_\Gamma^2 \\ &+\frac{\alpha_2}{2}\|w\|_\Gamma^2 + \frac{\gamma_2}{2}k_2(t)\|w\|_\Gamma^2 - \frac{\gamma_1}{2}\int_\Gamma k_1'\Box\frac{\partial w}{\partial \nu}d\Gamma - \frac{\gamma_2}{2}\int_\Gamma k_2'\Box w d\Gamma. \end{aligned} \tag{2.13}$$

To get a general stability result, the following is needed.

Remark 2.1. 1) If $G_i(i = 1, 2)$ are strictly convex on $(0, r]$ and $G_i(0) = 0$, then,

$$G_i(\theta s) \le \theta G_i(s), \quad s \in (0, r] \ \text{ and } 0 \le \theta \le 1. \tag{2.14}$$

2) Let G^* be the convex conjugate of G in the sense of Young [37], then

$$G^*(s) = s(G')^{-1}(s) - G[(G')^{-1}(s)], \quad \text{if } s \in (0, G'(r)], \tag{2.15}$$

and G^* satisfies the following Young's inequality

$$ab \le G^*(a) + G(b), \quad \text{if } a \in (0, G'(r)],\ b \in (0, r]. \tag{2.16}$$

3) Let F be a convex function on $[c, d]$, and if $\varrho : \Omega \to [c, d]$ and p are integrable functions on Ω such that $p(x) \ge 0$ and $\int_\Omega p(x)dx = p_0 > 0$, then Jensen's inequality states that

$$F\Big(\frac{1}{p_0}\int_\Omega \varrho(x)p(x)dx\Big) \le \frac{1}{p_0}\int_\Omega F(\varrho(x))p(x)dx. \tag{2.17}$$

3. General decay

In this section, we study the asymptotic behavior of the solutions for system (1.1)–(1.6). To show the general decay property, we first prove the dissipative property. Multiplying (1.1) by w_t and using (2.4), (2.7), Young's inequality and the boundary conditions (2.2) and (2.3), we obtain the following.

Lemma 3.1. *([33]) The energy function $E(t)$ satisfies*

$$E'(t) \leq -\frac{\gamma_1}{2}\Big\|\frac{\partial w_t}{\partial \nu}\Big\|_\Gamma^2 + \frac{\gamma_1}{2}k_1'(t)\Big\|\frac{\partial w}{\partial \nu}\Big\|_\Gamma^2 - \frac{\gamma_1}{2}\int_\Gamma k_1''\Box\frac{\partial w}{\partial \nu}d\Gamma + \frac{\gamma_1}{2}k_1^2(t)\Big\|\frac{\partial w_0}{\partial \nu}\Big\|_\Gamma^2$$
$$-\frac{\gamma_2}{2}\|w_t\|_\Gamma^2 + \frac{\gamma_2}{2}k_2'(t)\|w\|_\Gamma^2 - \frac{\gamma_2}{2}\int_\Gamma k_2''\Box w d\Gamma + \frac{\gamma_2}{2}k_2^2(t)\|w_0\|_\Gamma^2. \tag{3.1}$$

Since $w_0 \neq 0$ on Γ, Lemma 3.1 says that $E(t)$ may not be nonincreasing. So, we introduce the modified energy functional $\mathcal{E}(t)$ by

$$\mathcal{E}(t) = E(t) + \frac{\gamma_1}{2}\Big\|\frac{\partial w_0}{\partial \nu}\Big\|_\Gamma^2\int_t^\infty k_1^2(s)ds + \frac{\gamma_2}{2}\|w_0\|_\Gamma^2\int_t^\infty k_2^2(s)ds. \tag{3.2}$$

Then, from (3.1), we have

$$\mathcal{E}'(t) = E'(t) - \frac{\gamma_1}{2}k_1^2(t)\Big\|\frac{\partial w_0}{\partial \nu}\Big\|_\Gamma^2 - \frac{\gamma_2}{2}k_2^2(t)\|w_0\|_\Gamma^2 \leq 0. \tag{3.3}$$

For suitable choice of N_1 and N_2, let us introduce the Lyapunov functional

$$L(t) := N_1E(t) + N_2\Upsilon(t),$$

where

$$\Upsilon(t) := \int_\Omega\Big(m\cdot\nabla w + \frac{1}{2}w\Big)w_t dx.$$

It is not difficult to see that $L(t)$ satisfies $q_0E(t) \leq L(t) \leq q_1E(t)$, for some positive constants q_0 and q_1.

Lemma 3.2. *Under the assumption (A), the functional $\Upsilon(t)$ satisfies*

$$\Upsilon'(t) \leq \frac{1}{2}\int_\Gamma(m\cdot\nu)|w_t|^2d\Gamma - \frac{1}{2}\|w_t\|^2 - \|\Delta v\|^2 - \frac{1}{2}\int_\Gamma(m\cdot\nu)|\Delta v|^2d\Gamma$$
$$-\Big(\frac{3}{2} - \epsilon\lambda_0\Big)a(w,w) - \Big(\frac{\alpha_1}{2} - \frac{\epsilon}{2}\Big)\Big\|\frac{\partial w}{\partial \nu}\Big\|_\Gamma^2 - \Big(\frac{\alpha_2}{2} - \frac{\epsilon}{2}\Big)\|w\|_\Gamma^2$$
$$-\Big(\frac{1}{2} - \frac{\epsilon\lambda_0}{\delta}\Big)\int_\Gamma(m\cdot\nu)[w_{x_1x_1}^2 + w_{x_2x_2}^2 + 2\lambda w_{x_1x_1}w_{x_2x_2} + 2(1-\lambda)w_{x_1x_2}^2]d\Gamma$$
$$+4\gamma_1^2C_\epsilon\Big(\Big\|\frac{\partial w_t}{\partial \nu}\Big\|_\Gamma^2 + k_1^2(t)\Big\|\frac{\partial w}{\partial \nu}\Big\|_\Gamma^2 + k_1^2(t)\Big\|\frac{\partial w_0}{\partial \nu}\Big\|_\Gamma^2 + C(\delta_1)\int_\Gamma g_1\Box\frac{\partial w}{\partial \nu}d\Gamma\Big)$$
$$+4\gamma_2^2C_\epsilon\Big(\|w_t\|_\Gamma^2 + k_2^2(t)\|w\|_\Gamma^2 + k_2^2(t)\|w_0\|_\Gamma^2 + C(\delta_2)\int_\Gamma g_2\Box w d\Gamma\Big), \tag{3.4}$$

for any $0 < \delta_i < 1$ $(i = 1, 2)$, *where*

$$C(\delta_i) = \int_0^\infty\frac{(-k_i'(s))^2}{g_i(s)}ds \ \ and\ \ g_i(t) = k_i''(t) - \delta_ik_i'(t) > 0. \tag{3.5}$$

Proof. According to [33, 34], from (2.5) and (2.6), we obtain

$$\begin{aligned}\Upsilon'(t) = &\frac{1}{2}\int_\Gamma (m\cdot\nu)|w_t|^2 d\Gamma - \frac{1}{2}\|w_t\|^2 - \|\Delta v\|^2 - \frac{1}{2}\int_\Gamma (m\cdot\nu)|\Delta v|^2 d\Gamma \\ &-\frac{3}{2}a(w,w) + \int_\Gamma (\mathcal{A}_1 w)\frac{\partial}{\partial\nu}\Big(m\cdot\nabla w + \frac{1}{2}w\Big)d\Gamma - \int_\Gamma (\mathcal{A}_2 w)\Big(m\cdot\nabla w + \frac{1}{2}w\Big)d\Gamma \\ &-\frac{1}{2}\int_\Gamma (m\cdot\nu)[w_{x_1x_1}^2 + w_{x_2x_2}^2 + 2\lambda w_{x_1x_1}w_{x_2x_2} + 2(1-\lambda)w_{x_1x_2}^2]d\Gamma. \end{aligned} \tag{3.6}$$

Applying Young's inequality, we get

$$\Big|\int_\Gamma (\mathcal{A}_1 w)\frac{\partial}{\partial\nu}\Big(m\cdot\nabla w + \frac{1}{2}w\Big)d\Gamma\Big| \le \epsilon\Big\|\frac{\partial}{\partial\nu}(m\cdot\nabla w)\Big\|_\Gamma^2 + C_\epsilon\Big\|\mathcal{A}_1 w + \alpha_1\frac{\partial w}{\partial\nu}\Big\|_\Gamma^2 - \Big(\frac{\alpha_1}{2} - \frac{\epsilon}{2}\Big)\Big\|\frac{\partial w}{\partial\nu}\Big\|_\Gamma^2, \tag{3.7}$$

$$\Big| - \int_\Gamma (\mathcal{A}_2 w)\Big(m\cdot\nabla w + \frac{1}{2}w\Big)d\Gamma\Big| \le \epsilon\|m\cdot\nabla w\|_\Gamma^2 + C_\epsilon\|\mathcal{A}_2 w - \alpha_2 w\|_\Gamma^2 - \Big(\frac{\alpha_2}{2} - \frac{\epsilon}{2}\Big)\|w\|_\Gamma^2, \tag{3.8}$$

where ϵ is a positive constant. Using the trace theory and the fact $m\cdot\nu \ge \delta$ on Γ, we obtain

$$\begin{aligned}&\Big\|\frac{\partial}{\partial\nu}(m\cdot\nabla w)\Big\|_\Gamma^2 + \|m\cdot\nabla w\|_\Gamma^2 \\ &\le \lambda_0 a(w,w) + \frac{\lambda_0}{\delta}\int_\Gamma (m\cdot\nu)[w_{x_1x_1}^2 + w_{x_2x_2}^2 + 2\lambda w_{x_1x_1}w_{x_2x_2} + 2(1-\lambda)w_{x_1x_2}^2]d\Gamma, \end{aligned} \tag{3.9}$$

where λ_0 is a positive constant. Noting that

$$(k_2' * w)(t) = w(t)[k_2(t) - k_2(0)] - \int_0^t k_2'(t-s)(w(t) - w(s))ds,$$

the boundary condition (2.3) can be written as

$$\mathcal{A}_2 w - \alpha_2 w = \gamma_2\{w_t + k_2(t)w - k_2(t)w_0 - \int_0^t k_2'(t-s)(w(t) - w(s))ds\}. \tag{3.10}$$

Similarly, we can show that

$$\mathcal{A}_1 w + \alpha_1\frac{\partial w}{\partial\nu} = -\gamma_1\Big\{\frac{\partial w_t}{\partial\nu} + k_1(t)\frac{\partial w}{\partial\nu} - k_1(t)\frac{\partial w_0}{\partial\nu} - \int_0^t k_1'(t-s)\Big(\frac{\partial w(t)}{\partial\nu} - \frac{\partial w(s)}{\partial\nu}\Big)ds\Big\}. \tag{3.11}$$

Using (3.6)–(3.11), we arrive at

$$\begin{aligned}\Upsilon'(t) \le &\frac{1}{2}\int_\Gamma (m\cdot\nu)|w_t|^2 d\Gamma - \frac{1}{2}\|w_t\|^2 - \|\Delta v\|^2 - \frac{1}{2}\int_\Gamma (m\cdot\nu)|\Delta v|^2 d\Gamma \\ &-\Big(\frac{3}{2} - \epsilon\lambda_0\Big)a(w,w) - \Big(\frac{\alpha_1}{2} - \frac{\epsilon}{2}\Big)\Big\|\frac{\partial w}{\partial\nu}\Big\|_\Gamma^2 - \Big(\frac{\alpha_2}{2} - \frac{\epsilon}{2}\Big)\|w\|_\Gamma^2 \\ &-\Big(\frac{1}{2} - \frac{\epsilon\lambda_0}{\delta}\Big)\int_\Gamma (m\cdot\nu)[w_{x_1x_1}^2 + w_{x_2x_2}^2 + 2\lambda w_{x_1x_1}w_{x_2x_2} + 2(1-\lambda)w_{x_1x_2}^2]d\Gamma \\ &+4\gamma_1^2 C_\epsilon\Big(\Big\|\frac{\partial w_t}{\partial\nu}\Big\|_\Gamma^2 + k_1^2(t)\Big\|\frac{\partial w}{\partial\nu}\Big\|_\Gamma^2 + k_1^2(t)\Big\|\frac{\partial w_0}{\partial\nu}\Big\|_\Gamma^2 + \Big\| - \int_0^t k_1'(t-s)\Big(\frac{\partial w(t)}{\partial\nu} - \frac{\partial w(s)}{\partial\nu}\Big)ds\Big\|_\Gamma^2\Big)\end{aligned}$$

$$+4\gamma_2^2 C_\epsilon\Big(\|w_t\|_\Gamma^2 + k_2^2(t)\|w\|_\Gamma^2 + k_2^2(t)\|w_0\|_\Gamma^2 + \Big\| -\int_0^t k_2'(t-s)(w(t)-w(s))ds\Big\|_\Gamma^2\Big). \tag{3.12}$$

Using the Cauchy-Schwarz inequality and (3.5), we have (see details in [25, 27])

$$\begin{aligned}&\Big\| -\int_0^t k_2'(t-s)(w(t)-w(s))ds\Big\|_\Gamma^2\\ \le& \int_0^t \frac{(-k_2'(s))^2}{g_2(s)}ds \int_\Gamma\int_0^t (k_2''(t-s) - \delta_2 k_2'(t-s))|w(t)-w(s)|^2 ds d\Gamma \le C(\delta_2)\int_\Gamma g_2 \Box w d\Gamma,\end{aligned} \tag{3.13}$$

and

$$\Big\| -\int_0^t k_1'(t-s)\Big(\frac{\partial w(t)}{\partial \nu} - \frac{\partial w(s)}{\partial \nu}\Big)ds\Big\|_\Gamma^2 \le C(\delta_1)\int_\Gamma g_1 \Box \frac{\partial w}{\partial \nu} d\Gamma. \tag{3.14}$$

Substituting (3.13) and (3.14) into (3.12), we have (3.4). □

Next, we define the functionals

$$K_1(t) = \int_0^t f_1(t-s)\Big\|\frac{\partial w(s)}{\partial \nu}\Big\|_\Gamma^2 ds \text{ and } K_2(t) = \int_0^t f_2(t-s)\|w(s)\|_\Gamma^2 ds,$$

where $f_i(t) = \int_t^\infty (-k_i'(s))ds,\ i = 1, 2.$

Lemma 3.3. *Under the assumption (A), the functionals $K_1(t)$ and $K_2(t)$ satisfy the estimates*

$$K_1'(t) \le 3k_1(0)\Big\|\frac{\partial w}{\partial \nu}\Big\|_\Gamma^2 + \frac{1}{2}\int_\Gamma k_1' \Box \frac{\partial w}{\partial \nu} d\Gamma, \tag{3.15}$$

$$K_2'(t) \le 3k_2(0)\|w\|_\Gamma^2 + \frac{1}{2}\int_\Gamma k_2' \Box w d\Gamma. \tag{3.16}$$

Proof. Taking the derivative of the functional $K_2(t)$ and using the fact $f_2'(t) = k_2'(t)$, we find that

$$\begin{aligned}K_2'(t) =& f_2(0)\|w\|_\Gamma^2 + \int_0^t k_2'(t-s)\|w(s)\|_\Gamma^2 ds\\ =& \int_0^t k_2'(t-s)\|w(s)-w(t)\|_\Gamma^2 ds + 2\int_\Gamma w(t)\int_0^t k_2'(t-s)(w(s)-w(t))ds d\Gamma + k_2(t)\|w\|_\Gamma^2.\end{aligned} \tag{3.17}$$

Using Young's inequality and (2.8), we obtain

$$\begin{aligned}&2\int_\Gamma w(t)\int_0^t k_2'(t-s)(w(s)-w(t))ds d\Gamma\\ \le& 2k_2(0)\|w\|_\Gamma^2 + \frac{\int_0^t -k_2'(s)ds}{2k_2(0)}\int_\Gamma\int_0^t (-k_2'(t-s))|w(s)-w(t)|^2 ds d\Gamma\\ \le& 2k_2(0)\|w\|_\Gamma^2 - \frac{1}{2}\int_\Gamma k_2' \Box w d\Gamma.\end{aligned} \tag{3.18}$$

From (3.17) and (3.18), we get the estimate (3.16). Similarly, we can obtain the estimate (3.15). □

Lemma 3.4. *Suppose that the assumption (A) holds. Then, for $N_1, N_2 > 0$ large enough, there exist positive constants β_1 and β_2, such that*

$$L'(t) \le -\beta_1(\|w_t\|^2 + a(w,w) + \|\Delta v\|^2) + \beta_2\Big(k_1^2(t)\Big\|\frac{\partial w_0}{\partial \nu}\Big\|_\Gamma^2 + k_2^2(t)\|w_0\|_\Gamma^2\Big) - \frac{\gamma_1}{4}\int_\Gamma k_1'\Box\frac{\partial w}{\partial \nu}d\Gamma$$
$$-(\frac{\alpha_1}{2} + 4\gamma_1 k_1(0))\Big\|\frac{\partial w}{\partial \nu}\Big\|_\Gamma^2 - \frac{\gamma_2}{4}\int_\Gamma k_2'\Box w d\Gamma - (\frac{\alpha_2}{2} + 4\gamma_2 k_2(0))\|w\|_\Gamma^2,\ \textit{for } t \ge t_0, \tag{3.19}$$

where t_0 was introduced in (2.10).

Proof. Combining (3.1), (3.4) and (3.5), we see that

$$L'(t) \le -\frac{N_2}{2}\|w_t\|^2 - N_2\|\Delta v\|^2 - \gamma_2\Big(\frac{N_1}{2} - 4\gamma_2 C_\epsilon N_2 - \frac{RN_2}{2\gamma_2}\Big)\|w_t\|_\Gamma^2 - \gamma_1(\frac{N_1}{2} - 4\gamma_1 C_\epsilon N_2)\Big\|\frac{\partial w_t}{\partial \nu}\Big\|_\Gamma^2$$
$$-N_2\Big(\frac{3}{2} - \epsilon\lambda_0\Big)a(w,w) - N_2\Big(\frac{\alpha_1}{2} - \frac{\epsilon}{2} - 4\gamma_1^2 C_\epsilon k_1^2(t)\Big)\Big\|\frac{\partial w}{\partial \nu}\Big\|_\Gamma^2 - N_2\Big(\frac{\alpha_2}{2} - \frac{\epsilon}{2} - 4\gamma_2^2 C_\epsilon k_2^2(t)\Big)\|w\|_\Gamma^2$$
$$-N_2\Big(\frac{1}{2} - \frac{\epsilon\lambda_0}{\delta}\Big)\int_\Gamma (m\cdot\nu)[w_{x_1x_1}^2 + w_{x_2x_2}^2 + 2\lambda w_{x_1x_1}w_{x_2x_2} + 2(1-\lambda)w_{x_1x_2}^2]d\Gamma$$
$$-\frac{\gamma_1\delta_1 N_1}{2}\int_\Gamma k_1'\Box\frac{\partial w}{\partial \nu}d\Gamma - \gamma_1\Big(\frac{N_1}{2} - 4\gamma_1 C_\epsilon C(\delta_1)N_2\Big)\int_\Gamma g_1\Box\frac{\partial w}{\partial \nu}d\Gamma$$
$$-\frac{\gamma_2\delta_2 N_1}{2}\int_\Gamma k_2'\Box w d\Gamma - \gamma_2\Big(\frac{N_1}{2} - 4\gamma_2 C_\epsilon C(\delta_2)N_2\Big)\int_\Gamma g_2\Box w d\Gamma$$
$$+k_1^2(t)\Big(\frac{\gamma_1 N_1}{2} + 4\gamma_1^2 C_\epsilon N_2\Big)\Big\|\frac{\partial w_0}{\partial \nu}\Big\|_\Gamma^2 + k_2^2(t)\Big(\frac{\gamma_2 N_1}{2} + 4\gamma_2^2 C_\epsilon N_2\Big)\|w_0\|_\Gamma^2,$$

where $R = \max\{m(x)\cdot\nu(x) : x \in \Gamma\}$. We first fix $\epsilon > 0$ small such that

$$\frac{3}{2} - \epsilon\lambda_0 > 0,\ \ \frac{\alpha_i}{2} - \frac{\epsilon}{2} > 0,\ \ \text{and}\ \ \frac{1}{2} - \frac{\epsilon\lambda_0}{\delta} > 0,\ i = 1,2.$$

Next, we apply the fact $\lim_{t\to\infty} k_i(t) = 0$ $(i = 1,2)$ and choose N_2 large enough so that

$$N_2\Big(\frac{\alpha_i}{2} - \frac{\epsilon}{2} - 4\gamma_i^2 C_\epsilon k_i^2(t)\Big) > \frac{\alpha_i}{2} + 4\gamma_i k_i(0),\ i = 1,2,$$

for all $t > t_0$. From (2.8), (2.9) and (3.5), we have

$$-\delta_i k_i'(t) \le k_i''(t) - \delta_i k_i'(t) = g_i(t)\ \Rightarrow\ \frac{-\delta_i k_i'(t)}{g_i(t)} \le 1\ \Rightarrow\ \frac{\delta_i(-k_i'(t))^2}{g_i(t)} \le -k_i'(t),\ i = 1,2. \tag{3.20}$$

Integrating (3.20) and using (2.8), we obtain

$$\delta_i C(\delta_i) = \delta_i\int_0^\infty \frac{(-k_i'(s))^2}{g_i(s)}ds \le k_i(0),\ i = 1,2.$$

By the Lebesgue dominated convergence theorem, we find that $\delta_i C(\delta_i) \to 0$ as $\delta_i \to 0$. Then, there exists $0 < \delta_0 < 1$ such that, if $\delta_i < \delta_0$, then $4\delta_i\gamma_i C_\epsilon C(\delta_i)N_2 < \frac{1}{8}$ $(i = 1,2)$. Finally, taking N_1 large enough so that

$$N_1 > \max\Big\{8\gamma_1 C_\epsilon N_2,\ \Big(8\gamma_2 C_\epsilon + \frac{R}{\gamma_2}\Big)N_2,\ \frac{1}{2\delta_0}\Big\},$$

and choosing $\delta_i = \frac{1}{2N_1} < \delta_0$ $(i = 1,2)$, we have the estimate (3.19). □

Now, we are ready to prove our main result.

Theorem 3.1. *Suppose that the assumption (A) holds. Then there exist positive constants* $\epsilon_0, \sigma_1, \sigma_2, \kappa_1$ *and* κ_2 *such that the energy functional satisfies, for all* $t \geq t_0$,

$$E(t) \leq \sigma_1\Big\{1 + \int_{t_0}^{t} e^{\sigma_2 \int_{t_0}^{s} \xi(\eta)d\eta}\Big(k_1^2(s)\Big\|\frac{\partial w_0}{\partial \nu}\Big\|_\Gamma^2 + k_2^2(s)\|w_0\|_\Gamma^2\Big)ds\Big\}e^{-\sigma_2 \int_{t_0}^{t} \xi(s)ds}, \ \textit{if } G \textit{ is linear}, \tag{3.21}$$

$$E(t) \leq \kappa_1 H_1^{-1}\Big(\frac{\kappa_2\Big(1 + \big\|\frac{\partial w_0}{\partial \nu}\big\|_\Gamma^2\big(\int_{t_0}^{t} G(k_1(s))\xi(s)ds\big) + \|w_0\|_\Gamma^2\big(\int_{t_0}^{t} G(k_2(s))\xi(s)ds\big)\Big)}{t\xi(t)}\Big)$$
$$-\frac{\gamma_1}{2}\Big\|\frac{\partial w_0}{\partial \nu}\Big\|_\Gamma^2\Big(\int_t^\infty k_1^2(s)ds\Big) - \frac{\gamma_2}{2}\|w_0\|_\Gamma^2\Big(\int_t^\infty k_2^2(s)ds\Big), \ \textit{if } G \textit{ is nonlinear}, \tag{3.22}$$

where $H_1(t) = tG'(\epsilon_0 t)$, $G(t) = \min\{G_1(t), G_2(t)\}$ *and* $\xi(t) = \min\{\xi_1(t), \xi_2(t)\}$.

Proof. From (2.13) and (3.19), there exist positive constants β_3 and β_4 such that, for $t \geq t_0$

$$L'(t) \leq -\beta_3 E(t) - \beta_4\Big(\int_\Gamma k_1' \square \frac{\partial w}{\partial \nu} d\Gamma + \int_\Gamma k_2' \square w d\Gamma\Big) + \beta_2\Big(k_1^2(t)\Big\|\frac{\partial w_0}{\partial \nu}\Big\|_\Gamma^2 + k_2^2(t)\|w_0\|_\Gamma^2\Big). \tag{3.23}$$

Applying (2.12) and (3.1), we see that, for all $t \geq t_0$

$$\begin{aligned}
&\beta_4 \int_\Gamma \int_0^{t_0} \Big(-k_1'(s)\Big|\frac{\partial w(t)}{\partial \nu} - \frac{\partial w(t-s)}{\partial \nu}\Big|^2 - k_2'(s)|w(t) - w(t-s)|^2\Big)dsd\Gamma \\
\leq\ & \frac{\beta_4}{c_0} \int_\Gamma \int_0^{t_0} \Big(k_1''(s)\Big|\frac{\partial w(t)}{\partial \nu} - \frac{\partial w(t-s)}{\partial \nu}\Big|^2 + k_2''(s)|w(t) - w(t-s)|^2\Big)dsd\Gamma \\
\leq\ & \frac{\beta_4}{c_0\gamma_0}\Big(\gamma_1 k_1^2(t)\Big\|\frac{\partial w_0}{\partial \nu}\Big\|_\Gamma^2 + \gamma_2 k_2^2(t)\|w_0\|_\Gamma^2 - 2E'(t)\Big),
\end{aligned} \tag{3.24}$$

where $c_0 = \min\{c_1, c_2\}$ and $\gamma_0 = \min\{\gamma_1, \gamma_2\}$.

Let $\Phi(t) = L(t) + \frac{2\beta_4}{c_0\gamma_0}E(t)$, which is equivalent to $E(t)$. Using (3.23) and (3.24), we obtain for all $t \geq t_0$

$$\begin{aligned}
\Phi'(t) \leq\ & -\beta_3 E(t) + \beta_5\Big(k_1^2(t)\Big\|\frac{\partial w_0}{\partial \nu}\Big\|_\Gamma^2 + k_2^2(t)\|w_0\|_\Gamma^2\Big) \\
& -\beta_4\Big(\int_\Gamma \int_{t_0}^{t} k_1'(s)\Big|\frac{\partial w(t)}{\partial \nu} - \frac{\partial w(t-s)}{\partial \nu}\Big|^2 dsd\Gamma + \int_\Gamma \int_{t_0}^{t} k_2'(s)|w(t) - w(t-s)|^2 dsd\Gamma\Big),
\end{aligned} \tag{3.25}$$

where $\beta_5 = \max\{\beta_2 + \frac{\beta_4\gamma_1}{c_0\gamma_0},\ \beta_2 + \frac{\beta_4\gamma_2}{c_0\gamma_0}\}$.

We consider the following two cases.

1) G is linear: Multiplying (3.25) by the nonincreasing function $\xi(t)$ and using (2.9) and (3.1), we have

$$\begin{aligned}
\xi(t)\Phi'(t) \leq\ & -\beta_3\xi(t)E(t) + \beta_5\xi(t)\Big(k_1^2(t)\Big\|\frac{\partial w_0}{\partial \nu}\Big\|_\Gamma^2 + k_2^2(t)\|w_0\|_\Gamma^2\Big) \\
& +\beta_4\Big(\int_\Gamma \int_{t_0}^{t} k_1''(s)\Big|\frac{\partial w(t)}{\partial \nu} - \frac{\partial w(t-s)}{\partial \nu}\Big|^2 dsd\Gamma + \int_\Gamma \int_{t_0}^{t} k_2''(s)|w(t) - w(t-s)|^2 dsd\Gamma\Big) \\
\leq\ & -\beta_3\xi(t)E(t) + \beta_6\Big(k_1^2(t)\Big\|\frac{\partial w_0}{\partial \nu}\Big\|_\Gamma^2 + k_2^2(t)\|w_0\|_\Gamma^2\Big) - \frac{2\beta_4}{\gamma_0}E'(t), \ \forall t \geq t_0,
\end{aligned}$$

where $\beta_6 = \max\{\beta_5\xi_0 + \frac{\beta_4\gamma_1}{\gamma_0}, \beta_5\xi_0 + \frac{\beta_4\gamma_2}{\gamma_0}\}$ and $\xi(t) \le \xi_0$, for some $\xi_0 > 0$. This gives

$$(\xi(t)\Phi(t) + \frac{2\beta_4}{\gamma_0}E(t))' \le -\beta_3\xi(t)E(t) + \beta_6\Big(k_1^2(t)\Big\|\frac{\partial w_0}{\partial \nu}\Big\|_\Gamma^2 + k_2^2(t)\|w_0\|_\Gamma^2\Big),\ \forall t \ge t_0,$$

where $\xi(t)$ is a nonincreasing function. Hence, using the fact that $I(t) = \xi(t)\Phi(t) + \frac{2\beta_4}{\gamma_0}E(t) \sim E(t)$, we deduce that

$$I'(t) \le -\beta_7\xi(t)I(t) + \beta_6\Big(k_1^2(t)\Big\|\frac{\partial w_0}{\partial \nu}\Big\|_\Gamma^2 + k_2^2(t)\|w_0\|_\Gamma^2\Big),\ \forall t \ge t_0, \tag{3.26}$$

where β_7 is a positive constant. We introduce

$$J(t) = I(t) - \beta_6 e^{-\beta_7\int_{t_0}^t \xi(s)ds}\Big(\int_{t_0}^t k_1^2(s)e^{\beta_7\int_{t_0}^s \xi(\eta)d\eta}ds\Big\|\frac{\partial w_0}{\partial \nu}\Big\|_\Gamma^2 + \int_{t_0}^t k_2^2(s)e^{\beta_7\int_{t_0}^s \xi(\eta)d\eta}ds\|w_0\|_\Gamma^2\Big). \tag{3.27}$$

From (3.26), we have

$$J'(t) \le -\beta_7\xi(t)J(t),\ \forall t \ge t_0.$$

Integrating this over (t_0, t), we obtain

$$J(t) \le J(t_0)e^{-\beta_7\int_{t_0}^t \xi(s)ds},\ \forall t \ge t_0.$$

Using the fact that $I(t) \sim E(t)$ and (3.27), we get the estimate (3.21).
2) G is nonlinear: First, we construct the functional

$$\Psi(t) = L(t) + \gamma_1 K_1(t) + \gamma_2 K_2(t),$$

which is nonnegative. From (2.13), (3.15), (3.16) and (3.19), we obtain

$$\Psi'(t) \le -\rho_0 E(t) + \beta_2\Big(k_1^2(t)\Big\|\frac{\partial w_0}{\partial \nu}\Big\|_\Gamma^2 + k_2^2(t)\|w_0\|_\Gamma^2\Big),$$

where ρ_0 is some positive constant. Integrating this over (t_0, t), we arrive at

$$\rho_0\int_{t_0}^t E(s)ds \le \Psi(t_0) + \beta_2\Big(k_1(0)\Big\|\frac{\partial w_0}{\partial \nu}\Big\|_\Gamma^2\int_{t_0}^t k_1(s)ds + k_2(0)\|w_0\|_\Gamma^2\int_{t_0}^t k_2(s)ds\Big).$$

Therefore, from (2.8), we conclude that

$$\int_{t_0}^t E(s)ds < \infty.$$

Then, we define $\zeta_1(t)$ and $\zeta_2(t)$ by, for constants θ_1 and $\theta_2 \in (0, 1)$,

$$\zeta_1(t) := \theta_1\int_{t_0}^t \Big\|\frac{\partial w(t)}{\partial \nu} - \frac{\partial w(t-s)}{\partial \nu}\Big\|_\Gamma^2 ds,\ \ \zeta_2(t) := \theta_2\int_{t_0}^t \|w(t) - w(t-s)\|_\Gamma^2 ds \in (0, 1).$$

Using (2.9), (2.14), (2.17) and the fact that $\xi_1(t)$ is a positive nonincreasing function, we find that

$$-\int_{t_0}^t k_1'(s)\Big\|\frac{\partial w(t)}{\partial \nu} - \frac{\partial w(t-s)}{\partial \nu}\Big\|_\Gamma^2 ds \le \int_{t_0}^t G_1^{-1}\Big(\frac{k_1''(s)}{\xi_1(s)}\Big)\Big\|\frac{\partial w(t)}{\partial \nu} - \frac{\partial w(t-s)}{\partial \nu}\Big\|_\Gamma^2 ds$$

$$\leq \frac{\zeta_1(t)}{\theta_1} G_1^{-1}\Big(\theta_1 \int_{t_0}^t \frac{k_1''(s)}{\xi_1(s)\zeta_1(t)} \Big\|\frac{\partial w(t)}{\partial \nu} - \frac{\partial w(t-s)}{\partial \nu}\Big\|_\Gamma^2 ds\Big)$$
$$\leq \frac{1}{\theta_1} G_1^{-1}\Big(\theta_1 \int_{t_0}^t \frac{k_1''(s)}{\xi_1(s)} \Big\|\frac{\partial w(t)}{\partial \nu} - \frac{\partial w(t-s)}{\partial \nu}\Big\|_\Gamma^2 ds\Big)$$
$$\leq \frac{1}{\theta_1} G_1^{-1}\Big(\frac{1}{\xi_1(t)} \int_{t_0}^t k_1''(s) \Big\|\frac{\partial w(t)}{\partial \nu} - \frac{\partial w(t-s)}{\partial \nu}\Big\|_\Gamma^2 ds\Big)$$
$$\leq \frac{1}{\theta_1} G_1^{-1}\Big(\frac{1}{\xi(t)} \int_\Gamma k_1'' \square \frac{\partial w}{\partial \nu} d\Gamma\Big), \tag{3.28}$$

where $\xi(t) = \min\{\xi_1(t), \xi_2(t)\}$.

Similarly, we can prove that

$$-\int_{t_0}^t k_2'(s)\|w(t) - w(t-s)\|_\Gamma^2 ds \leq \frac{1}{\theta_2} G_2^{-1}\Big(\frac{1}{\xi(t)} \int_\Gamma k_2'' \square w d\Gamma\Big). \tag{3.29}$$

Combining (2.8), (3.2), (3.25), (3.28) and (3.29), we see that, for all $t \geq t_0$,

$$\Phi'(t) \leq -\beta_3 \mathcal{E}(t) + \beta_8\Big(k_1(t)\Big\|\frac{\partial w_0}{\partial \nu}\Big\|_\Gamma^2 + k_2(t)\|w_0\|_\Gamma^2\Big)$$
$$+\frac{\beta_4}{\theta_1} G_1^{-1}\Big(\frac{1}{\xi(t)} \int_\Gamma k_1'' \square \frac{\partial w}{\partial \nu} d\Gamma\Big) + \frac{\beta_4}{\theta_2} G_2^{-1}\Big(\frac{1}{\xi(t)} \int_\Gamma k_2'' \square w d\Gamma\Big), \tag{3.30}$$

where $\beta_8 = \max\{\beta_5 k_1(0) + \frac{\beta_3\gamma_1}{2}\int_0^\infty k_1(s)ds,\ \beta_5 k_2(0) + \frac{\beta_3\gamma_2}{2}\int_0^\infty k_2(s)ds\}$. Now, for $\epsilon_0 < r$, we define the functional

$$R(t) := \Phi(t) G'(\epsilon_0 \frac{\mathcal{E}(t)}{\mathcal{E}(0)}),$$

where $\mathcal{E}(t)$ is the modified energy given in (3.2). Using (2.11), (2.15), (2.16), (3.1), (3.3), (3.30) and the fact that $\mathcal{E}' \leq 0,\ G' > 0$ and $G'' > 0$, we obtain for all $t \geq t_0$

$$R'(t) \leq -\beta_3 G'(\epsilon_0 \frac{\mathcal{E}(t)}{\mathcal{E}(0)})\mathcal{E}(t) + \beta_8\Big\|\frac{\partial w_0}{\partial \nu}\Big\|_\Gamma^2 G'(\epsilon_0\frac{\mathcal{E}(t)}{\mathcal{E}(0)})k_1(t) + \beta_8\|w_0\|_\Gamma^2 G'(\epsilon_0\frac{\mathcal{E}(t)}{\mathcal{E}(0)})k_2(t)$$
$$+\frac{\beta_4}{\theta_1} G'(\epsilon_0\frac{\mathcal{E}(t)}{\mathcal{E}(0)}) G^{-1}\Big(\frac{1}{\xi(t)}\int_\Gamma k_1''\square\frac{\partial w}{\partial \nu}d\Gamma\Big) + \frac{\beta_4}{\theta_2} G'(\epsilon_0\frac{\mathcal{E}(t)}{\mathcal{E}(0)}) G^{-1}\Big(\frac{1}{\xi(t)}\int_\Gamma k_2''\square w d\Gamma\Big)$$
$$\leq -\Big[\beta_3\mathcal{E}(0) - (\beta_8\Big\|\frac{\partial w_0}{\partial \nu}\Big\|_\Gamma^2 + \beta_8\|w_0\|_\Gamma^2 + \frac{2\beta_4}{\theta_0})\epsilon_0\Big]\frac{\mathcal{E}(t)}{\mathcal{E}(0)} G'(\epsilon_0\frac{\mathcal{E}(t)}{\mathcal{E}(0)})$$
$$+\beta_8\Big\|\frac{\partial w_0}{\partial \nu}\Big\|_\Gamma^2 G(k_1(t)) + \beta_8\|w_0\|_\Gamma^2 G(k_2(t)) + \frac{\beta_4}{\theta_0\xi(t)}\Big(\int_\Gamma k_1''\square\frac{\partial w}{\partial \nu}d\Gamma + \int_\Gamma k_2''\square w d\Gamma\Big)$$
$$\leq -\Big[\beta_3\mathcal{E}(0) - (\beta_8\Big\|\frac{\partial w_0}{\partial \nu}\Big\|_\Gamma^2 + \beta_8\|w_0\|_\Gamma^2 + \frac{2\beta_4}{\theta_0})\epsilon_0\Big]\frac{\mathcal{E}(t)}{\mathcal{E}(0)} G'(\epsilon_0\frac{\mathcal{E}(t)}{\mathcal{E}(0)})$$
$$+\beta_8\Big\|\frac{\partial w_0}{\partial \nu}\Big\|_\Gamma^2 G(k_1(t)) + \beta_8\|w_0\|_\Gamma^2 G(k_2(t)) - \frac{2\beta_4}{\theta_0\gamma_0\xi(t)}\mathcal{E}'(t),$$

where $\theta_0 = \min\{\theta_1,\ \theta_2\}$ and $\gamma_0 = \min\{\gamma_1,\ \gamma_2\}$. Choosing ϵ_0 such that $\rho_1 = \beta_3\mathcal{E}(0) - (\beta_8\big\|\frac{\partial w_0}{\partial \nu}\big\|_\Gamma^2 + \beta_8\|w_0\|_\Gamma^2 + \frac{2\beta_4}{\theta_0})\epsilon_0 > 0$, we have

$$R'(t) \leq -\rho_1\frac{\mathcal{E}(t)}{\mathcal{E}(0)} G'(\epsilon_0\frac{\mathcal{E}(t)}{\mathcal{E}(0)}) + \beta_8\Big\|\frac{\partial w_0}{\partial \nu}\Big\|_\Gamma^2 G(k_1(t)) + \beta_8\|w_0\|_\Gamma^2 G(k_2(t)) - \frac{2\beta_4}{\theta_0\gamma_0\xi(t)}\mathcal{E}'(t).$$

Then, multiplying this by $\xi(t)$, we get

$$\xi(t)R'(t) \leq -\rho_1\xi(t)\frac{\mathcal{E}(t)}{\mathcal{E}(0)}G'(\epsilon_0\frac{\mathcal{E}(t)}{\mathcal{E}(0)}) + \beta_8\Big(\|w_0\|_\Gamma^2 G(k_2(t)) + \Big\|\frac{\partial w_0}{\partial\nu}\Big\|_\Gamma^2 G(k_1(t))\Big)\xi(t) - \frac{2\beta_4}{\theta_0\gamma_0}\mathcal{E}'(t). \quad (3.31)$$

Taking $\mathcal{F}(t) = \xi(t)R(t) + \frac{2\beta_4}{\theta_0\gamma_0}\mathcal{E}(t)$ and using (3.31) and $\xi' \leq 0$, we arrive at

$$\mathcal{F}'(t) \leq -\rho_1\xi(t)H_1(\frac{\mathcal{E}(t)}{\mathcal{E}(0)}) + \beta_8\Big(\Big\|\frac{\partial w_0}{\partial\nu}\Big\|_\Gamma^2 G(k_1(t)) + \|w_0\|_\Gamma^2 G(k_2(t))\Big)\xi(t),\ \forall t \geq t_0, \quad (3.32)$$

where $H_1(t) = tG'(\epsilon_0 t)$. Applying (3.32) and the fact that $\xi' \leq 0,\ \mathcal{E}' \leq 0$ and $H_1' \geq 0$, we find that

$$\begin{aligned}\Big[t\xi(t)H_1(\frac{\mathcal{E}(t)}{\mathcal{E}(0)})\Big]' &\leq \xi(t)H_1(\frac{\mathcal{E}(t)}{\mathcal{E}(0)})\\ &\leq -\frac{1}{\rho_1}\mathcal{F}'(t) + \frac{\beta_8}{\rho_1}\Big(\Big\|\frac{\partial w_0}{\partial\nu}\Big\|_\Gamma^2 G(k_1(t)) + \|w_0\|_\Gamma^2 G(k_2(t))\Big)\xi(t),\ \forall t \geq t_0.\end{aligned}$$

Integrating this over (t_0, t), we see that

$$\begin{aligned}t\xi(t)H_1(\frac{\mathcal{E}(t)}{\mathcal{E}(0)}) &\leq t_0\xi(t_0)H_1(\frac{\mathcal{E}(t_0)}{\mathcal{E}(0)}) + \frac{1}{\rho_1}\mathcal{F}(t_0) + \frac{\beta_8}{\rho_1}\int_{t_0}^t\Big(\Big\|\frac{\partial w_0}{\partial\nu}\Big\|_\Gamma^2 G(k_1(s)) + \|w_0\|_\Gamma^2 G(k_2(s))\Big)\xi(s)ds\\ &\leq \rho_2\Big(1 + \int_{t_0}^t(\Big\|\frac{\partial w_0}{\partial\nu}\Big\|_\Gamma^2 G(k_1(s)) + \|w_0\|_\Gamma^2 G(k_2(s)))\xi(s)ds\Big),\end{aligned}$$

where $\rho_2 = \max\{t_0\xi(t_0)H_1(\frac{\mathcal{E}(t_0)}{\mathcal{E}(0)}) + \frac{1}{\rho_1}\mathcal{F}(t_0),\ \frac{\beta_8}{\rho_1}\}$. Therefore, we conclude that

$$\mathcal{E}(t) \leq \mathcal{E}(0)H_1^{-1}\Big(\frac{\rho_2\Big(1 + \int_{t_0}^t(\Big\|\frac{\partial w_0}{\partial\nu}\Big\|_\Gamma^2 G(k_1(s)) + \|w_0\|_\Gamma^2 G(k_2(s)))\xi(s)ds\Big)}{t\xi(t)}\Big),\ \forall t \geq t_0.$$

Hence, applying (3.2), (3.22) is established. □

Examples. *We provide examples to explain the decay of energy [32, 36].*
1) For $k_1'(t) = k_2'(t) = -e^{-t^q}$ with $0 < q < 1$, we obtain $k_i''(t) = G_i(-k_i'(t))(i = 1, 2)$, where $G_1(t) = G_2(t) = \frac{qt}{[\ln(\frac{1}{t})]^{\frac{1}{q}-1}}$. Since

$$G_1'(t) = G_2'(t) = \frac{(1-q) + q\ln(\frac{1}{q})}{[\ln(\frac{1}{t})]^{\frac{1}{q}}} \text{ and } G_1''(t) = G_2''(t) = \frac{(1-q)[\ln(\frac{1}{q}) + \frac{1}{q}]}{[\ln(\frac{1}{t})]^{\frac{1}{q}+1}},$$

the functions G_1 and G_2 satisfy the condition (2.9) on $(0, r]$ for any $0 < r < 1$.
2) Let $k_i'(t) = -\frac{a_i}{(1+t)^2}$, where $a_i > 0, (i = 1, 2)$, be chosen so that assumption (A) holds. We choose $a = \min\{a_1, a_2\}$, then $k_i''(t) = b_iG_i(-k_i'(t))$. We select $b = \min\{b_1, b_2\}$, $G = \min\{G_1, G_2\}$ and $\xi(t) = \min\{\xi_1(t), \xi_2(t)\}$, then $G(t) = t^{\frac{3}{2}}, \xi(t) = b$.

4. Conclusions

The von Karman plates with memory on the boundary is also widely used in communication and signal processing. In particular, understanding the energy decay that occurs when a signal passes through a boundary can contribute to improving the performance of the communication system. This type of equation plays an important role in explaining various physical phenomena in the real world. In this paper, we study the von Karman plate system with general type of relaxation functions on the boundary. Here, we consider the resolvent kernels $k_i(i = 1, 2)$, namely $k_i''(t) \geq -\xi_i(t)G_i(-k_i'(t))$, where G_i are convex and increasing functions near the origin and ξ_i are positive nonincreasing functions. Using some properties of convex functions without the assumption that initial value $w_0 \equiv 0$ on the boundary, we prove the general decay rate result. These general decay estimates improve earlier results in the literature.

Use of AI tools declaration

The author declares she has not used Artificial Intelligence(AI) tools in the creation of this article.

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Conflict of interest

The author declares that she has no conflict of interest.

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