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Review

A comprehensive review of Grüss-type fractional integral inequality

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Abstract: A survey of results on Grüss-type inequalities associated with a variety of fractional integral and differential operators is presented. The fractional differential operators includes, Riemann-Liouville fractional integral operators, Riemann-Liouville fractional integrals of a function with respect to another function, Katugampola fractional integral operators, Hadamard's fractional integral operators, k-fractional integral operators, Raina's fractional integral operators, tempered fractional integral operators, conformable fractional integrals operators, proportional fractional integrals operators, generalized Riemann-Liouville fractional integral operators, Caputo-Fabrizio fractional integrals operators, Saigo fractional integral operators, quantum integral operators, and Hilfer fractional differential operators.

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1. Introduction

Fractional calculus is like an extended version of regular calculus that allows us to deal with numbers that are not whole, like 1.5 or 2.3. This might not sound like a big deal, but it is incredibly useful in many fields. When we want to understand how things change or accumulate over time, fractional calculus helps us do that more accurately, especially when things are complicated and do not follow normal rules. These fractional calculations come in handy when we are dealing with stuff like how liquids flow, how materials deform, or how we control things like robots or machines. Inequalities, in the context of fractional calculus, are like special rules that help us understand when things are bigger or smaller than each other, but with these non-whole numbers involved. These rules are important because they help us figure out if systems with fractional calculus are stable and work the way they should. Thus, in a nutshell, fractional calculus and inequalities help us make sense of the world in a more precise and practical way. Thus, the term convexity and inequalities in the frame of fractional calculus have been recommended as an engrossing area for researchers due to their vital role and fruitful importance in numerous branches of science. Integral inequalities have remarkable uses in probability, optimization theory, information technology, stochastic processes, statistics, integral operator theory and numerical integration. For the applications, see references [1–8].

In [9], a comprehensive and up-to-date review on Hermite-Hadamard-type inequalities for different kinds of convexities and different kinds of fractional integral operators is presented. In this review paper, we aim to discuss and present the up-to-date review of the Grüss type inequality via different fractional integral operators.

In [10] (see also [11]), the Grüss inequality is defined as the integral inequality that establishes a connection between the integral of the product of two functions and the product of the integrals. The inequality is as follows.

Theorem 1.1. *If* $\Omega, \Pi : [x_1, x_2] \to \mathbb{R}$ *are two continuous functions satisfying* $\mathfrak{m} \leq \Omega(t) \leq \mathfrak{M}$ *and* $\mathfrak{p} \leq \Pi(t) \leq \mathfrak{P}$, $t \in [x_1, x_2]$, $\mathfrak{m}, \mathfrak{M}, \mathfrak{p}, \mathfrak{P} \in \mathbb{R}$, *then*

$$\left|\frac{1}{x_2 - x_1} \int_{x_1}^{x_2} \Omega(s)\Pi(s)ds - \frac{1}{(x_2 - x_1)^2} \int_{x_1}^{x_2} \Omega(s)ds \int_{x_1}^{x_2} \Pi(s)ds\right| \leq \frac{1}{4}(\mathfrak{M} - \mathfrak{m})(\mathfrak{P} - \mathfrak{p}).$$

Our objective in this paper is to present a comprehensive and up-to-date review on Grüss-type inequalities for different kinds of fractional integral operators. In each section and subsection, we first introduce the basic definitions of fractional integral operators and then include the results on Grüss-type inequalities. We believe that the collection of almost all existing in the literature Grüss-type inequalities in one file will help new researchers in the field learn about the available work on the topic before developing new results. We present the results without proof but instead provide a complete reference for the details of each result elaborated in this survey for the convenience of the reader.

The remainder of this review paper is as follows. In Sections 2–15, we summarize Grüss-type integral inequalities and especially for Riemann-Liouville fractional integral operators in Section 2, for Riemann-Liouville fractional integrals of a function with respect to another function in Section 3, in Section 4 for Katugampola fractional integral operators, in Section 5 for Hadamard's fractional integral operators, in Section 6 for *k*-fractional integral operators, in Section 7 for Raina's fractional integral operators, in Section 8 for tempered fractional integral operators, in Section 9 for conformable fractional integrals operators, in Section 10 for proportional fractional integrals operators, in Section 11 for generalized Riemann-Liouville fractional integral operators, in Section 12 for Caputo-Fabrizio fractional integrals operators, for Saigo fractional integral operators in Section 13, in Section 14 for quantum integral operators and in Section 15 for Hilfer fractional differential operators.

Throughout this survey the following assumptions are used:

(H) Assume that $\Omega, \Pi : I \to \mathbb{R}$ are integrable functions on I for which there exist constants $\mathfrak{m}, \mathfrak{M}, \mathfrak{p}, \mathfrak{P} \in \mathbb{R}$, such that

$$\mathfrak{m} \leq \Omega(t) \leq \mathfrak{M}, \quad \mathfrak{p} \leq \Pi(t) \leq \mathfrak{P}, \quad t \in I.$$

(H_1) There exist two integrable functions $Q_1, Q_2 : [0, \infty) \to \mathbb{R}$ such that

$$Q_1(t) \leq \Omega(t) \leq Q_2(t) \quad \text{for all} \quad t \in [0, \infty).$$

(H_2) There exist two integrable functions $R_1, R_2 : [0, \infty) \to \mathbb{R}$ such that

$$R_1(t) \leq \Pi(t) \leq R_2(t) \quad \text{for all} \quad t \in [0, \infty).$$

2. Grüss-type integral inequalities via Riemann-Liouville fractional integral

In this subsection we give generalizations for Grüss-type inequalities by using the Riemann-Liouville fractional integrals. The first result deals with some inequalities using one fractional parameter.

Definition 2.1. *[12] A real valued function $\Omega(t), t \geq 0$ is said to be in*

(i) the space $C_\mu, \mu \in \mathbb{R}$ if there exists a real number $p > \mu$ such that $\Omega(t) = t^p \Omega_1(t)$, where $\Omega_1(t) \in C([0, \infty), \mathbb{R})$,

(i) the space $C^n_\mu, \mu \in \mathbb{R}$ if $\Omega^{(n)} \in C_\mu$.

Definition 2.2. *[12] The Riemann-Liouville integral operator of fractional order $\alpha \geq 0$, for an integrable function Ω is defined by*

$$J^\alpha \Omega(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Omega(s) ds, \quad \alpha > 0, \ t > 0,$$

and $J^0 \Omega(t) = \Omega(t)$.

Theorem 2.1. *[12] Assume that (H) holds on $[0, \infty)$. Then for all $t > 0$ and $\alpha > 0$ we have:*

$$\Big| \frac{t^\alpha}{\Gamma(\alpha+1)} J^\alpha \Omega(t)\Pi(t) - J^\alpha \Omega(t) J^\alpha \Pi(t) \Big| \leq \Big(\frac{t^\alpha}{2\Gamma(\alpha+1)} \Big)^2 (\mathfrak{M} - \mathfrak{m})(\mathfrak{P} - \mathfrak{p}).$$

In the next result two real positive parameters are used.

Theorem 2.2. *[12] Assume that (H) holds on $[0, \infty)$. Then for all $t > 0$ and $\alpha > 0, \beta > 0$ we have:*

$$\Big(\frac{t^\alpha}{\Gamma(\alpha+1)} J^\beta \Omega(t)\Pi(t) + \frac{t^\beta}{\Gamma(\beta+1)} J^\alpha \Omega(t)\Pi(t) - J^\alpha \Omega(t) J^\beta \Pi(t) - J^\beta \Omega(t) J^\alpha \Pi(t) \Big)^2$$

$$\leq \Big[\Big(\mathfrak{M}\frac{t^{\alpha}}{\Gamma(\alpha+1)} - J^{\alpha}\Omega(t)\Big)\Big(J^{\beta}\Omega(t) - \mathfrak{m}\frac{t^{\beta}}{\Gamma(\beta+1)}\Big)$$
$$+\Big(J^{\alpha}\Omega(t) - \mathfrak{m}\frac{t^{\alpha}}{\Gamma(\alpha+1)}\Big)\Big(\mathfrak{M}\frac{t^{\beta}}{\Gamma(\beta+1)} - J^{\beta}\Omega(t)\Big)\Big]$$
$$\times\Big[\Big(\mathfrak{P}\frac{t^{\alpha}}{\Gamma(\alpha+1)} - J^{\alpha}\Pi(t)\Big)\Big(J^{\beta}\Pi(t) - \mathfrak{p}\frac{t^{\beta}}{\Gamma(\beta+1)}\Big)$$
$$+\Big(J^{\alpha}\Pi(t) - \mathfrak{p}\frac{t^{\alpha}}{\Gamma(\alpha+1)}\Big)\Big(\mathfrak{P}\frac{t^{\beta}}{\Gamma(\beta+1)} - J^{\beta}\Pi(t)\Big)\Big].$$

Next, we present some fractional integral inequalities of Grüss type by using the Riemann-Liouville fractional integral. The constants appeared as bounds of the functions Ω and Π, are replaced by four integrable functions.

Theorem 2.3. *[13] Assume that $\Omega : [0,\infty) \to \mathbb{R}$ is an integrable function satisfying (H_1). Then, for $t > 0$, $\alpha,\beta > 0$, we have:*

$$J^{\beta}Q_1(t)J^{\alpha}\Omega(t) + J^{\alpha}Q_2(t)J^{\beta}\Omega(t) \geq J^{\alpha}Q_2(t)J^{\beta}Q_1(t) + J^{\alpha}\Omega(t)J^{\beta}\Omega(t).$$

Theorem 2.4. *[13] Suppose that $\Omega, \Pi : [0,\infty) \to \mathbb{R}$ are two integrable functions satisfying (H_1) and (H_2). Then, for $t > 0$, α, $\beta > 0$, the fractional integral inequalities hold:*

$$(i) \quad J^{\beta}R_1(t)J^{\alpha}\Omega(t) + J^{\alpha}Q_2(t)J^{\beta}\Pi(t) \geq J^{\beta}R_1(t)J^{\alpha}Q_2(t) + J^{\alpha}\Omega(t)J^{\beta}\Pi(t).$$
$$(ii) \quad J^{\beta}Q_1(t)J^{\alpha}\Pi(t) + J^{\alpha}R_2(t)J^{\beta}\Omega(t) \geq J^{\beta}Q_1(t)J^{\alpha}R_2(t) + J^{\beta}\Omega(t)J^{\alpha}\Pi(t).$$
$$(iii) \quad J^{\alpha}Q_2(t)J^{\beta}R_2(t) + J^{\alpha}\Omega(t)J^{\beta}\Pi(t) \geq J^{\alpha}Q_2(t)J^{\beta}\Pi(t) + J^{\beta}R_2(t)J^{\alpha}\Omega(t).$$
$$(iv) \quad J^{\alpha}Q_1(t)J^{\beta}R_1(t) + J^{\alpha}\Omega(t)J^{\beta}\Pi(t) \geq J^{\alpha}Q_1(t)J^{\beta}\Pi(t) + J^{\beta}R_1(t)J^{\alpha}\Omega(t).$$

Theorem 2.5. *[13] Assume that $\Omega, \Pi : [0,\infty) \to \mathbb{R}$ are two integrable functions satisfying (H_1) and (H_2). Then for all $t > 0$, $\alpha > 0$, we have:*

$$\left|\frac{t^{\alpha}}{\Gamma(\alpha+1)}J^{\alpha}\Omega(t)\Pi(t) - J^{\alpha}\Omega(t)J^{\alpha}\Pi(t)\right| \leq \sqrt{T(\Omega, Q_1, Q_2)T(\Pi, R_1, R_2)},$$

where $T(y,z,w)$ is defined by

$$T(y,z,w) = (J^{\alpha}w(t) - J^{\alpha}y(t))(J^{\alpha}y(t) - J^{\alpha}z(t)) + \frac{t^{\alpha}}{\Gamma(\alpha+1)}J^{\alpha}z(t)y(t) - J^{\alpha}z(t)J^{\alpha}y(t)$$
$$+\frac{t^{\alpha}}{\Gamma(\alpha+1)}J^{\alpha}w(t)y(t) - J^{\alpha}w(t)J^{\alpha}y(t) + J^{\alpha}z(t)J^{\alpha}w(t) - \frac{t^{\alpha}}{\Gamma(\alpha+1)}J^{\alpha}z(t)w(t).$$

In the next theorem we give an Ostrowski-Grüss type inequality of fractional type via Riemann-Liouville fractional integral.

Theorem 2.6. *[14] Let $\Omega : [x_1, x_2] \to \mathbb{R}$ be a differentiable mapping on (x_1, x_2) and $|\Omega'(x)| \leq M$ for all $x \in [x_1, x_2]$. Then*

$$\left|\frac{1}{2}\Omega(x) - (\alpha+1)\Gamma(\alpha)\frac{(x_2-x)^{1-\alpha}}{2(x_2-x_1)}J^{\alpha}_{x_1}\Omega(x_2) + \frac{1}{2}J^{\alpha-1}_{x_1}((x_2-x)^{1-\alpha}\Gamma(\alpha)\Omega(x_2))\right.$$

$$+\frac{(x_2-x)^{2-\alpha}}{2(x_2-x_1)}\Gamma(\alpha)J_{x_1}^{\alpha-1}\Omega(x_2)+\frac{(x_2-x)^{1-\alpha}(x-x_1)}{2(x_2-x_1)^{2-\alpha}}\Omega(x_1)\Bigg|$$
$$\leq \frac{M(x_2-x)^{1-\alpha}}{x_2-x_1}\Big[\frac{(x_2-x_1)^{\alpha}(x-x_1)+(x_2-x)^{\alpha}(x_1+x_2-2x)}{2\alpha}\Big],$$

where $x_1 \leq x < x_2$.

3. Grüss-type integral inequalities for Riemann-Liouville fractional integrals of a function with respect to another function

Definition 3.1. *[15] Let* $\psi : [0,\infty) \to \mathbb{R}$ *be positive, increasing function and also its derivative* ψ' *be continuous on* $[0,\infty)$ *and* $\psi(0) = 0$. *The fractional integral of Riemann-Liouville type of an integrable function* Ω *with respect to another function* ψ *is defined as*

$$I^{\alpha,\psi}\Omega(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (\psi(t)-\psi(s))^{\alpha-1}\psi'(s)\Omega(s)ds.$$

In the next we include Grüss type integral inequalities with the help of ψ-Riemann-Liouville fractional integral.

Theorem 3.1. *[16] Assume that* $\psi : [0,\infty) \to \mathbb{R}$ *is a positive, increasing function and also its derivative* ψ' *is continuous on* $[0,\infty)$ *and* $\psi(0) = 0$. *Assume that* $\Omega : [0,\infty) \to \mathbb{R}$ *is an integrable function satisfying* (H_1). *Then the following inequality holds:*

$$I^{\beta,\psi}Q_1(t)I^{\alpha,\psi}\Omega(t) + I^{\alpha,\psi}Q_2(t)I^{\beta,\psi}\Omega(t) \geq I^{\alpha,\psi}Q_2(t)I^{\beta,\psi}Q_1(t) + I^{\beta,\psi}\Omega(t)I^{\beta,\psi}\Omega(t).$$

Theorem 3.2. *[16] Let* ψ *be as in Theorem 3.1 and* Ω, Π *be two integrable functions satisfying* (H_1) *and* (H_2). *Then we have:*

(a) $I^{\beta,\psi}R_1(t)I^{\alpha,\psi}\Omega(t) + I^{\alpha,\psi}Q_2(t)I^{\beta,\psi}\Pi(t) \geq I^{\beta,\psi}R_1(t)I^{\alpha,\psi}Q_2(t) + I^{\alpha,\psi}\Omega(t)I^{\beta,\psi}\Pi(t)$.
(b) $I^{\beta,\psi}Q_1(t)I^{\alpha,\psi}\Pi(t) + I^{\alpha,\psi}R_2(t)I^{\beta,\psi}\Omega(t) \geq I^{\beta,\psi}Q_1(t)I^{\alpha,\psi}R_2(t) + I^{\beta,\psi}\Omega(t)I^{\alpha,\psi}\Pi(t)$.
(c) $I^{\alpha,\psi}Q_2(t)I^{\beta,\psi}R_2(t) + I^{\alpha,\psi}\Omega(t)I^{\beta,\psi}\Pi(t) \geq I^{\alpha,\psi}Q_2(t)I^{\beta,\psi}\Pi(t) + I^{\beta,\psi}R_2(t)I^{\alpha,\psi}\Omega(t)$.
(d) $I^{\alpha,\psi}Q_1(t)I^{\beta,\psi}R_1(t) + I^{\alpha,\psi}\Omega(t)I^{\beta,\psi}\Pi(t) \geq I^{\alpha,\psi}Q_1(t)I^{\beta,\psi}\Pi(t) + I^{\beta,\psi}R_1(t)I^{\alpha,\psi}\Omega(t)$.

Theorem 3.3. *[16] Let* ψ *be as in Theorem 3.1 and* Ω, Π *be two integrable functions satisfying* (H_1) *and* (H_2). *Then the following inequality holds:*

$$\left|\frac{\psi^{\alpha}(t)}{\Gamma(\alpha+1)}I^{\alpha,\psi}\Omega(t)\Pi(t) - I^{\alpha,\psi}\Omega(t)I^{\alpha,\psi}\Pi(t)\right| \leq \sqrt{T(\Omega,Q_1,Q_2)T(\Pi,R_1,R_2)},$$

where

$$\begin{aligned} T(y,z,w) &= (I^{\alpha,\psi}w(t)-I^{\alpha,\psi}y(t))(I^{\alpha,\psi}y(t)-I^{\alpha,\psi}z(t)) \\ &\quad +\frac{\psi^{\alpha}(t)}{\Gamma(\alpha+1)}I^{\alpha,\psi}v(t)I^{\alpha,\psi}y(t) - I^{\alpha,\psi}z(t)I^{\alpha,\psi}y(t) + \frac{\psi^{\alpha}(t)}{\Gamma(\alpha+1)}I^{\alpha,\psi}w(t)y(t) \\ &\quad -I^{\alpha,\psi}w(t)I^{\alpha,\psi}y(t) + I^{\alpha,\psi}z(t)I^{\alpha,\psi}w(t) - \frac{\psi^{\alpha}(t)}{\Gamma(\alpha+1)}I^{\alpha,\psi}z(t)w(t). \end{aligned}$$

4. Grüss-type fractional inequalities for Katugampola fractional integral operator

Now we define the space $X_c^p(x_1, x_2)$ in which Katugampola's fractional integrals are defined.

Definition 4.1. *[17] The space $X_c^p(x_1, x_2)$ $(c \in \mathbb{R}, 1 \le p < \infty)$ consists of those complex-valued Lebesgue measurable functions ϕ on (x_1, x_2) for which $\|\phi\|_{X_c^p} < \infty$, with*

$$\|\phi\|_{X_c^p} = \Big(\int_{x_1}^{x_2} |x^c \phi(x)|^p \frac{dx}{x} \Big)^{1/p} \quad (1 \le p < \infty),$$

and

$$\|\phi\|_{X_c^\infty} = \operatorname{ess}\sup_{x \in (x_1, x_2)} [x^c |\phi(x)|].$$

Definition 4.2. *[17] Let $\phi \in X_c^p(x_1, x_2)$, $\alpha > 0$ and $\beta, \rho, \eta, \kappa \in \mathbb{R}$. Then, the left- and right- sided fractional integrals of a function ϕ are defined respectively by*

$${}^{\rho}J^{\alpha,\beta}_{x_1+,\eta,\kappa}\phi(x) = \frac{\rho^{1-\beta} x^{\kappa}}{\Gamma(\alpha)} \int_{x_1}^{x} \frac{\tau^{\rho(\eta+1)-1}}{(x^\rho - \tau^\rho)^{1-\alpha}} \phi(\tau) d\tau, \quad 0 \le x_1 < x < x_2 \le \infty,$$

and

$${}^{\rho}J^{\alpha,\beta}_{x_2-,\eta,\kappa}\phi(x) = \frac{\rho^{1-\beta} x^{\rho\eta}}{\Gamma(\alpha)} \int_{x}^{x_2} \frac{\tau^{\kappa+\rho-1}}{(\tau^\rho - x^\rho)^{1-\alpha}} \phi(\tau) d\tau, \quad 0 \le x_1 < x < x_2 \le \infty,$$

if the integrals exist.

Now, we present several Grüss-type inequalities involving Katugampola's fractional integral.

Theorem 4.1. *[17] Assume that (H) holds on $[0, \infty)$. Then we have:*

$$\Big| \Lambda^{\rho,\beta}_{x,\kappa}(\alpha,\eta)\, {}^{\rho}J^{\alpha,\beta}_{\eta,\kappa}\Omega(x)\Pi(x) - {}^{\rho}J^{\alpha,\beta}_{\eta,\kappa}\Omega(x) {}^{\rho}J^{\alpha,\beta}_{\eta,\kappa}\Pi(x) \Big| \le \Big(\Lambda^{\rho,\beta}_{x,\kappa}(\alpha,\eta) \Big)^2 (\mathfrak{M} - \mathfrak{m})(\mathfrak{P} - \mathfrak{p}),$$

for all $\beta, \kappa \in \mathbb{R}$, $x > 0$, $\alpha > 0$, $\rho > 0$ and $\eta \ge 0$, where

$$\Lambda^{\rho,\beta}_{x,\kappa}(\alpha,\eta) = \frac{\Gamma(\eta+1)}{\Gamma(\eta+\alpha+1)} \rho^{-\beta} x^{\kappa+\rho(\eta+\alpha)}.$$

Theorem 4.2. *[17] Assume that (H) holds on $[0, \infty)$. Then for all $\beta, \kappa \in \mathbb{R}$, $x > 0$, $\alpha > 0$, $\gamma > 0$ and $\eta \ge 0$, we have:*

$$\begin{aligned} &\Big(\Lambda^{\rho,\beta}_{x,\kappa}(\alpha,\eta)\, {}^{\rho}J^{\gamma,\beta}_{\eta,\kappa}\Omega(x)\Pi(x) + \Lambda^{\rho,\beta}_{x,\kappa}(\gamma,\eta)\, {}^{\rho}J^{\alpha,\beta}_{\eta,\kappa}\Omega(x)\Pi(x) - {}^{\rho}J^{\alpha,\beta}_{\eta,\kappa}\Omega(x)\, {}^{\rho}J^{\alpha,\beta}_{\eta,\kappa}\Pi(x) \\ &- {}^{\rho}J^{\gamma,\beta}_{\eta,\kappa}\Omega(x)\, {}^{\rho}J^{\alpha,\beta}_{\eta,\kappa}\Pi(x) \Big)^2 \\ \le\ & \Big[\Big(\mathfrak{M}\Lambda^{\rho,\beta}_{x,\kappa}(\alpha,\eta) - {}^{\rho}J^{\alpha,\beta}_{\eta,\kappa}\Omega(x) \Big) \Big({}^{\rho}J^{\gamma,\beta}_{\eta,\kappa}\Omega(x) - \mathfrak{m}\Lambda^{\rho,\beta}_{x,\kappa}(\alpha,\eta) \Big) \\ &+ \Big({}^{\rho}J^{\alpha,\beta}_{\eta,\kappa}\Omega(x) - \mathfrak{m}\Lambda^{\rho,\beta}_{x,\kappa}(\alpha,\eta) \Big) \Big(\mathfrak{M}\Lambda^{\rho,\beta}_{x,\kappa}(\gamma,\eta) - {}^{\rho}J^{\gamma,\beta}_{\eta,\kappa}\Omega(x) \Big) \Big] \\ &\times \Big[\Big(\mathfrak{P}\Lambda^{\rho,\beta}_{x,\kappa}(\alpha,\eta) - {}^{\rho}J^{\alpha,\beta}_{\eta,\kappa}\Pi(x) \Big) \Big({}^{\rho}J^{\gamma,\beta}_{\eta,\kappa}\Pi(x) - \mathfrak{p}\Lambda^{\rho,\beta}_{x,\kappa}(\gamma,\eta) \Big) \\ &+ \Big({}^{\rho}J^{\alpha,\beta}_{\eta,\kappa}\Pi(x) - \mathfrak{p}\Lambda^{\rho,\beta}_{x,\kappa}(\alpha,\eta) \Big) \Big(\mathfrak{P}\Lambda^{\rho,\beta}_{x,\kappa}(\gamma,\eta) - {}^{\rho}J^{\gamma,\beta}_{\eta,\kappa}\Pi(x) \Big) \Big]. \end{aligned}$$

Theorem 4.3. *[17] Let $\alpha > 0, \beta, \rho, \eta, \kappa \in \mathbb{R}$, $\Omega, \Pi \in X_c^p(0, x)$ $x > 0$ and $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then we have:*

$$(a) \qquad \frac{1}{p}{}^{\rho}J_{\eta,\kappa}^{\alpha,\beta}\Omega^p(x) + \frac{1}{q}{}^{\rho}J_{\eta,\kappa}^{\alpha,\beta}\Pi^q(x) \geq \frac{\Gamma(\eta+\alpha+1)\rho^{\beta}}{\Gamma(\eta+1)x^{\rho(\eta+\alpha)+\kappa}}\Big({}^{\rho}J_{\eta,\kappa}^{\alpha,\beta}\Omega(x)\,{}^{\rho}J_{\eta,\kappa}^{\alpha,\beta}\Pi(x)\Big).$$

$$\begin{aligned}(b) \qquad & \frac{1}{p}{}^{\rho}J_{\eta,\kappa}^{\alpha,\beta}\Omega^p(x)\,{}^{\rho}J_{\eta,\kappa}^{\alpha,\beta}\Pi^p(x) + \frac{1}{q}{}^{\rho}J_{\eta,\kappa}^{\alpha,\beta}\Omega^q(x)\,{}^{\rho}J_{\eta,\kappa}^{\alpha,\beta}\Pi^q(x) \\ \geq\ & \Big({}^{\rho}J_{\eta,\kappa}^{\alpha,\beta}\Omega(x)\Pi(x)\Big)^2.\end{aligned}$$

$$\begin{aligned}(c) \qquad & \frac{1}{p}{}^{\rho}J_{\eta,\kappa}^{\alpha,\beta}\Omega^p(x)\,{}^{\rho}J_{\eta,\kappa}^{\alpha,\beta}\Pi^q(x) + \frac{1}{q}{}^{\rho}J_{\eta,\kappa}^{\alpha,\beta}\Omega^q(x)\,{}^{\rho}J_{\eta,\kappa}^{\alpha,\beta}\Pi^p(x) \\ \geq\ & \Big({}^{\rho}J_{\eta,\kappa}^{\alpha,\beta}(\Omega\Pi)^{p-1}(x)\Big)\Big({}^{\rho}J_{\eta,\kappa}^{\alpha,\beta}(\Omega\Pi)^{q-1}(x)\Big).\end{aligned}$$

$$(d) \qquad {}^{\rho}J_{\eta,\kappa}^{\alpha,\beta}\Omega^p(x)\,{}^{\rho}J_{\eta,\kappa}^{\alpha,\beta}\Pi^q(x) \geq \Big({}^{\rho}J_{\eta,\kappa}^{\alpha,\beta}\Omega(x)\Pi(x)\Big)\Big({}^{\rho}J_{\eta,\kappa}^{\alpha,\beta}\Omega^{p-1}(x)\Pi^{q-1}(x)\Big).$$

$$\begin{aligned}(e) \qquad & \frac{1}{p}{}^{\rho}J_{\eta,\kappa}^{\alpha,\beta}\Omega^p(x)\,{}^{\rho}J_{\eta,\kappa}^{\alpha,\beta}\Pi^2(x) + \frac{1}{q}{}^{\rho}J_{\eta,\kappa}^{\alpha,\beta}\Omega^2(x)\,{}^{\rho}J_{\eta,\kappa}^{\alpha,\beta}\Pi^q(x) \\ \geq\ & \Big({}^{\rho}J_{\eta,\kappa}^{\alpha,\beta}\Omega(x)\Pi(x)\Big)\Big({}^{\rho}J_{\eta,\kappa}^{\alpha,\beta}\Omega^{2/p}(x)\Pi^{2/p}(x)\Big).\end{aligned}$$

$$\begin{aligned}(f) \qquad & \frac{1}{p}{}^{\rho}J_{\eta,\kappa}^{\alpha,\beta}\Omega^2(x)\,{}^{\rho}J_{\eta,\kappa}^{\alpha,\beta}\Pi^q(x) + \frac{1}{q}{}^{\rho}J_{\eta,\kappa}^{\alpha,\beta}\Omega^q(x)\,{}^{\rho}J_{\eta,\kappa}^{\alpha,\beta}\Pi^2(x) \\ \geq\ & \Big({}^{\rho}J_{\eta,\kappa}^{\alpha,\beta}\Omega^{2/p}(x)\Pi^{2/p}(x)\Big)\Big({}^{\rho}J_{\eta,\kappa}^{\alpha,\beta}\Omega^{p-1}(x)\Pi^{q-1}(x)\Big).\end{aligned}$$

$$\begin{aligned}(g) \qquad & {}^{\rho}J_{\eta,\kappa}^{\alpha,\beta}\Omega^2(x)\,{}^{\rho}J_{\eta,\kappa}^{\alpha,\beta}\Big(\frac{\Pi^q(x)}{p} + \frac{\Pi^q(x)}{q}\Big) \\ \geq\ & \Big({}^{\rho}J_{\eta,\kappa}^{\alpha,\beta}\Omega^{2/p}(x)\Pi(x)\Big)\Big({}^{\rho}J_{\eta,\kappa}^{\alpha,\beta}\Omega^{2/q}(x)\Pi(x)\Big).\end{aligned}$$

Theorem 4.4. *[17] Assume that the assumptions of Theorem 4.3 are satisfied. In addition, let*

$$\mu = \min_{0\leq t\leq x}\frac{\Omega(t)}{\Pi(t)} \quad \textit{and} \quad \mathcal{M} = \max_{0\leq t\leq x}\frac{\Omega(t)}{\Pi(t)}.$$

Then we have:

$$(i) \quad 0 \leq \Big({}^{\rho}J_{\eta,\kappa}^{\alpha,\beta}\Omega^2(x)\,{}^{\rho}J_{\eta,\kappa}^{\alpha,\beta}\Pi^2(x)\Big) \leq \frac{(\mathcal{M}+\mu)^2}{4\mu\mathcal{M}}\Big({}^{\rho}J_{\eta,\kappa}^{\alpha,\beta}\Omega(x)\Pi(x)\Big)^2.$$

$$\begin{aligned}(ii) \quad 0 &\leq \sqrt{{}^{\rho}J_{\eta,\kappa}^{\alpha,\beta}\Omega^2(x)\,{}^{\rho}J_{\eta,\kappa}^{\alpha,\beta}\Pi^2(x)} - \Big({}^{\rho}J_{\eta,\kappa}^{\alpha,\beta}\Omega(x)\Pi(x)\Big) \\ &\leq \frac{(\sqrt{\mathcal{M}}-\sqrt{\mu})^2}{2\sqrt{\mu\mathcal{M}}}\Big({}^{\rho}J_{\eta,\kappa}^{\alpha,\beta}\Omega(x)\Pi(x)\Big).\end{aligned}$$

$$(iii) \quad 0 \leq {}^{\rho}J_{\eta,\kappa}^{\alpha,\beta}\Omega^2(x)\,{}^{\rho}J_{\eta,\kappa}^{\alpha,\beta}\Pi^2(x) - \Big({}^{\rho}J_{\eta,\kappa}^{\alpha,\beta}\Omega(x)\Pi(x)\Big)^2$$

$$\leq \frac{(\mathcal{M}-\mu)^2}{4\mu\mathcal{M}}\Big({}^{\rho}J^{\alpha,\beta}_{\eta,\kappa}\Omega(x)\Pi(x)\Big)^2.$$

Theorem 4.5. *[18] Assume that $\Omega : [0,\infty) \to \mathbb{R}$ is an integrable function satisfying (H_1). Then we have:*

$${}^{\rho}J^{\alpha,\beta}_{\eta,k}Q_2(t)\,{}^{\rho}J^{\delta,\lambda}_{\eta,k}\Omega(t) + {}^{\rho}J^{\alpha,\beta}_{\eta,k}\Omega(t)\,{}^{\rho}J^{\delta,\lambda}_{\eta,k}Q_1(t) \geq {}^{\rho}J^{\alpha,\beta}_{\eta,k}\Omega(t)\,{}^{\rho}J^{\delta,\lambda}_{\eta,k}\Omega(t) + {}^{\rho}J^{\alpha,\beta}_{\eta,k}Q_2(t)\,{}^{\rho}J^{\delta,\lambda}_{\eta,k}Q_1(t),$$

for all $t > 0, \alpha, \rho, \delta > 0, \beta, \eta, k, \lambda \in \mathbb{R}$.

Theorem 4.6. *[18] Suppose that $\Omega, \Pi : [0,\infty) \to \mathbb{R}$ are two integrable functions satisfying (H_1) and (H_2). Then for all $t > 0$ and $\alpha, \rho > 0, \beta, \eta, k \in \mathbb{R}$ we have:*

$$\Big[\Lambda^{\rho,\beta}_{t,k}(\alpha,\eta)\,{}^{\rho}J^{\alpha,\beta}_{\eta,k}\Omega(t)\Pi(t) - \Big({}^{\rho}J^{\alpha,\beta}_{\eta,k}\Omega(t)\,{}^{\rho}J^{\alpha,\beta}_{\eta,k}\Pi(t)\Big)\Big]^2 \leq T(\Omega, Q_1, Q_2)T(\Pi, R_1, R_2),$$

where

$$\begin{aligned} T(y,z,w) \;=\; & ({}^{\rho}J^{\alpha,\beta}_{\eta,k}w(t) - {}^{\rho}J^{\alpha,\beta}_{\eta,k}y(t))({}^{\rho}J^{\alpha,\beta}_{\eta,k}y(t) - {}^{\rho}J^{\alpha,\beta}_{\eta,k}z(t)) \\ & + \Lambda^{\rho,\beta}_{t,k}(\alpha,\eta)^{\rho}J^{\alpha,\beta}_{\eta,k}y(t)z(t) - {}^{\rho}J^{\alpha,\beta}_{\eta,k}y(t)\,{}^{\rho}J^{\alpha,\beta}_{\eta,k}z(t) \\ & + \Lambda^{\rho,\beta}_{t,k}(\alpha,\eta)\,{}^{\rho}J^{\alpha,\beta}_{\eta,k}y(t)w(t) - {}^{\rho}J^{\alpha,\beta}_{\eta,k}y(t)\,{}^{\rho}J^{\alpha,\beta}_{\eta,k}w(t) \\ & - \Lambda^{\rho,\beta}_{t,k}(\alpha,\eta)\,{}^{\rho}J^{\alpha,\beta}_{\eta,k}z(t)w(t) + {}^{\rho}J^{\alpha,\beta}_{\eta,k}z(t)\,{}^{\rho}J^{\alpha,\beta}_{\eta,k}w(t). \end{aligned}$$

Theorem 4.7. *[18] Suppose that $\Omega, \Pi : [0,\infty) \to \mathbb{R}$ are two integrable functions satisfying (H_1) and (H_2). Then for all $t > 0$ and $\alpha, \delta, \rho > 0, \beta, \lambda, \eta, k \in \mathbb{R}$ we have:*

(a) ${}^{\rho}J^{\delta,\lambda}_{\eta,k}\Omega(t)\,{}^{\rho}J^{\alpha,\beta}_{\eta,k}Q_2(t) + {}^{\rho}J^{\delta,\lambda}_{\eta,k}R_1(t)\,{}^{\rho}J^{\alpha,\beta}_{\eta,k}\Pi(t) \geq {}^{\rho}J^{\delta,\lambda}_{\eta,k}R_1(t)\,{}^{\rho}J^{\alpha,\beta}_{\eta,k}Q_2(t) + {}^{\rho}J^{\delta,\lambda}_{\eta,k}\Omega(t)\,{}^{\rho}J^{\alpha,\beta}_{\eta,k}\Pi(t).$
(b) ${}^{\rho}J^{\delta,\lambda}_{\eta,k}Q_1(t)\,{}^{\rho}J^{\alpha,\beta}_{\eta,k}\Omega(t) + {}^{\rho}J^{\alpha,\beta}_{\eta,k}R_2(t)\,{}^{\rho}J^{\delta,\lambda}_{\eta,k}\Pi(t) \geq {}^{\rho}J^{\delta,\lambda}_{\eta,k}Q_1(t)\,{}^{\rho}J^{\alpha,\beta}_{\eta,k}R_2(t) + {}^{\rho}J^{\delta,\lambda}_{\eta,k}\Pi(t)\,{}^{\rho}J^{\alpha,\beta}_{\eta,k}\Omega(t).$
(c) ${}^{\rho}J^{\alpha,\beta}_{\eta,k}Q_2(t)\,{}^{\rho}J^{\delta,\lambda}_{\eta,k}R_2(t) + {}^{\rho}J^{\alpha,\beta}_{\eta,k}\Pi(t)\,{}^{\rho}J^{\delta,\lambda}_{\eta,k}\Omega(t) \geq {}^{\rho}J^{\alpha,\beta}_{\eta,k}Q_2(t)\,{}^{\rho}J^{\delta,\lambda}_{\eta,k}\Omega(t) + {}^{\rho}J^{\delta,\lambda}_{\eta,k}R_2(t)\,{}^{\rho}J^{\alpha,\beta}_{\eta,k}\Pi(t).$
(d) ${}^{\rho}J^{\alpha,\beta}_{\eta,k}Q_1(t)\,{}^{\rho}J^{\delta,\lambda}_{\eta,k}R_1(t) + {}^{\rho}J^{\alpha,\beta}_{\eta,k}\Pi(t)\,{}^{\rho}J^{\delta,\lambda}_{\eta,k}\Omega(t) \geq {}^{\rho}J^{\alpha,\beta}_{\eta,k}Q_1(t)\,{}^{\rho}J^{\delta,\lambda}_{\eta,k}\Omega(t) + {}^{\rho}J^{\delta,\lambda}_{\eta,k}R_1(t)\,{}^{\rho}J^{\alpha,\beta}_{\eta,k}\Pi(t).$

5. Grüss-type fractional integral inequalities via Hadamard's fractional integral operator

Definition 5.1. *[15] The fractional integral of Hadamard type of order $\alpha \in \mathbb{R}^+$ of an integrable function $\Omega(t)$, for all $t > 1$ is defined as*

$${}_{H}J^{\alpha}\Omega(t) = \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\Omega(s)\frac{ds}{s}, \tag{5.1}$$

provided the integral exists. (Here $\log(\cdot) = \log_e(\cdot)$).

We present, by using Hadamard's fractional integral, some Grüss-type fractional integral inequalities.

Theorem 5.1. *[19] Assume that $\Omega : [1,\infty) \to \mathbb{R}$ is an integrable function satisfying (H_1). Then, for $t > 1$, $\alpha, \beta > 0$, we have*

$${}_{H}J^{\beta}Q_1(t)\,{}_{H}J^{\alpha}\Omega(t) + {}_{H}J^{\alpha}Q_2(t)\,{}_{H}J^{\beta}\Omega(t) \geq {}_{H}J^{\alpha}Q_2(t)\,{}_{H}J^{\beta}Q_1(t) + {}_{H}J^{\alpha}\Omega(t)\,{}_{H}J^{\beta}\Omega(t).$$

Theorem 5.2. *[19] Assume that* $\Omega : [1, \infty) \to \mathbb{R}$ *is an integrable function satisfying* (H_1)*. Let* $\theta_1, \theta_2 > 0$ *satisfying* $1/\theta_1 + 1/\theta_2 = 1$*. Then, for* $t > 1$*,* $\alpha, \beta > 0$*, we have*

$$\begin{aligned}
&\frac{1}{\theta_1}\frac{(\log t)^\beta}{\Gamma(\beta+1)}{}_HJ^\alpha\left((Q_2-\Omega)^{\theta_1}\right)(t)+\frac{1}{\theta_2}\frac{(\log t)^\alpha}{\Gamma(\alpha+1)}{}_HJ^\beta\left((\Omega-Q_1)^{\theta_2}\right)(t)\\
&\qquad+{}_HJ^\alpha Q_2(t){}_HJ^\beta Q_1(t)+{}_HJ^\alpha\Omega(t){}_HJ^\beta\Omega(t)\\
&\geq {}_HJ^\alpha Q_2(t){}_HJ^\beta\Omega(t)+{}_HJ^\alpha\Omega(t){}_HJ^\beta Q_1(t).
\end{aligned}$$

Theorem 5.3. *[19] Assume that* $\Omega : [1, \infty) \to \mathbb{R}$ *is an integrable function satisfying* (H_1)*. Let* $\theta_1, \theta_2 > 0$ *satisfying* $\theta_1 + \theta_2 = 1$*. Then, for* $t > 1$*,* $\alpha, \beta > 0$*, we have*

$$\begin{aligned}
&\theta_1\frac{(\log t)^\beta}{\Gamma(\beta+1)}{}_HJ^\alpha Q_2(t)+\theta_2\frac{(\log t)^\alpha}{\Gamma(\alpha+1)}{}_HJ^\beta\Omega(t)\\
&\quad\geq {}_HJ^\alpha(Q_2-\Omega)^{\theta_1}(t){}_HJ^\beta(\Omega-Q_1)^{\theta_2}(t)+\theta_1\frac{(\log t)^\beta}{\Gamma(\beta+1)}{}_HJ^\alpha\Omega(t)\\
&\quad+\theta_2\frac{(\log t)^\alpha}{\Gamma(\alpha+1)}{}_HJ^\beta Q_1(t).
\end{aligned}$$

Theorem 5.4. *[19] Assume that* $\Omega : [1, \infty) \to \mathbb{R}$ *is an integrable function satisfying* (H_1)*. Let* $p \geq q \geq 0$*,* $p \neq 0$*. Then, we have the following two inequalities, for any* $k > 0$*,* $t > 1$*,* $\alpha, \beta > 0$*,*

$$\begin{aligned}
&(i)\ {}_HJ^\alpha(Q_2-\Omega)^{\frac{q}{p}}(t)+\frac{q}{p}k^{\frac{q-p}{p}}{}_HJ^\alpha\Omega(t)\leq\frac{q}{p}k^{\frac{q-p}{p}}{}_HJ^\alpha Q_2(t)+\frac{p-q}{p}k^{\frac{q}{p}}\frac{(\log t)^\alpha}{\Gamma(\alpha+1)}.\\
&(ii)\ {}_HJ^\alpha(\Omega-Q_1)^{\frac{q}{p}}(t)+\frac{q}{p}k^{\frac{q-p}{p}}{}_HJ^\alpha Q_1(t)\leq\frac{q}{p}k^{\frac{q-p}{p}}{}_HJ^\alpha\Omega(t)+\frac{p-q}{p}k^{\frac{q}{p}}\frac{(\log t)^\alpha}{\Gamma(\alpha+1)}.
\end{aligned}$$

Theorem 5.5. *[19] Suppose that* $\Omega, \Pi : [1, \infty) \to \mathbb{R}$ *are two integrable functions satisfying* (H_1) *and* (H_2)*. Then, for* $t > 0$*,* $\alpha, \beta > 0$*, we have:*

(a) $\ {}_HJ^\beta R_1(t){}_HJ^\alpha\Omega(t)+{}_HJ^\alpha Q_2(t){}_HJ^\beta\Pi(t)\geq{}_HJ^\beta R_1(t){}_HJ^\alpha Q_2(t)+{}_HJ^\alpha\Omega(t){}_HJ^\beta\Pi(t).$

(b) $\ {}_HJ^\beta Q_1(t){}_HJ^\alpha\Pi(t)+{}_HJ^\alpha R_2(t){}_HJ^\beta\Omega(t)\geq{}_HJ^\beta Q_1(t){}_HJ^\alpha R_2(t)+{}_HJ^\beta\Omega(t){}_HJ^\alpha\Pi(t).$

(c) $\ {}_HJ^\beta R_2(t){}_HJ^\alpha Q_2(t)+{}_HJ^\alpha\Omega(t){}_HJ^\beta\Pi(t)\geq{}_HJ^\alpha Q_2(t){}_HJ^\beta\Pi(t)+{}_HJ^\beta R_2(t){}_HJ^\alpha\Omega(t).$

(d) $\ {}_HJ^\alpha Q_1(t){}_HJ^\beta R_1(t)+{}_HJ^\alpha\Omega(t){}_HJ^\beta\Pi(t)\geq{}_HJ^\alpha Q_1(t){}_HJ^\beta\Pi(t)+{}_HJ^\beta R_1(t){}_HJ^\alpha\Omega(t).$

Theorem 5.6. *[19] Suppose that* $\Omega, \Pi : [1, \infty) \to \mathbb{R}$ *are two integrable functions satisfying* (H_1) *and* (H_2)*. Let* $\theta_1, \theta_2 > 0$ *such that* $1/\theta_1 + 1/\theta_2 = 1$*. Then, for* $t > 1$*,* $\alpha, \beta > 0$*, the following inequalities hold:*

$$\begin{aligned}
(i)\ &\frac{1}{\theta_1}\frac{(\log t)^\beta}{\Gamma(\beta+1)}{}_HJ^\alpha(Q_2-\Omega)^{\theta_1}(t)+\frac{1}{\theta_2}\frac{(\log t)^\alpha}{\Gamma(\alpha+1)}{}_HJ^\beta(R_2-\Pi)^{\theta_2}(t)\\
&\geq{}_HJ^\alpha(Q_2-\Omega)(t){}_HJ^\beta(R_2-\Pi)(t).\\
(ii)\ &\frac{1}{\theta_1}{}_HJ^\alpha(Q_2-\Omega)^{\theta_1}(t){}_HJ^\beta(R_2-\Pi)^{\theta_1}(t)+\frac{1}{\theta_2}{}_HJ^\alpha(R_2-\Pi)^{\theta_2}(t){}_HJ^\beta(Q_2-\Omega)^{\theta_2}(t)\\
&\geq{}_HJ^\alpha(Q_2-\Omega)(R_2-\Pi)(t){}_HJ^\beta(Q_2-\Omega)(R_2-\Pi)(t).\\
(iii)\ &\frac{1}{\theta_1}\frac{(\log t)^\beta}{\Gamma(\beta+1)}{}_HJ^\alpha(\Omega-Q_1)^{\theta_1}(t)+\frac{1}{\theta_2}\frac{(\log t)^\alpha}{\Gamma(\alpha+1)}{}_HJ^\beta(\Pi-R_1)^{\theta_2}(t)\\
&\geq{}_HJ^\alpha(\Omega-Q_1)(t){}_HJ^\beta(\Pi-R_1)(t).
\end{aligned}$$

$$(iv)\ \frac{1}{\theta_1}{}_HJ^\alpha(\Omega-Q_1)^{\theta_1}(t){}_HJ^\beta(\Pi-R_1)^{\theta_1}(t)+\frac{1}{\theta_2}{}_HJ^\alpha(\Pi-R_1)^{\theta_2}(t){}_HJ^\beta(\Omega-Q_1)^{\theta_2}(t)$$
$$\geq {}_HJ^\alpha(\Omega-Q_1)(\Pi-R_1)(t){}_HJ^\beta(\Omega-Q_1)(\Pi-R_1)(t).$$

Theorem 5.7. *[19] Suppose that $\Omega,\Pi:[1,\infty)\to\mathbb{R}$ are two integrable functions satisfying (H_1) and (H_2). Let $\theta_1,\theta_2>0$ such that $\theta_1+\theta_2=1$. Then, for $t>1$, $\alpha,\beta>0$, we have:*

$$(a)\quad \theta_1\frac{(\log t)^\beta}{\Gamma(\beta+1)}{}_HJ^\alpha Q_2(t)+\theta_2\frac{(\log t)^\alpha}{\Gamma(\alpha+1)}{}_HJ^\beta R_2(t)$$
$$\geq {}_HJ^\alpha(Q_2-\Omega)^{\theta_1}(t){}_HJ^\beta(R_2-\Pi)^{\theta_2}(t)+\theta_1\frac{(\log t)^\beta}{\Gamma(\beta+1)}{}_HJ^\alpha\Omega(t)+\theta_2\frac{(\log t)^\alpha}{\Gamma(\alpha+1)}{}_HJ^\beta\Pi(t).$$

$$\begin{aligned}(b)\ &\theta_{1H}J^\alpha Q_2(t){}_HJ^\beta R_2(t)+\theta_{1H}J^\alpha\Omega(t){}_HJ^\beta\Pi(t)\\ &+\theta_{2H}J^\alpha R_2(t){}_HJ^\beta Q_2(t)+\theta_{2H}J^\alpha\Pi(t){}_HJ^\beta\Omega(t)\\ &\geq {}_HJ^\alpha(Q_2-\Omega)^{\theta_1}(R_2-\Pi)^{\theta_2}(t){}_HJ^\beta(R_2-\Pi)^{\theta_1}(Q_2-\Omega)^{\theta_2}(t)\\ &+\theta_{1H}J^\alpha Q_2(t){}_HJ^\beta\Pi(t)+\theta_{1H}J^\alpha\Omega(t){}_HJ^\beta R_2(t)\\ &+\theta_{2H}J^\alpha R_2(t){}_HJ^\beta\Omega(t)+\theta_{2H}J^\alpha\Pi(t){}_HJ^\beta Q_2(t).\end{aligned}$$

$$(c)\quad \theta_1\frac{(\log t)^\beta}{\Gamma(\beta+1)}{}_HJ^\alpha\Omega(t)+\theta_2\frac{(\log t)^\alpha}{\Gamma(\alpha+1)}{}_HJ^\beta\Pi(t)$$
$$\geq {}_HJ^\alpha(\Omega-Q_1)^{\theta_1}(t){}_HJ^\beta(\Pi-R_1)^{\theta_2}(t)+\theta_1\frac{(\log t)^\beta}{\Gamma(\beta+1)}{}_HJ^\alpha Q_1(t)+\theta_2\frac{(\log t)^\alpha}{\Gamma(\alpha+1)}{}_HJ^\beta R_1(t).$$

$$\begin{aligned}(d)\ &\theta_{1H}J^\alpha\Omega(t){}_HJ^\beta\Pi(t)+\theta_{1H}J^\alpha Q_1(t){}_HJ^\beta R_1(t)\\ &+\theta_{2H}J^\alpha\Pi(t){}_HJ^\beta\Omega(t)+\theta_{2H}J^\alpha R_1(t){}_HJ^\beta Q_1(t)\\ &\geq {}_HJ^\alpha(\Omega-Q_1)^{\theta_1}(\Pi-R_1)^{\theta_2}(t){}_HJ^\beta(\Pi-R_1)^{\theta_1}(\Omega-Q_1)^{\theta_2}(t)\\ &+\theta_{1H}J^\alpha\Omega(t){}_HJ^\beta R_1(t)+\theta_{1H}J^\alpha Q_1(t){}_HJ^\beta\Pi(t)\\ &+\theta_{2H}J^\alpha\Pi(t){}_HJ^\beta Q_1(t)+\theta_{2H}J^\alpha R_1(t){}_HJ^\beta\Omega(t).\end{aligned}$$

Theorem 5.8. *[19] Suppose that $\Omega,\Pi:[1,\infty)\to\mathbb{R}$ are two integrable functions satisfying (H_1) and (H_2). Then for all $t>1$, $\alpha>0$, we have*

$$\left|\frac{(\log t)^\alpha}{\Gamma(\alpha+1)}{}_HJ^\alpha\Omega(t)\Pi(t)-{}_HJ^\alpha\Omega(t){}_HJ^\alpha\Pi(t)\right|\leq|T(\Omega,Q_1,Q_2)|^{\frac{1}{2}}\,|T(\Pi,R_1,R_2)|^{\frac{1}{2}},$$

where $T(y,z,w)$ is defined by

$$\begin{aligned}T(y,z,w)\ =\ &({}_HJ^\alpha w(t)-{}_HJ^\alpha y(t))\,({}_HJ^\alpha y(t)-{}_HJ^\alpha z(t))\\ &+\frac{(\log t)^\alpha}{\Gamma(\alpha+1)}{}_HJ^\alpha z(t)y(t)-{}_HJ^\alpha z(t){}_HJ^\alpha y(t)\\ &+\frac{(\log t)^\alpha}{\Gamma(\alpha+1)}{}_HJ^\alpha w(t)y(t)-{}_HJ^\alpha w(t){}_HJ^\alpha y(t)\\ &+{}_HJ^\alpha z(t){}_HJ^\alpha w(t)-\frac{(\log t)^\alpha}{\Gamma(\alpha+1)}{}_HJ^\alpha z(t)w(t).\end{aligned}$$

6. Grüss-type fractional integral inequalities via k-fractional integral operators

In this section we present Grüss-type fractional integral inequalities concerning k-fractional integral operators.

6.1. k-fractional integral

k-fractional integral inequalities of Grüss-type are included in this section.

Definition 6.1. *[20] The k-fractional integral of the Riemann-Liouville type is defined as follows:*

$$ {}_kJ^{\alpha}_{x_1}\Omega(t) = \frac{1}{k\Gamma_k(\alpha)}\int_{x_1}^{t}(x-s)^{\frac{\alpha}{k}-1}\Omega(s)ds, \quad \alpha > 0,\ t > a. $$

Theorem 6.1. *[21] Assume that $\Omega : [0,\infty)$ is an integrable function satisfying (H_1). Then, for $t > 0$, $\alpha,\beta > 0$, $k > 0$, we have*

$$ {}_kJ^{\beta}Q_1(t){}_kJ^{\alpha}\Omega(t) + {}_kJ^{\alpha}Q_2(t){}_kJ^{\beta}\Omega(t) \geq {}_kJ^{\alpha}Q_2(t){}_kJ^{\beta}Q_1(t) + {}_kJ^{\alpha}\Omega(t){}_kJ^{\beta}\Omega(t). $$

Theorem 6.2. *[21] Assume that $\Omega : [0,\infty)$ is an integrable function satisfying (H_1). Let $\theta_1,\theta_2 > 0$ such that $1/\theta_1 + 1/\theta_2 = 1$. Then, we have for $t > 0$, $\alpha,\beta > 0$ and $k > 0$,*

$$ \begin{aligned} &\frac{1}{\theta_1}\frac{t^{\frac{\beta}{k}}}{\Gamma_k(\beta+k)}{}_kJ^{\alpha}\left((Q_2-\Omega)^{\theta_1}\right)(t) + \frac{1}{\theta_2}\frac{t^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}{}_kJ^{\beta}\left((\Omega-Q_1)^{\theta_2}\right)(t) \\ &\quad + {}_kJ^{\alpha}Q_2(t){}_kJ^{\beta}Q_1(t) + {}_kJ^{\alpha}\Omega(t){}_kJ^{\beta}\Omega(t) \\ &\geq {}_kJ^{\alpha}Q_2(t){}_kJ^{\beta}\Omega(t) + {}_kJ^{\alpha}\Omega(t){}_kJ^{\beta}Q_1(t). \end{aligned} $$

Theorem 6.3. *[21] Assume that $\Omega : [0,\infty)$ is an integrable function satisfying (H_1). Let $\theta_1,\theta_2 > 0$ such that $\theta_1 + \theta_2 = 1$. Then, for $t > 0$, $\alpha,\beta > 0$ and $k > 0$, we have*

$$ \begin{aligned} &\theta_1\frac{t^{\frac{\beta}{k}}}{\Gamma_k(\beta+k)}{}_kJ^{\alpha}Q_2(t) + \theta_2\frac{t^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}{}_kJ^{\beta}\Omega(t) \\ &\geq {}_kJ^{\alpha}(Q_2-\Omega)^{\theta_1}(t){}_kJ^{\beta}(\Omega-Q_1)^{\theta_2}(t) + \theta_1\frac{t^{\frac{\beta}{k}}}{\Gamma_k(\beta+k)}{}_kJ^{\alpha}\Omega(t) + \theta_2\frac{t^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}{}_kJ^{\beta}Q_1(t). \end{aligned} $$

Theorem 6.4. *[21] Assume that $\Omega, \Pi : [0,\infty) \to \mathbb{R}$ are two integrable functions satisfying (H_1) and (H_2). Then, for $t > 0$, $\alpha, \beta > 0$, $k > 0$ we have:*

(i) ${}_kJ^{\beta}R_1(t)\,{}_kJ^{\alpha}\Omega(t) + {}_kJ^{\alpha}Q_2(t)\,{}_kJ^{\beta}\Pi(t) \geq {}_kJ^{\beta}R_1(t)\,{}_kJ^{\alpha}Q_2(t) + {}_kJ^{\alpha}\Omega(t)\,{}_kJ^{\beta}\Pi(t)$.
(ii) ${}_kJ^{\beta}Q_1(t)\,{}_kJ^{\alpha}\Pi(t) + {}_kJ^{\alpha}R_2(t)\,{}_kJ^{\beta}\Omega(t) \geq {}_kJ^{\beta}Q_1(t)\,{}_kJ^{\alpha}R_2(t) + {}_kJ^{\beta}\Omega(t)\,{}_kJ^{\alpha}\Pi(t)$.
(iii) ${}_kJ^{\alpha}Q_2(t)\,{}_kJ^{\beta}R_2(t) + {}_kJ^{\alpha}\Omega(t)\,{}_kJ^{\beta}\Pi(t) \geq {}_kJ^{\alpha}Q_2(t)\,{}_kJ^{\beta}\Pi(t) + {}_kJ^{\beta}R_2(t)\,{}_kJ^{\alpha}\Omega(t)$.
(iv) ${}_kJ^{\alpha}Q_1(t)J^{\beta}R_1(t) + {}_kJ^{\alpha}\Omega(t)\,{}_kJ^{\beta}\Pi(t) \geq {}_kJ^{\alpha}Q_1(t)\,{}_kJ^{\beta}\Pi(t) + {}_kJ^{\beta}R_1(t)\,{}_kJ^{\alpha}\Omega(t)$.

Theorem 6.5. *[21] Assume that $\Omega, \Pi : [0,\infty) \to \mathbb{R}$ are two integrable functions satisfying (H_1) and (H_2). Then for all $t > 0$, $\alpha > 0$,$k > 0$, we have*

$$ \left|\frac{t^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\,{}_kJ^{\alpha}\Omega(t)\Pi(t) - {}_kJ^{\alpha}\Omega(t)\,{}_kJ^{\alpha}\Pi(t)\right| \leq \sqrt{T(\Omega,Q_1,Q_2)T(\Pi,R_1,R_2)}, $$

where

$$ T(y,z,w) = \left({}_kJ^{\alpha}w(t) - {}_kJ^{\alpha}y(t)\right)\left({}_kJ^{\alpha}y(t) - {}_kJ^{\alpha}z(t)\right) $$

$$
\begin{aligned}
&+\frac{t^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\ {}_kJ^{\alpha}z(t)y(t) -_k J^{\alpha}z(t)\ {}_kJ^{\alpha}y(t)\\
&+\frac{t^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}J^{\alpha}w(t)y(t) -_k J^{\alpha}w(t)\ {}_kJ^{\alpha}y(t)\\
&+_kJ^{\alpha}z(t)\ {}_kJ^{\alpha}w(t) - \frac{t^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\ {}_kJ^{\alpha}z(t)w(t).
\end{aligned}
$$

Theorem 6.6. *[22] Assume that (H) holds on $[x_1, x_2]$ and p be a positive function on $[x_1, x_2]$. Then for all $t > 0, k > 0, \alpha > 0$, we have*

$$
\left|\left({}_kJ^{\alpha}_{x_1}p(t)\right)\left({}_kJ^{\alpha}_{x_1}p(t)\Omega(t)\Pi(t)\right) - \left({}_kJ^{\alpha}_{x_1}p(t)\Omega(t)\right)\left({}_kJ^{\alpha}_{x_1}p(t)\Pi(t)\right)\right| \leq \frac{\left({}_kJ^{\alpha}_{x_1}p(t)\right)^2}{4}(\mathfrak{M}-\mathfrak{m})(\mathfrak{P}-\mathfrak{p}).
$$

Theorem 6.7. *[22] Let the assumptions of Theorem 6.6 be satisfied. Then, for all $t > 0, k > 0, \alpha, \beta > 0$, t he following inequality holds:*

$$
\begin{aligned}
&\Big[\left({}_kJ^{\alpha}_{x_1}p(t)\right)\left({}_kJ^{\beta}_{x_1}p(t)\Omega(t)\Pi(t)\right) + \left({}_kJ^{\beta}_{x_1}p(t)\right)\left({}_kJ^{\alpha}_{x_1}p(t)\Omega(t)\Pi(t)\right) - \left({}_kJ^{\alpha}_{x_1}p(t)\Omega(t)\right)\left({}_kJ^{\alpha}_{x_1}p(t)\Pi(t)\right)\\
&-\left({}_kJ^{\beta}_{x_1}p(t)\Omega(t)\right)\left({}_kJ^{\alpha}_{x_1}p(t)\Pi(t)\right)\Big]^2\\
\leq\ &\Big\{\Big[\mathfrak{M}\left({}_kJ^{\alpha}_{x_1}p(t)\right) - \left({}_kJ^{\alpha}_{x_1}p(t)\Omega(t)\right)\Big]\Big[\left({}_kJ^{\beta}_{x_1}p(t)\Omega(t)\right) - \mathfrak{m}\left({}_kJ^{\beta}_{x_1}p(t)\right)\Big]\\
&+\Big[\left({}_kJ^{\alpha}_{x_1}p(t)\Omega(t)\right) - \mathfrak{m}\left({}_kJ^{\alpha}_{x_1}p(t)\right)\Big]\Big[M\left({}_kJ^{\beta}_{x_1}p(t)\right) - \left({}_kJ^{\beta}_{x_1}p\Omega(t)\right)\Big]\Big\}\\
&\times\Big\{\Big[\mathfrak{P}\left({}_kJ^{\alpha}_{x_1}p(t)\right) - \left({}_kJ^{\alpha}_{x_1}p(t)\Pi(t)\right)\Big]\Big[\left({}_kJ^{\beta}_{x_1}p(t)\Pi(t)\right) - \mathfrak{p}\left({}_kJ^{\beta}_{x_1}p(t)\right)\Big]\\
&+\Big[\left({}_kJ^{\alpha}_{x_1}p(t)\Pi(t)\right) - \mathfrak{p}\left({}_kJ^{\alpha}_{x_1}p(t)\right)\Big]\Big[\mathfrak{P}\left({}_kJ^{\beta}_{x_1}p(t)\right) - \left({}_kJ^{\beta}_{x_1}p(t)\Pi(t)\right)\Big]\Big\}.
\end{aligned}
$$

6.2. k-fractional integrals of a function with respect to another function

Definition 6.2. *[23] Let ψ be a positive and increasing function on $[x_1, x_2]$. Then the left-sided and right-sided generalized Riemann–Liouville fractional integrals of a function Ω with respect to another function ψ of order $\alpha > 0$ are defined by*

$$
J^{\alpha,\psi}_{x_1+,k}\Omega(t) = \frac{1}{k\Gamma_k(\alpha)}\int_{x_1}^{t}(\psi(t)-\psi(s))^{\frac{\alpha}{k}-1}\psi'(s)\Omega(s)ds,\ t > x_1,
$$

$$
J^{\alpha,\psi}_{x_2-,k}\Omega(t) = \frac{1}{k\Gamma_k(\alpha)}\int_{t}^{x_2}(\psi(s)-\psi(t))^{\frac{\alpha}{k}-1}\psi'(s)\Omega(s)ds,\ t < x_2.
$$

Theorem 6.8. *[23] Assume that $\Omega : [0,\infty) \to \mathbb{R}$ is an integrable function satisfying (H_1). Then, for $t > 0$, $\alpha, \beta > 0$, $k > 0$, we have:*

$$
J^{\beta,\psi}_{0+,k}Q_1(t)J^{\alpha,\psi}_{0+,k}\Omega(t) + J^{\alpha,\psi}_{0+,k}Q_2(t)J^{\beta,\psi}_{0+,k}\Omega(t) \geq J^{\alpha,\psi}_{0+,k}Q_2(t)J^{\beta,\psi}_{0+,k}Q_1(t) + J^{\alpha,\psi}_{0+,k}\Omega(t)J^{\beta,\psi}_{0+,k}\Omega(t).
$$

Theorem 6.9. *[23] Suppose that $\Omega, \Pi : [0,\infty) \to \mathbb{R}$ are two integrable functions satisfying (H_1) and (H_2). Assume that ψ is a positive and increasing function with $\psi(0) = 0$ and ψ' continuous on $[0,\infty)$. Then, for $t > 0$, α, $\beta > 0$, $k > 0$ we have:*

(i) $J^{\beta,\psi}_{0+,k}R_1(t)\ J^{\alpha,\psi}_{0+,k}\Omega(t) + J^{\alpha,\psi}_{0+,k}Q_2(t)\ J^{\beta,\psi}_{0+,k}\Pi(t) \geq J^{\beta,\psi}_{0+,k}R_2(t)\ J^{\alpha,\psi}_{0+,k}Q_2(t) + J^{\alpha,\psi}_{0+,k}\Omega(t)\ J^{\beta,\psi}_{0+,k}\Pi(t).$

(ii) $J^{\beta,\psi}_{0+,k}Q_1(t)\ J^{\alpha,\psi}_{0+,k}\Pi(t) + \ J^{\alpha,\psi}_{0+,k}R_2(t)\ J^{\beta,\psi}_{0+,k}\Omega(t) \geq \ J^{\beta,\psi}_{0+,k}Q_1(t)\ J^{\alpha,\psi}_{0+,k}R_2(t) + \ J^{\beta,\psi}_{0+,k}\Omega(t)\ J^{\alpha,\psi}_{0+,k}\Pi(t).$

(iii) $J^{\alpha,\psi}_{0+,k}Q_2(t)\ {}_kJ^{\beta,\psi}_{0+,k}R_2(t) + \ J^{\alpha,\psi}_{0+,k}\Omega(t)\ J^{\beta,\psi}_{0+,k}\Pi(t) \geq \ J^{\alpha,\psi}_{0+,k}Q_2(t)\ J^{\beta,\psi}_{0+,k}\Pi(t) + \ J^{\beta,\psi}_{0+,k}R_2(t)\ J^{\alpha,\psi}_{0+,k}\Omega(t).$

(iv) $J^{\alpha,\psi}_{0+,k}Q_1(t)J^{\beta,\psi}_{0+,k}R_1(t) + \ J^{\alpha,\psi}_{0+,k}\Omega(t)\ J^{\beta,\psi}_{0+,k}\Pi(t) \geq \ J^{\alpha,\psi}_{0+,k}Q_1(t)\ J^{\beta,\psi}_{0+,k}\Pi(t) + \ J^{\beta,\psi}_{0+,k}R_1(t)\ J^{\alpha,\psi}_{0+,k}\Omega(t).$

Theorem 6.10. *[23] Suppose that $\Omega, \Pi : [0,\infty) \to \mathbb{R}$ are two integrable functions satisfying (H_1) and (H_2). Assume that ψ is a positive and increasing function on $[0,\infty)$ such that $\psi(0) = 0$ and ψ' is continuous on $[0,\infty)$. Then for all $t > 0$, $\alpha > 0$,$k > 0$, we have*

$$\left|\frac{\psi(t)^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\ J^{\alpha,\psi}_{0+,k}\Omega(t)\Pi(t) - \ J^{\alpha,\psi}_{0+,k}\Omega(t)\ J^{\alpha,\psi}_{0+,k}\Pi(t)\right| \leq \sqrt{T(\Omega, Q_1, Q_2)T(\Pi, R_1, R_2)},$$

where

$$\begin{aligned} T(y,z,w) \ &= \ \Big(\ J^{\alpha,\psi}_{0+,k}w(t) - \ J^{\alpha}y(t)\Big)\Big(\ J^{\alpha,\psi}_{0+,k}y(t) - \ J^{\alpha,\psi}_{0+,k}z(t)\Big) \\ &+\frac{\psi(t)^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\ J^{\alpha,\psi}_{0+,k}z(t)y(t) - J^{\alpha,\psi}_{0+,k}z(t)\ J^{\alpha,\psi}_{0+,k}y(t) \\ &+\frac{\psi(t)^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}J^{\alpha,\psi}_{0+,k}w(t)y(t) - J^{\alpha,\psi}_{0+,k}w(t)\ J^{\alpha,\psi}_{0+,k}y(t) \\ &+J^{\alpha,\psi}_{0+,k}z(t)\ J^{\alpha,\psi}_{0+,k}w(t) - \frac{t^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\ J^{\alpha,\psi}_{0+,k}z(t)w(t). \end{aligned}$$

7. Grüss-type integral inequalities via Raina's fractional integral operators

Definition 7.1. *[24] The function Ω is said to be $L_{p,r}[x_1, x_2]$ if*

$$\Big(\int_{x_1}^{x_2} |\Omega(t)|^p t^r dt\Big)^{1/p} < \infty, \quad 1 < p < \infty, \ \ r > 0.$$

Definition 7.2. *[24] The Γ_k (generalized gamma function) is defined by*

$$\Gamma_k(z) = \lim_{n\to\infty} \frac{n!k^n(nk)^{\frac{z}{k}-1}}{(z)_{nk}}, \ \ k > 0.$$

Definition 7.3. *[24] The function $\mathcal{F}^{\sigma,k}_{\rho,\lambda}$ is defined by*

$$\begin{aligned} \mathcal{F}^{\sigma,k}_{\rho,\lambda}(z) \ &= \ \mathcal{F}^{(\sigma(0),\sigma(1),\ldots,),k}_{\rho,\lambda} \\ &= \ \sum_{m=0}^{\infty} \frac{\sigma(m)}{k\Gamma_k(\rho k m + \lambda)} z^m, \quad \rho, \lambda > 0, \ z \in \mathbb{C}, \ |z| < R, \end{aligned}$$

where $R \in \mathbb{R}^+$ and $\sigma = (\sigma(1), \ldots, \sigma(m), \ldots)$ is a bounded sequence of positive real numbers.

Definition 7.4. *[24] Let $k > 0$, $\lambda > 0$, $\rho > 0$ and $\omega \in \mathbb{R}$. Assume that $\psi : [x_1, x_2] \to \mathbb{R}$ is an increasing function for which ψ' is continuous on (x_1, x_2). Then the left and right generalized k-fractional integrals of the function Ω with respect to ψ on $[x_1, x_2]$ are defined by*

$$J^{\sigma,k,\psi}_{\rho,\lambda,a+;\omega}\Omega(z) = \int_{x_1}^{z} \frac{\psi'(t)}{(\psi(z)-\psi(t))^{1-\frac{\lambda}{k}}}\mathcal{F}^{\sigma,k}_{\rho,\lambda}[\omega(\psi(z)-\psi(t))^{\rho}]\Omega(t)dt, \ \ z > x_1$$

and

$$J^{\sigma,k,\psi}_{\rho,\lambda,x_2-;\omega}\Omega(z) = \int_z^{x_2} \frac{\psi'(t)}{(\psi(t)-\psi(z))^{1-\frac{\lambda}{k}}} \mathcal{F}^{\sigma,k}_{\rho,\lambda}[\omega(\psi(t)-\psi(z))^{\rho}]\Omega(t)dt, \quad z < x_2,$$

respectively.

Theorem 7.1. *[24] Let $\rho, \lambda, \delta > 0$, $\omega \in \mathbb{R}$, $\Omega \in L_{1,r}[x_1, x_2]$, and (H_1) holds. Then we have:*

$$\begin{aligned} & J^{\sigma,k,\psi}_{\rho,\lambda,0+;\omega}Q_2(x)J^{\sigma,k,\psi}_{\rho,\delta,0+;\omega}\Omega(x) + J^{\sigma,k,\psi}_{\rho,\delta,0+;\omega}Q_1(x)J^{\sigma,k,\psi}_{\rho,\lambda,0+;\omega}\Omega(x) \\ \geq \ & J^{\sigma,k,\psi}_{\rho,\delta,0+;\omega}Q_1(x)J^{\sigma,k,\psi}_{\rho,\lambda,0+;\omega}Q_2(x) + J^{\sigma,k,\psi}_{\rho,\delta,0+;\omega}\Omega(x)J^{\sigma,k,\psi}_{\rho,\lambda,0+;\omega}\Omega(x). \end{aligned}$$

Theorem 7.2. *[24] Under the assumptions of Theorem 7.1, we have:*

$$\begin{aligned} & J^{\sigma,k,\psi}_{\rho,\lambda,0+;\omega}Q_2(x)J^{\sigma,k,\psi}_{\rho,\delta,0+;\omega}\Omega(x) + J^{\sigma,k,\psi}_{\rho,\delta,0+;\omega}Q_1(x)J^{\sigma,k,\psi}_{\rho,\lambda,0+;\omega}\Omega(x) \\ \geq \ & J^{\sigma,k,\psi}_{\rho,\delta,0+;\omega}Q_1(x)J^{\sigma,k,\psi}_{\rho,\lambda,0+;\omega}Q_2(x) + J^{\sigma,k,\psi}_{\rho,\delta,0+;\omega}\Omega(x)J^{\sigma,k,\psi}_{\rho,\lambda,0+;\omega}\Omega(x). \end{aligned}$$

Theorem 7.3. *[24] Let $\rho, \lambda, \delta > 0$, $\omega \in \mathbb{R}$, $\Omega, \Pi \in L_{1,r}[x_1, x_2]$ satifying (H_1) and (H_2) for all $x \in [0, \infty)$. Then we have*

$$\begin{aligned} & \Big| J^{\sigma,k,\psi}_{\rho,\lambda,0+;\omega}\Omega(x)\Pi(x)\mathcal{A}_\delta(x) + J^{\sigma,k,\psi}_{\rho,\lambda,0+;\omega}\Omega(x)\Pi(x)\mathcal{A}_\lambda(x) \\ & -J^{\sigma,k,\psi}_{\rho,\lambda,0+;\omega}\Omega(x)J^{\sigma,k,\psi}_{\rho,\delta,0+;\omega}\Pi(x) - J^{\sigma,k,\psi}_{\rho,\delta,0+;\omega}\Omega(x)J^{\sigma,k,\psi}_{\rho,\lambda,0+;\omega}\Pi(x)\Big| \\ \leq \ & \Big(\frac{\mathcal{A}_\lambda(x)\mathcal{A}_\delta(x)}{2}\Big)^2(Q_2 - Q_1)(R_1 - R_2), \end{aligned}$$

where $\mathcal{A}_\lambda$ and $\mathcal{A}_\delta$ are defined as

$$\mathcal{A}_\lambda(z) = (\psi(z))^{\frac{\lambda}{k}}\mathcal{F}^{\sigma,k}_{\rho,\lambda+1}(\omega(\psi(z))^{\rho}) \ \text{ and } \ \mathcal{A}_\delta = (\psi(z))^{\frac{\delta}{k}}\mathcal{F}^{\sigma,k}_{\rho,\delta+1}(\omega(\psi(z))^{\rho}),$$

respectively.

Theorem 7.4. *[24] Under the assumptions of Theorem 7.3, we have*

$$\begin{aligned} (i) \quad & J^{\sigma,k,\psi}_{\rho,\lambda,0+;\omega}Q_2(x)J^{\sigma,k,\psi}_{\rho,\delta,0+;\omega}\Pi(x) + J^{\sigma,k,\psi}_{\rho,\delta,0+;\omega}R_1(x)J^{\sigma,k,\psi}_{\rho,\lambda,0+;\omega}\Omega(x) \\ \geq \ & J^{\sigma,k,\psi}_{\rho,\delta,0+;\omega}R_1(x)J^{\sigma,k,\psi}_{\rho,\lambda,0+;\omega}Q_2(x) + J^{\sigma,k,\psi}_{\rho,\delta,0+;\omega}\Pi(x)J^{\sigma,k,\psi}_{\rho,\lambda,0+;\omega}\Omega(x), \end{aligned}$$

$$\begin{aligned} (ii) \quad & J^{\sigma,k,\psi}_{\rho,\lambda,0+;\omega}R_1(x)J^{\sigma,k,\psi}_{\rho,\lambda,0+;\omega}\Omega(x) + J^{\sigma,k,\psi}_{\rho,\lambda,0+;\omega}\Pi(x)J^{\sigma,k,\psi}_{\rho,\delta,0+;\omega}Q_1(x) \\ \geq \ & J^{\sigma,k,\psi}_{\rho,\lambda,0+;\omega}\Pi(x)J^{\sigma,k,\psi}_{\rho,\delta,0+;\omega}\Omega(x) + J^{\sigma,k,\psi}_{\rho,\lambda,0+;\omega}R_1(x)J^{\sigma,k,\psi}_{\rho,\delta,0+;\omega}Q_1(x), \end{aligned}$$

$$\begin{aligned} (iii) \quad & J^{\sigma,k,\psi}_{\rho,\lambda,0+;\omega}Q_2(x)J^{\sigma,k,\psi}_{\rho,\delta,0+;\omega}\Pi(x) + J^{\sigma,k,\psi}_{\rho,\lambda,0+;\omega}\Omega(x)J^{\sigma,k,\psi}_{\rho,\lambda,0+;\omega}R_1(x) \\ \geq \ & J^{\sigma,k,\psi}_{\rho,\lambda,0+;\omega}\Omega(x)J^{\sigma,k,\psi}_{\rho,\delta,0+;\omega}\Pi(x) + J^{\sigma,k,\psi}_{\rho,\delta,0+;\omega}R_1(x)J^{\sigma,k,\psi}_{\rho,\lambda,0+;\omega}Q_2(x), \end{aligned}$$

$$\begin{aligned} (iv) \quad & J^{\sigma,k,\psi}_{\rho,\lambda,0+;\omega}Q_1(x)J^{\sigma,k,\psi}_{\rho,\delta,0+;\omega}\Pi(x) + J^{\sigma,k,\psi}_{\rho,\lambda,0+;\omega}\Omega(x)J^{\sigma,k,\psi}_{\rho,\lambda,0+;\omega}R_1(x) \\ \geq \ & J^{\sigma,k,\psi}_{\rho,\lambda,0+;\omega}\Omega(x)J^{\sigma,k,\psi}_{\rho,\lambda,0+;\omega}\Pi(x) + J^{\sigma,k,\psi}_{\rho,\lambda,0+;\omega}Q_1(x)J^{\sigma,k,\psi}_{\rho,\lambda,0+;\omega}R_1(x). \end{aligned}$$

Now we present certain other associated fractional integral inequalities.

Theorem 7.5. *[24] Let $\alpha,\beta > 1$ and $a^{-1} + \beta^{-1} = 1$, and $\Omega, \Pi \in L_{1,r}[x_1, x_2]$. Then we have:*

$$
\begin{aligned}
(i)\qquad & a^{-1}\mathcal{A}_{\delta}(x)J^{\sigma,k,\psi}_{\rho,\lambda,0+;\omega}\Omega^{\alpha}(x) + \beta^{-1}\mathcal{A}_{\lambda}(x)J^{\sigma,k,\psi}_{\rho,\lambda,0+;\omega}\Pi^{\beta}(x)\\
\geq\ & J^{\sigma,k,\psi}_{\rho,\lambda,0+;\omega}\Omega(x)J^{\sigma,k,\psi}_{\rho,\lambda,0+;\omega}\Pi(x).\\
(ii)\qquad & a^{-1}J^{\sigma,k,\psi}_{\rho,\lambda,0+;\omega}\Omega^{\alpha}(x)J^{\sigma,k,\psi}_{\rho,\delta,0+;\omega}\Pi^{\alpha}(x) + \beta^{-1}J^{\sigma,k,\psi}_{\rho,\lambda,0+;\omega}\Pi^{\beta}J^{\sigma,k,\psi}_{\rho,\delta,0+;\omega}\Omega^{\beta}(x)\\
\geq\ & J^{\sigma,k,\psi}_{\rho,\lambda,0+;\omega}\Omega(x)\Pi(x)J^{\sigma,k,\psi}_{\rho,\delta,0+;\omega}\Omega(x)\Pi(x).\\
(iii)\qquad & a^{-1}J^{\sigma,k,\psi}_{\rho,\lambda,0+;\omega}\Omega^{\alpha}(x)J^{\sigma,k,\psi}_{\rho,\delta,0+;\omega}\Pi^{\beta}(x) + \beta^{-1}J^{\sigma,k,\psi}_{\rho,\delta,0+;\omega}\Omega^{\beta}(x)J^{\sigma,k,\psi}_{\rho,\lambda,0+;\omega}\Pi^{\alpha}(x)\\
\geq\ & J^{\sigma,k,\psi}_{\rho,\lambda,0+;\omega}\Omega(x)\Pi^{\alpha-1}(x)J^{\sigma,k,\psi}_{\rho,\delta,0+;\omega}\Omega(x)\Pi^{\beta-1}(x).\\
(iv)\qquad & a^{-1}J^{\sigma,k,\psi}_{\rho,\delta,0+;\omega}\Omega^{\alpha}(x)J^{\sigma,k,\psi}_{\rho,\lambda,0+;\omega}\Pi^{\beta}(x) + \beta^{-1}J^{\sigma,k,\psi}_{\rho,\delta,0+;\omega}\Pi^{\beta}(x)J^{\sigma,k,\psi}_{\rho,\lambda,0+;\omega}\Omega^{\alpha}(x)\\
\geq\ & J^{\sigma,k,\psi}_{\rho,\lambda,0+;\omega}\Omega^{\alpha-1}(x)\Pi^{\beta-1}(x)J^{\sigma,k,\psi}_{\rho,\delta,0+;\omega}\Omega(x)\Pi(x).\\
(v)\qquad & a^{-1}J^{\sigma,k,\psi}_{\rho,\lambda,0+;\omega}\Omega^{\alpha}(x)J^{\sigma,k,\psi}_{\rho,\delta,0+;\omega}\Pi^{2}(x) + \beta^{-1}J^{\sigma,k,\psi}_{\rho,\lambda,0+;\omega}\Pi^{\beta}(x)J^{\sigma,k,\psi}_{\rho,\lambda,0+;\omega}\Omega(x)\\
\geq\ & J^{\sigma,k,\psi}_{\rho,\lambda,0+;\omega}\Omega(x)\Pi(x)J^{\sigma,k,\psi}_{\rho,\delta,0+;\omega}\Omega^{2/\beta}(x)\Pi^{2/\alpha}(x).\\
(vi)\qquad & a^{-1}J^{\sigma,k,\psi}_{\rho,\lambda,0+;\omega}\Omega^{2}(x)J^{\sigma,k,\psi}_{\rho,\delta,0+;\omega}\Pi^{\beta}(x) + \beta^{-1}J^{\sigma,k,\psi}_{\rho,\delta,0+;\omega}\Omega^{\alpha}(x)J^{\sigma,k,\psi}_{\rho,\lambda,0+;\omega}\Pi^{2}(x)\\
\geq\ & J^{\sigma,k,\psi}_{\rho,\lambda,0+;\omega}\Omega^{2/\alpha}(x)\Pi^{2/\beta}(x)J^{\sigma,k,\psi}_{\rho,\delta,0+;\omega}\Omega^{\alpha-1}(x)\Pi^{\beta-1}(x).\\
(vii)\qquad & a^{-1}\mathcal{A}_{\delta}(x)J^{\sigma,k,\psi}_{\rho,\lambda,0+;\omega}\Omega^{2}(x)\Pi^{\beta}(x) + \beta^{-1}\mathcal{A}_{\lambda}(x)J^{\sigma,k,\psi}_{\rho,\delta,0+;\omega}\Pi^{\beta}(x)\Omega^{2}(x)\\
\geq\ & J^{\sigma,k,\psi}_{\rho,\lambda,0+;\omega}\Omega^{2/\alpha}(x)\Pi^{\beta-1}(x)J^{\sigma,k,\psi}_{\rho,\delta,0+;\omega}\Omega^{2/\beta}(x)\Pi^{\alpha-1}(x).
\end{aligned}
$$

Theorem 7.6. *[24] Assume that $\Omega, \Pi : [0,\infty) \to \mathbb{R}$ are two positive and integrable functions such that*

$$\mu = \min_{0\leq t\leq x}\frac{\Omega(t)}{\Omega(t)}, \quad \mathcal{M} = \max_{0\leq t\leq x}\frac{\Omega(t)}{\Pi(t)}.$$

Then we have

$$
\begin{aligned}
(a)\quad 0 \ \leq\ & J^{\sigma,k,\psi}_{\rho,\lambda,0+;\omega}\Omega^{2}(x)J^{\sigma,k,\psi}_{\rho,\lambda,0+;\omega}\Pi^{2}(x) \leq \frac{(\mu+\mathcal{M})^2}{4\mu\mathcal{M}}\Big(J^{\sigma,k,\psi}_{\rho,\lambda,0+;\omega}\Omega(x)\Pi(x)\Big)^2,\\
(b)\quad 0 \ \leq\ & \sqrt{J^{\sigma,k,\psi}_{\rho,\lambda,0+;\omega}\Omega^{2}(x)J^{\sigma,k,\psi}_{\rho,\lambda,0+;\omega}\Pi^{2}(x)} - J^{\sigma,k,\psi}_{\rho,\lambda,0+;\omega}\Omega(x)\Pi(x)\\
\leq\ & \frac{(\sqrt{\mathcal{M}}-\sqrt{\mu})^2}{2\sqrt{\mu\mathcal{M}}}\Big(J^{\sigma,k,\psi}_{\rho,\lambda,0+;\omega}\Omega(x)\Pi(x)\Big),\\
(c)\quad 0 \ \leq\ & J^{\sigma,k,\psi}_{\rho,\lambda,0+;\omega}\Omega^{2}(x)J^{\sigma,k,\psi}_{\rho,\lambda,0+;\omega}\Pi^{2}(x) - \Big(J^{\sigma,k,\psi}_{\rho,\lambda,0+;\omega}\Omega(x)\Pi(x)\Big)^2\\
\leq\ & \frac{(\mathcal{M}-\mu)^2}{4\mu\mathcal{M}}\Big(J^{\sigma,k,\psi}_{\rho,\lambda,0+;\omega}\Omega(x)\Pi(x)\Big)^2.
\end{aligned}
$$

8. Grüss-type inequalities via tempered fractional integrals concerning another function

In this section we define a generalized left sided tempered fractional integral with respect to another function. Then we present Grüss-type integral inequalities.

Definition 8.1. *[16] Suppose $\Omega \in L_1[0,\infty)$ and the function $\psi : [0,\infty) \to \mathbb{R}$ is positive, and increasing with continuous derivative and $\psi(0) = 0$. Then the Lebesgue real-valued measurable function Ω defined on $[0,\infty)$ is said to be in the space X_ψ^p, $(1 \le p < \infty)$ for which*

$$\|\Omega\|_{X_\psi^p} = \Big(\int_{x_1}^{x_2} |\Omega(t)|\psi'(t)dt\Big)^{1/p} < \infty, \quad 1 \le p < \infty.$$

When $p = \infty$, then

$$\|\Omega\|_{X_\psi^\infty} = \text{ess} \sup_{0 \le t < \infty} [\psi'(t)\Omega(t)].$$

Definition 8.2. *[25] Suppose that $\kappa, \xi \in \mathbb{C}$ with $\Re(\kappa) > 0$ and $\Re(\xi) \ge 0$. The tempered fractional left sided integral is defined by*

$$({}_{x_1}J^{\kappa,\xi}\Omega)(t) = \frac{1}{\Gamma(\kappa)} \int_{x_1}^{t} e^{-\xi(t-s)}(t-s)^{\kappa-1}\Omega(s)ds, \quad t > x_1.$$

Definition 8.3. *[26] Let Ω be an integrable function in the space $X_\psi^p(0,\infty)$ and assume that ψ : $[0,\infty) \to \mathbb{R}$ is positive, and increasing with continuous derivative and $\psi(0) = 0$. Then the generalized left sided tempered fractional integral of a function Ω with respect to another function ψ is defined by*

$$({}^{\psi}J^{\kappa,\xi}\Omega)(t) = \frac{1}{\Gamma(\kappa)} \int_{0}^{t} e^{-\xi(\psi(t)-\psi(s))}(\psi(t)-\psi(s))^{\kappa-1}\psi'(s)\Omega(s)ds, \quad t > 0,$$

where $\xi > 0$, $\kappa \in \mathbb{C}$ with $\Re(\kappa) > 0$.

Theorem 8.1. *[27] Suppose that $\Omega \in X_\psi^p(0,\infty)$ and assume that $\psi : [0,\infty) \to \mathbb{R}$ is positive, and increasing with continuous derivative and $\psi(0) = 0$. Moreover, we assume that (H_1) holds. Then for $t > 0$, $\kappa, \lambda > 0$, we have*

$$\begin{aligned} & {}^{\psi}J^{\kappa,\xi}Q_2(t)\,{}^{\psi}J^{\lambda,\xi}\Omega(t) + {}^{\psi}J^{\kappa,\xi}\Omega(t)\,{}^{\psi}J^{\lambda,\xi}Q_1(t) \\ \ge\ & {}^{\psi}J^{\kappa,\xi}Q_2(t)\,{}^{\psi}J^{\lambda\xi}Q_1(t) + {}^{\psi}J^{\kappa,\lambda}\Omega(t)\,{}^{\psi}J^{\lambda,\xi}\Omega(t). \end{aligned}$$

Theorem 8.2. *[27] Assume that $\Omega, \Pi : [0,\infty) \to \mathbb{R}$ are two integrable functions satisfying (H_1) and (H_2). In addition, we suppose that $\psi : [0,\infty) \to \mathbb{R}$ is positive, and increasing with continuous derivative and $\psi(0) = 0$. Then, for $t > 0$ and $\kappa, \lambda > 0$, the following inequalities hold:*

$$\begin{aligned} (a) \qquad & {}^{\psi}J^{\kappa,\xi}Q_2(t)\,{}^{\psi}J^{\lambda,\xi}\Pi(t) + {}^{\psi}J^{\lambda,\xi}\Omega(t)\,{}^{\psi}J^{\kappa,\xi}R_1(t) \\ \ge\ & {}^{\psi}J^{\kappa,\xi}Q_2(t)\,{}^{\psi}J^{\lambda,\xi}R_1(t) + {}^{\psi}J^{\kappa,\xi}\Omega(t)\,{}^{\psi}J^{\lambda,\xi}\Pi(t). \end{aligned}$$

$$\begin{aligned} (b) \qquad & {}^{\psi}J^{\lambda,\xi}Q_1(t)\,{}^{\psi}J^{\kappa,\xi}\Pi(t) + {}^{\psi}J^{\kappa,\xi}R_2(t)\,{}^{\psi}J^{\lambda,\xi}\Omega(t) \\ \ge\ & {}^{\psi}J^{\kappa,\xi}Q_1(t)\,{}^{\psi}J^{\lambda,\xi}R_2(t) + {}^{\psi}J^{\kappa,\xi}\Omega(t)\,{}^{\psi}J^{\lambda,\xi}\Pi(t). \end{aligned}$$

$$(c)\qquad {}^{\psi}J^{\kappa,\xi}Q_2(t)\,{}^{\psi}J^{\lambda,\xi}R_2(t)+{}^{\psi}J^{\kappa,\xi}\Omega(t)\,{}^{\psi}J^{\lambda,\xi}\Pi(t)$$
$$\geq\ {}^{\psi}J^{\kappa,\xi}Q_2(t)\,{}^{\psi}J^{\lambda,\xi}\Pi(t)+{}^{\psi}J^{\kappa,\xi}\Omega(t)\,{}^{\psi}J^{\lambda,\xi}R_2(t).$$

$$(d)\qquad {}^{\psi}J^{\kappa,\xi}Q_1(t)\,{}^{\psi}J^{\lambda,\xi}R_1(t)+{}^{\psi}J^{\kappa,\xi}\Omega(t)\,{}^{\psi}J^{\lambda,\xi}\Pi(t)$$
$$\geq\ {}^{\psi}J^{\kappa,\xi}Q_1(t)\,{}^{\psi}J^{\lambda,\xi}\Pi(t)+{}^{\psi}J^{\kappa,\xi}\Omega(t)\,{}^{\psi}J^{\lambda,\xi}R_1(t).$$

We present in the next certain other types of inequalities for tempered fractional integral.

Theorem 8.3. *[27] Assume that the assumptions on Theorem 8.2 hold. If $p,q>1$ are such that $\frac{1}{p}+\frac{1}{q}=1$, then, for $t>0$ we have:*

$$(i)\qquad \frac{1}{p}{}^{\psi}J^{\kappa,\xi}\Omega^{p}(t)\,{}^{\psi}J^{\lambda,\xi}\Pi^{p}(t)+\frac{1}{q}{}^{\psi}J^{\kappa,\xi}\Pi^{q}(t)\,{}^{\psi}J^{\lambda,\xi}\Omega^{q}(t)$$
$$\geq\ {}^{\psi}J^{\kappa,\xi}\Omega(t)\Pi(t)\,{}^{\psi}J^{\lambda,\xi}\Pi(t)\Omega(t).$$

$$(ii)\qquad \frac{1}{p}{}^{\psi}J^{\kappa,\xi}\Omega^{p}(t)\,{}^{\psi}J^{\lambda,\xi}\Pi^{p}(t)+\frac{1}{q}{}^{\psi}J^{\kappa,\xi}\Pi^{q}(t)\,{}^{\psi}J^{\lambda,\xi}\Omega^{p}(t)$$
$$\geq\ {}^{\psi}J^{\lambda,\xi}\Pi^{q-1}(t)\Omega^{p-1}(t)\,{}^{\psi}J^{\kappa,\xi}\Pi(t)\Omega(t).$$

$$(iii)\qquad \frac{1}{p}{}^{\psi}J^{\kappa,\xi}\Omega^{p}(t)\,{}^{\psi}J^{\lambda,\xi}\Pi^{2}(t)+\frac{1}{q}{}^{\psi}J^{\kappa,\xi}\Pi^{q}(t)\,{}^{\psi}J^{\lambda,\xi}\Omega^{2}(t)$$
$$\geq\ {}^{\psi}J^{\lambda,\xi}\Omega^{2/p}(t)\Pi^{2/q}(t)\,{}^{\psi}J^{\kappa,\xi}\Omega(t)\Pi(t).$$

$$(iv)\qquad \frac{1}{p}{}^{\psi}J^{\kappa,\xi}\Omega^{2}(t)\,{}^{\psi}J^{\lambda,\xi}\Pi^{q}(t)+\frac{1}{q}{}^{\psi}J^{\kappa,\xi}\Pi^{2}(t)\,{}^{\psi}J^{\lambda,\xi}\Omega^{p}(t)$$
$$\geq\ {}^{\psi}J^{\lambda,\xi}\Omega^{p-1}(t)\Pi^{q-1}(t)\,{}^{\psi}J^{\kappa,\xi}\Omega^{2/p}(t)\Pi^{2/q}(t).$$

Theorem 8.4. *[27] Assume that the assumptions on Theorem 8.2 hold. If $p,q>1$ are such that $\frac{1}{p}+\frac{1}{q}=1$, then, for $t>0$ we have:*

$$(a)\qquad p\,{}^{\psi}J^{\kappa,\xi}\Omega(t)\,{}^{\psi}J^{\lambda,\xi}\Pi(t)+q\,{}^{\psi}J^{\kappa,\xi}\Pi(t)\,{}^{\psi}J^{\lambda,\xi}\Omega(t)$$
$$\geq\ {}^{\psi}J^{\kappa,\xi}\Omega^{p}(t)\Pi^{q}(t)\,{}^{\psi}J^{\lambda,\xi}\Omega^{q}(t)\Pi^{p}(t).$$

$$(b)\qquad p\,{}^{\psi}J^{\kappa,\xi}\Omega^{p-1}(t)\,{}^{\psi}J^{\lambda,\xi}\Omega(t)\Pi^{q}(t)+q\,{}^{\psi}J^{\kappa,\xi}\Omega^{q-1}(t)\,{}^{\psi}J^{\lambda,\xi}\Omega^{q}(t)\Pi(t)$$
$$\geq\ {}^{\psi}J^{\kappa,\xi}\Pi^{q}(t)\,{}^{\psi}J^{\lambda,\xi}\Omega^{p}(t).$$

$$(c)\qquad p\,{}^{\psi}J^{\kappa,\xi}\Omega(t)\,{}^{\psi}J^{\lambda,\xi}\Pi^{2/p}(t)+q\,{}^{\psi}J^{\kappa,\xi}\Pi^{q}(t)\,{}^{\psi}J^{\lambda,\xi}\Omega^{2/q}(t))$$
$$\geq\ {}^{\psi}J^{\lambda,\xi}\Omega^{p}(t)\Pi(t)\,{}^{\psi}J^{\kappa,\xi}\Pi^{q}(t)\Omega^{2}(t).$$

$$(d)\qquad p\,{}^{\psi}J^{\kappa,\xi}\Omega^{2/p}(t)\Pi^{q}(t)\,{}^{\psi}J^{\lambda,\xi}\Pi^{q-1}(t)+q\,{}^{\psi}J^{\kappa,\xi}\Pi^{q-1}(t)\,{}^{\psi}J^{\lambda,\xi}\Omega^{2/q}(t)\Pi^{p}(t)$$
$$\geq\ {}^{\psi}J^{\lambda,\xi}\Omega^{2}(t)\,{}^{\psi}J^{\kappa,\xi}\Pi^{2}(t).$$

Theorem 8.5. *[27] Assume that the assumptions on Theorem 8.2 hold. Let $p,q>1$ be such that $\frac{1}{p}+\frac{1}{q}=1$. Suppose that*

$$\mathcal{K}=\min_{0\leq s\leq t}\frac{\Omega(t)}{\Pi(t)}\quad\textit{and}\quad\mathcal{H}=\max_{0\leq s\leq t}\frac{\Omega(t)}{\Pi(t)}.$$

Then, for $t > 0$ we have:

$$(i) \qquad {}^{\psi}J^{\kappa,\xi}\Omega^2(t)\,{}^{\psi}J^{\kappa,\xi}\Pi^2(t) \leq \frac{(\mathcal{K}+\mathcal{H})^2}{4\mathcal{K}\mathcal{H}}\Big({}^{\psi}J^{\kappa,\xi}\Omega(t)\Pi(t)\Big)^2,$$

$$\begin{aligned}(ii)\quad 0 &\leq \sqrt{{}^{\psi}J^{\kappa,\xi}\Omega^2(t)\,{}^{\psi}J^{\kappa,\xi}\Pi^2(t)} - \Big({}^{\psi}J^{\kappa,\xi}\Omega(t)\Pi(t)\Big)\\ &\leq \frac{\sqrt{\mathcal{H}}-\sqrt{\mathcal{K}}}{2\sqrt{\mathcal{K}\mathcal{H}}}\Big({}^{\psi}J^{\kappa,\xi}\Omega(t)\Pi(t)\Big),\end{aligned}$$

$$\begin{aligned}(iii)\quad 0 &\leq {}^{\psi}J^{\kappa,\xi}\Omega^2(t)\,{}^{\psi}J^{\kappa,\xi}\Pi^2(t) - \Big({}^{\psi}J^{\kappa,\xi}\Omega(t)\Pi(t)\Big)^2\\ &\leq \frac{\mathcal{H}-\mathcal{K}}{4\mathcal{K}\mathcal{H}}\Big({}^{\psi}J^{\kappa,\xi}\Omega(t)\Pi(t)\Big)^2.\end{aligned}$$

9. Grüss type integral inequalities for conformable fractional integrals

In this section we deal with Grüss-type integral inequalities concerning conformable fractional integrals.

9.1. Grüss type integral inequalities for generalized η-conformable fractional integrals

We now introduced the definition of the generalized mixed η-conformable fractional integral.

Definition 9.1. *[28] Assume that $\Omega : [x_1, x_2] \to \mathbb{R}$ and $\alpha \in \mathbb{C}$, $\Re(\alpha) > 0$, $\rho > 0$, η is defined on $[x_1, x_2] \times [x_1, x_2]$. Then the mixed left η-conformable generalized fractional integral of Ω is defined by*

$$J_{\eta}^{\alpha,\rho}\Omega(x) = \frac{1}{\Gamma(\alpha)}\int_{x_1+\eta(x,x_1)}^{x_2}\Omega(s)\Big(\frac{(\eta(x_2,s))^{\rho} - (x_1 - x + \eta(x_2,x_1))^{\rho}}{\rho}\Big)^{\alpha-1}(\eta(x_2,s))^{\rho-1}ds,$$

and the mixed right η-conformable generalized fractional integral of Ω is defined by

$$J_{\eta}^{\alpha,\rho}\Omega(x) = \frac{1}{\Gamma(\alpha)}\int_{x_1}^{x_1+\eta(x,x_1)}\Omega(s)\Big(\frac{(\eta(s,x_1))^{\rho} - (x - b + \eta(x_2,x_1))^{\rho}}{\rho}\Big)^{\alpha-1}(\eta(s,x_1))^{\rho-1}ds.$$

Theorem 9.1. *[28] Assume that $\Omega : [0,\infty)$ is an integrable function satisfying (H_1) and $t > 0$, $\alpha,\beta,\rho > 0$. Then, we have:*

$$J_{\eta}^{\beta,\rho}Q_1(t)J_{\eta}^{\alpha,\rho}\Omega(t) + J_{\eta}^{\alpha,\rho}Q_2(t)J_{\eta}^{\beta,\rho}\Omega(t) \geq J_{\eta}^{\alpha,\rho}Q_2(t)J_{\eta}^{\beta,\rho}Q_1(t) + J_{\eta}^{\alpha,\rho}\Omega(t)J_{\eta}^{\beta,\rho}\Omega(t).$$

Theorem 9.2. *[28] Assume that $\Omega, \Pi : [0,\infty) \to \mathbb{R}$ are two integrable function satisfying (H_1) and (H_2) and $t > 0$, $\alpha,\beta,\rho > 0$. Then we have:*

(i). $J_{\eta}^{\beta,\rho}R_1(t)J_{\eta}^{\alpha,\rho}\Omega(t) + J_{\eta}^{\alpha,\rho}Q_2(t)J_{\eta}^{\beta,\rho}\Pi(t) \geq J_{\eta}^{\beta,\rho}R_1(t)J_{\eta}^{\alpha,\rho}Q_2(t) + J_{\eta}^{\alpha,\rho}\Omega(t)J_{\eta}^{\beta,\rho}\Pi(t).$
(ii). $J_{\eta}^{\beta,\rho}Q_1(t)J_{\eta}^{\alpha,\rho}\Pi(t) + J_{\eta}^{\alpha,\rho}R_2(t)J_{\eta}^{\beta,\rho}\Omega(t) \geq J_{\eta}^{\beta,\rho}Q_1(t)J_{\eta}^{\alpha,\rho}R_2(t) + J_{\eta}^{\beta,\rho}\Omega(t)J_{\eta}^{\alpha,\rho}\Pi(t).$
(iii). $J_{\eta}^{\alpha,\rho}Q_2(t)J_{\eta}^{\beta,\rho}R_2(t) + J_{\eta}^{\alpha,\rho}\Omega(t)J_{\eta}^{\beta,\rho}\Pi(t) \geq J_{\eta}^{\alpha,\rho}Q_2(t)J_{\eta}^{\beta,\rho}\Pi(t) + J_{\eta}^{\beta,\rho}R_2(t)J_{\eta}^{\alpha,\rho}\Omega(t).$
(iv). $J_{\eta}^{\alpha,\rho}Q_1(t)J_{\eta}^{\beta,\rho}R_1(t) + J_{\eta}^{\alpha,\rho}\Omega(t)J_{\eta}^{\beta,\rho}\Pi(t) \geq J_{\eta}^{\alpha,\rho}Q_1(t)J_{\eta}^{\beta,\rho}\Pi(t) + J_{\eta}^{\beta,\rho}R_1(t)J_{\eta}^{\alpha,\rho}\Omega(t).$

Theorem 9.3. *[28] Assume that $\Omega, \Pi : [0, \infty) \to \mathbb{R}$ are two integrable function satisfying (H_1) and (H_2) and $t > 0$, $\alpha, \beta, \rho > 0$. Then:*

$$\begin{aligned}
&\Big| J_{\eta}^{\alpha,\rho}\Omega(t)\Pi(t)\Big\{\frac{(\eta(x_2, x_1 + \eta(t, x_1))^{\rho} - (x_1 - t + \eta(x_2, x_1))^{\rho})^{\alpha}}{\Gamma(\alpha+1)\rho^{\alpha}} - \frac{(\eta(x_2, x_1)^{\rho} - (x_1 - t + \eta(x_2, x_1))^{\rho})^{\alpha}}{\Gamma(\alpha+1)\rho^{\alpha}}\Big\} \\
&-J_{\eta}^{\alpha,\rho}\Omega(t)J_{\eta}^{\alpha,\rho}\Pi(t)\Big| \\
\le\ & \sqrt{T(\Omega, Q_1, Q_2)T(\Pi, R_1, R_2)},
\end{aligned}$$

where

$$\begin{aligned}
& T(u, v, w) \\
=\ & (J_{\eta}^{\alpha,\rho}w(t) - J_{\eta}^{\alpha,\rho}u(t))(J_{\eta}^{\alpha,\rho}u(t) - J_{\eta}^{\alpha,\rho}v(t)) + J_{\eta}^{\alpha,\rho}v(t)u(t) \\
& \times\Big\{\frac{(\eta(x_2, x_1 + \eta(t, x_1))^{\rho} - (x_1 - t + \eta(x_2, x_1))^{\rho})^{\alpha}}{\Gamma(\alpha+1)\rho^{\alpha}} - \frac{(\eta(x_2, x_1)^{\rho} - (x_1 - t + \eta(x_2, x_1))^{\rho})^{\alpha}}{\Gamma(\alpha+1)\rho^{\alpha}}\Big\} \\
& -J_{\eta}^{\alpha,\rho}v(t)J_{\eta}^{\alpha,\rho}u(t) \\
& +J_{\eta}^{\alpha,\rho}w(t)\Big\{\frac{(\eta(x_2, x_1 + \eta(t, x_1))^{\rho} - (x_1 - t + \eta(x_2, x_1))^{\rho})^{\alpha}}{\Gamma(\alpha+1)\rho^{\alpha}} - \frac{(\eta(x_2, x_1)^{\rho} - (x_1 - t + \eta(x_2, x_1))^{\rho})^{\alpha}}{\Gamma(\alpha+1)\rho^{\alpha}}\Big\} \\
& -J_{\eta}^{\alpha,\rho}w(t)J_{\eta}^{\alpha,\rho}u(t) + J_{\eta}^{\alpha,\rho}v(t)J_{\eta}^{\alpha,\rho}w(t) \\
& -J_{\eta}^{\alpha,\rho}v(t)w(t)\Big\{\frac{(\eta(x_2, x_1 + \eta(t, x_1))^{\rho} - (x_1 - t + \eta(x_2, x_1))^{\rho})^{\alpha}}{\Gamma(\alpha+1)\rho^{\alpha}} - \frac{(\eta(x_2, x_1)^{\rho} - (x_1 - t + \eta(x_2, x_1))^{\rho})^{\alpha}}{\Gamma(\alpha+1)\rho^{\alpha}}\Big\}.
\end{aligned}$$

9.2. Grüss type integral inequalities for (k, s)-fractional conformable integrals

The (k, s)-fractional conformable integral operator is defined as

Definition 9.2. *[29] Let Ω be an integrable function, $\alpha \in \mathbb{C}$, $\Re(\alpha) > 0$ and $s > 0$. The (k, s)-fractional conformable integral operator is defined as*

$$I_k^{\alpha,s}\Omega(t) = \frac{s^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_{x_1}^{t}\Big[(t - x_1)^s - (x - x_1)^s\Big]^{\frac{\alpha}{k}-1}(x - x_1)^{s-1}\Omega(x)ds, \quad t \in [x_1, x_2].$$

Here, we present Grüss type inequalities involving the (k, s)-fractional conformable integral $I_k^{\alpha,s}$ defined in Definition 9.2.

Theorem 9.4. *[29] Assume that $\Omega : [0, \infty) \to \mathbb{R}$ is an integrable function satisfying (H_1) and $k, s, \alpha, \beta > 0$. Then we have:*

$$I_k^{\beta,s}Q_1(t)I_k^{\alpha,s}\Omega(t) + I_k^{\alpha,s}Q_2(t)I_k^{\beta,s}\Omega(t) \ge I_k^{\alpha,s}Q_2(t)I_k^{\beta,s}Q_1(t) + I_k^{\alpha,s}\Omega(t)I_k^{\beta,s}\Omega(t).$$

Theorem 9.5. *[29] Assume that $\Omega, \Pi : [0, \infty) \to \mathbb{R}$ are two integrable function satisfyingt (H_1) and (H_2) and $k, s, \alpha, \beta > 0$. Then we have:*

(i). $I_k^{\beta,s}R_1(t)I_k^{\alpha,s}\Omega(t) + I_k^{\alpha,s}Q_2(t)I_k^{\beta,s}\Pi(t) \ge I_k^{\beta,s}R_1(t)I_k^{\beta,s}Q_2(t) + I_k^{\alpha,s}\Omega(t)I_k^{\beta,s}\Pi(t)$.
(ii). $I_k^{\beta,s}Q_1(t)I_k^{\alpha,s}\Pi(t) + I_k^{\alpha,s}R_2(t)I_k^{\beta,s}\Omega(t) \ge I_k^{\beta,s}Q_1(t)I_k^{\beta,s}R_2(t) + I_k^{\alpha,s}\Pi(t)I_k^{\beta,s}\Omega(t)$.
(iii). $I_k^{\alpha,s}Q_2(t)I_k^{\beta,s}R_2(t) + I_k^{\alpha,s}\Omega(t)I_k^{\beta,s}\Pi(t) \ge I_k^{\alpha,s}Q_2(t)I_k^{\beta,s}\Pi(t) + I_k^{\beta,s}R_2(t)I_k^{\alpha,s}\Omega(t)$.
(iv). $I_k^{\alpha,s}Q_1(t)I_k^{\beta,s}R_1(t) + I_k^{\alpha,s}\Omega(t)I_k^{\beta,s}\Pi(t) \ge I_k^{\alpha,s}Q_1(t)I_k^{\beta,s}\Pi(t) + I_k^{\beta,s}R_1(t)I_k^{\alpha,s}\Omega(t)$.

Theorem 9.6. *[29] Assume that* $\Omega, \Pi : [0, \infty) \to \mathbb{R}$ *are two integrable function satisfyingt* (H_1) *and* (H_2) *and* $k, s, \alpha > 0$. *Then we have:*

$$\Big|\frac{s^{-\frac{\alpha}{k}}(t-x_1)^{\frac{s\alpha}{k}}}{\Gamma_k(\alpha+k)} I_k^{\alpha,s}(\Omega(t)\Pi(t)) - I_k^{\alpha,s}\Omega(t) I_k^{\alpha,s}\Pi(t)\Big| \geq \sqrt{T_k(\Omega, Q_1, Q_2) T_k(\Pi, R_1, R_2)},$$

where

$$\begin{aligned} T_k(x,y,z) &= (I_k^{\alpha,s}z(t) - I_k^{\alpha,s}x(t))(I_k^{\alpha,s}x(t) - I_k^{\alpha,s}y(t)) \\ &\quad + \frac{s^{-\frac{\alpha}{k}}(t-x_1)^{\frac{s\alpha}{k}}}{\Gamma_k(\alpha+k)} I_k^{\alpha,s}(y(t)x(t)) - I_k^{\alpha,s}y(t) I_k^{\alpha,s}x(t) \\ &\quad + \frac{s^{-\frac{\alpha}{k}}(t-x_1)^{\frac{s\alpha}{k}}}{\Gamma_k(\alpha+k)} I_k^{\alpha,s}(z(t)x(t)) - I_k^{\alpha,s}z(t) I_k^{\alpha,s}x(t) \\ &\quad - \frac{s^{-\frac{\alpha}{k}}(t-x_1)^{\frac{s\alpha}{k}}}{\Gamma_k(\alpha+k)} I_k^{\alpha,s}(y(t)z(t)) + I_k^{\alpha,s}y(t) I_k^{\alpha,s}z(t). \end{aligned}$$

Definition 9.3. *[30] Let* $\lambda \in \mathbb{C}, \Re(\lambda) > 0$. *We define the left and right sided fractional conformable integral operators as*

$$\begin{aligned} {}^{\lambda}_{x_1}J^{\mu}\Omega(x) &= \frac{1}{\Gamma(\lambda)} \int_{x_1}^{x} \Big(\frac{(x-x_1)^{\mu} - (t-x_1)^{\mu}}{\mu}\Big)^{\lambda-1} \frac{\Omega(t)}{(t-x_1)^{1-\mu}} dt, \\ {}^{\lambda}_{x_2}J^{\mu}\Omega(x) &= \frac{1}{\Gamma(\lambda)} \int_{x}^{x_2} \Big(\frac{(x_2-x)^{\mu} - (x_2-t)^{\mu}}{\mu}\Big)^{\lambda-1} \frac{\Omega(t)}{(x_2-t)^{1-\mu}} dt. \end{aligned}$$

For the results in this section we consider $x_1 = 0$.

Theorem 9.7. *[30] Suppose that* $\Omega : [0, \infty) \to \mathbb{R}$ *is an integrable function satisfying* (H_1). *Then for* $x, \alpha, \beta > 0$ *we have:*

$${}^{\beta}J^{\mu}Q_1(t)\, {}^{\alpha}J^{\mu}\Omega(t) + {}^{\alpha}J^{\mu}Q_2(t)\, {}^{\beta}J^{\mu}\Omega(t) \geq {}^{\alpha}J^{\mu}Q_2(t)\, {}^{\beta}J^{\mu}Q_1(t) + {}^{\alpha}J^{\mu}\Omega(t)\, {}^{\beta}J^{\mu}\Omega(t).$$

Theorem 9.8. *[30] Assume that* $\Omega, \Pi : [0, \infty) \to \mathbb{R}$ *are two integrable functions satisfying* (H_1) *and* (H_2). *Then for* $x, \alpha, \beta > 0$ *we have:*

(1) ${}^{\beta}J^{\mu}R_1(t)\, {}^{\alpha}J^{\mu}\Omega(t) + {}^{\alpha}J^{\mu}Q_2(t)\, {}^{\beta}J^{\mu}\Pi(t) \geq {}^{\alpha}J^{\mu}Q_2(t)\, {}^{\beta}J^{\mu}R_1(t) + {}^{\alpha}J^{\mu}\Omega(t)\, {}^{\beta}J^{\mu}\Pi(t)$.
(2) ${}^{\beta}J^{\mu}Q_1(t)\, {}^{\alpha}J^{\mu}\Pi(t) + {}^{\alpha}J^{\mu}R_2(t)\, {}^{\beta}J^{\mu}\Omega(t) \geq {}^{\alpha}J^{\mu}Q_1(t)\, {}^{\beta}J^{\mu}R_2(t) + {}^{\alpha}J^{\mu}\Omega(t)\, {}^{\beta}J^{\mu}\Pi(t)$.
(3) ${}^{\alpha}J^{\mu}Q_2(t)\, {}^{\beta}J^{\mu}R_2(t) + {}^{\alpha}J^{\mu}\Omega(t)\, {}^{\beta}J^{\mu}\Pi(t) \geq {}^{\alpha}J^{\mu}Q_2(t)\, {}^{\beta}J^{\mu}\Pi(t) + {}^{\beta}J^{\mu}R_2(t)\, {}^{\alpha}J^{\mu}\Omega(t)$.
(4) ${}^{\alpha}J^{\mu}Q_1(t)\, {}^{\beta}J^{\mu}R_1(t) + {}^{\alpha}J^{\mu}\Omega(t)\, {}^{\beta}J^{\mu}\Pi(t) \geq {}^{\alpha}J^{\mu}Q_1(t)\, {}^{\beta}J^{\mu}\Pi(t) + {}^{\alpha}J^{\mu}\Omega(t)\, {}^{\beta}J^{\mu}R_1(t)$.

Theorem 9.9. *[30] Assume that* $\Omega, \Pi : [0, \infty) \to \mathbb{R}$ *are two integrable functions satisfying* (H_1) *and* (H_2). *Then for* $x, \alpha, \beta > 0$ *we have:*

$$\left|\frac{t^{\mu\alpha}}{\mu^{\alpha}\Gamma(\alpha+1)}\, {}^{\alpha}J^{\mu}\Omega(t)\Pi(t) - {}^{\alpha}J^{\mu}\Omega(t)\, {}^{\alpha}J^{\mu}\Pi(t)\right| \leq \sqrt{T(\Omega, Q_1, Q_2) T(\Pi, R_1 R_2)},$$

where

$$T(y,z,w) = ({}^{\alpha}J^{\mu}w(t) - {}^{\alpha}J^{\mu}y(t))({}^{\alpha}J^{\mu}y(t) - {}^{\alpha}J^{\mu}z(t))$$

$$+\frac{t^{\mu\alpha}}{\mu^{\alpha}\Gamma(\alpha+1)}{}^{\alpha}J^{\mu}z(t)y(t)-{}^{\alpha}J^{\mu}z(t)\,{}^{\alpha}J^{\mu}y(t)$$
$$+\frac{t^{\mu\alpha}}{\mu^{\alpha}\Gamma(\alpha+1)}{}^{\alpha}J^{\mu}w(t)y(t)-{}^{\alpha}J^{\mu}w(t)\,{}^{\alpha}J^{\mu}y(t)$$
$$+{}^{\alpha}J^{\mu}z(t)\,{}^{\alpha}J^{\mu}w(t)-\frac{t^{\mu\alpha}}{\mu^{\alpha}\Gamma(\alpha+1)}{}^{\alpha}J^{\mu}z(t)w(t).$$

10. Grüss-type integrals inequalities via generalized proportional gractional operators

Definition 10.1. *[31] The proportional fractional integrals, left- and right-sided, of a function Ω of order α and $\sigma \in (0,1]$ are defined by*

$$I^{\alpha,\sigma}_{x_1}\Omega(t)=\frac{1}{\sigma^{\alpha}\Gamma(\alpha)}\int_{x_1}^{t}e^{\frac{\sigma-1}{\sigma}(t-s)}(t-s)^{\alpha-1}\Omega(s)ds,$$

and

$$I^{\alpha,\sigma}_{x_2}\Omega(t)=\frac{1}{\sigma^{\alpha}\Gamma(\alpha)}\int_{t}^{x_2}e^{\frac{\sigma-1}{\sigma}(s-t)}(s-t)^{\alpha-1}\Omega(s)ds,$$

where $\alpha \in \mathbb{C}$, $\Re(\alpha)>0$.

In what follows, we present Grüss-type inequality with the help of the proportional fractional integral defined above.

Theorem 10.1. *[31] Assume that $\Omega:[0,\infty)\to\mathbb{R}$ is an integrable function satisfying (H_1). Then:*

$$I^{\beta,\sigma}Q_1(t)I^{\alpha,\sigma}\Omega(t)+I^{\alpha,\sigma}Q_2(t)I^{\beta,\sigma}\Omega(t)\geq I^{\alpha,\sigma}Q_2(t)I^{\beta,\sigma}Q_1(t)+I^{\beta,\sigma}\Omega(t)I^{\beta,\sigma}\Omega(t).$$

Theorem 10.2. *[31] Let $\sigma\in(0,1]$. Suppose that $\Omega,\Pi:[0,\infty)\to\mathbb{R}$ are two integrable functions satisfying (H_1) and (H_2). Then the following inequalities hold:*

(a) $I^{\beta,\sigma}R_1(t)I^{\alpha,\sigma}\Omega(t)+I^{\alpha,\sigma}Q_2(t)I^{\beta,\sigma}\Pi(t)\geq I^{\beta,\sigma}R_1(t)I^{\alpha,\sigma}Q_2(t)+I^{\alpha,\sigma}\Omega(t)I^{\beta,\sigma}\Pi(t).$
(b) $I^{\beta,\sigma}Q_1(t)I^{\alpha,\sigma}\Pi(t)+I^{\alpha,\sigma}R_2(t)I^{\beta,\sigma}\Omega(t)\geq I^{\beta,\sigma}Q_1(t)I^{\alpha,\sigma}R_2(t)+I^{\beta,\sigma}\Omega(t)I^{\alpha,\sigma}\Pi(t).$
(c) $I^{\alpha,\sigma}Q_2(t)I^{\beta,\sigma}R_2(t)+I^{\alpha,\sigma}\Omega(t)I^{\beta,\sigma}\Pi(t)\geq I^{\alpha,\sigma}Q_2(t)I^{\beta,\sigma}\Pi(t)+I^{\beta,\sigma}R_2(t)I^{\alpha,\sigma}\Omega(t).$
(d) $I^{\alpha,\sigma}Q_1(t)I^{\beta,\sigma}R_1(t)+I^{\alpha,\sigma}\Omega(t)I^{\beta,\sigma}\Pi(t)\geq I^{\alpha,\sigma}Q_1(t)I^{\beta,\sigma}\Pi(t)+I^{\beta,\sigma}R_1(t)I^{\alpha,\sigma}\Omega(t).$

Theorem 10.3. *[31] Let $x>0$, $\alpha,\beta>0$, and $p,q>1$ satisfying $\frac{1}{p}+\frac{1}{q}=1$, and $\Omega,\Pi:[0,\infty)\to\mathbb{R}$ be two positive integrable functions. Then we have:*

$$(i)\qquad \frac{1}{p}J^{\alpha,\sigma}_{0+}\Omega^{p}(x)\,J^{\beta,\sigma}_{0+}\Pi^{p}(x)+\frac{1}{q}J^{\alpha,\sigma}_{0+}\Pi^{q}(x)\,J^{\beta,\sigma}_{0+}\Omega^{q}(x)$$
$$\geq\Big(J^{\alpha,\sigma}_{0+}\Omega(x)\Pi(x)\Big)\Big(J^{\beta,\sigma}_{0+}\Omega(x)\Pi(x)\Big).$$

$$(ii)\qquad \frac{1}{p}J^{\beta,\sigma}_{0+}\Pi^{q}(x)\,J^{\alpha,\sigma}_{0+}\Omega^{p}(x)+\frac{1}{q}J^{\beta,\sigma}_{0+}\Omega^{p}(x)\,J^{\alpha,\sigma}_{0+}\Pi^{q}(x)$$
$$\geq\Big(J^{\beta,\sigma}_{0+}\Omega^{p-1}\Pi^{q-1}(x)\Big)\Big(J^{\alpha,\sigma}_{0+}\Omega(x)\Pi(x)\Big).$$

$$(iii) \qquad \frac{1}{p} J_{0+}^{\beta,\sigma} \Pi^2(x)\ J_{0+}^{\alpha,\sigma} \Omega^p(x) + \frac{1}{q} J_{0+}^{\beta,\sigma} \Omega^2(x)\ J_{0+}^{\alpha,\sigma} \Pi^q(x)$$
$$\geq \Big(J_{0+}^{\beta,\sigma} \Omega^{2/q}(x) \Pi^{2/p}(x)\Big)\Big(J_{0+}^{\alpha,\sigma} \Omega(x) \Pi(x)\Big).$$

$$(iv) \qquad \frac{1}{p} J_{0+}^{\beta,\sigma} \Pi^q(x)\ J_{0+}^{\alpha,\sigma} \Omega^2(x) + \frac{1}{q} J_{0+}^{\beta,\sigma} \Omega^p(x)\ J_{0+}^{\alpha,\sigma} \Pi^2(x)$$
$$\geq \Big(J_{0+}^{\beta,\sigma} \Omega^{p-1}(x) \Pi^{q-1}(x)\Big)\Big(J_{0+}^{\alpha,\sigma} \Omega^{2/p}(x) \Pi^{2/p}(x)\Big).$$

Theorem 10.4. *[31] Let the assumptions of Theorem 10.1 be hold. In addition, let*

$$\mu = \min_{0 \leq t \leq x} \frac{\Omega(t)}{\Pi(t)} \quad and \quad \mathcal{M} = \max_{0 \leq t \leq x} \frac{\Omega(t)}{\Pi(t)}.$$

Then, we have:

$$(a) \quad 0 \leq \Big(J_{0+}^{\alpha,\sigma} \Omega^2(x)\ J_{0+}^{\alpha,\sigma} \Pi^2(x)\Big) \leq \frac{(\mathcal{M}+\mu)^2}{4\mu\mathcal{M}} \Big(J_{0+}^{\alpha,\sigma} \Omega(x)\Pi(x)\Big)^2.$$

$$(b) \quad 0 \leq \sqrt{J_{0+}^{\alpha,\sigma} \Omega^2(x)\ J_{0+}^{\alpha,\sigma} \Pi^2(x)} - \Big(J_{0+}^{\alpha,\sigma} \Omega(x)\Pi(x)\Big)$$
$$\leq \frac{(\sqrt{\mathcal{M}} - \sqrt{\mu})^2}{2\sqrt{\mu\mathcal{M}}} \Big(J_{0+}^{\alpha,\sigma} \Omega(x)\Pi(x)\Big).$$

$$(c) \quad 0 \leq J_{0+}^{\alpha,\sigma} \Omega^2(x)\ J_{0+}^{\alpha,\sigma} \Pi^2(x) - \Big(J_{0+}^{\alpha,\sigma} \Omega(x)\Pi(x)\Big)^2$$
$$\leq \frac{(\mathcal{M}-\mu)^2}{4\mu\mathcal{M}} \Big(J_{0+}^{\alpha,\sigma} \Omega(x)\Pi(x)\Big)^2.$$

Definition 10.2. *[32] Assume that Ω is integrable and ψ is a strictly increasing continuous function on $[x_1, x_2]$. For $\sigma \in (0,1]$, $\alpha \in \mathbb{C}$, $\Re(\alpha) \geq 0$, $k \in \mathbb{R}^+$, we define the left- and right-sided proportional k-fractional integrals, respectively, as*

$${}^{k,\psi}I_{x_1}^{\alpha,\sigma} \Omega(t) = \frac{1}{\sigma^{\frac{\alpha}{k}} k\Gamma_k(\alpha)} \int_{x_1}^{t} e^{\frac{\sigma-1}{\sigma}(\psi(t)-\psi(s))} (\psi(t) - \psi(s))^{\frac{\alpha}{k}-1} \psi'(s)\Omega(s) ds,$$

and

$${}^{k,\psi}I_{x_2}^{\alpha,\sigma} \Omega(t) = \frac{1}{\sigma^{\frac{\alpha}{k}} k\Gamma_k(\alpha)} \int_{t}^{x_2} e^{\frac{\sigma-1}{\sigma}(\psi(s)-\psi(t))} (\psi(s) - \psi(t))^{\frac{\alpha}{k}-1} \psi'(s)\Omega(s) ds.$$

In what follows, we present Grüss-type inequality with the help of the generalized k-fractional integral.

Theorem 10.5. *[32] Let $\Omega, \Pi : [0, \infty) \to \mathbb{R}$ be positive integrable functions. satisfying $(H_1), (H_2)$ with positive integrable functions Q_1, Q_2, R_1, R_2 and ψ be a strictly increasing continuous function. Tthen the following inequality also holds:*

$$\left| \frac{\frac{1}{\alpha}[\psi(t) - \psi(0))^{\frac{\alpha}{k}}}{\sigma^{\frac{\alpha}{k}} \Gamma_k(\alpha)} {}^{k,\psi}I^{\alpha,\sigma} \Omega(t)\Pi(t) - {}^{k,\psi}I^{\alpha,\sigma} \Omega(t)\ {}^{k,\psi}I^{\alpha,\sigma} \Pi(t) \right|$$

$$\leq \sqrt{T(\Omega, Q_1, Q_2)(t)T(\Pi, R_1, R_2)(t)},$$

where

$$T(u,v,w)(t) = \frac{\frac{1}{\alpha}[\psi(t)-\psi(0))^{\frac{\alpha}{k}}}{4\sigma^{\frac{\alpha}{k}}\Gamma_k(\alpha)} \frac{\left({}^{k,\psi}I^{\alpha,\sigma}\{(v+w)u\}(t)\right)^2}{{}^{k,\psi}I^{\alpha,\sigma}\Omega(t)\Pi(t)} - \left({}^{k,\psi}I^{\alpha,\sigma}(u)(t)\right)^2.$$

11. Grüss type integral inequalities for generalized Riemann-Liouville fractional integral operators

Definition 11.1. *A function* Ω *is said to be* $L_{p,s}[x_1, x_2]$ *if*

$$\left(\int_{x_1}^{x_2} |\Omega(t)|^p t^s dt\right)^{1/p} < \infty, \quad 1 \leq p < \infty, \quad s \geq 0.$$

Definition 11.2. *[33] Let* $\Omega \in L_{1,s}[0,\infty)$*. The Riemann-Liouville generalized fractional integral of* Ω *of order* $\alpha > 0$ *and* $s \geq 0$ *is defined by*

$$I^{\alpha,s}\Omega(t) = \frac{(s+1)^{1-\alpha}}{\Gamma(\alpha)} \int_{x_1}^{t} (t^{s+1} - \tau^{s+1})^{\alpha-1} \tau^s \Omega(\tau) d\tau, \quad t \in [x_1, x_2].$$

In this section, we present some Grüss type inequalities via the fractional integral defined in Definition 11.2.

Theorem 11.1. *[34] Let* $\Omega \in L_{1,s}[x_1, x_2]$ *satisfying* (H_1) *and* $k > 0,\ s \geq 0,\ \alpha, \beta > 0$*. Then we have the following inequality:*

$$I^{\beta,s}Q_1(t)I^{\alpha,s}\Omega(t) + I^{\alpha,s}Q_2(t)I^{\beta,s}\Omega(t) \geq I^{\alpha,s}Q_2(t)I^{\beta,s}Q_1(t) + I^{\alpha,s}\Omega(t)I^{\beta,s}\Omega(t).$$

Theorem 11.2. *[34] Assume that* $\Omega, \Pi : [0,\infty) \to \mathbb{R}$ *are two integrable functions satisfying* (H_1) *and* (H_2) *and* $k > 0,\ s \geq 0,\ \alpha, \beta > 0$*. Then we have the following inequalities:*

(i) $I^{\beta,s}R_1(t)I^{\alpha,s}\Omega(t) + I^{\alpha,s}Q_2(t)I^{\beta,s}\Pi(t) \geq I^{\beta,s}R_1(t)I^{\alpha,s}Q_2(t) + I^{\alpha,s}\Omega(t)I^{\beta,s}\Pi(t)$.
(ii) $I^{\beta,s}Q_1(t)I^{\alpha,s}\Pi(t) + I^{\alpha,s}R_2(t)I^{\beta,s}\Omega(t) \geq I^{\beta,s}Q_1(t)I^{\alpha,s}R_2(t) + I^{\alpha,s}\Pi(t)I^{\beta,s}\Omega(t)$.
(iii) $I^{\alpha,s}Q_2(t)I^{\beta,s}R_2(t) + I^{\alpha,s}\Omega(t)I^{\beta,s}\Pi(t) \geq I^{\alpha,s}Q_2(t)I^{\beta,s}\Pi(t) + I^{\beta,s}R_2(t)I^{\alpha,s}\Omega(t)$.
(iv) $I^{\alpha,s}Q_1(t)I^{\beta,s}R_1(t) + I^{\alpha,s}\Omega(t)I^{\beta,s}\Pi(t) \geq I^{\alpha,s}Q_1(t)I^{\beta,s}\Pi(t) + I^{\beta,s}R_1(t)I^{\alpha,s}\Omega(t)$.

Theorem 11.3. *[35] Under the assumptions of Theorem 11.2 we have for all* $t \in [x_1, x_2]$*,* $s \geq 0$ *and* $\alpha > 0$

$$\left|\frac{(s+1)^{-\alpha}t^{(s+1)\alpha}}{\Gamma(\alpha+1)} I^{\alpha,s}\Omega(t)\Pi(t) - I^{\alpha,s}\Omega(t)I^{\alpha,s}\Pi(t)\right| \leq \sqrt{T(\Omega, Q_1, Q_1)T(\Pi, R_1, R_2)},$$

where

$$\begin{aligned} T(x,y,z) &= (I^{\alpha,s}z(t) - I^{\alpha,s}x(t))(I^{\alpha,s}x(t) - I^{\alpha,s}y(t)) \\ &\quad + \frac{(s+1)^{-\alpha}t^{(s+1)\alpha}}{\Gamma(\alpha+1)} I^{\alpha,s}y(t)x(t) - I^{\alpha,s}y(t)I^{\alpha,s}x(t) \\ &\quad + \frac{(s+1)^{-\alpha}t^{(s+1)\alpha}}{\Gamma(\alpha+1)} I^{\alpha,s}z(t)x(t) - I^{\alpha,s}z(t)I^{\alpha,s}x(t) \\ &\quad - \frac{(s+1)^{-\alpha}t^{(s+1)\alpha}}{\Gamma(\alpha+1)} I^{\alpha,s}y(t)z(t) + I^{\alpha,s}y(t)I^{\alpha,s}x(t). \end{aligned}$$

Now we define (k, s)-Riemann-Liouville fractional integral.

Definition 11.3. *[33] Let $\Omega : [x_1, x_2] \to \mathbb{R}$ be a continuous function. Then (k, s)-Riemann-Liouville fractional integral of Ω of order $\alpha > 0$ is defined by*

$$J^{\alpha,s}_{x_1,k}\Omega(t) = \frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \int_{x_1}^{t} (t^{s+1} - \tau^{s+1})^{\frac{\alpha}{k}-1} \tau^s \Omega(\tau) d\tau, \quad t \in [x_1, x_2],$$

where $k > 0, s \in \mathbb{R} \setminus \{-1\}$.

Now, for the generalized (k, s)-Riemann-Liouville fractional integral defined above, we give some Grüss type inequalities.

Theorem 11.4. *[35] Let $\Omega \in L_{1,s}[x_1, x_2]$ satisfying (H_1) and $k > 0,\ s \geq 0,\ \alpha, \beta > 0$. Then we have the following inequality:*

$$J^{\beta,s}_{x_1,k}Q_1(t)J^{\alpha,s}_{x_1,k}\Omega(t) + J^{\alpha,s}_{x_1,k}Q_2(t)\Omega(t) \geq J^{\alpha,s}_{x_1,k}Q_2(t)J^{\beta,s}_{x_1,k}Q_1(t) + J^{\alpha,s}_{x_1,k}\Omega(t)J^{\beta,s}_{x_1,k}\Omega(t).$$

Theorem 11.5. *[35] Assume that $\Omega, \Pi : [0, \infty) \to \mathbb{R}$are two integrable functions satisfying (H_1) and (H_2) and $k > 0,\ s \geq 0,\ \alpha, \beta > 0$. Then we have:*

(i) $J^{\beta,s}_{x_1,k}R_1(t)J^{\alpha,s}_{x_1,k}\Omega(t) + J^{\alpha,s}_{x_1,k}Q_2(t)J^{\beta,s}_{x_1,k}\Pi(t) \geq J^{\beta,s}_{x_1,k}R_1(t)J^{\alpha,s}_{x_1,k}Q_2(t) + J^{\alpha,s}_{x_1,k}\Omega(t)J^{\beta,s}_{x_1,k}\Pi(t).$

(ii) $J^{\beta,s}_{x_1,k}Q_1(t)J^{\alpha,s}_{x_1,k}\Pi(t) + J^{\alpha,s}_{x_1,k}R_2(t)J^{\beta,s}_{x_1,k}\Omega(t) \geq J^{\beta,s}_{x_1,k}Q_1(t)J^{\alpha,s}_{x_1,k}R_2(t) + J^{\alpha,s}_{x_1,k}\Pi(t)J^{\beta,s}_{x_1,k}\Omega(t).$

(iii) $J^{\alpha,s}_{x_1,k}Q_2(t)J^{\beta,s}_{x_1,k}R_2(t) + J^{\alpha,s}_{x_1,k}\Omega(t)J^{\beta,s}_{x_1,k}\Pi(t) \geq J^{\alpha,s}_{x_1,k}Q_2(t)J^{\beta,s}_{x_1,k}\Pi(t) + J^{\beta,s}_{x_1,k}R_2(t)J^{\alpha,s}_{x_1,k}\Omega(t).$

(iv) $J^{\alpha,s}_{x_1,k}Q_1(t)J^{\beta,s}_{x_1,k}R_1(t) + J^{\alpha,s}_{x_1,k}\Omega(t)J^{\beta,s}_{x_1,k}\Pi(t) \geq J^{\alpha,s}_{x_1,k}Q_1(t)J^{\beta,s}_{x_1,k}\Pi(t) + J^{\beta,s}_{x_1,k}R_1(t)J^{\alpha,s}_{x_1,k}\Omega(t).$

Theorem 11.6. *[35] Under the assumptions of Theorem 11.5 we have for all $t \in [x_1, x_2]$, $s \geq 0$ and $\alpha > 0$*

$$\left| \frac{(s+1)^{-\frac{\alpha}{k}} t^{(s+1)\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)} J^{\alpha,s}_{x_1,k}\Omega(t)\Pi(t) - J^{\alpha,s}_{x_1,k}\Omega(t)J^{\alpha,s}_{x_1,k}\Pi(t) \right| \leq \sqrt{T^s_k(\Omega, Q_1, Q_1)T^s_k(\Pi, R_1, R_2)},$$

where

$$\begin{aligned} T^s_k(x, y, z) &= (J^{\alpha,s}_{x_1,k}z(t) - J^{\alpha,s}_{x_1,k}x(t))(J^{\alpha,s}_{x_1,k}x(t) - J^{\alpha,s}_{x_1,k}y(t)) \\ &\quad + \frac{(s+1)^{-\frac{\alpha}{k}} t^{(s+1)\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)} J^{\alpha,s}_{x_1,k}y(t)x(t) - J^{\alpha,s}_{x_1,k}y(t)J^{\alpha,s}_{x_1,k}x(t) \\ &\quad + \frac{(s+1)^{-\frac{\alpha}{k}} t^{(s+1)\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)} J^{\alpha,s}_{x_1,k}z(t)x(t) - J^{\alpha,s}_{x_1,k}z(t)J^{\alpha,s}_{x_1,k}x(t) \\ &\quad - \frac{(s+1)^{-\frac{\alpha}{k}} t^{(s+1)\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)} J^{\alpha,s}_{x_1,k}y(t)z(t) + J^{\alpha,s}_{x_1,k}y(t)J^{\alpha,s}_{x_1,k}x(t). \end{aligned}$$

12. Grüss-type fractional inequalities via Caputo-Fabrizio integral operator

In this section, we present the Grüss-type fractional integral inequalities involving the Caputo-Fabrizio fractional integral.

Definition 12.1. *[36] Assume that $\alpha \in \mathbb{R}$ such that $0 < \alpha < 1$. We define the Caputo-Fabrizio fractional integral of a function Ω of order α by*

$$I^{\alpha}_{0,t}\Omega(t) = \frac{1}{\alpha}\int_0^t e^{-\left(\frac{1-\alpha}{\alpha}\right)(t-s)}\Omega(s)ds.$$

Theorem 12.1. *[37] Assume that $\Omega : [0,\infty) \to \mathbb{R}$ is an integrable function satisfying (H_1). Then, for $t > 0$, $\alpha,\beta > 0$, $k > 0$, we have:*

$$I^{\beta}_{0,t}Q_1(t)I^{\alpha}_{0,t}\Omega(t) + I^{\alpha}_{0,t}Q_2(t)I^{\beta}_{0,t}\Omega(t) \geq I^{\alpha}_{0,t}Q_2(t)I^{\beta}_{0,t}Q_1(t) + I^{\alpha}_{0,t}\Omega(t)I^{\beta}_{0,t}\Omega(t).$$

Theorem 12.2. *[37] Suppose that $\Omega, \Pi : [0,\infty) \to \mathbb{R}$ are two integrable functions satisfying (H_1) and (H_2). Then, for $t > 0$, α, $\beta > 0$, $k > 0$ we have the inequalities:*

(a) $I^{\beta}_{0,t}R_1(t)\, I^{\alpha}_{0,t}\Omega(t) + I^{\alpha}_{0,t}Q_2(t)\, I^{\beta}_{0,t}\Pi(t) \geq I^{\beta}_{0,t}R_1(t)\, I^{\alpha}_{0,t}Q_2(t) + I^{\alpha}_{0,t}\Omega(t)\, I^{\beta}_{0,t}\Pi(t).$
(b) $I^{\beta}_{0,t}Q_1(t)\, I^{\alpha}_{0,t}\Pi(t) + I^{\alpha}_{0,t}R_2(t)\, I^{\beta}_{0,t}\Omega(t) \geq I^{\beta}_{0,t}Q_1(t)\, I^{\alpha}_{0,t}R_2(t) + I^{\beta}_{0,t}\Omega(t)\, I^{\alpha}_{0,t}\Pi(t).$
(c) $I^{\alpha}_{0,t}Q_2(t)\, {}_kI^{\beta}_{0,t}R_2(t) + I^{\alpha}_{0,t}\Omega(t)\, I^{\beta}_{0,t}\Pi(t) \geq I^{\alpha}_{0,t}Q_2(t)\, I^{\beta}_{0,t}\Pi(t) + I^{\beta}_{0,t}R_2(t)\, I^{\alpha}_{0,t}\Omega(t).$
(d) $I^{\alpha}_{0,t}Q_1(t)I^{\beta}_{0,t}R_1(t) + I^{\alpha}_{0,t}\Omega(t)\, I^{\beta}_{0,t}\Pi(t) \geq I^{\alpha}_{0,t}Q_1(t)\, I^{\beta}_{0,t}\Pi(t) + I^{\beta}_{0,t}R_1(t)\, I^{\alpha}_{0,t}\Omega(t).$

Theorem 12.3. *[37] Assume that $\Omega, \Pi : [0,\infty) \to \mathbb{R}$ are two integrable functions satisfying (H_1) and (H_2). Then for all $t > 0$, $\alpha > 0$, we have:*

$$\left|\left(\frac{1}{1-\alpha}\left[1 - e^{-\left(\frac{1-\alpha}{\alpha}\right)t}\right]\right) I^{\alpha}_{0,t}\Omega(t)\Pi(t) - I^{\alpha}_{0,t}\Omega(t)\, I^{\alpha}_{0,t}\Pi(t)\right| \leq \sqrt{T(\Omega, Q_1, Q_2)T(\Pi, R_1, R_2)},$$

where

$$\begin{aligned} T(y,z,w) &= \left(I^{\alpha}_{0,t}w(t) - J^{\alpha}y(t)\right)\left(I^{\alpha}_{0,t}y(t) - I^{\alpha}_{0,t}z(t)\right) \\ &\quad +\left(\frac{1}{1-\alpha}\left[1 - e^{-\left(\frac{1-\alpha}{\alpha}\right)t}\right]\right)I^{\alpha}_{0,t}z(t)y(t) - I^{\alpha}_{0,t}z(t)\, I^{\alpha}_{0,t}y(t) \\ &\quad +\left(\frac{1}{1-\alpha}\left[1 - e^{-\left(\frac{1-\alpha}{\alpha}\right)t}\right]\right)I^{\alpha}_{0,t}w(t)y(t) - I^{\alpha}_{0,t}w(t)\, I^{\alpha}_{0,t}y(t) \\ &\quad +I^{\alpha}_{0,t}z(t)\, I^{\alpha}_{0,t}w(t) + \left(\frac{1}{1-\alpha}\left[1 - e^{-\left(\frac{1-\alpha}{\alpha}\right)t}\right]\right)I^{\alpha}_{0,t}z(t)w(t). \end{aligned}$$

13. Grüss-type fractional inequalities for Saigo fractional integral operator

In this section, Grüss-type fractional integral inequalities are presented via Saigo fractional integer operator.

Definition 13.1. *[38] Assume that $\alpha > 0, \beta, \eta \in \mathbb{R}$. The Saigo fractional integral $I^{\alpha,\beta,\eta}_{0,x}[\Omega(x)]$ of order α for a real-valued continuous function Ω is defined by*

$$I^{\alpha,\beta,\eta}_{0,x}[\Omega(x)] = \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}\, {}_2F_1\left(\alpha+\beta, -\eta; \alpha; 1 - \frac{t}{x}\right)\Omega(t)dt,$$

where ${}_2F_1$ is the Gaussian hypergeometric function defined by

$${}_2F_1(a,b;c;x) = \sum_{n=0}^{\infty}\frac{(a)_n(b)_n x^n}{(c)_n n!},$$

and $(a)_n$ *is the Pochhammer symbol*

$$(a)_0 = 1,\ (a)_n = a(a+1)\cdots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}.$$

Theorem 13.1. *[39] Assume that* $\Omega, \Pi : [0,\infty) \to \mathbb{R}$ *are two integrable functions satisfying* (H_1) *and* (H_2). *Then for all* $x > 0$, $\alpha > \max\{0, -\beta\}, \beta < 1, \beta - 1 < \eta < 0$, *one has*

$$\begin{aligned}&\left|\frac{\Gamma(1-\beta+\eta)}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)x^{\beta}} I_{0,x}^{\alpha,\beta,\eta}[\Omega(x)\Pi(x)] - I_{0,x}^{\alpha,\beta,\eta}[\Omega(x)] I_{0,x}^{\alpha,\beta,\eta}[\Pi(x)]\right| \\ \leq\ & \sqrt{T(\Omega, Q_1(x), Q_2(x)) T(\Pi, R_1(x), R_2(x))},\end{aligned}$$

where

$$\begin{aligned} T(a,b,c) \ =\ & (I_{0,x}^{\alpha,\beta,\eta}[c(x)] - I_{0,x}^{\alpha,\beta,\eta}[a(x)])(I_{0,x}^{\alpha,\beta,\eta}[a(x)] - I_{0,x}^{\alpha,\beta,\eta}[b(x)]) \\ & + \frac{\Gamma(1-\beta+\eta)}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)x^{\beta}} I_{0,x}^{\alpha,\beta,\eta}[b(x)a(x)] - I_{0,x}^{\alpha,\beta,\eta}[b(x)] I_{0,x}^{\alpha,\beta,\eta}[a(x)] \\ & + \frac{\Gamma(1-\beta+\eta)}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)x^{\beta}} I_{0,x}^{\alpha,\beta,\eta}[c(x)a(x)] - I_{0,x}^{\alpha,\beta,\eta}[c(x)] I_{0,x}^{\alpha,\beta,\eta}[a(x)] \\ & + I_{0,x}^{\alpha,\beta,\eta}[b(x)] I_{0,x}^{\alpha,\beta,\eta}[c(x)] + \frac{\Gamma(1-\beta+\eta)}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)x^{\beta}} I_{0,x}^{\alpha,\beta,\eta}[b(x)c(x)]. \end{aligned}$$

Theorem 13.2. *[39] Suppose that* $\Omega, \Pi : [0,\infty) \to \mathbb{R}$ *are two integrable functions satisfying* (H_1) *and* (H_2). *Then for all* $x > 0$, $\alpha > \max\{0, -\beta\}$, $\psi > \max\{0 - \phi\}, \beta < 1, \beta - 1 < \eta < 0, \phi < 1, \phi - 1 < \zeta < 0$, *we have:*

$$\begin{aligned} (a) \qquad & I_{0,x}^{\psi,\phi,\zeta}[Q_1(x)] I_{0,x}^{\alpha,\beta,\eta}[\Omega(x)] + I_{0,x}^{\alpha,\beta,\eta}[Q_2(x)] I_{0,x}^{\psi,\phi,\zeta}[\Omega(x)] \\ \geq\ & I_{0,x}^{\psi,\phi,\zeta}[Q_2(x)] I_{0,x}^{\alpha,\beta,\eta}[Q_1(x)] + I_{0,x}^{\alpha,\beta,\eta}[\Omega(x)] I_{0,x}^{\psi,\phi,\zeta}[\Omega(x)]. \end{aligned}$$

$$\begin{aligned} (b) \qquad & I_{0,x}^{\psi,\phi,\zeta}[R_1(x)] I_{0,x}^{\alpha,\beta,\eta}[\Omega(x)] + I_{0,x}^{\alpha,\beta,\eta}[Q_2(x)] I_{0,x}^{\psi,\phi,\zeta}[\Pi(x)] \\ \geq\ & I_{0,x}^{\psi,\phi,\zeta}[R_1(x)] I_{0,x}^{\alpha,\beta,\eta}[Q_2(x)] + I_{0,x}^{\alpha,\beta,\eta}[\Omega(x)] I_{0,x}^{\psi,\phi,\zeta}[\Pi(x)]. \end{aligned}$$

$$\begin{aligned} (c) \qquad & I_{0,x}^{\psi,\phi,\zeta}[Q_1(x)] I_{0,x}^{\alpha,\beta,\eta}[\Pi(x)] + I_{0,x}^{\alpha,\beta,\eta}[R_2(x)] I_{0,x}^{\psi,\phi,\zeta}[\Omega(x)] \\ \geq\ & I_{0,x}^{\psi,\phi,\zeta}[Q_1(x)] I_{0,x}^{\alpha,\beta,\eta}[R_2(x)] + I_{0,x}^{\psi,\phi,\zeta}[u\Omega(x)] I_{0,x}^{\alpha,\beta,\eta}[\Pi(x)]. \end{aligned}$$

$$\begin{aligned} (d) \qquad & I_{0,x}^{\alpha,\beta,\eta}[Q_2(x)] I_{0,x}^{\psi,\phi,\zeta}[R_2(x)] + I_{0,x}^{\alpha,\beta,\eta}[\Omega(x)] I_{0,x}^{\psi,\phi,\zeta}[\Pi(x)] \\ \geq\ & I_{0,x}^{\alpha,\beta,\eta}[Q_2(x)] I_{0,x}^{\psi,\phi,\zeta}[\Pi(x)] + I_{0,x}^{\psi,\phi,\zeta}[R_2(x)] I_{0,x}^{\alpha,\beta,\eta}[\Omega(x)]. \end{aligned}$$

$$\begin{aligned} (e) \qquad & I_{0,x}^{\alpha,\beta,\eta}[Q_1(x)] I_{0,x}^{\psi,\phi,\zeta}[R_1(x)] + I_{0,x}^{\alpha,\beta,\eta}[\Omega(x)] I_{0,x}^{\psi,\phi,\zeta}[\Pi(x)] \\ \geq\ & I_{0,x}^{\alpha,\beta,\eta}[Q_1(x)] I_{0,x}^{\psi,\phi,\zeta}[\Pi(x)] + I_{0,x}^{\psi,\phi,\zeta}[R_1(x)] I_{0,x}^{\alpha,\beta,\eta}[\Omega(x)]. \end{aligned}$$

We define a fractional integral $K^{\alpha,\beta,\eta}$ associated with the Gauss hypergeometric function as follows:

Definition 13.2. *[40] Let* $\Omega \in C_{\mu}$. *For* $\alpha > \max\{0, -(\eta+1)\}$, $\eta - \beta > -1, \beta < 1$, *we define a fractional integral* $K^{\alpha,\beta,\eta} f$ *as follows:*

$$K^{\alpha,\beta,\eta}\Omega(t) = \frac{\Gamma(1-\beta)\Gamma(\alpha+\eta+1)}{\Gamma(\eta-\beta+1)} t^{\beta} I_{0+}^{\alpha,\beta,\eta}\Omega(t),$$

where $I_{0+}^{\alpha,\beta,\eta} f$ is the right-hand sided Gauss hypergeometric fractional integral of order α defined in Definition 13.1.

We present integral inequalities of Grüss type for the above defined hypergeometric fractional integral.

Theorem 13.3. *[40] Let $\Omega, \Pi \in C_\mu$ satisfying the condition (H) on $[0, \infty)$. Then for all $t > 0$, $\alpha > \max\{0, -(\eta + 1)\}$, $\eta - \beta > -1, \beta < 1$, we have*

$$|K^{\alpha,\beta,\eta}\Omega(t)\Pi(t) - K^{\alpha,\beta,\eta}\Omega(t)K^{\alpha,\beta,\eta}\Pi(t)| \leq \frac{1}{4}(\mathfrak{M} - \mathfrak{m})(\mathfrak{P} - \mathfrak{p}).$$

Theorem 13.4. *[40] Let Ω and Π be two synchronous functions on $[0, \infty)$. Then the following inequality holds:*

$$K^{\alpha,\beta,\eta}\Omega(t)\Pi(t) \geq K^{\alpha,\beta,\eta}\Omega(t)K^{\alpha,\beta,\eta}\Pi(t).$$

Another fractional integral operator $K^{\alpha,\beta,\eta,\delta}$ associated with the Gauss hypergeometric function is defined as follows.

Definition 13.3. *[41] Let $\Omega \in C_\mu$. For $\alpha > \max\{0, -(\delta + \eta + 1)\}$, $\eta - \beta > -1, \beta < 1, \delta > -1$ we define a fractional integral $K^{\alpha,\beta,\eta,\delta}\Omega$ as follows:*

$$K^{\alpha,\beta,\eta,\delta}\Omega(t) = \frac{\Gamma(1-\beta)\Gamma(\alpha+\delta+\eta+1)}{\Gamma(\eta-\beta+1)\Gamma(\delta+1)} t^{\beta+\beta} I_{0+}^{\alpha,\beta,\eta,\delta}\Omega(t),$$

where $I_{0+}^{\alpha,\beta,\eta,\delta}\Omega$ is the right-hand sided Gauss hypergeometric fractional integral of order α defined by

$$I_{0,x}^{\alpha,\beta,\eta,\delta}[\Omega(x)] = \frac{x^{-\alpha-\beta-2\delta}}{\Gamma(\alpha)} \int_0^x t^\delta (x-t)^{\alpha-1} \, {}_2F_1\Big(\alpha+\beta+\delta, -\eta; \alpha; 1 - \frac{t}{x}\Big)\Omega(t)dt,$$

and ${}_2F_1$ is the Gaussian hypergeometric function defined in Definition 13.1.

We establish two Grüss-type fractional integral inequalities involving the Gauss hypergeometric function.

Theorem 13.5. *[41] Assume that $\Omega, \Pi : [x_1, x_2] \to \mathbb{R}$ are two integrable functions satisfying the condition (H) on $[0, \infty)$. Then, for all $x \in [0, \infty)$, $\alpha > 0$, $\delta > -1$, and $\beta, \eta \in \mathbb{R}$ with $\alpha + \beta + \delta \geq 0$ and $\eta \leq 0$, we have:*

$$|K^{\alpha,\beta,\eta;\delta}\Omega(t)\Pi(t) - K^{\alpha,\beta,\eta,\delta}\Omega(t)K^{\alpha,\beta,\eta,\delta}\Pi(t)| \leq \frac{1}{4}(\mathfrak{M} - \mathfrak{m})(\mathfrak{P} - \mathfrak{p}).$$

Theorem 13.6. *[41] Suppose that $\Omega, \Pi : [0, \infty) \to \mathbb{R}$ are two synchronous functions (i.e $(\Omega(t) - \Omega(s))(\Pi(t) - \Pi(s)) \geq 0,\ t, s \in [0, \infty)$). Then, for all $x \in [0, \infty)$, $\alpha > 0$, $\delta > -1$, and $\beta, \eta \in \mathbb{R}$ with $\alpha + \beta + \delta \geq 0$ and $\eta \leq 0$, we have:*

$$K^{\alpha,\beta,\eta,\delta}\Omega(t)\Pi(t) \geq K^{\alpha,\beta,\eta,\delta}\Omega(t)K^{\alpha,\beta,\eta,\delta}\Pi(t).$$

Now we give some Grüss-type inequalities for generalized hypergeometric function fractional order integral operators. We start with the following definitions.

Definition 13.4. *[42] Let $\alpha, \alpha', \beta, \beta', \gamma \in \mathbb{R}$ and $\gamma > 0$. Then the Saigo and Maeda fractional integral operator $I_t^{\alpha,\alpha',\beta,\beta',\gamma}[\Omega(x)]$ of order α for a real-valued continuous function Ω is defined by*

$$I_t^{\alpha,\alpha',\beta,\beta',\gamma}[\Omega(x)] = \frac{x^{-\alpha}}{\Gamma(\gamma)} \int_0^x (x-t)^{\gamma-1} t^{-\alpha'} F_3\Big(\alpha, \alpha', \beta, \beta', \gamma, 1 - \frac{t}{x}, 1 - \frac{x}{t}\Big)\Omega(t)dt,$$

where F_3 is the Appell hypergeometric function defined by

$$F_3(\alpha, \alpha', \beta, \beta', \gamma, x, y) = \sum_{n=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\alpha)_n(\alpha')_n(\beta)_n(\beta')_n x^m y^n}{(\gamma)_{m+n} m! n!}, \quad \max\{|x|, |y|\} < 1,$$

and $(a)_n$ is the Pochhammer symbol.

Definition 13.5. *[43] Assume that $\alpha, \alpha', \beta, \beta', \gamma \in \mathbb{R}$ such that*

$$\gamma > \max\{0, \alpha + \alpha' + \beta - 1, \alpha + \alpha' - 1, \alpha' + \beta - 1\} \ \textit{and} \ \beta' > \max\{-1, \alpha' - 1\}.$$

Then we define a fractional integral operator

$$(S_t^{\alpha,\alpha',\beta,\beta',\gamma}\Omega)(x) = \frac{\Gamma(1+\gamma-\alpha-\alpha')\Gamma(1+\gamma-\alpha'-\beta)\Gamma(1+\beta')}{\Gamma(1+\gamma-\alpha-\alpha'-\beta)\Gamma(1+\beta'-\alpha')} x^{\alpha+\alpha'-\gamma}(I_t^{\alpha,\alpha',\beta,\beta',\gamma}\Omega)(x),$$

where $I_t^{\alpha,\alpha',\beta,\beta',\gamma}$ is the Saigo-Maeda fractional integral of order γ.

The main results for Grüss inequalities are given now.

Theorem 13.7. *[43] Assume that $h : [0,\infty) \to \mathbb{R}$ is an integrable function satisfying the condition $m_1 \leq h(x) \leq M_1$ for all $x \in [0,\infty)$. Then for $\alpha, \alpha', \beta, \beta', \gamma \in \mathbb{R}$ we have:*

$$\begin{aligned} &\Big|(S_t^{\alpha,\alpha',\beta,\beta',\gamma}h^2)(x) - \Big((S_t^{\alpha,\alpha',\beta,\beta',\gamma}h)(x)\Big)^2 \\ =\ &\Big(M_1 - (S_t^{\alpha,\alpha',\beta,\beta',\gamma}h)(x)\Big)\Big((S_t^{\alpha,\alpha',\beta,\beta',\gamma}h)(x) - m\Big)(M_1 - h)(h - m_1)(x), \end{aligned}$$

provided $\alpha, \alpha', \beta, \beta', \gamma > 0$.

Theorem 13.8. *[43] Assume that (H) holds on $[0,\infty)$. In addition, let $\Omega, \Pi \in C_\mu$. Then for $\alpha, \alpha', \beta, \beta', \gamma \in \mathbb{R}$ and $\alpha, \alpha', \beta, \beta', \gamma > 0$ we have:*

$$\Big|(S^{\alpha,\alpha',\beta,\beta',\gamma}\Omega(x)\Pi(x) - (S^{\alpha,\alpha',\beta,\beta',\gamma}\Omega)(x)(S^{\alpha,\alpha',\beta,\beta',\gamma}\Pi)(x)\Big| \leq \frac{1}{4}(\mathfrak{M} - \mathfrak{m})(\mathfrak{P} - \mathfrak{p}), \ \forall x \in [0,\infty).$$

Theorem 13.9. *[43] Let Ω and Π be two synchronous functions on $[0,\infty)$ and let $v, w : [0,\infty) \to [0,\infty)$. Then for all $t > 0$,*

$$\begin{aligned} &\Big(S_t^{\alpha,\alpha',\beta,\beta',\gamma}v\Omega\Pi\Big)(x)\Big(S_t^{\alpha,\alpha',\beta,\beta',\gamma}w\Big)(x) + \Big(S_t^{\alpha,\alpha',\beta,\beta',\gamma}w\Omega\Pi\Big)(x)\Big(S_t^{\alpha,\alpha',\beta,\beta',\gamma}v\Big)(x) \\ \geq\ &\Big(S_t^{\alpha,\alpha',\beta,\beta',\gamma}w\Pi\Big)(x)\Big(S_t^{\alpha,\alpha',\beta,\beta',\gamma}v\Omega\Big)(x) + \Big(S_t^{\alpha,\alpha',\beta,\beta',\gamma}w\Omega\Big)(x)\Big(S_t^{\alpha,\alpha',\beta,\beta',\gamma}v\Pi\Big)(x). \end{aligned}$$

14. Quantum Grüss-type integral inequalities

In this section we present Grüss-type integral inequalities via quantum calculus.

14.1. q-Grüss inequality involving the Riemann-Liouville fractional q-integrals

Definition 14.1. *[44] The Jakson's q-derivarive and q-integral of a function* Ω *defined on J are, respectively, given by*

$$D_q\Omega(t) = \frac{\Omega(t) - \Omega(tq)}{t(1-q)}, \quad t \neq 0, \quad q \neq 1,$$

$$\int_0^t \Omega(s) d_q s = t(1-q) \sum_{k=0}^{\infty} q^k \Omega(tq^k).$$

Definition 14.2. *[45] The Riemann-Liouville fractional q-integral operator of a function* Ω *of order* α *is given by*

$$I_q^\alpha \Omega(t) = \frac{t^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^t \left(\frac{qs}{t}, q\right)_{\alpha-1} \Omega(s) d_q s, \quad \alpha > 0, \quad 0 < q < 1,$$

where

$$(a,q)_\alpha = \frac{(a;q)_\infty}{(aq^\alpha;q)_\infty}, \quad \alpha \in \mathbb{R}$$

and

$$(\alpha,q)_\infty = \prod_{j=0}^{\infty}(1-\alpha q^j).$$

Now, we present some q-Grüss integral inequalities.

Theorem 14.1. *[46] Assume that* $\Omega, \Pi : [0,\infty) \to \mathbb{R}$ *are two integrable functions satisfyingt* (H_1) *and* (H_2)*. Then, for* $t > 0$ *and* $\alpha > 0$*, we have:*

$$\left|\frac{t^\alpha}{\Gamma_q(\alpha+1)} I_q^\alpha \Omega(t)\Pi(t) - I_q^\alpha \Omega(t) I_q^\alpha \Pi(t)\right| \leq \sqrt{T_q(\Omega, Q_1.Q_2) T_q(\Pi, R_1, R_2)},$$

where

$$\begin{aligned} T_q(u,v,w) &= (I_q^\alpha w(t) - I_q^\alpha u(t))(I_q^\alpha u(t) - I_q^\alpha v(t)) + \frac{t^\alpha}{\Gamma_q(\alpha+1)} I_q^\alpha v(t)u(t) - I_q^\alpha v(t) I_q^\alpha u(t) \\ &\quad + I_q^\alpha v(t) I_q^\alpha w(t) - \frac{t^\alpha}{\Gamma_q(\alpha+1)} I_q^\alpha v(t)w(t). \end{aligned}$$

14.2. Quantum Grüss-type integral inequalities on finite intervals

Definition 14.3. *[47] Assume* $\Omega : J \to \mathbb{R}$ *is a continuous function and let* $x \in J$*. Then the expression*

$${}_{x_1}D_q\Omega(x) = \frac{\Omega(x) - \Omega(qx + (1-q)x_1)}{(1-q)(x-x_1)}, \; t \neq x_1, \quad {}_{x_1}D_q\Omega(x_1) = \lim_{x \to x_1} {}_{x_1}D_q\Omega(x),$$

is called the q-derivative on J of function Ω *at x.*

Definition 14.4. *[47] Assume* $\Omega : J \to \mathbb{R}$ *is a continuous function. Then the q-integral on J is defined by*

$$\int_{x_1}^x \Omega(t)_{x_1} d_q t = (1-q)(x-x_1) \sum_{n=0}^{\infty} q^n \Omega(q^n x + (1-q^n)x_1)$$

for $x \in J$. Moreover, if $c \in (a, x)$ then the definite q-integral on J is defined by

$$\begin{aligned}\int_c^x \Omega(t)_{x_1} d_q t &= \int_{x_1}^x \Omega(t)_{x_1} d_q t - \int_{x_1}^c \Omega(t)_{x_1} d_q t \\ &= (1-q)(x-x_1)\sum_{n=0}^{\infty} q^n \Omega(q^n x + (1-q^n)x_1) \\ &\quad -(1-q)(c-x_1)\sum_{n=0}^{\infty} q^n \Omega(q^n c + (1-q^n)x_1).\end{aligned}$$

Theorem 14.2. *[47] Assume $\Omega, \Pi : J \to \mathbb{R}$ are continuous functions on J satisfying the vondition (H). Then we have the inequality*

$$\left|\frac{1}{x_2-x_1}\int_{x_1}^{x_2} \Omega(x)\Pi(x)_{x_1} d_q x - \left(\frac{1}{x_2-x_1}\int_{x_1}^{x_2} \Omega(x)_{x_1} d_q x\right)\left(\frac{1}{x_2-x_1}\int_{x_1}^{x_2} \Pi(x)_{x_1} d_q x\right)\right| \\ \leq \frac{1}{4}(\mathfrak{M}-\mathfrak{m})(\mathfrak{P}-\mathfrak{p}).$$

Now, we are going to present the q-Grüss-Čebyšev integral inequality on interval $[x_1, x_2]$.

Theorem 14.3. *[47] Let $\Omega, \Pi : J \to \mathbb{R}$ be L_1, L_2-Lipschitzian continuous functions on $[x_1, x_2]$, so that*

$$|\Omega(u) - \Omega(v)| \leq L_1|u-v|, \qquad |\Pi(u) - \Pi(v)| \leq L_2|u-v|,$$

for all $u, v \in [x_1, x_2]$. Then we have:

$$\left|\frac{1}{x_2-x_1}\int_{x_1}^{x_2} \Omega(x)\Pi(x)_{x_1} d_q x - \left(\frac{1}{x_2-x_1}\int_{x_1}^{x_2} \Omega(x)_{x_1} d_q x\right)\left(\frac{1}{x_2-x_1}\int_{x_1}^{x_2} \Pi(x)_{x_1} d_q x\right)\right| \\ \leq \frac{qL_1L_2}{(1+q+q^2)(1+q)^2}(x_2-x_1)^2.$$

14.3. Quantum symmetric analogue of Grüss inequality

Let $q \in (0, 1)$ and let I be any interval of $\mathbb{R}$ containing 0, and denote by I_q the set

$$I_q = qI = \{qX : X \in I\}; \;\; I_q \subseteq I.$$

Definition 14.5. *[44] Let $\Omega : I \to \mathbb{R}$. The q-symmetric difference operator of Ω is defined by*

$$(\tilde{D}_q\Omega)(t) = \frac{\Omega(qt) - \Omega(q^{-1}t)}{(q-q^{-1})t}; \;\; t \in I_q \setminus \{0\},$$

and

$$(\tilde{D}_q\Omega)(t) = \Omega'(0), \; t = 0.$$

Definition 14.6. *[44] Suppose that $x_1, x_2 \in I$ and $x_1 < x_2$. For $\Omega : I \to \mathbb{R}$ and for $q \in (0, 1)$, the q-symmetric integral of Ω is given by*

$$\int_{x_1}^{x_2} \Omega(t)\tilde{d}_q t = \int_0^{x_2} \Omega(t)\tilde{d}_q t - \int_0^{x_1} \Omega(t)\tilde{d}_q t,$$

where

$$\int_0^x \Omega(t)\tilde{d}_q t = x(1-q^2)\sum_{n=0}^{\infty} q^{2n}\Omega(q^{2n+1}x), \quad x \in I,$$

provided that the series converges at $x = x_1$ and $x = x_2$.

Now, the concepts of q-symmetric derivative and q-symmetric integral are extended on finite intervals. We fix $s \in \mathbb{N} \cup \{0\}$. Let $J_s = [t_s, t_{s+1}] \subset \mathbb{R}$ be an interval containing 0 and $0 < q_s < 1$ be a constant. For a function $\Omega : I_s \to \mathbb{R}$, we define the q_s-symmetric derivative at a point $t \in I_s$ as follows:

Definition 14.7. *[48] Assume that $\Omega : I_s \to \mathbb{R}$ is continuous and $t \in I_s$. The q_s-symmetric derivative of Ω at t is defined as*

$$(D_{q_s}\Omega)(t) = \frac{\Omega(q_s^{-1}t + (1-q_s^{-1})t_s) - \Omega(q_s t + (1-q_s)t_s)}{(q_s^{-1} - q_s)(t - t_s)}; \quad t \neq t_s,$$

$$(D_{q_s}\Omega)(t_s) \lim_{t \to t_s}(D_{q_s}\Omega)(t).$$

Definition 14.8. *[48] Assume that $\Omega : I_s \to \mathbb{R}$ is a continuous function. The q_s-symmetric integral is defined as*

$$\int_{t_s}^{t} \Omega(s) d_{q_s} t = (t - t_s)(1 - q_s^2)\sum_{n=0}^{\infty} q_s^{2n}\Omega(q_s^{2n+1}t + (1 - q_s^{2n+1})t_s).$$

Now, we present q_s-symmetric analogue of Grüss-Chebyshev integral inequality.

Theorem 14.4. *[48] Let Ω and $\Pi : J = [x_1, x_2] \to \mathbb{R}$ be L_1, L_2-Lipschitzian continuous functions on $[x_1, x_2]$ so that*

$$|\Omega(u) - \Omega(v)| \leq L_1|u - v|, \quad |\Pi(u) - \Pi(v)| \leq L_2|u - v|,$$

for all $u, v \in [x_1, x_2]$. Then:

$$\begin{aligned} &\Big|\frac{1}{x_2 - x_1}\int_{x_1}^{x_2} \Omega(x)\Pi(x) d_{q_s}x - \Big(\frac{1}{x_2 - x_1}\int_{x_1}^{x_2} \Omega(x) d_{q_s}x\Big)\Big(\frac{1}{x_2 - x_1}\int_{x_1}^{x_2} \Pi(x) d_{q_s}x\Big)\Big| \\ \leq\ & \frac{L_1 L_2 q_s^4 (x_2 - x_1)^2}{(1 + q_s^2 + q_s^4)(1 + q_s^2)^2}. \end{aligned}$$

14.4. Grüss-type fractional integral inequalities via fractional quantum calculus

The following concepts are adapted by Ref. ([49]). We state a q-shifting operator as

$${}_{x_1}\Phi_q(m) = qm + (1-q)x_1, \quad 0 < q < 1, \quad m, x_1 \in \mathbb{R}.$$

The q-analog is stated by

$$(m; q)_0 = 1, \quad (m; q)_k = \prod_{i=1}^{k-1}(1 - q^i m), \quad k \in \mathbb{N} \cup \{\infty\}.$$

The q number is stated by

$$[m]_q = \frac{1 - q^m}{1 - q}, \quad m \in \mathbb{R}.$$

The q-Gamma function is stated by

$$\Gamma_q(t) = \frac{{}_0(1 - {}_0\Phi_q(1))_q^{(t-1)}}{(1-q)^{t-1)}}, \quad t \in \mathbb{R} \setminus \{0, -1, -2, \ldots\}.$$

Here, we add some definitions regarding fractional q-calculus, namely the Riemann-Liouville fractional q-integral.

Definition 14.9. *[49] Let $\alpha \geq 0$ and function Ω be a continuous stated on $[x_1, x_2]$. Then $({}_{x_1}I_q^0\Omega)(t) = \Omega(t)$ is given by*

$$\begin{aligned}({}_{x_1}I_q^\alpha\Omega)(t) &= \frac{1}{\Gamma_q(\alpha)}\int_{x_1}^t {}_{x_1}(t - {}_{x_1}\Phi_q(s))_q^{\alpha-1}\Omega(s)\ {}_{x_1}d_q s \\ &= \frac{(1-q)(t-x_1)}{\Gamma_q(\alpha)}\sum_{i=0}^{\infty} q^i\ {}_{x_1}(t - {}_{x_1}\Phi_q^{i+1}(t))_q^{\alpha-1}\Omega({}_{x_1}\Phi_q^i(t)).\end{aligned}$$

Now, we present the fractional q-Grüss integral inequality on the interval $[x_1, x_2]$.

Theorem 14.5. *[50] Let $\Omega, \Pi : [x_1, x_2] \to \mathbb{R}$ be continuous functions satisfying (H). For $0 < q < 1$ and $\alpha > 0$, we have the inequality*

$$\begin{aligned}&\Big|\frac{\Gamma_q(\alpha+1)}{(x_2-x_1)^\alpha}({}_{x_1}I_q^\alpha\Omega(s)\Pi(s))(b) - \Big(\frac{\Gamma_q(\alpha+1)}{(x_2-x_1)^\alpha}({}_{x_1}I_q^\alpha\Omega(s))(b)\Big)\Big(\frac{\Gamma_q(\alpha+1)}{(x_2-x_1)^\alpha}({}_{x_1}I_q^\alpha\Pi(s))(b)\Big)\Big| \\ \leq\ & \frac{1}{4}(\mathfrak{M} - \mathfrak{m})(\mathfrak{P} - \mathfrak{p}).\end{aligned}$$

Next, we give the fractional q-Grüss-Čebyšev integral inequality on the interval $[x_1, x_2]$.

Theorem 14.6. *[50] Let $\Omega, \Pi : [x_1, x_2] \to \mathbb{R}$ be L_1-, L_2-Lipschitzian continuous functions, so that*

$$|\Omega(u) - \Omega(v)| \leq L_1|u - v|, \qquad |\Pi(u) - \Pi(v)| \leq L_2|u - v|,$$

for all $u, v \in [x_1, x_2]$, $0 < q < 1$, $L_1, L_2 > 0$, and $\alpha > 0$. Then we have the inequality

$$\begin{aligned}&\Big|\frac{(x_2-x_1)^\alpha}{\Gamma_q(\alpha+1)}({}_{x_1}I_q^\alpha\Omega(s)\Pi(s))(x_2) - ({}_{x_1}I_q^\alpha\Omega(s))(x_2)({}_{x_1}I_q^\alpha\Pi(s))(x_2)\Big| \\ \leq\ & \frac{L_1L_2(x_2-x_1)^{2\alpha+2}}{\Gamma_q(\alpha+2)\Gamma_q(\alpha+3)}\Big((1+q)[\alpha+1]_q - [\alpha+2]_q\Big).\end{aligned}$$

15. Grüss-type inequalities via fractional Hilfer derivative

In this section we give Grüss-type integral inequalities via fractional Hilfer derivative operators.

15.1. Grüss-type integral inequalities via k-Hilfer fractional derivative operator

In this section we present several integral inequalities for the k-Hilfer fractional derivative operator.

Definition 15.1. *[51] Let $\Omega \in L^1[x_1, x_2]$, $\Omega \star K_{(1-\eta)(\eta-\xi)} \in AC^n[x_1, x_2]$, $n-1 < \xi < n$, $0 < \eta \leq 1$, $n \in \mathbb{N}$. Then the following*

$$({}^k D^{\xi,\eta}_{x_1+}\Omega)(x) = I^{\eta(n-\xi)}_{x_1+,k} \frac{d^n}{dx^n}(I^{(1-\eta)(n-\xi)}_{x_1+,k}\Omega(x)),$$

is called the Hiler k-fractional derivative.

Theorem 15.1. *[51] Let $k > 0$ and $(D^{\xi+\eta(n-\xi)}_{x_1+,k}\Pi)$ be a positive function on $[0, \infty)$, and let $({}^k D^{\xi,\eta}_{x_1+}\Omega)$ denote the Hilfer k-fractional derivative of order ξ, $0 < \xi < 1$, and type $0 < \eta \leq 1$. Suppose that:*

There exist $(D^{\xi+\eta(n-\xi)}_{x_1+,k}R_1)$, $(D^{\xi+\eta(n-\xi)}_{x_1+,k}R_2)$ such that

$$(D^{\xi+\eta(n-\xi)}_{x_1+,k}R_1)(\xi) \leq (D^{\xi+\eta(n-\xi)}_{x_1+,k}\Pi)(\xi) \leq (D^{\xi+\eta(n-\xi)}_{x_1+,k}R_2)(\xi),$$

for all $\xi \in [0, \infty)$.

Then

$$\begin{aligned} &({}^k D^{\xi,\eta}_{x_1+}R_1)(\xi)({}^k D^{\xi,\eta}_{x_1+}\Pi)(\xi) + ({}^k D^{\xi,\eta}_{x_1+}R_2)(\xi)({}^k D^{\xi,\eta}_{x_1+}\Pi)(\xi) \\ \geq\ &({}^k D^{\xi,\eta}_{x_1+}R_1)(\xi)({}^k D^{\xi,\eta}_{x_1+}R_2)(\xi) + ({}^k D^{\xi,\eta}_{x_1+}\Pi)(\xi)({}^k D^{\xi,\eta}_{x_1+}\Pi)(\xi). \end{aligned}$$

15.2. Grüss-type inequalities via generalized k-fractional Hilfer-Katugampola derivative

In this section, we present inequalities of the Grüss-type via k-fractional Hilfer-Katugampola generalized derivative.

Definition 15.2. *[52] Let $n - 1 < \alpha \leq n$, $0 \leq \beta \leq 1$, $n \in \mathbb{N}$, $\rho > 0, k > 0$ and $\Omega \in M_q[x_1, x_2] = \Big\{\Omega : \|\Omega_q\| = \Big(\int_{x_1}^{x_2} |\Omega(t)|^q dt\Big)^{1/q} < \infty\Big\}$, $1 < q < \infty$. The generalized k-fractional Hilfer-Katugampola derivatives (left-sided and right-sided) are defined as*

$$\mathcal{D}^{\alpha,\gamma,\rho}_{x_1,k}\Omega(t) = \frac{\rho^{1-\frac{\gamma-\alpha}{k}}}{k\Gamma_k(\gamma-\alpha)}\int_{x_1}^{t}(t^\rho - y^\rho)^{\frac{\gamma-\alpha}{k}-1}y^{\rho-1}\Omega^{(\gamma)}(y)dy, \quad t > x_1,$$

$$\mathcal{D}^{\alpha,\gamma,\rho}_{x_2,k}\Omega(t) = \frac{\rho^{1-\frac{\gamma-\alpha}{k}}}{k\Gamma_k(\gamma-\alpha)}\int_{t}^{x_2}(y^\rho - t^\rho)^{\frac{\gamma-\alpha}{k}-1}y^{\rho-1}\Omega^{(\gamma)}(y)dy, \quad t < x_2,$$

where $\gamma = \alpha + \beta(kn - \alpha)$, $\alpha > 0$.

Theorem 15.2. *[52] Let $\rho, \delta, \alpha, \gamma, k, a > 0$ and $\Omega \in M_q[x_1, x_2]$ be positive integrable function on $[x_1, x_2]$ satisfying (H_1). Then we have:*

$$\begin{aligned} &\mathcal{D}^{\delta,\gamma,\rho}_{x_1,k}\Omega(t)\,\mathcal{D}^{\alpha,\gamma,\rho}_{x_1,k}Q_2(t) + \mathcal{D}^{\delta,\gamma,\rho}_{x_1,k}Q_1(t)\,\mathcal{D}^{\alpha,\gamma,\rho}_{x_1,k}\Omega(t) \\ \geq\ &\mathcal{D}^{\delta,\gamma,\rho}_{x_1,k}Q_1(t)\,\mathcal{D}^{\alpha,\gamma,\rho}_{x_1,k}Q_2(t) + \mathcal{D}^{\delta,\gamma,\rho}_{x_1,k}\Omega(t)\,\mathcal{D}^{\alpha,\gamma,\rho}_{x_1,k}\Omega(t). \end{aligned}$$

Theorem 15.3. *[52] Let $\rho, \delta, \alpha, \gamma, k, a > 0$ and $\Omega, \Pi \in M_q[x_1, x_2]$ be positive integrable functions on $[x_1, x_2]$ satisfying (H_1) and (H_2). Then we have:*

$$(a) \qquad \mathcal{D}^{\delta,\gamma,\rho}_{x_1,k}R_1(t)\,\mathcal{D}^{\alpha,\gamma,\rho}_{x_1,k}\Omega(t) + \mathcal{D}^{\delta,\gamma,\rho}_{x_1,k}\Pi(t)\,\mathcal{D}^{\alpha,\gamma,\rho}_{x_1,k}Q_2(t)$$

$$\geq \mathcal{D}_{x_1,k}^{\delta,\gamma,\rho} R_1(t)\, \mathcal{D}_{x_1,k}^{\alpha,\gamma,\rho} Q_2(t) + \mathcal{D}_{x_1,k}^{\delta,\gamma,\rho} \Pi(t)\, \mathcal{D}_{x_1,k}^{\alpha,\gamma,\rho} \Omega(t).$$

$$(b) \quad \mathcal{D}_{x_1,k}^{\delta,\gamma,\rho} Q_1(t)\, \mathcal{D}_{x_1,k}^{\alpha,\gamma,\rho} \Pi(t) + \mathcal{D}_{x_1,k}^{\delta,\gamma,\rho} \Omega(t)\, \mathcal{D}_{x_1,k}^{\alpha,\gamma,\rho} R_2(t) \geq \mathcal{D}_{x_1,k}^{\delta,\gamma,\rho} Q_1(t)\, \mathcal{D}_{x_1,k}^{\alpha,\gamma,\rho} R_2(t) + \mathcal{D}_{x_1,k}^{\delta,\gamma,\rho} \Pi(t)\, \mathcal{D}_{x_1,k}^{\alpha,\gamma,\rho} \Omega(t).$$

$$(c) \quad \mathcal{D}_{x_1,k}^{\delta,\gamma,\rho} R_2(t)\, \mathcal{D}_{x_1,k}^{\alpha,\gamma,\rho} Q_2(t) + \mathcal{D}_{x_1,k}^{\delta,\gamma,\rho} \Pi(t)\, \mathcal{D}_{x_1,k}^{\alpha,\gamma,\rho} \Omega(t) \geq \mathcal{D}_{x_1,k}^{\delta,\gamma,\rho} \Pi(t)\, \mathcal{D}_{x_1,k}^{\alpha,\gamma,\rho} Q_2(t) + \mathcal{D}_{x_1,k}^{\delta,\gamma,\rho} R_2(t)\, \mathcal{D}_{x_1,k}^{\alpha,\gamma,\rho} \Omega(t).$$

$$(d) \quad \mathcal{D}_{x_1,k}^{\delta,\gamma,\rho} R_1(t)\, \mathcal{D}_{x_1,k}^{\alpha,\gamma,\rho} Q_1(t) + \mathcal{D}_{x_1,k}^{\delta,\gamma,\rho} \Pi(t)\, \mathcal{D}_{x_1,k}^{\alpha,\gamma,\rho} \Omega(t) \geq \mathcal{D}_{x_1,k}^{\delta,\gamma,\rho} \Pi(t)\, \mathcal{D}_{x_1,k}^{\alpha,\gamma,\rho} Q_1(t) + \mathcal{D}_{x_1,k}^{\delta,\gamma,\rho} R_1(t)\, \mathcal{D}_{x_1,k}^{\alpha,\gamma,\rho} \Omega(t).$$

Theorem 15.4. *[52] Let $\rho, \delta, \alpha, \gamma, k, a > 0$ and $\Omega, \Pi \in M_q[x_1, x_2]$ be positive integrable functions on $[x_1, x_2]$ satisfying (H_1) and (H_2). If $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then we have:*

$$(a) \quad \frac{1}{p}\mathcal{D}_{x_1,k}^{\delta,\gamma,\rho}(\Pi(t))^p\, \mathcal{D}_{x_1,k}^{\alpha,\gamma,\rho}(\Omega(t))^p + \frac{1}{q}\mathcal{D}_{x_1,k}^{\delta,\gamma,\rho}(\Omega(t))^q\, \mathcal{D}_{x_1,k}^{\alpha,\gamma,\rho}(\Pi(t))^q \geq \mathcal{D}_{x_1,k}^{\delta,\gamma,\rho}\Omega(t)\Pi(t)\, \mathcal{D}_{x_1,k}^{\alpha,\gamma,\rho}\Omega(t)\Pi(t).$$

$$(b) \quad \frac{1}{p}\mathcal{D}_{x_1,k}^{\delta,\gamma,\rho}(\Pi(t))^p\, \mathcal{D}_{x_1,k}^{\alpha,\gamma,\rho}(\Omega(t))^p + \frac{1}{q}\mathcal{D}_{x_1,k}^{\delta,\gamma,\rho}(\Omega(t))^q\, \mathcal{D}_{x_1,k}^{\alpha,\gamma,\rho}(\Pi(t))^q \geq \mathcal{D}_{x_1,k}^{\delta,\gamma,\rho}\Omega^{p-1}(t)\Pi^{q-1}(t)\, \mathcal{D}_{x_1,k}^{\alpha,\gamma,\rho}\Omega(t)\Pi(t).$$

$$(c) \quad \frac{1}{p}\mathcal{D}_{x_1,k}^{\delta,\gamma,\rho}(\Pi(t))^p\, \mathcal{D}_{x_1,k}^{\alpha,\gamma,\rho}(\Omega(t))^2 + \frac{1}{q}\mathcal{D}_{x_1,k}^{\delta,\gamma,\rho}(\Omega(t))^q\, \mathcal{D}_{x_1,k}^{\alpha,\gamma,\rho}(\Pi(t))^2 \geq \mathcal{D}_{x_1,k}^{\delta,\gamma,\rho}\Omega^{2/p}(t)\Pi^{2/q}(t)\, \mathcal{D}_{x_1,k}^{\alpha,\gamma,\rho}\Omega(t)\Pi(t).$$

$$(d) \quad \frac{1}{p}\mathcal{D}_{x_1,k}^{\delta,\gamma,\rho}(\Pi(t))^q\, \mathcal{D}_{x_1,k}^{\alpha,\gamma,\rho}(\Omega(t))^2 + \frac{1}{q}\mathcal{D}_{x_1,k}^{\delta,\gamma,\rho}(\Omega(t))^p\, \mathcal{D}_{x_1,k}^{\alpha,\gamma,\rho}(\Pi(t))^2 \geq \mathcal{D}_{x_1,k}^{\delta,\gamma,\rho}\Omega^{p-1}(t)\Pi^{q-1}(t)\, \mathcal{D}_{x_1,k}^{\alpha,\gamma,\rho}\Omega^{2/p}(t)\Pi^{2/q}(t).$$

Theorem 15.5. *[52] Let $\rho, \delta, \alpha, \gamma, k, a > 0$ and $\Omega, \Pi \in M_q[x_1, x_2]$ be positive integrable functions on $[x_1, x_2]$. Let*

$$\mu = \min_{0 \leq y \leq t} \frac{\Omega(y)}{\Pi(y)}, \quad \mathcal{M} = \max_{0 \leq y \leq t} \frac{\Omega(y)}{\Pi(y)}.$$

Then we have:

$$(i) \quad \frac{(\mu + \mathcal{M})^2}{4\mu\mathcal{M}}\mathcal{D}_{x_1,k}^{\alpha,\gamma,\rho}[\Omega(t)\Pi(t)]^2 \geq \mathcal{D}_{x_1,k}^{\alpha,\gamma,\rho}[\Pi(t)]^2\, \mathcal{D}_{x_1,k}^{\alpha,\gamma,\rho}[\Omega(t)]^2.$$

$$(ii) \quad \frac{\sqrt{\mu} - \sqrt{\mathcal{M}}}{2\sqrt{\mu\mathcal{M}}}\mathcal{D}_{x_1,k}^{\alpha,\gamma,\rho}[\Omega(t)\Pi(t)] \geq \sqrt{\mathcal{D}_{x_1,k}^{\alpha,\gamma,\rho}[\Pi(t)]^2\, \mathcal{D}_{x_1,k}^{\alpha,\gamma,\rho}[\Omega(t)]^2} - \mathcal{D}_{x_1,k}^{\alpha,\gamma,\rho}[\Omega(t)\Pi(t)] \geq 0.$$

$$(iii) \quad \frac{\mu - \mathcal{M}}{4\mu\mathcal{M}}\mathcal{D}_{x_1,k}^{\alpha,\gamma,\rho}[\Omega(t)\Pi(t)]^2 \geq \mathcal{D}_{x_1,k}^{\alpha,\gamma,\rho}[\Pi(t)]^2\, \mathcal{D}_{x_1,k}^{\alpha,\gamma,\rho}[\Omega(t)]^2 - [\mathcal{D}_{x_1,k}^{\alpha,\gamma,\rho}[\Omega(t)\Pi(t)]]^2 \geq 0.$$

16. Conclusions

Our objective in this paper was to present a comprehensive and up-to-date review on Grüss-type inequalities for fractional differential operators. We presented results including inequalities of the Grüss-type for different kinds of fractional integral and differential operators. Grüss-type inequalities for fractional integrals of Riemann-Liouville, Katugampola, Hadamard's, Raina's, tempered, conformable, proportional, Caputo-Fabrizio, Saigo's are included. Moreover Grüss-type inequalities concerning Hilfer fractional differential operators and quantum Grüss-type integral inequalities are also presented. We believe that the present survey will provide a platform for the researchers working on Grüss-type inequalities to learn about the available work on the topic before developing the new results. Future research regarding this review paper is fascinating. Our review paper might inspire a good number of additional studies.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they do not have conflict of interest regarding this manuscript.

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