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Research article

$N(\kappa)$-paracontact metric manifolds admitting the Fischer-Marsden conjecture

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Abstract: We characterize $N(\kappa)$-paracontact metric manifolds (NKPMM) M^{2n+1} satisfying the Fischer-Marsden conjecture. We demostrate that, if an M^{2n+1} satisfies the Fischer-Marsden equation, then either M^{2n+1} with $\kappa > -1$ is a non-Einstein manifold or M^{2n+1} is locally isometric to $\mathbb{E}^{n+1} \times \mathbb{H}^n(-4)$ for $n > 1$. For the 3-dimensional case, we show that M^3 is an Einstein manifold.

Keywords: $N(\kappa)$-paracontact metric manifolds; Fischer-Marsden conjecture; Einstein manifold
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1. Introduction

Paracontact geometry equipped with the nullity distribution contributes a crucial part in the development of modern paracontact geometry. The pioneering work of Kaneyuki and Williams [1] opened the door to the study of paracontact geometry for researchers. The para-Kähler manifolds with its applications in pseudo-Riemannian geometry and mathematical physics have motivated researchers to concentrate on paracontact geometry. In [2], Zamkovoy presented a systematic research of paracontact metric manifolds (PMMs). The geometrical and physical properties of PMMs have been studied by many researchers. Calvaruso et al. [3] have investigated paracontact metric structures on the unit tangent bundle. The properties of bi-paracontact structure and Legendre foliations have been explored in [4]. Blaga [5] has studied the properties of Lorentzian para-Sasakian manifolds endowed with η-Ricci solitons. Three-dimensional paracontact metric manifolds have studied in [6–9]. Cappelletti-Montano et al. [10] introduced the notion of paracontact (κ, μ)-spaces and obtained their various properties, where κ and μ are real constants. After that, the properties of paracontact metric

(κ, μ)-spaces have been studied in [11–14]. The classification of paracontact metric (κ, μ)-spaces with non-trivial examples were given in [15, 16].

Let M^{2n+1} be a $(2n + 1)$-dimensional PMM, and g is a pseudo-Riemannian metric of M^{2n+1}. If the set of all pseudo-Riemannian metrics of unit volume on M^{2n+1} is represented by $\mathcal{G}$, then we have

$$\mathcal{L}_g(g^*) + g(g^*, S_g) + \triangle_g(tr_g g^*) = div(div(g^*)),$$

where g^* represents the $(0, 2)$-type symmetric bilinear tensor, $\triangle_g$ is the negative Laplacian of the pseudo-Riemannian metric g, S_g is the Ricci tensor corresponding to g and $'div'$ and $'tr'$ are used for divergence and trace, respectively. Here, $\mathcal{L}_g$ is the linearized scalar curvature operator. Let $\mathcal{L}_g^*$ represent the formal L^2-adjoint of $\mathcal{L}_g$. Then, the aforementioned equation assumes the form

$$\mathcal{L}_g^*(\lambda) = Hess_g\lambda - (\triangle_g\lambda)g - \lambda S_g, \tag{1.1}$$

where $Hess_g\lambda$ is the Hessian of the smooth function λ corresponding to g and is defined by the relation $Hess_g\lambda(\mathfrak{U}_1, \mathfrak{U}_2) = g(\nabla_{\mathfrak{U}_1} D\lambda, \mathfrak{U}_2)$, $\forall\ \mathfrak{U}_1, \mathfrak{U}_2 \in \mathfrak{X}(M^{2n+1})$. Here, D denotes the gradient operator and $\mathfrak{X}(M^{2n+1})$ the collection of all smooth vector fields of M^{2n+1}. In the present paper, we represent $\mathcal{L}_g^*(\lambda) = 0$ as the Fischer-Marsden equation (briefly, by FME). The doublet (g, λ) satisfying the equation $\mathcal{L}_g^*(\lambda) = 0$, for $\lambda \neq 0$, is known as the non-trivial solution of the FME. Bourguignon [17], Fischer and Marsden [18] considered a Riemannian manifold satisfying the FME and proved that the scalar curvature of the manifold is constant. In 2000, Corvino [19] showed that a complete Riemannian manifold with the warped product metric $g^* = g - \lambda^2 dt^2$ is Einstein if and only if the doublet (g, λ) is a non-trivial solution of the FME. Recently, Al-Dayal et al. [20] have considered a semi-Riemannian manifold satisfying the Fischer-Marsden equation and proved that the manifold under consideration is a quasi-Einstein manifold. In 1974, Fischer and Marsden [18] conjectured that, if a compact Riemannian n-manifold concedes (g, λ) with $\lambda \neq 0$, we attain an Einstein manifold. The first counter example of the Fischer-Marsden conjecture (in short, FMC) was given by Kobayashi [21]. In [22], Cernea and Guan established that a closed homogeneous Riemannian manifold (M, g) satisfying the equation $\mathcal{L}_g^*(\lambda) = 0$ is locally isometric to $E \times S^m$, where E and S^m denote the Einstein manifold and the Euclidean sphere, respectively. Recently, Patra and Ghosh [23] proved that if a K-contact (or a (κ, μ)-contact) metric manifold has $\mathcal{L}_g^*(\lambda) = 0$, the manifold is Einstein (or locally isometric to the sphere S^{2n+1}). Prakasha et al. [24] considered a $(2n+1)$-dimensional $(\kappa, \mu)'$-almost Kenmotsu manifold (M, g) admits $(g, \lambda \neq 0)$ and proved that (M, g) is locally isometric to $\mathbb{H}^{n+1}(\alpha) \times_f \mathbb{R}^n$ or $\mathbb{B}^{n+1}(\alpha') \times_{f'} \mathbb{R}^n$. In [25–27], Chaubey et al. studied the properties of Kenmotsu manifolds, generalized Sasakian-space-forms and cosymplectic manifolds satisfying the Fischer-Marsden conjecture. Deshmukh et al. [28] explored the Fischer-Marsden conjecture in Riemannian manifolds. Very recently, Suh et al. [29–31] and Venkatesha et al. [32] have explored the properties of FMC on the hypersurfaces of space-forms.

The above studies inspire us to characterize $N(\kappa)$-paracontact metric manifolds (in brief, NKPMMs) satisfying the FME, that is, $\mathcal{L}_g^*(\lambda) = 0$. Following an overview in Section 1, in Section 2 we gather the basic known results and definitions of PMMs. Section 3 deals with the study of three-dimensional NKPMMs satisfying the Fischer-Marsden equation and prove that the manifold under consideration is Einstein. In Section 4, we characterize an NKPMM satisfying the equation $\mathcal{L}_g^*(\lambda) = 0$ for $n > 1$. It is proved that either there does not exist an Einstein NKPMM with $\kappa > -1$, or the manifold M^{2n+1} is locally isometric to the product of a hyperbolic space $\mathbb{H}^n(-4)$ and a Euclidean space $\mathbb{E}^{n+1}$.

2. Paracontact metric manifolds

Let M^{2n+1} be a $(2n+1)$-dimensional differentiable manifold of class C^∞. Then, a triplet (ϕ, ξ, η) defined on M^{2n+1} and satisfying the relations

$$\phi^2(\mathfrak{U}_1) + \eta(\mathfrak{U}_1)\xi = \mathfrak{U}_1, \quad \eta(\xi) = 1, \quad \forall\ \mathfrak{U}_1 \in \mathfrak{X}(M^{2n+1}), \tag{2.1}$$

where ϕ is a tensor field of type $(1,1)$, η a tensor field of type $(0,1)$ and the Reeb vector field ξ, is known as an almost paracontact structure on M^{2n+1}. The manifold M^{2n+1} equipped with the structure (ϕ, ξ, η) is called an almost paracontact manifold. It is noticed that the structure tensor ϕ induces an almost paracomplex structure J on the horizontal distribution $\mathcal{D} = ker(\eta)$, that is, the eigensubbundles $\mathcal{D}^+$ and $\mathcal{D}^-$ have equal dimension n corresponding to the eigenvalues $+1$ and -1 of J, respectively. If M^{2n+1} admits a pseudo-Riemannian metric g of type $(0,2)$ such that the relations

$$g(\mathfrak{U}_1, \xi) = \eta(\mathfrak{U}_1), \quad g(\phi\mathfrak{U}_1, \phi\mathfrak{U}_2) + g(\mathfrak{U}_1, \mathfrak{U}_2) = \eta(\mathfrak{U}_1)\eta(\mathfrak{U}_2), \tag{2.2}$$

hold for all $\mathfrak{U}_1,\ \mathfrak{U}_2\ \in \mathfrak{X}(M^{2n+1})$, then (M^{2n+1}, g) is known as an almost PMM. From (2.1) and (2.2), it follows that the following relations

$$rank\,\phi = 2n, \quad \phi\xi = 0, \quad \eta \circ \phi = 0, \quad g(\phi\mathfrak{U}_1, \mathfrak{U}_2) + g(\mathfrak{U}_1, \phi\mathfrak{U}_2) = 0, \tag{2.3}$$

hold for all $\mathfrak{U}_1,\ \mathfrak{U}_2\ \in \mathfrak{X}(M^{2n+1})$. An almost PMM M^{2n+1} with $g(\mathfrak{U}_1, \phi\mathfrak{U}_2) = d\eta(\mathfrak{U}_1, \mathfrak{U}_2)$ becomes a PMM. Here d represents the exterior derivative operator.

An almost paracontact metric structure with $[\phi, \phi] - 2d\eta \otimes \xi = 0$ is said to be normal, where $[\phi, \phi]$ represents the Nijenhuis tensor corresponding to the structure tensor ϕ. In [2], Zamkovoy proved that an almost PMM M^{2n+1} possesses at least a (locally) ϕ-basis, that is, the set $\{E_1, E_2, E_3, ..., E_n, \phi E_1, \phi E_2, \phi E_3, ..., \phi E_n, \xi\}$ represents a (locally) pseudo-orthonormal basis of the vector fields, where $E_1, E_2, E_3, ..., E_n, \xi$ and $\phi E_1, \phi E_2, \phi E_3, ..., \phi E_n$ are space-like and time-like vector fields, respectively. In M^{2n+1}, the ϕ-basis is determined by a (locally) pseudo-orthonormal basis of $ker(\eta)$. If possible, we suppose that e_3 is time-like and $\{e_2, e_3\}$ a pseudo-orthonormal basis of $ker(\eta)$. Then, from Eq (2.2) we conclude that $\phi e_2 \in ker(\eta)$ is time-like and orthonormal to e_2. Thus $\phi e_2 = \pm e_3$ and hence we consider $\{e_2, \pm e_3, \xi\}$ to be a ϕ-basis on M^3. For more details, we refer to [7]. On a PMM M^{2n+1}, a symmetric and trace-free $(1,1)$-type tensor h, defined by $h = \frac{1}{2}\mathcal{L}_\xi\phi$, satisfies

$$h\xi = 0, \quad \phi h + h\phi = 0, \quad trh = 0, \quad trh\phi = 0, \tag{2.4}$$

$$\nabla_{\mathfrak{U}_1}\xi = -\phi\mathfrak{U}_1 + \phi h\,\mathfrak{U}_1, \tag{2.5}$$

for all $\mathfrak{U}_1 \in \mathfrak{X}(M^{2n+1})$, where ∇ and trh denote the Levi-Civita connection and the trace of the operator h, respectively. An almost paracontact metric structure is said to be a K-paracontact structure if ξ is Killing, that is, $h = 0$. An almost PMM is said to be a para-Sasakian manifold if and only if

$$(\nabla_{\mathfrak{U}_1}\phi)(\mathfrak{U}_2) = -g(\mathfrak{U}_1, \mathfrak{U}_2)\xi + \eta(\mathfrak{U}_2)\mathfrak{U}_1,$$

for all $\mathfrak{U}_1,\ \mathfrak{U}_2\ \in \mathfrak{X}(M^{2n+1})$. A normal PMM is para-Sasakian and satisfies

$$R(\mathfrak{U}_1, \mathfrak{U}_2)\xi = \eta(\mathfrak{U}_1)\mathfrak{U}_2 - \eta(\mathfrak{U}_2)\mathfrak{U}_1, \quad \forall\ \ \mathfrak{U}_1,\ \mathfrak{U}_2\ \in \mathfrak{X}(M^{2n+1}).$$

The converse is not true. Here R denotes the curvature tensor corresponding to ∇.

Next, we consider that the Reeb vector field ξ of a $(2n+1)$-dimensional PMM M^{2n+1} belongs to the (κ, μ)-nullity distribution.

Definition 2.1. *A PMM M^{2n+1} is said to be a paracontact (κ, μ)-manifold if*

$$R(\mathfrak{U}_1, \mathfrak{U}_2)\xi = \kappa\{\eta(\mathfrak{U}_2)\mathfrak{U}_1 - \eta(\mathfrak{U}_1)\mathfrak{U}_2\} + \mu\{\eta(\mathfrak{U}_2)h\mathfrak{U}_1 - \eta(\mathfrak{U}_1)h\mathfrak{U}_2\},$$

for all $\mathfrak{U}_1$, $\mathfrak{U}_2 \in \mathfrak{X}(M^{2n+1})$, where κ and μ are real constants [12].

As a particular case, the paracontact metric (κ, μ)-manifold with $\mu = 0$ reduces to an NKPMM. Hence, the above equation becomes

$$R(\mathfrak{U}_1, \mathfrak{U}_2)\xi = \kappa\{\eta(\mathfrak{U}_2)\mathfrak{U}_1 - \eta(\mathfrak{U}_1)\mathfrak{U}_2\}. \tag{2.6}$$

In light of Eqs (2.2), (2.5) and (2.6), we have

$$S(\mathfrak{U}_1, \xi) = 2n\kappa\eta(\mathfrak{U}_1), \tag{2.7}$$

$$R(\xi, \mathfrak{U}_1)\mathfrak{U}_2 = \kappa\{g(\mathfrak{U}_1, \mathfrak{U}_2)\xi - \eta(\mathfrak{U}_2)\mathfrak{U}_1\}, \tag{2.8}$$

$$(\nabla_{\mathfrak{U}_1}\eta)(\mathfrak{U}_2) = g(\phi\mathfrak{U}_2, \mathfrak{U}_1) - g(\phi h\mathfrak{U}_2, \mathfrak{U}_1),$$

where S denotes the Ricci tensor of M^{2n+1}. For $dim\, M = 3$, the NKPMM M^3 satisfies the following relations

$$Q\mathfrak{U}_1 = \left(\frac{r}{2} - \kappa\right)\mathfrak{U}_1 + \left(3\kappa - \frac{r}{2}\right)\eta(\mathfrak{U}_1)\xi, \tag{2.9}$$

$$S(\mathfrak{U}_1, \xi) = 2\kappa\eta(\mathfrak{U}_1), \tag{2.10}$$

for each $\mathfrak{U}_1$, $\mathfrak{U}_2 \in \mathfrak{X}(M^{2n+1})$, where the Ricci operator associated with the Ricci tensor S is Q, that is, $S(\cdot, \cdot) = g(Q\cdot, \cdot)$ and r denotes the scalar curvature of M^{2n+1} [9].

3. Three-dimensional NKPMM satisfying the Fischer-Marsden conjecture

This section deals with the study of the Fischer-Marsden conjecture within the framework of a three-dimensional NKPMM. In this section, we represent M^3 as a three-dimensional NKPMM. We recall the following results.

Lemma 3.1. *A three-dimensional paracontact metric (κ, μ)-manifold is Einsteinian if and only if $\kappa = \mu = 0$ (see Corollary 4.14, [10]).*

De et al. [33] showed that the following results hold on a three-dimensional NKPMM.

Lemma 3.2. *i) If and only if the manifold is an Einstein manifold, an M^3 is Ricci semisymmetric.*
ii) If and only if the manifold has constant curvature κ, an M^3 is Ricci semisymmetric.
iii) An M^3 is Riccisymmetric if and only if the manifold is of constant curvature κ.

Before proving our main results, we prove the following propositions.

Proposition 3.1. *If a PMM M^{2n+1} satisfies the FMC, then we have*

$$R(\mathfrak{U}_1, \mathfrak{U}_2)D\lambda = (\mathfrak{U}_1\lambda)Q\mathfrak{U}_2 - (\mathfrak{U}_2\lambda)Q\mathfrak{U}_1 + \lambda\{(\nabla_{\mathfrak{U}_1}Q)(\mathfrak{U}_2) - (\nabla_{\mathfrak{U}_2}Q)(\mathfrak{U}_1)\} + (\mathfrak{U}_1 f)\mathfrak{U}_2 - (\mathfrak{U}_2 f)\mathfrak{U}_1, \tag{3.1}$$

for all vector fields $\mathfrak{U}_1$ and $\mathfrak{U}_2$ of M^{2n+1}, where $f = -\frac{r\lambda}{2n}$.

Proof. Assume that there is a non-trivial solution (g, λ) to the equation $\mathcal{L}_g^*(\lambda) = 0$. Then, from Eq (1.1), we have

$$-(\triangle_g \lambda)g + Hess_g \lambda - \lambda S_g = 0,$$

where $\triangle_g \lambda = -\frac{r\lambda}{2n}$. Thus, the FME can be written as

$$\nabla_{\mathfrak{U}_1} D\lambda = \lambda Q\mathfrak{U}_1 + f\mathfrak{U}_1, \ \ \mathfrak{U}_1 \in \mathfrak{X}(M^{2n+1}), \quad f = -\frac{r\lambda}{2n}. \tag{3.2}$$

Equation (3.2)'s covariant derivative along the vector field $\mathfrak{U}_2$ results in

$$\nabla_{\mathfrak{U}_2}\nabla_{\mathfrak{U}_1} D\lambda = (\mathfrak{U}_2\lambda)Q\mathfrak{U}_1 + \lambda\{(\nabla_{\mathfrak{U}_2}Q)(\mathfrak{U}_1) + Q(\nabla_{\mathfrak{U}_2}\mathfrak{U}_1)\} + (\mathfrak{U}_2 f)\mathfrak{U}_1 + f\nabla_{\mathfrak{U}_2}\mathfrak{U}_1. \tag{3.3}$$

Interchanging $\mathfrak{U}_1$ and $\mathfrak{U}_2$ in (3.3) and using the obtained equation, (3.2) and (3.3) in $R(\mathfrak{U}_1, \mathfrak{U}_2)D\lambda = [\nabla_{\mathfrak{U}_1}, \nabla_{\mathfrak{U}_2}]D\lambda - \nabla_{[\mathfrak{U}_1,\mathfrak{U}_2]}D\lambda$, we immediately get the required result. □

Proposition 3.2. *On M^3, we have*

$$(\nabla_\xi Q)(\mathfrak{U}_2) - (\nabla_{\mathfrak{U}_2}Q)(\xi) = \frac{dr(\xi)}{2}(\mathfrak{U}_2 - \eta(\mathfrak{U}_2)\xi) - \frac{r-6\kappa}{2}(\phi\mathfrak{U}_2 - \phi h\mathfrak{U}_2). \tag{3.4}$$

Proof. From Eq (2.10), we have $Q\xi = 2\kappa\xi$, where κ is a real constant. Taking the covariant derivative of this equation along the vector field $\mathfrak{U}_2$, and using Eqs (2.3), (2.5) and (2.9), we obtain

$$(\nabla_{\mathfrak{U}_2}Q)(\xi) = \frac{r-6\kappa}{2}(\phi\mathfrak{U}_2 - \phi h\mathfrak{U}_2). \tag{3.5}$$

Differentiating Eq (2.9) once more along the Reeb vector field ξ, we have

$$\begin{aligned}(\nabla_\xi Q)(\mathfrak{U}_2) + Q(\nabla_\xi\mathfrak{U}_2) \ &= \ \frac{dr(\xi)}{2}(\mathfrak{U}_2 - \eta(\mathfrak{U}_2)\xi) + \left(\frac{r}{2} - \kappa\right)\nabla_\xi\mathfrak{U}_2 \\ &\quad + \left(3\kappa - \frac{r}{2}\right)\{(\nabla_\xi\eta)(\mathfrak{U}_2) + \eta(\nabla_\xi\mathfrak{U}_2)\xi + \eta(\mathfrak{U}_2)\nabla_\xi\xi\}.\end{aligned}$$

In light of Eqs (2.2)–(2.5), (2.9) and (2.10), the above equation becomes

$$(\nabla_\xi Q)(\mathfrak{U}_2) = \frac{dr(\xi)}{2}(\mathfrak{U}_2 - \eta(\mathfrak{U}_2)\xi). \tag{3.6}$$

Thus, in view of (3.5) and (3.6), we get the statement of Proposition 3.2. □

From Eq (3.5), we have

$$(div\, Q)(\xi) = 0 \quad \Longrightarrow \quad dr(\xi) = 0, \tag{3.7}$$

where $div\, Q$ denotes the divergence of Ricci operator Q. This equation shows that the scalar curvature r is locally constant along the vector field ξ.

Now, we are going to prove the main result of this section. Changing $\mathfrak{U}_1$ by ξ in (3.1), we obtain

$$R(\xi, \mathfrak{U}_2)D\lambda = (\xi\lambda)Q\mathfrak{U}_2 - (\mathfrak{U}_2\lambda)Q\xi + \lambda\{(\nabla_\xi Q)(\mathfrak{U}_2) - (\nabla_{\mathfrak{U}_2}Q)(\xi)\} + (\xi f)\mathfrak{U}_2 - (\mathfrak{U}_2 f)\xi. \tag{3.8}$$

Taking the inner product of (3.8) with $\mathfrak{U}_1$ and then calling Eqs (2.2), (2.3), (2.10) and (3.4), we obtain

$$\begin{aligned}g(R(\xi, \mathfrak{U}_2)D\lambda, \mathfrak{U}_1) \ &= \ (\xi\lambda)S(\mathfrak{U}_2, \mathfrak{U}_1) - 2\kappa(\mathfrak{U}_2\lambda)\eta(\mathfrak{U}_1) - (\mathfrak{U}_2 f)\eta(\mathfrak{U}_1) \\ &\quad + \lambda\{\frac{dr(\xi)}{2}[g(\mathfrak{U}_1, \mathfrak{U}_2) - \eta(\mathfrak{U}_1)\eta(\mathfrak{U}_2)] - \frac{r-6\kappa}{2}[g(\phi\mathfrak{U}_2, \mathfrak{U}_1) \\ &\quad - g(\phi h\mathfrak{U}_2, \mathfrak{U}_1)]\} + (\xi f)g(\mathfrak{U}_2, \mathfrak{U}_1).\end{aligned} \tag{3.9}$$

In view of Eqs (2.1) and (2.8), we get

$$g(R(\xi, \mathfrak{U}_2)D\lambda, \mathfrak{U}_1) = \kappa\{(\mathfrak{U}_2\lambda)\eta(\mathfrak{U}_1) - (\xi\lambda)g(\mathfrak{U}_2, \mathfrak{U}_1)\}. \tag{3.10}$$

This result also holds well for the $(2n+1)$-dimensional NKPMM. Eq (3.9) along with Eq (3.10) gives

$$\begin{aligned}&(\xi\lambda)S(\mathfrak{U}_2, \mathfrak{U}_1) - 2\kappa(\mathfrak{U}_2\lambda)\eta(\mathfrak{U}_1) + \lambda\{\frac{dr(\xi)}{2}[g(\mathfrak{U}_1, \mathfrak{U}_2) - \eta(\mathfrak{U}_1)\eta(\mathfrak{U}_2)]\\ &-\frac{r-6\kappa}{2}[g(\phi\mathfrak{U}_2, \mathfrak{U}_1) - g(\phi h\mathfrak{U}_2, \mathfrak{U}_1)]\} - (\mathfrak{U}_2 f)\eta(\mathfrak{U}_1)\\ &+(\xi f)g(\mathfrak{U}_2, \mathfrak{U}_1) - \kappa\{(\mathfrak{U}_2\lambda)\eta(\mathfrak{U}_1) - (\xi\lambda)g(\mathfrak{U}_2, \mathfrak{U}_1)\} = 0.\end{aligned} \tag{3.11}$$

Let $\{e_i,\ i = 1, 2, 3\}$ be a local orthonormal basis on M^3. Setting $\mathfrak{U}_1 = \mathfrak{U}_2 = e_i$ in (3.11) and summing for i, $i = 1, 2, 3$, we conclude that

$$\begin{aligned}&\sum_{i=1}^{3}(\xi\lambda)S(e_i, e_i) - \sum_{i=1}^{3}2\kappa(e_i\lambda)\eta(e_i) + \sum_{i=1}^{3}\lambda\{\frac{dr(\xi)}{2}[g(e_i, e_i) - \eta(e_i)\eta(e_i)]\\ &-\frac{r-6\kappa}{2}[g(\phi e_i, e_i) - g(\phi h e_i, e_i)]\} - \sum_{i=1}^{3}(e_i f)\eta(e_i)\\ &+\sum_{i=1}^{3}(\xi f)g(e_i, e_i) - \sum_{i=1}^{3}\kappa\{(e_i\lambda)\eta(e_i) - (\xi\lambda)g(e_i, e_i)\} = 0,\end{aligned}$$

which becomes

$$r(\xi\lambda) - 2\kappa(\xi\lambda) + \lambda dr(\xi) + 2(\xi f) + 2\kappa(\xi\lambda) = 0.$$

It is obvious that, on M^3, $2f = -r\lambda$, and hence it gives us

$$2(\xi f) + r(\xi\lambda) = -(\xi r)\lambda.$$

From Eqs (3.7) and (3.11), if $(div\, Q)(\xi) = 0$, then we have

$$\begin{aligned}&(\xi\lambda)S(\mathfrak{U}_2, \mathfrak{U}_1) - 2\kappa(\mathfrak{U}_2\lambda)\eta(\mathfrak{U}_1) - \frac{\lambda}{2}(r-6\kappa)\{g(\phi\mathfrak{U}_2, \mathfrak{U}_1) - g(\phi h\mathfrak{U}_2, \mathfrak{U}_1)\}\\ &-(\mathfrak{U}_2 f)\eta(\mathfrak{U}_1) + (\xi f)g(\mathfrak{U}_2, \mathfrak{U}_1) + \kappa\{(\xi\lambda)g(\mathfrak{U}_2, \mathfrak{U}_1) - (\mathfrak{U}_2\lambda)\eta(\mathfrak{U}_1)\} = 0.\end{aligned} \tag{3.12}$$

Setting $\mathfrak{U}_1 = \xi$ in (3.12) and using the Eqs (2.1)–(2.4) and (2.10), we get

$$3\kappa[(\xi\lambda)\eta(\mathfrak{U}_2) - (\mathfrak{U}_2\lambda)] - [(\mathfrak{U}_2 f) - (\xi f)\eta(\mathfrak{U}_2)] = 0.$$

By replacing $\mathfrak{U}_1$ in Eq (3.12) with $\phi\mathfrak{U}_1$ and then using Eq (2.3), we discover

$$(\xi\lambda)S(\mathfrak{U}_2, \phi\mathfrak{U}_1) = \frac{\lambda}{2}(r-6\kappa)\{g(\phi\mathfrak{U}_2, \phi\mathfrak{U}_1) - g(\phi h\mathfrak{U}_2, \phi\mathfrak{U}_1)\} - [(\xi f) + \kappa(\xi\lambda)]g(\mathfrak{U}_2, \phi\mathfrak{U}_1). \tag{3.13}$$

Interchanging $\mathfrak{U}_1$ and $\mathfrak{U}_2$ in (3.13), we get

$$(\xi\lambda)S(\mathfrak{U}_1, \phi\mathfrak{U}_2) = \frac{\lambda}{2}(r-6\kappa)\{g(\phi\mathfrak{U}_1, \phi\mathfrak{U}_2) - g(\phi h\mathfrak{U}_1, \phi\mathfrak{U}_2)\} - [(\xi f) + \kappa(\xi\lambda)]g(\mathfrak{U}_1, \phi\mathfrak{U}_2). \tag{3.14}$$

Adding (3.13) and (3.14), we find

$$\lambda(r-6\kappa)[g(\phi\mathfrak{U}_1,\phi\mathfrak{U}_2)-g(\phi h\mathfrak{U}_1,\phi\mathfrak{U}_2)]=0.$$

Therefore, $\lambda \neq 0$ as we are interested in the Fischer-Marsden equation's non-trivial solution. Now we divide our study into two cases as:

Case I. We suppose that $r \neq 6\kappa$. Then, the above equation along with (2.5) reflects that $\nabla\xi = 0$. This result together with Eq (2.6) shows that $R(\mathfrak{U}_1,\mathfrak{U}_2)\xi = 0$, and hence we have $\mu = 0$ and $\kappa = 0$. These results and Lemma 3.1 infer that the three-dimensional NKPMM obeying the FME is Einstein.

Case II. Let us assume that $r = 6\kappa$. That is, the scalar curvature of the three-dimensional NKPMM satisfying $\mathcal{L}_g^*(\lambda) = 0$ is constant. This shows that

$$(\mathfrak{U}_2 f) = -3\kappa(\mathfrak{U}_2\lambda). \tag{3.15}$$

In consequence of $r = 6\kappa$ and (3.15), Eq (3.12) reduces to

$$(\xi\lambda)\{S(\mathfrak{U}_2,\mathfrak{U}_1)-2\kappa g(\mathfrak{U}_2,\mathfrak{U}_1)\}=0. \tag{3.16}$$

This shows that either $S = 2\kappa g$ or $(\xi\lambda) = 0$. If possible, we consider that $(\xi\lambda) = 0$, and hence $g(\xi, D\lambda) = 0$. Differentiating $g(\xi, D\lambda) = 0$ covariantly along $\mathfrak{U}_1$, we find

$$g(\nabla_{\mathfrak{U}_1}\xi, D\lambda)+g(\xi,\nabla_{\mathfrak{U}_1}D\lambda)=0.$$

The above equation along with Eqs (2.2), (2.5), (2.10) and (3.2) give

$$-g(\phi\mathfrak{U}_1,D\lambda)+g(\phi h\mathfrak{U}_1,D\lambda)+(2\kappa\lambda+f)\eta(\mathfrak{U}_1)=0. \tag{3.17}$$

Substituting $\mathfrak{U}_1 = \xi$ in (3.17) and using Eqs (2.3) and (2.4), we get

$$f=-2\kappa\lambda. \tag{3.18}$$

From Eqs (3.2) and (3.18), we conclude that $r = 4\kappa$, which contradicts our hypothesis. Hence, $(\xi\lambda) \neq 0$, and thus Eq (3.16) gives $S = 2\kappa g$. By considering the above discussions and Lemma 3.1, we state:

Theorem 3.1. *An M^3 satisfying the Fischer-Marsden conjecture is Einstein.*

In light of Lemma 3.2 and Theorem 3.1, we can state the following:

Corollary 3.1. *Let the Fischer-Marsden conjecture hold on an M^3. Then, the following conditions are equivalent:*
(i) M^3 is Einstein,
(ii) M^3 is Ricci semisymmetric,
(iii) M^3 is a space of constant curvature,
(iv) M^3 is Ricci symmetric.

4. The Fischer-Marsden conjecture on an NKPMM

The aim of this section is to study the properties of a $(2n+1)$-dimensional NKPMM satisfying the Fischer-Marsden conjecture. We will utilize the following result to support our main finding.

Lemma 4.1. *[34] Let M^{2n+1} be a PMM and suppose that $R(\mathfrak{U}_1, \mathfrak{U}_2)\xi = 0$ for all vector fields $\mathfrak{U}_1$, $\mathfrak{U}_2$. Then, locally, M^{2n+1} is the product of a flat $(n+1)$-dimensional manifold and an n-dimensional manifold of negative curvature equal to -4 for $n > 1$.*

In [10], Cappelletti-Montano et al. characterized $(2n+1)$-dimensional paracontact metric (κ, μ)-manifolds and proved many interesting results. It is observed that the Ricci operator Q of a $(2n+1)$-dimensional NKPMM satisfies the following relation

$$Q = -2(n-1)I + 2(n-1)h + [2(n-1) + 2\kappa n]\eta \otimes \xi, \tag{4.1}$$

for $\kappa \neq -1$, where I is an identity operator on M^{2n+1}. Throughout this section, we suppose that $\kappa \neq -1$. The symmetric tensor field h also satisfies

$$\begin{aligned}(\nabla_{\mathfrak{U}_1} h)(\mathfrak{U}_2) - (\nabla_{\mathfrak{U}_2} h)(\mathfrak{U}_1) &= -(1+\kappa)\{2g(\mathfrak{U}_1, \phi\mathfrak{U}_2)\xi + \eta(\mathfrak{U}_1)\phi\mathfrak{U}_2 \\ &\quad -\eta(\mathfrak{U}_2)\phi\mathfrak{U}_1\} + \eta(\mathfrak{U}_1)\phi h\mathfrak{U}_2 - \eta(\mathfrak{U}_2)\phi h\mathfrak{U}_1,\end{aligned} \tag{4.2}$$

for all $\mathfrak{U}_1$, $\mathfrak{U}_2 \in \mathfrak{X}(M^{2n+1})$. From (4.1), we have

$$(\nabla_{\mathfrak{U}_1} Q)(\mathfrak{U}_2) = 2(n-1)(\nabla_{\mathfrak{U}_1} h)(\mathfrak{U}_2) + [2(n-1) + 2\kappa n]\{g(\nabla_{\mathfrak{U}_1}\xi, \mathfrak{U}_2)\xi + \eta(\mathfrak{U}_2)\nabla_{\mathfrak{U}_1}\xi\},$$

which gives

$$\begin{aligned}(\nabla_{\mathfrak{U}_1} Q)(\mathfrak{U}_2) - (\nabla_{\mathfrak{U}_2} Q)(\mathfrak{U}_1) &= 2(n-1)\{(\nabla_{\mathfrak{U}_1} h)(\mathfrak{U}_2) - (\nabla_{\mathfrak{U}_2} h)(\mathfrak{U}_1)\} \\ &\quad +[2(n-1) + 2\kappa n]\{g(\nabla_{\mathfrak{U}_1}\xi, \mathfrak{U}_2)\xi + \eta(\mathfrak{U}_2)\nabla_{\mathfrak{U}_1}\xi \\ &\quad -g(\nabla_{\mathfrak{U}_2}\xi, \mathfrak{U}_1)\xi - \eta(\mathfrak{U}_1)\nabla_{\mathfrak{U}_2}\xi\}.\end{aligned} \tag{4.3}$$

In consequence of Eqs (2.1)–(2.3), (2.5), (3.1), (4.2) and (4.3), we get

$$\begin{aligned}g(R(\xi, \mathfrak{U}_2)D\lambda, \mathfrak{U}_1) &= (\xi\lambda)S(\mathfrak{U}_2, \mathfrak{U}_1) + 2\lambda(n-1)\{g((\nabla_\xi h)(\mathfrak{U}_2) \\ &\quad -(\nabla_{\mathfrak{U}_2} h)(\xi), \mathfrak{U}_1)\} + \lambda[2(n-1) + 2\kappa n]\{g(\nabla_\xi\xi, \mathfrak{U}_2)\eta(\mathfrak{U}_1) \\ &\quad +g(\nabla_\xi\xi, \mathfrak{U}_1)\eta(\mathfrak{U}_2) - g(\nabla_{\mathfrak{U}_2}\xi, \xi)\eta(\mathfrak{U}_1) - g(\nabla_{\mathfrak{U}_2}\xi, \mathfrak{U}_1)\} \\ &\quad +(\xi f)g(\mathfrak{U}_2, \mathfrak{U}_1) - (\mathfrak{U}_2 f)\eta(\mathfrak{U}_1) - 2n\kappa(\mathfrak{U}_2\lambda)\eta(\mathfrak{U}_1) \\ &= (\xi\lambda)S(\mathfrak{U}_2, \mathfrak{U}_1) - 2n\kappa(\mathfrak{U}_2\lambda)\eta(\mathfrak{U}_1) + (\xi f)g(\mathfrak{U}_2, \mathfrak{U}_1) \\ &\quad -(\mathfrak{U}_2 f)\eta(\mathfrak{U}_1) - 2(n-1)\lambda\{(1+\kappa)g(\phi\mathfrak{U}_2, \mathfrak{U}_1) \\ &\quad -g(\phi h\mathfrak{U}_2, \mathfrak{U}_1)\} - \lambda[2(n-1) + 2n\kappa]g(\nabla_{\mathfrak{U}_2}\xi, \mathfrak{U}_1).\end{aligned} \tag{4.4}$$

From Eqs (3.10) and (4.4), we have

$$\begin{aligned}&(\xi\lambda)S(\mathfrak{U}_2, \mathfrak{U}_1) - 2n\kappa(\mathfrak{U}_2\lambda)\eta(\mathfrak{U}_1) + (\xi f)g(\mathfrak{U}_2, \mathfrak{U}_1) - (\mathfrak{U}_2 f)\eta(\mathfrak{U}_1) \\ &-2(n-1)\lambda\{(1+\kappa)g(\phi\mathfrak{U}_2, \mathfrak{U}_1) - g(\phi h\mathfrak{U}_2, \mathfrak{U}_1)\} + \kappa\{(\xi\lambda)g(\mathfrak{U}_2, \mathfrak{U}_1) \\ &-(\mathfrak{U}_2\lambda)\eta(\mathfrak{U}_1)\} + \lambda[2(n-1) + 2n\kappa]\{g(\phi\mathfrak{U}_2, \mathfrak{U}_1) - g(\phi h\mathfrak{U}_2, \mathfrak{U}_1)\} = 0.\end{aligned} \tag{4.5}$$

Contracting Eq (4.1), we obtain

$$r = 2n(\kappa - 2(n-1)) \tag{4.6}$$

and hence Eq (3.2) becomes

$$f = (2(n-1) - \kappa)\lambda, \quad \kappa \neq -1,$$

which infers

$$(\mathfrak{U}_1 f) = (2(n-1) - \kappa)(\mathfrak{U}_1\lambda). \tag{4.7}$$

Using Eq (4.7) in (4.5), we obtain

$$\begin{aligned}&(\xi\lambda)S(\mathfrak{U}_2,\mathfrak{U}_1) + (2(n-1)-\kappa)\{(\xi\lambda)g(\mathfrak{U}_2,\mathfrak{U}_1) - (\mathfrak{U}_2\lambda)\eta(\mathfrak{U}_1)\}\\&-2n\kappa(\mathfrak{U}_2\lambda)\eta(\mathfrak{U}_1) - 2(n-1)\lambda\{(1+\kappa)g(\phi\mathfrak{U}_2,\mathfrak{U}_1) - g(\phi h\mathfrak{U}_2,\mathfrak{U}_1)\}\\&+\kappa\{(\xi\lambda)g(\mathfrak{U}_2,\mathfrak{U}_1) - (\mathfrak{U}_2\lambda)\eta(\mathfrak{U}_1)\} + \lambda[2(n-1)\\&+2n\kappa]\{g(\phi\mathfrak{U}_2,\mathfrak{U}_1) - g(\phi h\mathfrak{U}_2,\mathfrak{U}_1)\} = 0.\end{aligned} \tag{4.8}$$

Setting $\mathfrak{U}_1 = \xi$ in (4.8) and using Eqs (2.1)–(2.4) and (2.7), we get

$$[n\kappa + (n-1)]\{(\mathfrak{U}_2\lambda) - (\xi\lambda)\eta(\mathfrak{U}_2)\} = 0.$$

This shows that either $n\kappa + n - 1 = 0$ or $(\mathfrak{U}_2\lambda) - (\xi\lambda)\eta(\mathfrak{U}_2) = 0$.

Case I. We suppose that $n\kappa + n - 1 = 0$. Thus, $\kappa = -1 + \frac{1}{n} > -1$ for $n > 1$.

Cappelletti-Montano et al. [10] investigated several results of $(2n+1)$-dimensional paracontact metric (κ,μ)-manifolds for $\kappa > -1$. They proved that a three-dimensional NKPMM with $\kappa > -1$ is an η-Einstein manifold. They also showed that there is no paracontact (κ,μ)-manifold for $\kappa > -1$ and $n > 1$ that can be Einstein.

Case II. If possible, we suppose that $(\mathfrak{U}_2\lambda) - (\xi\lambda)\eta(\mathfrak{U}_2) = 0$ for $n > 1$ on M^{2n+1}. Thus we have $D\lambda = (\xi\lambda)\xi$. By covariantly differentiating this outcome along the vector field $\mathfrak{U}_1$, we discover

$$\nabla_{\mathfrak{U}_1} D\lambda = \mathfrak{U}_1(\xi\lambda)\xi + (\xi\lambda)\nabla_{\mathfrak{U}_1}\xi.$$

According to Eq (3.2), the previous equation has the following form:

$$\lambda Q\mathfrak{U}_1 + f\mathfrak{U}_1 = \mathfrak{U}_1(\xi\lambda)\xi + (\xi\lambda)\nabla_{\mathfrak{U}_1}\xi. \tag{4.9}$$

Taking a local frame field and contracting Eq (4.9), we get

$$\lambda r + (2n+1)\{2(n-1) - \kappa\}\lambda = \xi(\xi\lambda). \tag{4.10}$$

Again replacing $\mathfrak{U}_1$ by ξ in Eq (3.2) and taking the inner product with ξ, we find

$$\xi(\xi\lambda) = f + 2n\kappa\lambda. \tag{4.11}$$

Equations (4.10) and (4.11) along with Eqs (3.2) and (4.6) give us

$$2n\kappa\lambda = 0.$$

By the hypothesis $\lambda \neq 0$, we have $\kappa = 0$ for $n > 1$ and, from Eq (2.6), we get $R(\mathfrak{U}_1,\mathfrak{U}_2)\xi = 0$. This result along with the Lemma 4.1 tell us that M^{2n+1}, $n > 1$, is locally isometric to the product of the Euclidean space $\mathbb{E}^{n+1}$ and a hyperbolic space $\mathbb{H}^n(-4)$ of constant curvature -4. Thus, we are in a position to state the following:

Theorem 4.1. *Let a $(2n+1)$-dimensional $N(k)$-paracontact metric manifold M^{2n+1} with $n > 1$ satisfy the equation $\mathcal{L}^*_g(\lambda) = 0$. Then, either M^{2n+1} is locally isometric to $\mathbb{E}^{n+1} \times \mathbb{H}^n$ or there does not exist an Einstein NKPMM with $\kappa > -1$ for M^{2n+1}.*

5. Conclusions

The notion of the Fischer-Marsden conjecture on Riemannian manifolds was introduced by Fischer and Marsden [18], and it has been further extended by Bourguignon [17]. This conjecture on some classes of almost contact metric manifolds has been explored by many researchers. In this manuscript, we defined the Fischer-Marsden conjecture on semi-Riemannian manifolds, and in particular, we studied the non-trivial solutions of the Fischer-Marsden equation on $N(k)$-paracontact metric manifolds. This manuscript may open a door for researchers to explore the non-trivial solutions of the Fischer-Marsden equation on the classes of semi-Riemannian manifolds.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in writing the paper.

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Conflict of interest

The authors declare that there is no conflict of interest in this paper.

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