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Research article

Characteristics of planar sextic indirect-PH curves

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Abstract: By a fractional quadratic transformation, an indirect-PH curve can have rational offsets. In this paper, I study properties of planar sextic indirect-PH curves, in terms of their Bézier control polygon legs. With our results, sextic Bézier curves can be efficiently tested whether they are indirect-PH curves. The main strategy to achieve our results is using complex representation of planar parametric curve. Sextic indirect-PH curves can be classified into three classes according to different factorizations of their hodographs. Necessary and sufficient conditions for all classes of sextic indirect-PH curves can be described by non-linear complex systems. By analyzing these non-linear systems, algebraic conditions for a sextic Bézier curve to be an indirect-PH curve are first discussed, then geometric characteristics in terms of legs of its control polygon are revealed.

Keywords: geometric design; characteristics; Pythagorean hodograph; control polygon
Mathematics Subject Classification: 65D17

1. Introduction

Bézier curves are fundamental in computer aided geometric design (CAGD) and they are broadly applied in computer aided design and manufacturing (CAD/CAM) [1]. However, a Bézier curve, in general, may not have rational offset curves [2], so offset approximation methods were proposed [3–5]. A class of planar polynomial curves that possesses rational offset curves are studied in this paper. These curves provide a number of advantages in engineering applications, such as robotics, computer-numerical-control (CNC) machining, motion planning, railway design, shape blending, etc. For example, their arc-lengths can simply be computed by evaluating rational polynomials (avoiding the need of numerical integrations), which speeds up algorithms for CNC machining [6].

A polynomial curve is called a Pythagorean hodograph (PH) [7] curve if the Euclidean norm of its derivative (also called hodograph) is also a polynomial. Therefore, offsets of a PH curve are rational curves and can be exactly represented in CAD systems [7]. So far, PH curves have been widely studied, e.g., rational PH curves [8, 9], Minkowski PH curves [10], spatial PH curves and higher dimensional PH curves [9, 11], etc. Moreover, Hermite interpolation and curve design using various PH curves are proposed [12–18]. More details and examples can be found in books and surveys [1, 19–21]. Interestingly, PH curves and polynomial minimal surfaces are intrinsically related and their algebraic and geometric relationships have been well studied [22, 23].

However, a curve that has rational offsets is not necessary to become a PH curve. A properly-parameterized planar polynomial curve has rational offsets if and only if the squared norm of its hodograph has at most two complex roots (with nonzero imaginary part) of odd multiplicity [24]. These curves can be categorized into two cases, one are PH curves [7] and the others are indirect-PH curves [24, 25].

In order to analyze and construct curves with rational offsets, algebraic methods such as the complex variable model [24, 26] and quaternion model [27, 28] were usually used. Polynomial curves are represented using Bézier form in CAD systems because Bézier control polygons provide an intuitive way to dealing with curves. However, algebraic structure of polynomial curves with rational offsets is not simply transferred to intuitive constraints on their control polygons. Farouki and Sakkalis [7] introduced an elegant geometric characteristic for cubic PH curves, which are two constraints on the lengths of legs and two interior angles of their Bézier control polygons. By introducing hyperbolic functions, a class of quasi-Bézier curves possess PH property if their control polygons have similar geometric properties with cubic PH curves [29].

Wang and Fang [30] derived geometric characteristics of quartic PH curves, which are also legs and angles of their Bézier control polygons. This gives a geometric approach for construction of PH quartics. Algebraic and geometric characteristics of PH curves of degree five to seven are further studied [31–34].

Lu et al. studied cubic indirect-PH curves [25], which is extended to quartic indirect-PH curves [35]. Identification and construction of quintic indirect-PH curves can be divided into two classes, which were discussed separately [36, 37]. The usage of control polygons was also proposed for designing planar C^2 PH quintic spline curves in [38]. Due to the increase in degree, the factorization of polynomials varies. The higher the degree, the more complex the complex system needs to be considered. Therefore, whether there is a unified method for identification of indirect-PH curves of arbitrary degree is an open problem.

In this paper, I try to take the relevant work one step further to study sextic indirect-PH curves. The complex variable model is employed to analyze characteristics of sextic indirect-PH curves. These curves are first classified into three classes according to different factorizations of their hodographs, and then necessary and sufficient conditions on their Bézier control polygons are studied, respectively. One class of sextic indirect-PH curves are regular curves, while the other two classes may exhibit cusps. The algebraic conditions for sextic indirect-PH curves are non-linear systems. Our main idea is treating these non-linear systems using methods solving linear systems, e.g., Gaussian elimination, geometric methods, etc. Furthermore, the geometric characteristics of these curves are presented, and the results are in terms of legs of their Bézier control polygons.

The rest of this paper is organized as follows: In Section 2, I recall some basic concepts and properties of planar polynomial curves with rational offsets. Section 3 presents necessary and sufficient conditions for a planar sextic curve to be an indirect-PH curve. Finally, I conclude the paper in Section 4.

2. Preliminaries

Let $\mathbf{i} = \sqrt{-1}$ be the imaginary unit. For a complex number $\boldsymbol{c} \in \mathbb{C}$, I denote $\overline{\boldsymbol{c}}$ the conjugate complex of $\boldsymbol{c}$, $\mathrm{Re}(\boldsymbol{c})$ and $\mathrm{Im}(\boldsymbol{c})$ the real and imaginary parts of $\boldsymbol{c}$, respectively, and $\|\boldsymbol{c}\|$ the Euclidean norm of $\boldsymbol{c}$, i.e., $\|\boldsymbol{c}\|^2 = \mathrm{Re}^2(\boldsymbol{c}) + \mathrm{Im}^2(\boldsymbol{c})$.

In the complex plane $\mathbb{C}$, a planar curve $\boldsymbol{P}(t) = (x(t), y(t))$ can be expressed as a complex function $\boldsymbol{P}(t) = x(t) + \mathbf{i}y(t)$. This complex representation is quite useful for analysis and construction of PH curves [26, 30]. It is also used to present necessary and sufficient conditions for a properly-parameterized polynomial curve to have rational offsets [24]:

Theorem 2.1. *Let $\boldsymbol{P}(t) = x(t) + \mathbf{i}y(t)$ be a properly-parameterized polynomial curve, it has rationally parameterized offsets if and only if its hodograph $\boldsymbol{P}'(t) = x'(t) + \mathbf{i}y'(t)$ satisfies*

$$\boldsymbol{P}'(t) = \rho(t)\boldsymbol{R}(t)\boldsymbol{W}^2(t), \tag{2.1}$$

where $\rho(t)$ is a polynomial of t with real coefficients, $\boldsymbol{R}(t)$ and $\boldsymbol{W}(t)$ have forms

$$\begin{aligned} \boldsymbol{R}(t) &= \lambda t + 1 + \mathbf{i}\mu t, \\ \boldsymbol{W}(t) &= u(t) + \mathbf{i}v(t), \end{aligned} \tag{2.2}$$

$\lambda, \mu \in \mathbb{R}$, $u(t)$ and $v(t)$ are polynomial of t with real coefficients and they are relatively prime.

If $\mu = 0$, I have $\|\boldsymbol{P}'(t)\|^2 = \rho^2(t)\|\boldsymbol{W}^2(t)\|^2$, i.e., the Euclid norm of its hodograph is a real polynomial $\rho(t)\|\boldsymbol{W}^2(t)\|$, then it is a PH curve and it has rational offsets [7]. However, following the decomposition (2.1), a planar polynomial curve with rational offset curves may not be a PH curve, i.e., the case $\mu \neq 0$, which means $\boldsymbol{R}(t)$ is a polynomial with complex coefficients. Let

$$\begin{aligned} b &= \sqrt{(\lambda + 1)^2 + \mu^2}, \\ c &= \sqrt{(\lambda + 2)^2 + \mu^2}, \end{aligned}$$

and $B_i^n(t) = \binom{n}{i}(1-t)^{n-i}t^i$, $i = 0, \ldots, n$, be Bernstein polynomials, after a quadratic parameter transformation [24, 25]

$$t(s) = \frac{B_1^2(s) + (c - 1 + b)B_2^2(s)}{(c + 1 - b)B_0^2(s) + (1 + b)B_1^2(s) + (c - 1 + b)B_2^2(s)},$$

the offsets of this curve can be rational curves because

$$(\lambda t + 1)^2 + (\mu t)^2 = \left(\frac{(c + 1 - b)B_0^2(s) + \frac{c^2 - (1-b)^2}{2}B_1^2(s) + b(c - 1 + b)B_2^2(s)}{(c + 1 - b)B_0^2(s) + (1 + b)B_1^2(s) + (c - 1 + b)B_2^2(s)} \right)^2.$$

Such a curve is called an indirect-PH curve [24, 25]. I study properties of sextic indirect-PH curves in this paper.

Let $\boldsymbol{P}_i = x_i + \mathbf{i}y_i$, $i = 0, \ldots, n$, be control points in the complex plane, a sextic Bézier curve is defined by

$$\boldsymbol{P}(t) = \sum_{i=0}^{6} \boldsymbol{P}_i B_i^6(t),$$

whose hodograph can also be expressed in Bernstein form

$$\boldsymbol{P}'(t) = 6\sum_{i=0}^{5} \Delta\boldsymbol{P}_i B_i^5(t), \tag{2.3}$$

where $\Delta\boldsymbol{P}_i = \boldsymbol{P}_{i+1} - \boldsymbol{P}_i$ is the first forward-difference of the i-th control point. Besides, I denote the second forward-difference of the i-th control point as $\Delta^2\boldsymbol{P}_i$, i.e., $\Delta^2\boldsymbol{P}_i = \Delta\boldsymbol{P}_{i+1} - \Delta\boldsymbol{P}_i$.

3. Characteristics of sextic indirect-PH curves

Following Theorem 2.1, a sextic indirect-PH curve's hodograph has form (2.1), and the degrees of $\rho(t)$, $\boldsymbol{R}(t)$, $\boldsymbol{W}(t)$ are 0, 1, 2, or 2, 1, 1, or 4, 1, 0, which are called class I, class II, class III sextic indirect-PH curves, respectively, in following discussion. Notably, all the polynomials will be written in Bernstein form in this section. I may suppose that the three classes of sextic indirect-PH curves are mutually exclusive, e.g., when $\boldsymbol{W}(t)$ is a degree 2 complex polynomial, it must have no real polynomial factors. In the rest of this section, I study all three classes of sextic indirect-PH curves, respectively.

3.1. Class I sextic indirect-PH curve

Let $\boldsymbol{P}_i$, $i = 0, \ldots, 6$, be Bézier control points of a planar sextic curve, and $\boldsymbol{M}_i$, $i = 0, 1$, be defined in determinant form,

$$\boldsymbol{M}_0 = \left\| \begin{matrix} \begin{vmatrix} \Delta^2\boldsymbol{P}_0 & 4\Delta^2\boldsymbol{P}_1 \\ \begin{vmatrix} \Delta\boldsymbol{P}_0 & 5\Delta\boldsymbol{P}_1 \\ \Delta^2\boldsymbol{P}_0 & 4\Delta^2\boldsymbol{P}_1 \end{vmatrix} & \begin{vmatrix} \Delta\boldsymbol{P}_0 & 10\Delta\boldsymbol{P}_2 \\ \Delta^2\boldsymbol{P}_0 & 6\Delta^2\boldsymbol{P}_2 \end{vmatrix} \end{vmatrix} & \begin{vmatrix} \Delta^2\boldsymbol{P}_0 & \Delta^2\boldsymbol{P}_4 \\ \begin{vmatrix} \Delta\boldsymbol{P}_0 & 5\Delta\boldsymbol{P}_1 \\ \Delta^2\boldsymbol{P}_0 & 4\Delta^2\boldsymbol{P}_1 \end{vmatrix} & \begin{vmatrix} \Delta\boldsymbol{P}_0 & \Delta\boldsymbol{P}_5 \\ \Delta^2\boldsymbol{P}_0 & 0 \end{vmatrix} \end{vmatrix} \\ \begin{vmatrix} \Delta^2\boldsymbol{P}_0 & \Delta^2\boldsymbol{P}_4 \\ \begin{vmatrix} \Delta\boldsymbol{P}_0 & 5\Delta\boldsymbol{P}_1 \\ \Delta^2\boldsymbol{P}_0 & 4\Delta^2\boldsymbol{P}_1 \end{vmatrix} & \begin{vmatrix} \Delta\boldsymbol{P}_0 & \Delta\boldsymbol{P}_5 \\ \Delta^2\boldsymbol{P}_0 & 0 \end{vmatrix} \end{vmatrix} & \begin{vmatrix} 4\Delta^2\boldsymbol{P}_3 & \Delta^2\boldsymbol{P}_4 \\ \begin{vmatrix} \Delta\boldsymbol{P}_0 & 5\Delta\boldsymbol{P}_4 \\ \Delta^2\boldsymbol{P}_0 & \Delta^2\boldsymbol{P}_4 \end{vmatrix} & \begin{vmatrix} \Delta\boldsymbol{P}_0 & \Delta\boldsymbol{P}_5 \\ \Delta^2\boldsymbol{P}_0 & 0 \end{vmatrix} \end{vmatrix} \end{matrix} \right\|,$$

$$\boldsymbol{M}_1 = \left\| \begin{matrix} \begin{vmatrix} \Delta^2\boldsymbol{P}_0 & 4\Delta^2\boldsymbol{P}_3 \\ \begin{vmatrix} \Delta\boldsymbol{P}_0 & 5\Delta\boldsymbol{P}_1 \\ \Delta^2\boldsymbol{P}_0 & 4\Delta^2\boldsymbol{P}_1 \end{vmatrix} & \begin{vmatrix} \Delta\boldsymbol{P}_0 & 5\Delta\boldsymbol{P}_4 \\ \Delta^2\boldsymbol{P}_0 & \Delta^2\boldsymbol{P}_4 \end{vmatrix} \end{vmatrix} & \begin{vmatrix} \Delta^2\boldsymbol{P}_0 & \Delta^2\boldsymbol{P}_4 \\ \begin{vmatrix} \Delta\boldsymbol{P}_0 & 5\Delta\boldsymbol{P}_1 \\ \Delta^2\boldsymbol{P}_0 & 4\Delta^2\boldsymbol{P}_1 \end{vmatrix} & \begin{vmatrix} \Delta\boldsymbol{P}_0 & \Delta\boldsymbol{P}_5 \\ \Delta^2\boldsymbol{P}_0 & 0 \end{vmatrix} \end{vmatrix} \\ \begin{vmatrix} 6\Delta^2\boldsymbol{P}_2 & \Delta^2\boldsymbol{P}_4 \\ \begin{vmatrix} \Delta\boldsymbol{P}_0 & 10\Delta\boldsymbol{P}_3 \\ \Delta^2\boldsymbol{P}_0 & 4\Delta^2\boldsymbol{P}_3 \end{vmatrix} & \begin{vmatrix} \Delta\boldsymbol{P}_0 & \Delta\boldsymbol{P}_5 \\ \Delta^2\boldsymbol{P}_0 & 0 \end{vmatrix} \end{vmatrix} & \begin{vmatrix} 4\Delta^2\boldsymbol{P}_3 & \Delta^2\boldsymbol{P}_4 \\ \begin{vmatrix} \Delta\boldsymbol{P}_0 & 5\Delta\boldsymbol{P}_4 \\ \Delta^2\boldsymbol{P}_0 & \Delta^2\boldsymbol{P}_4 \end{vmatrix} & \begin{vmatrix} \Delta\boldsymbol{P}_0 & \Delta\boldsymbol{P}_5 \\ \Delta^2\boldsymbol{P}_0 & 0 \end{vmatrix} \end{vmatrix} \end{matrix} \right\|,$$

then I give four complex numbers z_i, $i = 0, \ldots, 3$,

$$z_0 = \frac{\boldsymbol{M}_1^2\Delta\boldsymbol{P}_0}{\boldsymbol{M}_0^2\Delta\boldsymbol{P}_5}, \tag{3.1}$$

$$z_1 = \frac{\boldsymbol{M}_0}{\boldsymbol{M}_1}\sqrt{6\Delta\boldsymbol{P}_5}, \tag{3.2}$$

$$z_2 = \frac{15\boldsymbol{M}_0^2\Delta\boldsymbol{P}_4 - 3\boldsymbol{M}_1^2\Delta\boldsymbol{P}_0}{2\boldsymbol{M}_0^2\sqrt{6\Delta\boldsymbol{P}_5}}, \tag{3.3}$$

$$z_3 = \sqrt{6\Delta\boldsymbol{P}_5}. \tag{3.4}$$

A necessary and sufficient condition for a class I sextic indirect-PH curve can be described as following theorem.

Theorem 3.1. *A planar sextic Bézier curve is a class I sextic indirect-PH curve if and only if*

$$\begin{aligned} 6\Delta \boldsymbol{P}_0 &= z_0 z_1^2, \\ 30\Delta \boldsymbol{P}_1 &= 4z_0 z_1 z_2 + z_1^2, \\ 60\Delta \boldsymbol{P}_2 &= 2z_0 z_1 z_3 + 4z_0 z_2^2 + 4z_1 z_2, \\ 60\Delta \boldsymbol{P}_3 &= 4z_0 z_2 z_3 + 2z_1 z_3 + 4z_2^2, \\ 30\Delta \boldsymbol{P}_4 &= z_0 z_3^2 + 4z_2 z_3, \\ 6\Delta \boldsymbol{P}_5 &= z_3^2. \end{aligned} \tag{3.5}$$

Proof. Notably, the polynomials $\boldsymbol{R}(t)$ and $\boldsymbol{W}(t)$ of (2.1) can be written in Bernstein form. Hence, a planar sextic Bézier curve with control points $\boldsymbol{P}_i$, $i = 0, \ldots, 6$ is an indirect-PH curve if and only if there are $z_i \in \mathbb{C}$, $i = 0, \ldots, 3$, such that

$$\boldsymbol{P}'(t) = [z_0(1-t) + t][z_1(1-t)^2 + 2z_2(1-t)t + z_3 t^2]^2,$$

which is expanded as

$$\begin{aligned} \boldsymbol{P}'(t) =& z_0 z_1^2 (1-t)^5 + (4z_0 z_1 z_2 + z_1^2)(1-t)^4 t \\ &+ [z_0(2z_1 z_3 + 4z_2^2) + 4z_1 z_2](1-t)^3 t^2 \\ &+ [4z_0 z_2 z_3 + (2z_1 z_3 + 4z_2^2)](1-t)^2 t^3 \\ &+ (z_0 z_3^2 + 4z_2 z_3)(1-t)t^4 + z_3^2 t^5, \end{aligned}$$

thus, I may derive (3.5) by matching its coefficients with (2.3).

Although the control points are known, z_i are all unknowns. It is necessary to solve all of them to make sure whether the curve is an indirect-PH curve or not. Notably, the quadric polynomial $\boldsymbol{W}(t) = z_1(1-t)^2 + 2z_2(1-t)t + z_3 t^2$ is a common factor of $\boldsymbol{P}'(t)$ and $\boldsymbol{P}''(t)$, I consider a system of equations

$$\boldsymbol{P}'(t) = 6\sum_{i=0}^{5} \Delta \boldsymbol{P}_i B_i^5(t) = 0, \tag{3.6}$$

$$\boldsymbol{P}''(t) = 30\sum_{i=0}^{4} \Delta^2 \boldsymbol{P}_i B_i^4(t) = 0. \tag{3.7}$$

Multiplying Eq (3.7) by $1-t$ first, then eliminating the item $(1-t)^5$ with (3.6), and dividing a non-zero factor t, I get a system of two quartic equations,

$$\begin{aligned} &30\sum_{i=0}^{4} \Delta^2 \boldsymbol{P}_i B_i^4(t) = 0, \\ &\begin{vmatrix} \Delta \boldsymbol{P}_0 & 5\Delta \boldsymbol{P}_1 \\ \Delta^2 \boldsymbol{P}_0 & 4\Delta^2 \boldsymbol{P}_1 \end{vmatrix}(1-t)^4 + \begin{vmatrix} \Delta \boldsymbol{P}_0 & 10\Delta \boldsymbol{P}_2 \\ \Delta^2 \boldsymbol{P}_0 & 6\Delta^2 \boldsymbol{P}_2 \end{vmatrix}(1-t)^3 t + \begin{vmatrix} \Delta \boldsymbol{P}_0 & 10\Delta \boldsymbol{P}_3 \\ \Delta^2 \boldsymbol{P}_0 & 4\Delta^2 \boldsymbol{P}_3 \end{vmatrix}(1-t)^2 t^2 \\ &+ \begin{vmatrix} \Delta \boldsymbol{P}_0 & 5\Delta \boldsymbol{P}_4 \\ \Delta^2 \boldsymbol{P}_0 & \Delta^2 \boldsymbol{P}_4 \end{vmatrix}(1-t)t^3 + \begin{vmatrix} \Delta \boldsymbol{P}_0 & \Delta \boldsymbol{P}_5 \\ \Delta^2 \boldsymbol{P}_0 & 0 \end{vmatrix} t^4 = 0. \end{aligned}$$

Furthermore, by eliminating the items $(1-t)^4$ and t^4, respectively, I get a system of two cubic equations,

$$\begin{aligned}
&\begin{Vmatrix} \Delta^2 \boldsymbol{P}_0 & 4\Delta^2 \boldsymbol{P}_1 \\ \begin{vmatrix} \Delta \boldsymbol{P}_0 & 5\Delta \boldsymbol{P}_1 \\ \Delta^2 \boldsymbol{P}_0 & 4\Delta^2 \boldsymbol{P}_1 \end{vmatrix} & \begin{vmatrix} \Delta \boldsymbol{P}_0 & 10\Delta \boldsymbol{P}_2 \\ \Delta^2 \boldsymbol{P}_0 & 6\Delta^2 \boldsymbol{P}_2 \end{vmatrix} \end{Vmatrix} (1-t)^3 + \begin{Vmatrix} \Delta^2 \boldsymbol{P}_0 & 6\Delta^2 \boldsymbol{P}_2 \\ \begin{vmatrix} \Delta \boldsymbol{P}_0 & 5\Delta \boldsymbol{P}_1 \\ \Delta^2 \boldsymbol{P}_0 & 4\Delta^2 \boldsymbol{P}_1 \end{vmatrix} & \begin{vmatrix} \Delta \boldsymbol{P}_0 & 10\Delta \boldsymbol{P}_3 \\ \Delta^2 \boldsymbol{P}_0 & 4\Delta^2 \boldsymbol{P}_3 \end{vmatrix} \end{Vmatrix} (1-t)^2 t \\
&+ \begin{Vmatrix} \Delta^2 \boldsymbol{P}_0 & 4\Delta^2 \boldsymbol{P}_3 \\ \begin{vmatrix} \Delta \boldsymbol{P}_0 & 5\Delta \boldsymbol{P}_1 \\ \Delta^2 \boldsymbol{P}_0 & 4\Delta^2 \boldsymbol{P}_1 \end{vmatrix} & \begin{vmatrix} \Delta \boldsymbol{P}_0 & 5\Delta \boldsymbol{P}_4 \\ \Delta^2 \boldsymbol{P}_0 & \Delta^2 \boldsymbol{P}_4 \end{vmatrix} \end{Vmatrix} (1-t)t^2 + \begin{Vmatrix} \Delta^2 \boldsymbol{P}_0 & \Delta^2 \boldsymbol{P}_4 \\ \begin{vmatrix} \Delta \boldsymbol{P}_0 & 5\Delta \boldsymbol{P}_1 \\ \Delta^2 \boldsymbol{P}_0 & 4\Delta^2 \boldsymbol{P}_1 \end{vmatrix} & \begin{vmatrix} \Delta \boldsymbol{P}_0 & \Delta \boldsymbol{P}_5 \\ \Delta^2 \boldsymbol{P}_0 & 0 \end{vmatrix} \end{Vmatrix} t^3 = 0, \\
&\begin{Vmatrix} \Delta^2 \boldsymbol{P}_0 & \Delta^2 \boldsymbol{P}_4 \\ \begin{vmatrix} \Delta \boldsymbol{P}_0 & 5\Delta \boldsymbol{P}_1 \\ \Delta^2 \boldsymbol{P}_0 & 4\Delta^2 \boldsymbol{P}_1 \end{vmatrix} & \begin{vmatrix} \Delta \boldsymbol{P}_0 & \Delta \boldsymbol{P}_5 \\ \Delta^2 \boldsymbol{P}_0 & 0 \end{vmatrix} \end{Vmatrix} (1-t)^3 + \begin{Vmatrix} 4\Delta^2 \boldsymbol{P}_1 & \Delta^2 \boldsymbol{P}_4 \\ \begin{vmatrix} \Delta \boldsymbol{P}_0 & 10\Delta \boldsymbol{P}_2 \\ \Delta^2 \boldsymbol{P}_0 & 6\Delta^2 \boldsymbol{P}_2 \end{vmatrix} & \begin{vmatrix} \Delta \boldsymbol{P}_0 & \Delta \boldsymbol{P}_5 \\ \Delta^2 \boldsymbol{P}_0 & 0 \end{vmatrix} \end{Vmatrix} (1-t)^2 t \\
&+ \begin{Vmatrix} 6\Delta^2 \boldsymbol{P}_2 & \Delta^2 \boldsymbol{P}_4 \\ \begin{vmatrix} \Delta \boldsymbol{P}_0 & 10\Delta \boldsymbol{P}_3 \\ \Delta^2 \boldsymbol{P}_0 & 4\Delta^2 \boldsymbol{P}_3 \end{vmatrix} & \begin{vmatrix} \Delta \boldsymbol{P}_0 & \Delta \boldsymbol{P}_5 \\ \Delta^2 \boldsymbol{P}_0 & 0 \end{vmatrix} \end{Vmatrix} (1-t)t^2 + \begin{Vmatrix} 4\Delta^2 \boldsymbol{P}_3 & \Delta^2 \boldsymbol{P}_4 \\ \begin{vmatrix} \Delta \boldsymbol{P}_0 & 5\Delta \boldsymbol{P}_4 \\ \Delta^2 \boldsymbol{P}_0 & \Delta^2 \boldsymbol{P}_4 \end{vmatrix} & \begin{vmatrix} \Delta \boldsymbol{P}_0 & \Delta \boldsymbol{P}_5 \\ \Delta^2 \boldsymbol{P}_0 & 0 \end{vmatrix} \end{Vmatrix} t^3 = 0.
\end{aligned}$$

In the same way, I may further reduce the degree of above system by first eliminating $(1-t)^3$ and t^3, and then dividing t and $(1-t)$, respectively. Thus, I will get a system of two quadric equations, which both shall be $k\boldsymbol{W}(t) = 0$, for some $k \in \mathbb{R}$.

Therefore, I can compute $\frac{z_1}{\boldsymbol{M}_0} = \frac{z_3}{\boldsymbol{M}_1}$, and suppose that $z_3 = \sqrt{6\Delta \boldsymbol{P}_5}$, which gives an expression of z_1 as (3.2). Moreover, I substitute z_1 into the first equation of the system (3.5) to get (3.1). At last, I get (3.3) by substituting z_0, z_1 and z_3 into the fourth equation of the system (3.5).

In contrast, if z_i defined by (3.1)–(3.4) make the system (3.5) holds, it is obvious that the curve is an indirect-PH curve. □

Notably, there are no real roots for $\boldsymbol{P}'(t) = 0$, therefore, a class I sextic indirect-PH curve is a regular curve, i.e., there are no cusps on the curve. To further indicate the properties of its control polygon, I introduce some auxiliary points, as shown in Figure 1. Let points $\boldsymbol{Q}_i$, $i = 0, \ldots, 5$, are defined as follows,

$$\begin{aligned}
\boldsymbol{Q}_0 &= \boldsymbol{P}_1 + \frac{z_1^2}{30} = \boldsymbol{P}_2 - \frac{2z_0 z_1 z_2}{15}, \\
\boldsymbol{Q}_1 &= \boldsymbol{P}_2 + \frac{z_1 z_2}{15}, \\
\boldsymbol{Q}_2 &= \boldsymbol{Q}_1 + \frac{z_0 z_2^2}{30} = \boldsymbol{P}_3 - \frac{z_0 z_1 z_3}{30}, \\
\boldsymbol{Q}_3 &= \boldsymbol{P}_3 + \frac{z_2^2}{15}, \\
\boldsymbol{Q}_4 &= \boldsymbol{Q}_3 + \frac{z_1 z_3}{30} = \boldsymbol{P}_4 - \frac{z_0 z_2 z_3}{15}, \\
\boldsymbol{Q}_5 &= \boldsymbol{P}_4 + \frac{2z_2 z_3}{15} = \boldsymbol{P}_5 - \frac{z_0 z_3^2}{30},
\end{aligned}$$

then, I have

$$z_0 = \frac{\Delta \boldsymbol{P}_0}{5(\boldsymbol{Q}_0 - \boldsymbol{P}_1)} = \frac{2(\boldsymbol{P}_2 - \boldsymbol{Q}_0)}{5(\boldsymbol{Q}_1 - \boldsymbol{P}_2)} = \frac{2\Delta \boldsymbol{Q}_1}{\boldsymbol{Q}_3 - \boldsymbol{P}_3} = \frac{\boldsymbol{P}_3 - \boldsymbol{Q}_2}{2\Delta \boldsymbol{Q}_3} = \frac{5(\boldsymbol{P}_4 - \boldsymbol{Q}_4)}{2(\boldsymbol{Q}_5 - \boldsymbol{P}_4)} = \frac{5(\boldsymbol{P}_5 - \boldsymbol{Q}_4)}{\Delta \boldsymbol{P}_5}.$$

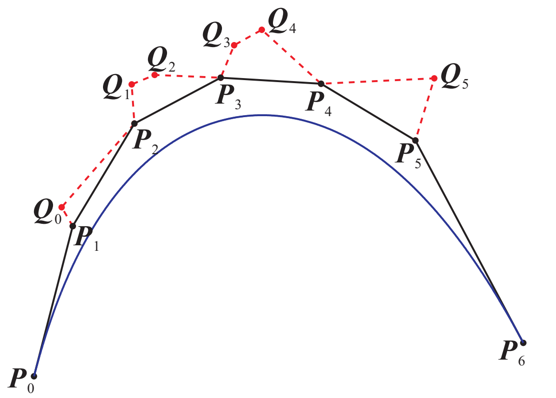

Figure 1. An example of a class I sextic indirect-PH curve with its Bézier control points and auxiliary points.

Figure 2 gives examples for testing whether a given Bézier curve is a class I sextic indirect-PH curve or not. To do that, I have to do approximate computation, because the data for control polygons in practical operation are not always keeping accurate. Farouki et al. [39] proposed a robust PH identification by choosing a relative tolerance ϵ for algebraic constraints for control polygon, that is $10^2 - 10^3$ larger than machine unit, this is $\eta \approx 2.22 \times 10^{-16}$.

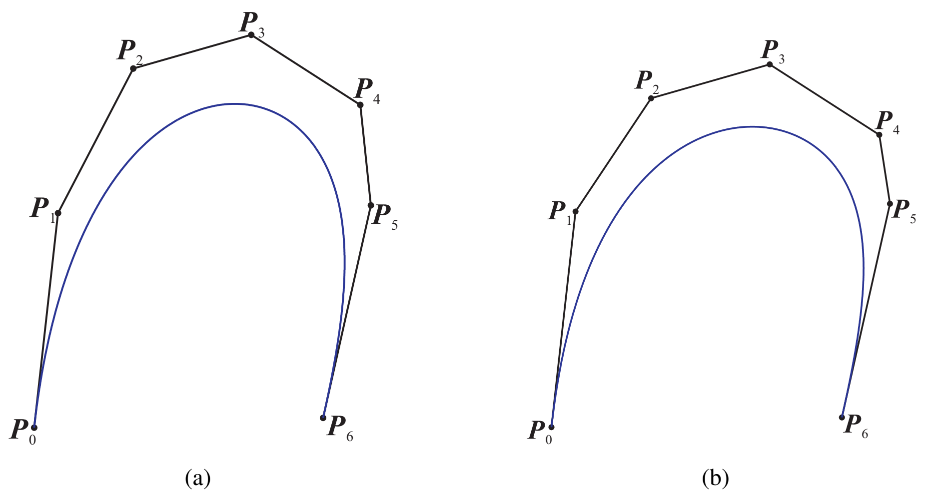

Figure 2. Test of class I sextic indirect-PH curves. (a) The curve is a class I sextic indirect-PH curve. (b) The curve is not a class I sextic indirect-PH curve.

In Figure 2(a), the control points are $\boldsymbol{P}_0 = 0$, $\boldsymbol{P}_1 = 0.7500 + 6.7292\mathbf{i}$, $\boldsymbol{P}_2 = 3.1083 + 11.2625\mathbf{i}$, $\boldsymbol{P}_3 = 6.8083 + 12.3208\mathbf{i}$, $\boldsymbol{P}_4 = 10.2250 + 10.1208\mathbf{i}$, $\boldsymbol{P}_5 = 10.5583 + 6.9708\mathbf{i}$ and $\boldsymbol{P}_6 = 9.0583 + 0.3042\mathbf{i}$. I compute $z_0 = 2 - 1.5\mathbf{i}$, $z_1 = 2 + 3.5\mathbf{i}$, $z_2 = 3 + 2\mathbf{i}$, $z_3 = 4 - 5\mathbf{i}$, thus the differences between the right- and left-hand sides of equations in (3.5) are 3.5527×10^{-15}, 3.8658×10^{-13}, 6.8109×10^{-13}, 6.3870×10^{-13}, 1.7764×10^{-14}, 0, respectively, so the curve is a class I sextic indirect-PH curve.

In Figure 2(b), I have $\boldsymbol{P}_0 = 0$, $\boldsymbol{P}_1 = 0.7500 + 6.7292\mathbf{i}$, $\boldsymbol{P}_2 = 3.1083 + 10.2625\mathbf{i}$, $\boldsymbol{P}_3 = 6.8083 + 11.3208\mathbf{i}$, $\boldsymbol{P}_4 = 10.2250 + 9.1208\mathbf{i}$, $\boldsymbol{P}_5 = 10.5583 + 6.9708\mathbf{i}$ and $\boldsymbol{P}_6 = 9.0583 + 0.3042\mathbf{i}$. I have $z_0 = -4.0983 - 1.5577\mathbf{i}$, $z_1 = -1.5866 + 2.5978\mathbf{i}$, $z_2 = 8.2558 - 4.8334\mathbf{i}$ and $z_3 = 4 - 5\mathbf{i}$. The differences between the right- and left-hand sides of equations in (3.5) are 1.2809×10^{-14}, 599.3607, 1715.3, 952.5519, 2.2749×10^{-14} and 0, respectively, the curve is not a class I sextic indirect-PH curve.

3.2. Class II sextic indirect-PH curve

Let

$$a_0 = \begin{vmatrix} \begin{vmatrix} \mathrm{Im}(\Delta \boldsymbol{P}_0\overline{\Delta \boldsymbol{P}_5}) & 5\mathrm{Im}(\Delta \boldsymbol{P}_4\overline{\Delta \boldsymbol{P}_5}) \\ 5\mathrm{Im}(\Delta \boldsymbol{P}_0\overline{\Delta \boldsymbol{P}_1}) & \mathrm{Im}(\Delta \boldsymbol{P}_0\overline{\Delta \boldsymbol{P}_5}) \end{vmatrix} & \begin{vmatrix} 10\mathrm{Im}(\Delta \boldsymbol{P}_3\overline{\Delta \boldsymbol{P}_5}) & 5\mathrm{Im}(\Delta \boldsymbol{P}_4\overline{\Delta \boldsymbol{P}_5}) \\ 5\mathrm{Im}(\Delta \boldsymbol{P}_0\overline{\Delta \boldsymbol{P}_4}) & \mathrm{Im}(\Delta \boldsymbol{P}_0\overline{\Delta \boldsymbol{P}_5}) \end{vmatrix} \\ \begin{vmatrix} \mathrm{Im}(\Delta \boldsymbol{P}_0\overline{\Delta \boldsymbol{P}_5}) & 5\mathrm{Im}(\Delta \boldsymbol{P}_1\overline{\Delta \boldsymbol{P}_5}) \\ 5\mathrm{Im}(\Delta \boldsymbol{P}_0\overline{\Delta \boldsymbol{P}_1}) & 10\mathrm{Im}(\Delta \boldsymbol{P}_0\overline{\Delta \boldsymbol{P}_2}) \end{vmatrix} & \begin{vmatrix} \mathrm{Im}(\Delta \boldsymbol{P}_0\overline{\Delta \boldsymbol{P}_5}) & 5\mathrm{Im}(\Delta \boldsymbol{P}_4\overline{\Delta \boldsymbol{P}_5}) \\ 5\mathrm{Im}(\Delta \boldsymbol{P}_0\overline{\Delta \boldsymbol{P}_1}) & \mathrm{Im}(\Delta \boldsymbol{P}_0\overline{\Delta \boldsymbol{P}_5}) \end{vmatrix} \end{vmatrix},$$

$$a_1 = \frac{1}{2}\begin{vmatrix} \begin{vmatrix} 5\mathrm{Im}(\Delta \boldsymbol{P}_1\overline{\Delta \boldsymbol{P}_5}) & 5\mathrm{Im}(\Delta \boldsymbol{P}_4\overline{\Delta \boldsymbol{P}_5}) \\ 10\mathrm{Im}(\Delta \boldsymbol{P}_0\overline{\Delta \boldsymbol{P}_2}) & \mathrm{Im}(\Delta \boldsymbol{P}_0\overline{\Delta \boldsymbol{P}_5}) \end{vmatrix} & \begin{vmatrix} 10\mathrm{Im}(\Delta \boldsymbol{P}_3\overline{\Delta \boldsymbol{P}_5}) & 5\mathrm{Im}(\Delta \boldsymbol{P}_4\overline{\Delta \boldsymbol{P}_5}) \\ 5\mathrm{Im}(\Delta \boldsymbol{P}_0\overline{\Delta \boldsymbol{P}_4}) & \mathrm{Im}(\Delta \boldsymbol{P}_0\overline{\Delta \boldsymbol{P}_5}) \end{vmatrix} \\ \begin{vmatrix} \mathrm{Im}(\Delta \boldsymbol{P}_0\overline{\Delta \boldsymbol{P}_5}) & 10\mathrm{Im}(\Delta \boldsymbol{P}_2\overline{\Delta \boldsymbol{P}_5}) \\ 5\mathrm{Im}(\Delta \boldsymbol{P}_0\overline{\Delta \boldsymbol{P}_1}) & 10\mathrm{Im}(\Delta \boldsymbol{P}_0\overline{\Delta \boldsymbol{P}_3}) \end{vmatrix} & \begin{vmatrix} \mathrm{Im}(\Delta \boldsymbol{P}_0\overline{\Delta \boldsymbol{P}_5}) & 5\mathrm{Im}(\Delta \boldsymbol{P}_4\overline{\Delta \boldsymbol{P}_5}) \\ 5\mathrm{Im}(\Delta \boldsymbol{P}_0\overline{\Delta \boldsymbol{P}_1}) & \mathrm{Im}(\Delta \boldsymbol{P}_0\overline{\Delta \boldsymbol{P}_5}) \end{vmatrix} \end{vmatrix}, \tag{3.8}$$

$$a_2 = \begin{vmatrix} \begin{vmatrix} 10\mathrm{Im}(\Delta \boldsymbol{P}_2\overline{\Delta \boldsymbol{P}_5}) & 5\mathrm{Im}(\Delta \boldsymbol{P}_4\overline{\Delta \boldsymbol{P}_5}) \\ 10\mathrm{Im}(\Delta \boldsymbol{P}_0\overline{\Delta \boldsymbol{P}_3}) & \mathrm{Im}(\Delta \boldsymbol{P}_0\overline{\Delta \boldsymbol{P}_5}) \end{vmatrix} & \begin{vmatrix} 10\mathrm{Im}(\Delta \boldsymbol{P}_3\overline{\Delta \boldsymbol{P}_5}) & 5\mathrm{Im}(\Delta \boldsymbol{P}_4\overline{\Delta \boldsymbol{P}_5}) \\ 5\mathrm{Im}(\Delta \boldsymbol{P}_0\overline{\Delta \boldsymbol{P}_4}) & \mathrm{Im}(\Delta \boldsymbol{P}_0\overline{\Delta \boldsymbol{P}_5}) \end{vmatrix} \\ \begin{vmatrix} \mathrm{Im}(\Delta \boldsymbol{P}_0\overline{\Delta \boldsymbol{P}_5}) & 10\mathrm{Im}(\Delta \boldsymbol{P}_3\overline{\Delta \boldsymbol{P}_5}) \\ 5\mathrm{Im}(\Delta \boldsymbol{P}_0\overline{\Delta \boldsymbol{P}_1}) & 5\mathrm{Im}(\Delta \boldsymbol{P}_0\overline{\Delta \boldsymbol{P}_4}) \end{vmatrix} & \begin{vmatrix} \mathrm{Im}(\Delta \boldsymbol{P}_0\overline{\Delta \boldsymbol{P}_5}) & 5\mathrm{Im}(\Delta \boldsymbol{P}_4\overline{\Delta \boldsymbol{P}_5}) \\ 5\mathrm{Im}(\Delta \boldsymbol{P}_0\overline{\Delta \boldsymbol{P}_1}) & \mathrm{Im}(\Delta \boldsymbol{P}_0\overline{\Delta \boldsymbol{P}_5}) \end{vmatrix} \end{vmatrix}.$$

Let

$$z_0 = \frac{a_2\Delta \boldsymbol{P}_0}{a_0\Delta \boldsymbol{P}_5}\left(\frac{36a_0\boldsymbol{A}\Delta \boldsymbol{P}_5 - 2a_0a_2\boldsymbol{B}^2}{324\Delta \boldsymbol{P}_0\Delta \boldsymbol{P}_5 - a_0a_2\boldsymbol{A}\boldsymbol{B}}\right)^2, \tag{3.9}$$

$$z_1 = \frac{324\Delta \boldsymbol{P}_0\Delta \boldsymbol{P}_5 - a_0a_2\boldsymbol{A}\boldsymbol{B}}{36a_0\boldsymbol{A}\Delta \boldsymbol{P}_5 - 2a_0a_2\boldsymbol{B}^2}\sqrt{\frac{6\Delta \boldsymbol{P}_5}{a_2}}, \tag{3.10}$$

$$z_2 = \sqrt{\frac{6\Delta \boldsymbol{P}_5}{a_2}}, \tag{3.11}$$

where

$$\boldsymbol{A} = \frac{30\Delta \boldsymbol{P}_1 a_0 - 12a_1\Delta \boldsymbol{P}_0}{a_0^2},$$

$$\boldsymbol{B} = \frac{30\Delta \boldsymbol{P}_4 a_2 - 12a_1\Delta \boldsymbol{P}_5}{a_2^2}.$$

Then a characteristic of class II sextic indirect-PH curves is given as following theorem.

Theorem 3.2. *A planar sextic Bézier curve is a class II sextic indirect-PH curve if and only if*

$$\begin{aligned} 6\Delta \boldsymbol{P}_0 &= a_0z_0z_1^2, \\ 30\Delta \boldsymbol{P}_1 &= a_0(z_1^2 + 2z_0z_1z_2) + 2a_1z_0z_1^2, \\ 60\Delta \boldsymbol{P}_2 &= a_0(z_0z_2^2 + 2z_1z_2) + 2a_1(z_1^2 + 2z_0z_1z_2) + z_0z_1^2, \\ 60\Delta \boldsymbol{P}_3 &= a_0z_2^2 + 2a_1(z_0z_2^2 + 2z_1z_2) + a_2(z_1^2 + 2z_0z_1z_2), \\ 30\Delta \boldsymbol{P}_4 &= 2a_1z_2^2 + a_2(z_0z_2^2 + 2z_1z_2), \\ 6\Delta \boldsymbol{P}_5 &= a_2z_2^2. \end{aligned} \tag{3.12}$$

Proof. According to Theorem 2.1, a planar curve is a class II sextic indirect-PH curve if and only if there are $a_i \in \mathbb{R}$, $z_i \in \mathbb{C}$, $i = 0, 1, 2$, such that its hodograph has form

$$\boldsymbol{P}'(t) = [a_0(1-t)^2 + 2a_1(1-t)t + a_2t^2][z_0(1-t) + t][z_1(1-t) + z_2t]^2,$$

which is expanded as

$$\begin{aligned}\boldsymbol{P}'(t) =& a_0z_0z_1^2(1-t)^5 + [a_0(z_1^2 + 2z_0z_1z_2) + 2a_1z_0z_1^2](1-t)^4t\\ &+ [a_0(z_0z_2^2 + 2z_1z_2) + 2a_1(z_1^2 + 2z_0z_1z_2) + z_0z_1^2](1-t)^3t^2\\ &+ [a_0z_2^2 + 2a_1(z_0z_2^2 + 2z_1z_2) + a_2(z_1^2 + 2z_0z_1z_2)](1-t)^2t^3\\ &+ [2a_1z_2^2 + a_2(z_0z_2^2 + 2z_1z_2)](1-t)t^4 + a_2z_2^2t^5,\end{aligned}$$

thus the conditions (3.12) are immediately followed by matching its coefficients with (2.3).

For a given Bézier curve, I have to find such a_i, z_i, $i = 0, 1, 2$, to make (3.12) hold. Note that $\rho(t) = a_0(1-t)^2 + 2a_1(1-t)t + a_2t^2$ is a common factor of $\boldsymbol{P}'(t)$ and $\overline{\boldsymbol{P}'(t)}$, so I consider a system of equations

$$\begin{aligned}\boldsymbol{P}'(t) &= 6\sum_{i=0}^{5} \Delta\boldsymbol{P}_i B_i^5(t) = 0,\\ \overline{\boldsymbol{P}'(t)} &= 6\sum_{i=0}^{5} \overline{\Delta\boldsymbol{P}_i} B_i^5(t) = 0.\end{aligned}$$

To reduce the degree of the system, I eliminate the items $(1-t)^5$ and t^5, and divide non-zero items t and $1-t$, respectively. Moreover, because

$$\Delta\boldsymbol{P}_i\overline{\Delta\boldsymbol{P}_j} - \Delta\boldsymbol{P}_j\overline{\Delta\boldsymbol{P}_i} = 2\mathrm{Im}(\Delta\boldsymbol{P}_i\overline{\Delta\boldsymbol{P}_j}),$$

I derive

$$\begin{aligned}&\mathrm{Im}(\Delta\boldsymbol{P}_0\overline{\Delta\boldsymbol{P}_5})(1-t)^4 + 5\mathrm{Im}(\Delta\boldsymbol{P}_1\overline{\Delta\boldsymbol{P}_5})(1-t)^3t + 10\mathrm{Im}(\Delta\boldsymbol{P}_2\overline{\Delta\boldsymbol{P}_5})(1-t)^2t^2 + 10\mathrm{Im}(\Delta\boldsymbol{P}_3\overline{\Delta\boldsymbol{P}_5})(1-t)t^3\\ &\quad + 5\mathrm{Im}(\Delta\boldsymbol{P}_4\overline{\Delta\boldsymbol{P}_5})t^4 = 0,\\ &5\mathrm{Im}(\Delta\boldsymbol{P}_0\overline{\Delta\boldsymbol{P}_1})(1-t)^4 + 10\mathrm{Im}(\Delta\boldsymbol{P}_0\overline{\Delta\boldsymbol{P}_2})(1-t)^3t + 10\mathrm{Im}(\Delta\boldsymbol{P}_0\overline{\Delta\boldsymbol{P}_3})(1-t)^2t^2 + 5\mathrm{Im}(\Delta\boldsymbol{P}_0\overline{\Delta\boldsymbol{P}_4})(1-t)t^3\\ &\quad + \mathrm{Im}(\Delta\boldsymbol{P}_0\overline{\Delta\boldsymbol{P}_5})t^4 = 0.\end{aligned}$$

Again, eliminating the items $(1-t)^4$ and t^4, respectively, I get

$$\begin{aligned}&\begin{vmatrix}\mathrm{Im}(\Delta\boldsymbol{P}_0\overline{\Delta\boldsymbol{P}_5}) & 5\mathrm{Im}(\Delta\boldsymbol{P}_4\overline{\Delta\boldsymbol{P}_5})\\ 5\mathrm{Im}(\Delta\boldsymbol{P}_0\overline{\Delta\boldsymbol{P}_1}) & \mathrm{Im}(\Delta\boldsymbol{P}_0\overline{\Delta\boldsymbol{P}_5})\end{vmatrix}(1-t)^3 + \begin{vmatrix}5\mathrm{Im}(\Delta\boldsymbol{P}_1\overline{\Delta\boldsymbol{P}_5}) & 5\mathrm{Im}(\Delta\boldsymbol{P}_4\overline{\Delta\boldsymbol{P}_5})\\ 10\mathrm{Im}(\Delta\boldsymbol{P}_0\overline{\Delta\boldsymbol{P}_2}) & \mathrm{Im}(\Delta\boldsymbol{P}_0\overline{\Delta\boldsymbol{P}_5})\end{vmatrix}(1-t)^2t\\ &+ \begin{vmatrix}10\mathrm{Im}(\Delta\boldsymbol{P}_2\overline{\Delta\boldsymbol{P}_5}) & 5\mathrm{Im}(\Delta\boldsymbol{P}_4\overline{\Delta\boldsymbol{P}_5})\\ 10\mathrm{Im}(\Delta\boldsymbol{P}_0\overline{\Delta\boldsymbol{P}_3}) & \mathrm{Im}(\Delta\boldsymbol{P}_0\overline{\Delta\boldsymbol{P}_5})\end{vmatrix}(1-t)t^2 + \begin{vmatrix}10\mathrm{Im}(\Delta\boldsymbol{P}_3\overline{\Delta\boldsymbol{P}_5}) & 5\mathrm{Im}(\Delta\boldsymbol{P}_4\overline{\Delta\boldsymbol{P}_5})\\ 5\mathrm{Im}(\Delta\boldsymbol{P}_0\overline{\Delta\boldsymbol{P}_4}) & \mathrm{Im}(\Delta\boldsymbol{P}_0\overline{\Delta\boldsymbol{P}_5})\end{vmatrix}t^3 = 0,\\ &\begin{vmatrix}\mathrm{Im}(\Delta\boldsymbol{P}_0\overline{\Delta\boldsymbol{P}_5}) & 5\mathrm{Im}(\Delta\boldsymbol{P}_1\overline{\Delta\boldsymbol{P}_5})\\ 5\mathrm{Im}(\Delta\boldsymbol{P}_0\overline{\Delta\boldsymbol{P}_1}) & 10\mathrm{Im}(\Delta\boldsymbol{P}_0\overline{\Delta\boldsymbol{P}_2})\end{vmatrix}(1-t)^3 + \begin{vmatrix}\mathrm{Im}(\Delta\boldsymbol{P}_0\overline{\Delta\boldsymbol{P}_5}) & 10\mathrm{Im}(\Delta\boldsymbol{P}_2\overline{\Delta\boldsymbol{P}_5})\\ 5\mathrm{Im}(\Delta\boldsymbol{P}_0\overline{\Delta\boldsymbol{P}_1}) & 10\mathrm{Im}(\Delta\boldsymbol{P}_0\overline{\Delta\boldsymbol{P}_3})\end{vmatrix}(1-t)^2t\end{aligned}$$

$$+\begin{vmatrix} \mathrm{Im}(\Delta \boldsymbol{P}_0 \overline{\Delta \boldsymbol{P}_5}) & 10\mathrm{Im}(\Delta \boldsymbol{P}_3 \overline{\Delta \boldsymbol{P}_5}) \\ 5\mathrm{Im}(\Delta \boldsymbol{P}_0 \overline{\Delta \boldsymbol{P}_1}) & 5\mathrm{Im}(\Delta \boldsymbol{P}_0 \overline{\Delta \boldsymbol{P}_4}) \end{vmatrix} (1-t)t^2 + \begin{vmatrix} \mathrm{Im}(\Delta \boldsymbol{P}_0 \overline{\Delta \boldsymbol{P}_5}) & 5\mathrm{Im}(\Delta \boldsymbol{P}_4 \overline{\Delta \boldsymbol{P}_5}) \\ 5\mathrm{Im}(\Delta \boldsymbol{P}_0 \overline{\Delta \boldsymbol{P}_1}) & \mathrm{Im}(\Delta \boldsymbol{P}_0 \overline{\Delta \boldsymbol{P}_5}) \end{vmatrix} t^3 = 0.$$

I further reduce the degree of the system using the same method, thus I can get a system of two quadric equations. If the given curve is a sextic indirect-PH curve, then both quadric equations shall be compatible, that is they both have form $k\rho(t) = 0$, for some $k \in \mathbb{R}$. Therefore, I may give a_i, $i = 0, 1, 2$, as (3.8).

Now, I substitute a_i, $i = 0, 1, 2$, into the second and fifth equations of (3.12), and I get

$$z_1^2 + 2z_0z_1z_2 = \boldsymbol{A} = \frac{30a_0\Delta \boldsymbol{P}_1 - 12a_1\Delta \boldsymbol{P}_0}{a_0^2},$$
$$z_0z_2^2 + 2z_1z_2 = \boldsymbol{B} = \frac{30a_2\Delta \boldsymbol{P}_4 - 12a_1\Delta \boldsymbol{P}_5}{a_2^2}.$$

Moreover, it is known that $a_0z_0z_1^2 = 6\Delta \boldsymbol{P}_0$, and $a_2z_2^2 = 6\Delta \boldsymbol{P}_5$, thus I get $z_2 = \sqrt{\frac{6\Delta \boldsymbol{P}_5}{a_2}}$, and

$$a_0z_1^3 - a_0\boldsymbol{A}z_1 + 12\Delta \boldsymbol{P}_0z_2 = 0,$$
$$2a_0z_1^3z_2 - a_0\boldsymbol{B}z_1^2 + 6\Delta \boldsymbol{P}_0z_2^2 = 0.$$

By eliminating the item z_1^3 and the constant item, respectively, I can derive

$$3a_0z_2z_1^2 - 2a_0\boldsymbol{B}z_1 + a_0\boldsymbol{A}z_2 = 0,$$
$$a_0\boldsymbol{B}z_1^2 - 2a_0\boldsymbol{A}z_2z_1 + 18\Delta \boldsymbol{P}_0z_2^2 = 0,$$

which gives a solution of z_1 as (3.10). Finally, by substituting it into $a_0z_0z_1^2 = 6\Delta \boldsymbol{P}_0$, I get a solution of z_0 as (3.9).

In contrast, if the given a_i, z_i, $i = 0, 1, 2$, make the system (3.12) hold, it is obvious that the given curve is a class II sextic indirect-PH curve. □

Figure 3 gives an example of a class II sextic indirect-PH curve. In this figure, auxiliary points $\boldsymbol{Q}_i$, $i = 0, \ldots, 5$, are given as follow:

$$\boldsymbol{Q}_0 = \boldsymbol{P}_1 + \frac{a_1z_0z_1^2}{15} = \boldsymbol{P}_2 - \frac{a_0\boldsymbol{A}}{30},$$
$$\boldsymbol{Q}_1 = \boldsymbol{P}_2 + \frac{a_2z_0z_1^2}{60},$$
$$\boldsymbol{Q}_2 = \boldsymbol{Q}_1 + \frac{a_1\boldsymbol{A}}{30} = \boldsymbol{P}_3 - \frac{a_0\boldsymbol{B}}{60},$$
$$\boldsymbol{Q}_3 = \boldsymbol{P}_3 + \frac{a_2\boldsymbol{A}}{60},$$
$$\boldsymbol{Q}_4 = \boldsymbol{Q}_3 + \frac{a_1\boldsymbol{B}}{30} = \boldsymbol{P}_4 - \frac{a_0z_2^2}{60},$$
$$\boldsymbol{Q}_5 = \boldsymbol{P}_4 + \frac{a_2\boldsymbol{B}}{30} = \boldsymbol{P}_5 - \frac{a_1z_2^2}{15}.$$

Therefore, I may further get

$$\begin{aligned}
&2\Delta \boldsymbol{P}_0 : 5(\boldsymbol{Q}_0 - \boldsymbol{P}_1) : 20(\boldsymbol{Q}_1 - \boldsymbol{P}_2)\\
=&(\boldsymbol{P}_2 - \boldsymbol{Q}_0) : \Delta \boldsymbol{Q}_1 : 2(\boldsymbol{Q}_3 - \boldsymbol{P}_3)\\
=&2(\boldsymbol{P}_3 - \boldsymbol{Q}_2) : \Delta \boldsymbol{Q}_3 : (\boldsymbol{Q}_5 - \boldsymbol{P}_4)\\
=&20(\boldsymbol{P}_4 - \boldsymbol{Q}_3) : 5(\boldsymbol{P}_5 - \boldsymbol{Q}_5) : 2\Delta \boldsymbol{P}_5\\
=&a_0 : a_1 : a_2.
\end{aligned}$$

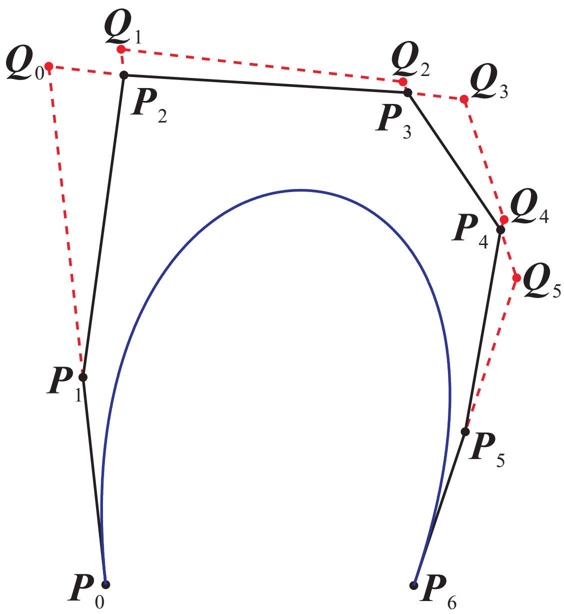

Figure 3. An example of a class II sextic indirect-PH curve with its Bézier control points and auxiliary points.

In Figure 4(a), I assign $\boldsymbol{P}_0 = 0$, $\boldsymbol{P}_1 = -0.1667 + 1.5167\mathbf{i}$, $\boldsymbol{P}_2 = 0.1167 + 3.8797\mathbf{i}$, $\boldsymbol{P}_3 = 2.3157 + 3.8716i\mathbf{i}$, $\boldsymbol{P}_4 = 3.2626+2.7807\mathbf{i}$, $\boldsymbol{P}_5 = 3.0484+1.0170\mathbf{i}$, $\boldsymbol{P}_6 = 2.4465-0.7830\mathbf{i}$. I have $a_0 = -238.5820$, $a_1 = -954.3280$, $a_2 = -572.5968$, $z_0 = 2 - 0.8\mathbf{i}$, $z_1 = -0.1144 + 0.0687\mathbf{i}$, $z_2 = 0.1144 + 0.0824\mathbf{i}$, the differences between the right- and left-hand sides of equations in (3.12) are 1.6653×10^{-16}, 1.1102×10^{-16}, 1.9861×10^{-15}, 8.8818×10^{-16}, 4.4409×10^{-16}, 0, respectively, so the curve can be classified as a class II sextic indirect-PH curve. Moreover, it is clear the real roots of $\rho(t) = 0$ are the parameter values of the cusps. However, in this example, the curve is a regular curve because $a_1^2 - a_0 a_2 < 0$.

In Figure 4(b), I assign $\boldsymbol{P}_0 = 0$, $\boldsymbol{P}_1 = -0.1667 + 1.5167\mathbf{i}$, $\boldsymbol{P}_2 = 0.1167 + 2.8797\mathbf{i}$, $\boldsymbol{P}_3 = 2.3157 + 2.8716i\mathbf{i}$, $\boldsymbol{P}_4 = 3.2626+1.7807\mathbf{i}$, $\boldsymbol{P}_5 = 3.0484+1.0170\mathbf{i}$, $\boldsymbol{P}_6 = 2.4465-0.7830\mathbf{i}$. I have $a_0 = -982.4731$, $a_1 = -2022.3$, $a_2 = -2093.1$, $z_0 = 1.2214 + 1.0446\mathbf{i}$, $z_1 = -0.0356 + 0.0673\mathbf{i}$, $z_2 = 0.0599 + 0.0431\mathbf{i}$, the differences between the right- and left-hand sides of equations in (3.12) are 7.5758×10^{-16}, 4.7429×10^{-16}, 1.4714, 1.1718, 0, 4.4409×10^{-16}, respectively, so the curve is not a class II sextic indirect-PH curve.

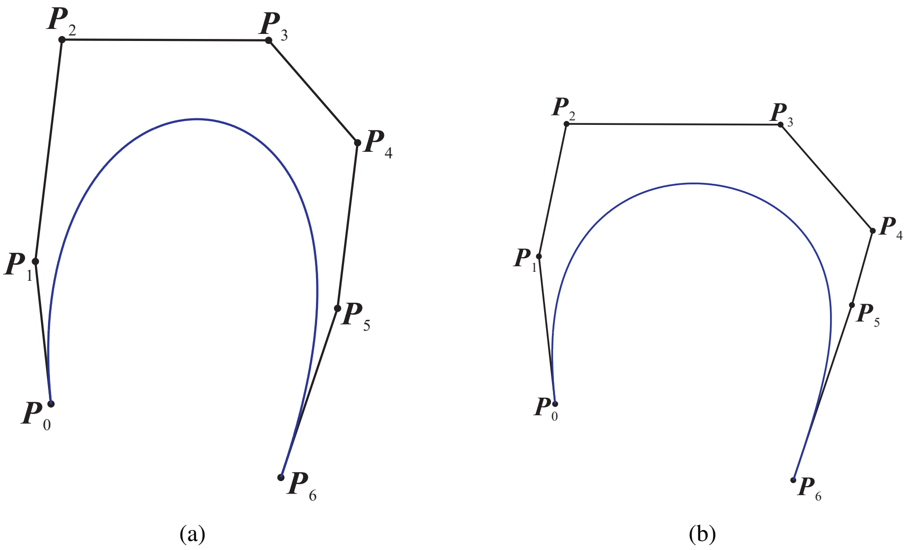

Figure 4. Test of class II sextic indirect-PH curves. (a) The curve is a class II sextic indirect-PH curve. (b) The curve is not a class II sextic indirect-PH curve.

3.3. Class III sextic indirect-PH curve

For a class III sextic indirect-PH curve, let $\boldsymbol{P}_i$, $i = 0, \ldots, 6$, be its Bézier control points, then auxiliary points $\boldsymbol{Q}_i$, $i = 0, 1, 2, 3$, are constructed such that points $\boldsymbol{Q}_0$ and $\boldsymbol{Q}_3$ are on lines $\boldsymbol{P}_0\boldsymbol{P}_1$ and $\boldsymbol{P}_5\boldsymbol{P}_6$, respectively, and they further satisfy $\boldsymbol{P}_0\boldsymbol{P}_1 /\!/ \boldsymbol{P}_2\boldsymbol{Q}_1 /\!/ \boldsymbol{P}_3\boldsymbol{Q}_2 /\!/ \boldsymbol{P}_4\boldsymbol{Q}_3$ and $\boldsymbol{Q}_0\boldsymbol{P}_2 /\!/ \boldsymbol{Q}_1\boldsymbol{P}_3 /\!/ \boldsymbol{Q}_2\boldsymbol{P}_4 /\!/ \boldsymbol{P}_5\boldsymbol{P}_6$, see Figure 5. Thus, a necessary and sufficient condition for sextic indirect-PH curves is given as following theorem.

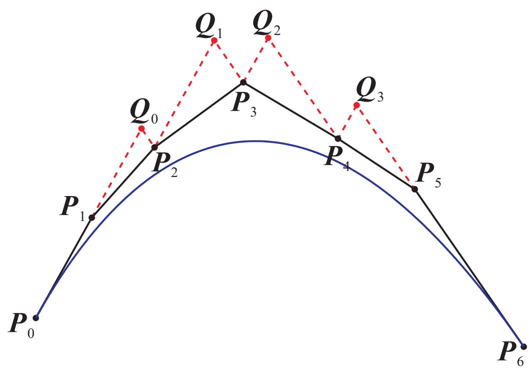

Figure 5. An example of a class III sextic indirect-PH curve with its Bézier control points and auxiliary points.

Theorem 3.3. *A planar Bézier curve is a class III sextic indirect-PH curve if and only if*

$$
\begin{aligned}
&12\Delta\boldsymbol{P}_0 : 15(\boldsymbol{Q}_0 - \boldsymbol{P}_1) : 20(\boldsymbol{Q}_1 - \boldsymbol{P}_2) : 30(\boldsymbol{Q}_2 - \boldsymbol{P}_3) : 60(\boldsymbol{Q}_3 - \boldsymbol{P}_4)\\
=&60(\boldsymbol{P}_2 - \boldsymbol{Q}_0) : 30(\boldsymbol{P}_3 - \boldsymbol{Q}_1) : 20(\boldsymbol{P}_4 - \boldsymbol{Q}_2) : 15(\boldsymbol{P}_5 - \boldsymbol{Q}_3) : 12\Delta\boldsymbol{P}_5.
\end{aligned}
\tag{3.13}
$$

Proof. The hodograph of a class III sextic indirect-PH curve has form

$$
\boldsymbol{P}'(t) = [a_0(1-t)^4 + 4a_1(1-t)^3 t + 6a_2(1-t)^2 t^2 + 4a_3(1-t)t^3 + a_4 t^4][z_0(1-t) + z_1 t],
$$

for some $a_i \in \mathbb{R}$, $i = 0, \ldots, 4$, and $z_j \in \mathbb{C}$, $j = 0, 1$, which is expanded as

$$\begin{aligned}\boldsymbol{P}'(t) =& a_0 z_0 (1-t)^5 + (a_0 z_1 + 4a_1 z_0)(1-t)^4 t + (4a_1 z_1 + 6a_2 z_0)(1-t)^3 t^2 + (6a_2 z_1 + 4a_3 z_0)(1-t)^2 t^3 \\ &+ a_4(4a_3 z_1 + z_0)(1-t)t^4 + a_4 z_1 t^5.\end{aligned}$$

By matching its coefficients with (2.3), I get

$$\begin{aligned}6\Delta\boldsymbol{P}_0 &= a_0 z_0, \\ 30\Delta\boldsymbol{P}_1 &= a_0 z_1 + 4a_1 z_0, \\ 60\Delta\boldsymbol{P}_2 &= 4a_1 z_1 + 6a_2 z_0, \\ 60\Delta\boldsymbol{P}_3 &= 6a_2 z_1 + 4a_3 z_0, \\ 30\Delta\boldsymbol{P}_4 &= 4a_3 z_1 + a_4 z_0, \\ 6\Delta\boldsymbol{P}_5 &= a_4 z_1.\end{aligned}$$

Let

$$\begin{aligned}\boldsymbol{Q}_0 &= \boldsymbol{P}_1 + \frac{2a_1}{15} z_0 = \boldsymbol{P}_2 - \frac{a_0}{30} z_1, \\ \boldsymbol{Q}_1 &= \boldsymbol{P}_2 + \frac{a_2}{10} z_0 = \boldsymbol{P}_3 - \frac{a_1}{15} z_1, \\ \boldsymbol{Q}_2 &= \boldsymbol{P}_3 + \frac{a_3}{15} z_0 = \boldsymbol{P}_4 - \frac{a_2}{10} z_1, \\ \boldsymbol{Q}_3 &= \boldsymbol{P}_4 + \frac{a_4}{30} z_0 = \boldsymbol{P}_5 - \frac{2a_3}{15} z_1,\end{aligned} \tag{3.14}$$

then

$$\begin{aligned}&12\Delta\boldsymbol{P}_0 : 15(\boldsymbol{Q}_0 - \boldsymbol{P}_1) : 20(\boldsymbol{Q}_1 - \boldsymbol{P}_2) : 30(\boldsymbol{Q}_2 - \boldsymbol{P}_3) : 60(\boldsymbol{Q}_3 - \boldsymbol{P}_4) = a_0 : a_1 : a_2 : a_3 : a_4, \\ &60(\boldsymbol{P}_2 - \boldsymbol{Q}_0) : 30(\boldsymbol{P}_3 - \boldsymbol{Q}_1) : 20(\boldsymbol{P}_4 - \boldsymbol{Q}_2) : 15(\boldsymbol{P}_5 - \boldsymbol{Q}_3) : 12(\Delta\boldsymbol{P}_5) = a_0 : a_1 : a_2 : a_3 : a_4,\end{aligned}$$

which immediately gives (3.13).

In contrast, if there are points $\boldsymbol{Q}_i$, $i = 0, 1, 2, 3$, such that (3.13) holds, then a_i, $i = 0, \ldots, 4$, are not difficult to be determined, e.g., I may suppose $a_0 = 1$ or $a_4 = 1$. Following (3.14), I can get

$$\begin{aligned}z_0 &= \frac{6\Delta\boldsymbol{P}_0}{a_0} = \frac{15(\boldsymbol{Q}_0 - \boldsymbol{P}_1)}{2a_1} = \frac{10(\boldsymbol{Q}_1 - \boldsymbol{P}_2)}{a_2} = \frac{15(\boldsymbol{Q}_2 - \boldsymbol{P}_3)}{a_3} = \frac{30(\boldsymbol{Q}_3 - \boldsymbol{P}_4)}{a_4}, \\ z_1 &= \frac{30(\boldsymbol{P}_2 - \boldsymbol{Q}_0)}{a_0} = \frac{15(\boldsymbol{P}_3 - \boldsymbol{Q}_1)}{a_1} = \frac{10(\boldsymbol{P}_4 - \boldsymbol{Q}_2)}{a_2} = \frac{15(\boldsymbol{P}_5 - \boldsymbol{Q}_3)}{2a_3} = \frac{6\Delta\boldsymbol{P}_5}{a_4}.\end{aligned}$$

□

In Figure 6(a), I give an example that $\boldsymbol{P}_0 = 0$, $\boldsymbol{P}_1 = 0.6782 + 1.5224\mathbf{i}$, $\boldsymbol{P}_2 = 1.1537 + 1.8361\mathbf{i}$, $\boldsymbol{P}_3 = 1.7648 + 2.4542\mathbf{i}$, $\boldsymbol{P}_4 = 2.5129 + 1.8728\mathbf{i}$, $\boldsymbol{P}_5 = 3.0569 + 1.5867\mathbf{i}$, $\boldsymbol{P}_6 = 4.0777 + 0.1101\mathbf{i}$. Thus, I have

$$\begin{aligned}&12\Delta\boldsymbol{P}_0 : 15(\boldsymbol{Q}_0 - \boldsymbol{P}_1) : 20(\boldsymbol{Q}_1 - \boldsymbol{P}_2) : 30(\boldsymbol{Q}_2 - \boldsymbol{P}_3) : 60(\boldsymbol{Q}_3 - \boldsymbol{P}_4) \\ =&60(\boldsymbol{P}_2 - \boldsymbol{Q}_0) : 30(\boldsymbol{P}_3 - \boldsymbol{Q}_1) : 20(\boldsymbol{P}_4 - \boldsymbol{Q}_2) : 15(\boldsymbol{P}_5 - \boldsymbol{Q}_3) : 12\Delta\boldsymbol{P}_5 \\ =&2 : 1 : 2 : 1 : 2,\end{aligned}$$

which means the curve is a class III sextic indirect-PH curve.

In Figure 6(b), I assign $\boldsymbol{P}_0 = 0$, $\boldsymbol{P}_1 = 0.6782+1.5224\mathbf{i}$, $\boldsymbol{P}_2 = 1.1537+2.0361\mathbf{i}$, $\boldsymbol{P}_3 = 1.7648+2.3542\mathbf{i}$, $\boldsymbol{P}_4 = 2.5129 + 2.0728\mathbf{i}$, $\boldsymbol{P}_5 = 3.0569 + 1.5867\mathbf{i}$, $\boldsymbol{P}_6 = 4.0777 + 0.1101\mathbf{i}$ and I have

$$\begin{aligned}
&12\Delta\boldsymbol{P}_0 : 15(\boldsymbol{Q}_0 - \boldsymbol{P}_1) : 20(\boldsymbol{Q}_1 - \boldsymbol{P}_2) : 30(\boldsymbol{Q}_2 - \boldsymbol{P}_3) : 60(\boldsymbol{Q}_3 - \boldsymbol{P}_4)\\
&= 1.6551 : 0.9989 : 1.3326 : 1.3314 : 1,\\
&60(\boldsymbol{P}_2 - \boldsymbol{Q}_0) : 30(\boldsymbol{P}_3 - \boldsymbol{Q}_1) : 20(\boldsymbol{P}_4 - \boldsymbol{Q}_2) : 15(\boldsymbol{P}_5 - \boldsymbol{Q}_3) : 12\Delta\boldsymbol{P}_5\\
&= 0.7346 : 0.6990 : 0.8673 : 0.5663 : 1,
\end{aligned}$$

which are not equal, so the curve is not a class III sextic indirect-PH curve.

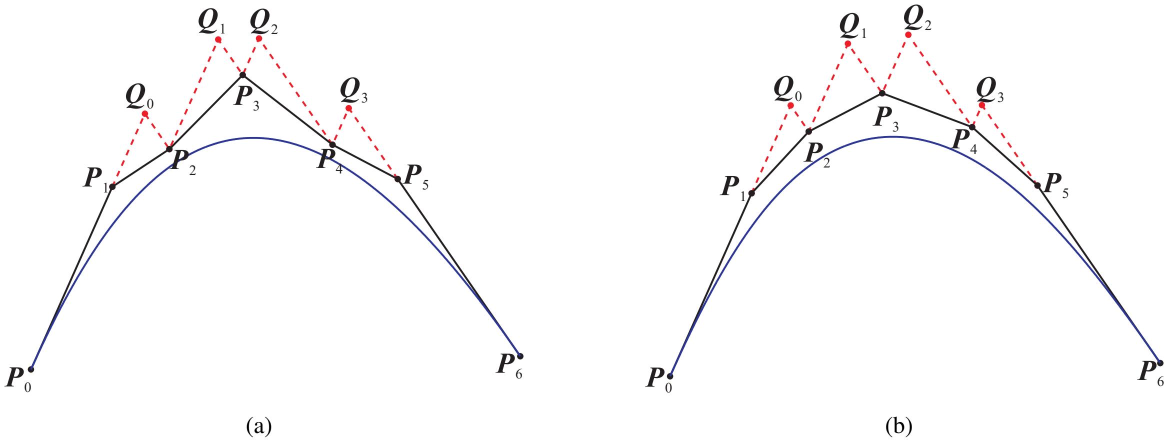

Figure 6. Test of class III sextic indirect-PH curves. (a) The curve is a class III sextic indirect-PH curve. (b) The curve is not a class III sextic indirect-PH curve.

4. Conclusions

In this paper, I study characteristics of sextic indirect-PH curves. Following our results, whether a sextic polynomial curve is an indirect-PH curve or not can be determined by its Bézier control polygon. The methods I used are quite fundamental, which can be extended to study geometric properties of other polynomial curves with rational offsets.

Note that several open problems need to be addressed for future research to advance. First, the geometric construction of sextic indirect-PH curves is not yet clear. For example, given endpoints with specified continuous constraints, constructing a curve that meets the indirect-PH property is possible. Second, it requires developing new theories and techniques to give a unified approach to identifying indirect PH curves of an arbitrary degree. Finally, how to apply indirect-PH curves to construct spline curves can further enable designers to complete free geometry modeling.

Use of AI tools declaration

The author declares that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares to have no competing interests.

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