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Research article

Spectral solutions for the time-fractional heat differential equation through a novel unified sequence of Chebyshev polynomials

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Abstract: In this article, we propose two numerical schemes for solving the time-fractional heat equation (TFHE). The proposed methods are based on applying the collocation and tau spectral methods. We introduce and employ a new set of basis functions: The unified Chebyshev polynomials (UCPs) of the first and second kinds. We establish some new theoretical results regarding the new UCPs. We employ these results to derive the proposed algorithms and analyze the convergence of the proposed double expansion. Furthermore, we compute specific integer and fractional derivatives of the UCPs in terms of their original UCPs. The derivation of these derivatives will be the fundamental key to deriving the proposed algorithms. We present some examples to verify the efficiency and applicability of the proposed algorithms.

Keywords: Chebyshev polynomials; recurrence relation; spectral methods; fractional differential equations; convergence analysis
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1. Introduction

Many different areas of mathematics and applied sciences can benefit from Chebyshev polynomials (CPs). For example, the area of approximation theory frequently employs CPs. Moreover, they are very helpful in numerical analysis. Spectral methods can utilize CPs and their various combinations as basis functions to obtain numerical solutions for various differential equations. These methods have the potential to achieve both rapid convergence and highly efficient solutions. For some articles that employ different types of CPs, see, for example, [1–5].

Fractional differential equations (FDEs) are crucial in different disciplines of the applied sciences.

In fact, they describe many phenomena that cannot be described by ordinary differential equations. This is due to their ability to model complex phenomena involving memory and hereditary properties. For example, they model several biological and physiological processes, such as tumor growth and the behavior of neurons (see [6]). These equations are also used to simulate anomalous diffusion, wave propagation in complex media and electromagnetic phenomena (see [7]). In addition, the complicated mechanical reaction of viscoelastic materials under stress or strain has been frequently modeled using FDEs (see [8]). The use of fractional calculus has been seen in the domain of signal processing, namely in the areas of denoising, filtering and feature extraction; see, for example, [9].

Due to the importance of partial FDEs and the non-availability of solving them analytically in most cases, a lot of effort has been put into creating trustworthy numerical and analytical methods for treating these types of equations. Researchers have presented various methods, such as the Adomian decomposition method [10], the operational matrix methods [11, 12] and the splines method [13], to solve different partial FDEs.

The various types of time-FDEs have been studied by many researchers. For example, the authors in [14] employed a finite difference method for treating the time-fractional diffusion equation. In [15], the authors followed an approach for treating the time-fractional Fisher's equations. The authors in [16] followed another approach for treating some types of time-fractional PDE. An integral method was followed in [17] for treating some time-FDEs. In [18], the authors treated other space-time FDEs. The authors in [19] combined the dual reciprocity method and the Laplace transformation approach with the singular boundary method to obtain solutions to anomalous heat conduction issues in functionally graded materials. In [20], the authors introduced a novel localized collocation method utilizing fundamental solutions to analyze long-term anomalous heat conduction in functionally graded materials. In [21], the authors followed a quadratic spline collocation method for the time fractional subdiffusion equation. Another approach is followed in [22] to handle the fractional-diffusion equation. In [23], the authors followed a certain collocation method for the time tempered fractional diffusion equation.

Among the important time-FDEs is TFHE. Researchers have utilized different numerical algorithms to solve this equation. For instance, the authors in [24] utilized an implicit difference scheme to handle the TFHE. The authors of [25] followed an approach for treating the TFHE. In [26], the authors employed another collocation algorithm to treat the same equation. Because the Caputo derivative is not local, solving TFHE is notoriously hard and takes a long time. For this reason, fast and parallel numerical solutions for these kinds of TFHEs are desirable [27, 28].

Various types of DEs, including PDEs, can be treated numerically using different versions of spectral methods due to their high accuracy and versatility. When compared to other numerical approaches used to solve PDEs, spectral methods have many benefits. They provide high precision and efficient solutions since the error drops exponentially as you add more terms to the proposed expansion. In these methods, the numerical solution is expressed as combinations of different special functions, which are called basis functions. The choice of suitable basis functions depends on the spectral method that will be applied. For some books regarding the different spectral methods and their applications, one can be referred to [29–33]. There are three celebrated spectral approaches. The Galerkin method has some restrictions on choosing the basis functions; see, for example, [34–38], where such restrictions do not exist when collocation and tau methods are applied; see, for example, [39–43].

In this paper, we will introduce a new type of polynomial that generalizes the first and second kinds of CPs. These polynomials are new and differ from the existing generalizing polynomials of CPs, such as Gegenbauer and Jacobi polynomials. This motivates us to study and employ such polynomials. Furthermore, we have two advantages to using these polynomials:

- Several solutions can be obtained if these polynomials are used as basis functions due to the presence of two parameters.
- If these polynomials are used to treat TFHE, it will be shown that the Chebyshev first and second kinds of approximations are not always the best among the other approximations. This demonstrates the benefit of introducing such polynomials.

The article's primary aims can be summed up as follows:

(i) We introduce a new type of polynomials, named unified Chebyshev polynomials (UCPs), that unify CPs of the first and second kinds.

(ii) We implement some new formulas related to the UCPs and the shifted ones that are essential for our suggested algorithms.

(iii) We utilize the introduced polynomials along with the collocation and tau spectral methods to treat the TFHE.

The rest of the paper is organized as follows: The next section gives an overview of CPs and a new set of UCPs, as well as some definitions of fractional calculus. In Section 3, we present new formulas for the UCPs and the shifted ones that are necessary for our proposed algorithms. In Section 4.1, we employ the spectral tau method to treat the TFHE. In Section 4.2, we employ another collocation procedure to solve the same type of equation. In Section 5, we deeply discuss convergence and error analysis. In Section 6, we present several numerical examples that involve tables, figures and comparisons. Finally, Section 7 reports some conclusions.

2. Some fundamentals and preliminaries

In this section, the fractional differential operator in the Caputo sense is presented, and some useful properties are utilized throughout the paper. In addition, some essentials regarding the CPs are presented.

2.1. Some definitions of fractional calculus

Definition 2.1. *[6, 8] The fractional differential operator in Caputo sense is defined as*

$$(D^{\nu} f)(t) = \begin{cases} \dfrac{1}{\Gamma(n-\nu)} \displaystyle\int_0^t (t-\tau)^{n-\nu-1} f^{(n)}(\tau)\, d\tau, & \nu > 0,\ t > 0, \\[2ex] f^{(n)}(t), & \nu = n, \end{cases} \tag{2.1}$$

where $n-1 \le \nu \le n, n \in \mathbb{N}$.

Here are some properties that are satisfied by D^ν for $n-1 \leq \nu \leq n$,

$$D^\nu(\mu_1 f(t) + \mu_2 g(t)) = \mu_1\, D^\nu f(t) + \mu_2\, D^\nu g(t), \tag{2.2}$$

$$D^\nu\, t^k = \begin{cases} \dfrac{\Gamma(k+1)}{\Gamma(k+1-\nu)}\, t^{k-\nu}, & k \in \mathbb{N}, k \geq \lceil \nu \rceil, \\ 0, & k < \lceil \nu \rceil, \end{cases} \tag{2.3}$$

where $\lceil \nu \rceil$ denotes the smallest integer greater than or equal to ν. For more properties of fractional derivatives, see, for example, [44].

2.2. An overview on CPs as a type of new UCPs

It is well known that the well-known four kinds of CPs are all particular types of Jacobi polynomials (see, [45]). All of these polynomials satisfy the following recurrence relation:

$$\phi_k(\theta) = 2\,\theta\,\phi_{k-1}(\theta) - \phi_{k-2}(\theta), \quad k \geq 2, \quad \theta \in [-1, 1], \tag{2.4}$$

but with the following different initial values:

$$T_0(\theta) = 1,\ T_1(\theta) = \theta, \quad U_0(\theta) = 1,\ U_1(\theta) = 2\theta,$$

$$V_0(\theta) = 1,\ V_1(\theta) = 2\theta - 1, \quad W_0(\theta) = 1,\ W_1(\theta) = 2\theta + 1,$$

where $T_i(\theta), U_i(\theta), V_i(\theta)$ and $W_i(\theta)$ denote, respectively, the four kinds of CPs, each of degree i.

Among the important properties of CPs is that $\phi_{-j}(\theta),\ j \geq 0$ can be expressed in terms of $\phi_j(\theta)$. In fact, we have

$$T_{-j}(\theta) = T_j(\theta), \quad U_{-j}(\theta) = -U_{j-2}(\theta),$$
$$V_{-j}(\theta) = V_{j-1}(\theta), \quad W_{-j}(\theta) = -W_{j-1}(\theta).$$

In this paper, we will introduce a new type of polynomials that unify the first- and second-kinds of CPs. These polynomials will be referred to as "Unified Chebsyhev polynomials (UCPs)". The sequence of UCP, $G_k^{A,B}(\theta),\ A, B > 0$, may be constructed using the recurrence relation:

$$G_{k+1}^{A,B}(\theta) = 2\,\theta\, G_{k-1}^{A,B}(\theta) - G_k^{A,B}(\theta), \quad G_0^{A,B}(\theta) = A,\ G_1^{A,B}(\theta) = B\,\theta,\ k \geq 1. \tag{2.5}$$

The first few UCPs $G_k^{A,B}(\theta),\ k = 2, 3, \ldots, 7$, are:

$$\begin{aligned}
G_2^{A,B}(\theta) &= 2B\,\theta^2 - A,\\
G_3^{A,B}(\theta) &= 4B\,\theta^3 - (2A + B)\,\theta,\\
G_4^{A,B}(\theta) &= 8B\,\theta^4 - 4(A + B)\,\theta^2 + A,\\
G_5^{A,B}(\theta) &= 16B\,\theta^5 - 4(2A + 3B)\,\theta^3 + (4A + B)\,\theta,\\
G_6^{A,B}(\theta) &= 32B\,\theta^6 - 8(2A + 4B)\,\theta^4 + 3(4A + 2B)\,\theta^2 - A,\\
G_7^{A,B}(\theta) &= 64B\,\theta^7 - 16(2A + 5B)\,\theta^5 + 8(4A + 3B)\,\theta^3 - (6A + B)\,\theta.
\end{aligned}$$

Remark 2.1. *It is evident that the UCPs are generalizations of both the first- and second-kind CPs. We have*

$$T_k(\theta) = G_k^{1,1}(\theta), \quad U_k(\theta) = G_k^{2,1}(\theta).$$

3. Some new formulas concerned with UCPs

In this section, we will establish some new formulas concerned with UCPs and their shifted polynomials on $[0, \ell]$ that will be employed in the next section to derive our proposed algorithms.

3.1. Some new results for the UCPs

We are going to state and prove a basic theorem regarding the UCPs. The UCPs can be expressed as a combination of two terms of CPs of the second kind, as we will demonstrate. The following theorem exhibits this important result.

Theorem 3.1. *The following expression for the UCPs is valid*

$$G_n^{A,B}(\theta) = \frac{1}{2}(B-2A)U_{n-2}(\theta) + \frac{B}{2}\,U_n(\theta),\ n \geq 0. \tag{3.1}$$

Proof. Consider the polynomial:

$$\xi_n(\theta) = \frac{1}{2}(B-2A)U_{n-2}(\theta) + \frac{B}{2}\,U_n(\theta).$$

It is easy to see that $\xi_0(\theta) = G_0^{A,B}(\theta)$ and $\xi_1(\theta) = G_1^{A,B}(\theta)$. Now, we are going to prove that $\xi_n(\theta) = G_n^{A,B}(\theta),\ \forall\, n \geq 2$. We will prove that they have the same recursive formula. That is, we will prove that

$$\xi_{n+2}(\theta) - 2\,\theta\,\xi_{n+1}(\theta) + \xi_n(\theta) = 0.$$

We have

$$\begin{aligned}\xi_{n+2}(\theta) - 2\,\theta\,\xi_{n+1}(\theta) + \xi_n(\theta) =& \frac{1}{2}(B-2A)U_n(\theta) + \frac{1}{2}BU_{n+2}(\theta)\\ &- 2\,\theta\left(\frac{1}{2}(B-2A)U_{n-1}(\theta) + \frac{1}{2}BU_{n+1}(\theta)\right)\\ &+ \frac{1}{2}(B-2A)U_{n-2}(\theta) + \frac{B}{2}\,U_n(\theta).\end{aligned}$$

Using the well-known formula

$$\theta\,U_n(\theta) = \frac{1}{2}\left(U_{n-1}(\theta) + U_{n+1}(\theta)\right),$$

we see that

$$\xi_{n+2}(\theta) - 2\,\theta\,\xi_{n+1}(\theta) + \xi_n(\theta) = 0.$$

Theorem 3.1 is now proved. $\square$

Theorem 3.2. *The analytic form of $G_n^{A,B}(\theta)$ is:*

$$G_i^{A,B}(\theta) = \sum_{r=0}^{\lfloor \frac{i}{2} \rfloor} \frac{(-1)^r 2^{i-2r-1}(B(i-2r) + 2Ar)(1+i-2r)_{r-1}}{r!}\theta^{i-2r}. \tag{3.2}$$

Proof. Formula (3.2) can be obtained directly from the expression in (3.1) along with the analytic form of $U_i(\theta)$ given by

$$U_i(\theta) = \sum_{r=0}^{\lfloor \frac{i}{2} \rfloor} \frac{(-1)^r\, 2^{i-2r}\, (i-r)!}{(i-2r)!\, r!}\, \theta^{i-2r}. \tag{3.3}$$

□

The following theorem gives the connection formula between the second-kind CPs $U_n(\theta)$ and the UCPs.

Theorem 3.3. *For every non-negative integer j, the following expression for $U_j(\theta)$ holds*

$$U_j(\theta) = 2 \sum_{r=0}^{\lfloor \frac{j}{2} \rfloor} \frac{c_{j-2r}\,(2A-B)^r}{B^{r+1}}\, G_{j-2r}(\theta), \tag{3.4}$$

where

$$c_i = \begin{cases} \dfrac{B}{2A}, & i = 0, \\ 1. & i \geq 1. \end{cases} \tag{3.5}$$

Proof. The proof can be easily accomplished by induction on j. □

Now, we will give the inversion formula to $G_n^{A,B}(\theta)$ in the following theorem.

Theorem 3.4. *For every non-negative integer m, the following formula holds*

$$\theta^m = \sum_{i=0}^{\lfloor \frac{m}{2} \rfloor} c_{m-2i}\, S_{i,m}\, G^{A,B}_{m-2i}(\theta), \tag{3.6}$$

where

$$S_{i,m} = A^i B^{-i-1} 2^{1-m+i} + \sum_{j=1}^{i} \frac{A^{i-j} B^{j-i-1} 2^{1-m-j+i} (m-j-i)(m-i+1)_{j-1}}{j!}. \tag{3.7}$$

Proof. The proof is based on making use of the inversion formula of $U_j(\theta)$ given by

$$\theta^j = 2^{-j} \sum_{m=0}^{\lfloor \frac{j}{2} \rfloor} \frac{(1+j-2m)\, j!}{(j-m+1)!\, m!} U_{j-2m}(\theta), \tag{3.8}$$

along with the connection formula that is given by (3.4). □

3.2. Some formulas of the shifted UCPs

In this paper, it is useful to define the so-called shifted UCPs on $[0, \ell]$ as

$$G^{A,B}_{n,\ell}(x) = G^{A,B}_{n}\left(\frac{2\,x}{\ell} - 1\right), \quad n \geq 1,\ 0 \leq x \leq \ell.$$

According to this definition, it is easy to see from (3.1) that $G^{A,B}_{n,\ell}(x)$ can be expressed in terms of the shifted second-kind CPs as in the following corollary:

Corollary 3.1. *For every positive integer n, one has*

$$G^{A,B}_{n,\ell}(x) = \frac{1}{2}(B - 2A)U_{n-2,\ell}(x) + \frac{B}{2}\,U_{n,\ell}(x),\ n \geq 0, \tag{3.9}$$

where $U_{n,\ell}(x) = U_{n,1}\left(\frac{2\,x}{\ell} - 1\right)$ *is the shifted CPs of second kind.*

Proof. Formula (3.9) is a direct consequence of (3.1) by replacing x by $\left(\frac{2\,x}{\ell} - 1\right)$. □

The following two corollaries are of interest hereafter. They are regarding the analytic and inversion formulas of the shifted polynomials $G^{A,B}_{n,\ell}(x)$.

Theorem 3.5. *Let n be a positive integer.* $G^{A,B}_{n,\ell}(x)$ *has the following analytic formula*

$$G^{A,B}_{n,\ell}(x) = \sum_{m=0}^{n} d^{(n)}_{m}\,\ell^{-m}\,x^{m}, \tag{3.10}$$

where

$$d^{(n)}_{m} = \begin{cases} (-1)^{n}(A + (B - A)\,n), & m = 0, \\ \dfrac{4^{m}(-1)^{n-m}(m + n - 1)!\left((B - A)\left(m^{2} + m + n^{2}\right) + A(2m + 1)n\right)}{(2m + 1)!\,(n - m)!}, & m = 1, 2, \ldots, n. \end{cases} \tag{3.11}$$

Proof. Using the analytical expansion of $U_{n,\ell}(x)$:

$$U_{n,\ell}(x) = \sum_{i=0}^{n}(-1)^{i}\left(\frac{4}{\ell}\right)^{n-i}\frac{(2n - i + 1)!}{i!(2n - 2i + 1)!}\,x^{n-i},\ n > 0, \tag{3.12}$$

together with (3.9), formula (3.10) can be obtained. □

In the following theorem, we give the inversion formula of $G^{A,B}_{n,\ell}(x)$ which will play a pivotal role in investigating the convergence analysis of the proposed expansion.

Theorem 3.6. *For every positive integer m, the following inversion formula holds*

$$x^{m} = \sum_{p=0}^{m} F_{p,m}\,G^{A,B}_{p,\ell}(x), \tag{3.13}$$

where

$$F_{p,m} = c_{p}(2m + 1)!\,\ell^{m}\sum_{k=0}^{\lfloor\frac{m-p}{2}\rfloor}\frac{4^{1-m}B^{-k-1}(2A - B)^{k}(2k + p + 1)}{(m - p - 2k)!(2k + m + p + 2)!}, \tag{3.14}$$

and c_p *is as defined in* (3.5).

Proof. The proof is based on making use of the inversion formula of the shifted second kind of CPs $U_{j,\ell}(x)$ on $[0, \ell]$ given by

$$x^m = (2m+1)!\,\ell^m\, 2^{1-2m} \sum_{p=0}^{m} \frac{(p+1)}{(m-p)!(m+p+2)!} U_{p,\ell}(x), \tag{3.15}$$

along with the connection formula

$$U_{j,\ell}(x) = 2 \sum_{r=0}^{\left\lfloor \frac{j}{2} \right\rfloor} \frac{c_{j-2r}\,(2A-B)^r}{B^{r+1}}\, G_{j-2r,\ell}(x), \tag{3.16}$$

which can be obtained form (3.4) only if $(\frac{2x}{\ell} - 1)$ is substituted instead of x. □

Remark 3.1. *It is worth mentioning here that when $B = 2A$, we get*

$$G_n^{A,2A}(x) = A\, U_n(x), \qquad G_{n,\ell}^{A,2A}(x) = A\, U_{n,\ell}(x),$$

while when $B = A$, we get

$$G_n^{A,A}(x) = A\, T_n(x), \qquad G_{n,\ell}^{A,A}(x) = A\, T_{n,\ell}(x).$$

The following two theorems are pivotal in deriving our proposed numerical algorithms. They consist of the integer and fractional derivatives of $G_{j,\ell}^{A,B}(x)$.

Theorem 3.7. *The qth derivative of $G_{j,\ell}^{A,B}(x)$ can be expressed as:*

$$D^q\, G_{n,\ell}^{A,B}(x) = \sum_{m=0}^{n-q} h_{m,n}^{(q)}\, G_{m,\ell}^{A,B}(x),\ n \geq q \geq 1, \tag{3.17}$$

where

$$h_{m,n}^{(q)} = \sum_{r=m}^{n-q} \frac{(r+q)!}{r!\,\ell^{r+q}}\, d_{r+q}^{(n)}\, F_{m,r}.$$

Proof. The proof is a direct consequence of Theorems 3.5 and 3.6. □

Theorem 3.8. *Let α be a positive real number. We have*

$$D^\alpha\, G_{n,\ell}^{A,B}(x) = x^{-\alpha} \sum_{m=0}^{n} \tilde{h}_{m,n}^{(\alpha)} G_{m,\ell}^{A,B}(x),\ n > \alpha > 0, \tag{3.18}$$

where

$$\tilde{h}_{m,n}^{(\alpha)} = \sum_{r=max(m,\lceil\alpha\rceil)}^{n} \frac{r!}{\ell^r\, \Gamma(r+1-\alpha)}\, d_r^{(n)}\, F_{m,r}. \tag{3.19}$$

Proof. The proof of (3.18) can be obtained by applying the fractional derivative D^α to (3.10) and using (2.2), relation (3.13) leads to (3.18) and the proof of theorem is complete. □

3.3. Some useful integral formulas involve the UCPs

In this part, we will develop some useful formulas that will be utilized in the derivation of our proposed algorithms.

Lemma 3.1. *For every real positive real number β, and positive integers n and m, the following integral formula holds:*

$$\int_0^{\ell} \omega_{\ell}(x) x^{\beta} U_{n,\ell}(x)\, dx = I(n,\beta,\ell), \quad n = 0,1,2,\ldots, \tag{3.20}$$

with

$$I(n,\beta,\ell) = \frac{\sqrt{\pi}\,(n+1)\,\ell^{\beta+2}\,\Gamma\left(\beta+\frac{3}{2}\right)\Gamma(\beta+1)}{2\,\Gamma(n+\beta+3)\,\Gamma(\beta+1-n)}, \tag{3.21}$$

and in general

$$\int_0^{\ell} \omega_{\ell}(x) x^{\beta}\, U_{n,\ell}(x) U_{m,\ell}(x)\, dx = I(n,m,\beta,\ell), \tag{3.22}$$

where

$$\omega_{\ell}(x) = \sqrt{x\,(\ell - x)},\; I(n,m,\beta,\ell) = \sum_{k=0}^{min(n,m)} I(|n-m| + 2k,\beta,\ell).$$

Proof. First, we prove (3.20) by induction. It is easy to see that this formula holds for $n = 0$. Now, suppose that (3.20) holds for all $n < m$, and we need to show that it holds at $n = m$. Making use of the recurrence relation

$$U_{n+1,\ell}(x) = 2(2\,x/\ell - 1) U_{n,\ell}(x) - U_{n-1,\ell}(x),$$

we get

$$\int_0^{\ell} \omega_{\ell}(x) x^{\beta} U_{n,\ell}(x)\, dx = \int_0^{\ell} \omega_{\ell}(x) x^{\beta} \left(\frac{4}{\ell} x\, U_{n-1,\ell}(x) - 2U_{n-1,\ell}(x) - U_{n-2,\ell}(x)\right) dx. \tag{3.23}$$

By using the induction hypothesis, we obtain

$$\int_0^{\ell} \omega_{\ell}(x) x^{\beta} U_{n,\ell}(x)\, dx = \frac{4}{\ell} I(n-1,\beta+1,\ell) - 2\,I(n-1,\beta,\ell) - I(n-2,\beta,\ell). \tag{3.24}$$

Substitution of (3.21) into (3.24) and performing some calculations lead to (3.20). Now, to prove (3.22), we make use of the well-known linearization formula

$$U_n(x) U_m(x) = \sum_{k=0}^{n} U_{m-n+2k}(x),\; m \geq n,$$

along with (3.20). □

Corollary 3.2. *For every real positive real number β, and positive integers n and m, the following integral formula holds:*

$$\int_0^{\ell} \omega_{\ell}(x)\, x^{\beta}\, G_{n,\ell}^{A,B}(x)\, U_{m,\ell}(x)\, dx = J(n,m,\beta,\ell), \quad n,m = 0,1,2,\ldots, \tag{3.25}$$

where

$$J(n,m,\beta,\ell) = \frac{1}{2}(B - 2A) I(n-2,m,\beta,\ell) + \frac{B}{2}\, I(n,m,\beta,\ell). \tag{3.26}$$

Proof. Formula (3.25) is a direct result of Lemma 3.1 along with the expression in (3.9). □

4. Tau and collocation solutions for TFHE

This section is concerned with analyzing two numerical algorithms for handling TFHE, providing a detailed explanation of the proposed algorithms, namely, the unified shifted Chebyshev tau method (USCTM) and the unified shifted Chebyshev collocation method (USCCM) that will be used for the following TFHE (see [24, 26, 46]):

$$\frac{\partial^\alpha y(x,t)}{\partial t^\alpha} = \frac{\partial^2 y(x,t)}{\partial x^2} + z(x,t),\ 0 < \alpha \le 1, \tag{4.1}$$

governed by the nonlocal conditions

$$y(x,0) - y(x,\ell_2) = f(x), \quad 0 < x < \ell_1, \tag{4.2}$$

and

$$y(0,t) = y(\ell_1,t) = 0, \quad 0 < t \le \ell_2. \tag{4.3}$$

In this instance, the known functions are $z(x,t)$ and $f(x)$, whereas the unknown function is given by $y(x,t)$. Now, consider the space:

$$\Omega = \operatorname{span}\{G^{A,B}_{n,\ell_1}(x)\, G^{A,B}_{m,\ell_2}(t) : n, m = 0, 1, \ldots, N\},$$

and suppose that $y(x,t) \in \Omega$ may be approximately represented as

$$y_N(x,t) = \sum_{n=0}^{N} \sum_{m=0}^{N} c_{n,m}\, G^{A,B}_{n,\ell_1}(x)\, G^{A,B}_{m,\ell_2}(t). \tag{4.4}$$

To be able to apply both tau and collocation methods, we sholud first get the residual of Eq (4.1). This residual may be obtained using the following formula:

$$R_N(x,t) = \frac{\partial^\alpha y_N(x,t)}{\partial t^\alpha} - \frac{\partial^2 y_N(x,t)}{\partial x^2} - z(x,t). \tag{4.5}$$

4.1. Tau approach for treating TFHE

This section provides a detailed explanation of the algorithm (USCTM) that will be used to handle TFHE (4.1) using tau spectral method. The tau method's main goal is to identify $y_N(x,t)$ such that

$$(t^\alpha R_N(x,t), U_{i,\ell_1}(x)\, U_{j,\ell_2}(t))_{\tilde{\omega}(x,t)} = 0, \quad 0 \le i \le N-2, \quad 0 \le j \le N-1, \tag{4.6}$$

where $\tilde{\omega}(x,t) = \omega_{\ell_1}(x)\, \omega_{\ell_2}(t)$.

Now, to be able to compute the integral in the left-hand side of (4.6), use the explicit representation of the fractional derivatives $D^\alpha G^{A,B}_{n,\ell}(x)$ in (3.18) to obtain the following two important integrals for a positive integer q and a positive real number α

$$\int_0^\ell \omega_\ell(t)\, x^\alpha D^\alpha G^{A,B}_{m,\ell}(t)\, U_{j,\ell}(t)\, dt = a^{(\alpha)}_{m,j}(\ell), \quad m, j = 0, 1, 2, \ldots, \tag{4.7}$$

with

$$a_{m,j}^{(\alpha)}(\ell) = \sum_{k=0}^{m} \tilde{h}_{k,m}^{(\alpha)}\, J(k, j, 0, \ell),$$

and

$$\int_0^{\ell} \omega_\ell(x)\, x^\alpha\, D^q G_{n,\ell}^{A,B}(x) U_{i,\ell}(x)\, dt = b_{n,i}^{(q)}(\alpha, \ell), \quad n, i = 0, 1, 2, \dots, \tag{4.8}$$

with

$$b_{n,i}^{(q)}(\alpha, \ell) = \sum_{k=0}^{n-q} h_{k,n}^{(q)}\, J(k, i, \alpha, \ell),$$

where $\tilde{h}_{k,m}^{(\alpha)}$ and $J(k, j, 0, \ell)$ can be computed from (3.19) and (3.26), respectively.

The nonlocal conditions (4.2) and (4.3) give

$$y_N(x_i, 0) - y_N(x_i, \ell_2) = f(x_i), \quad 0 \le i \le N, \tag{4.9}$$

$$y_N(0, t_j) = 0, \quad 0 \le j \le N - 1, \tag{4.10}$$

and

$$y_N(\ell_1, t_j) = 0, \quad 0 \le j \le N - 1, \tag{4.11}$$

where x_i and t_j, $i, j = 0, \dots, M,$ are the zeros of $G_{n,\ell_1}^{A,B}(x)$ and $G_{m,\ell_2}^{A,B}(t)$, respectively. The integrals in (4.7) and (4.8) enable one to write (4.6)–(4.11) as follows:

$$\sum_{n=0}^{N} \sum_{m=0}^{N} c_{n,m} H_{n,m,i,j} = z_{i,j}, \quad 0 \le i \le N - 2, \quad 0 \le j \le N - 1, \tag{4.12}$$

and

$$\left.\begin{aligned} \sum_{n=0}^{N} \sum_{m=0}^{N} c_{n,m} K_{n,m,i} &= f_i, \quad 0 \le i \le N,\\ \sum_{n=0}^{N} \sum_{m=0}^{N} c_{n,m} L_{n,m,j} &= 0, \quad 0 \le j \le N - 1,\\ \sum_{n=0}^{N} \sum_{m=0}^{N} c_{n,m} M_{n,m,j} &= 0, \quad 0 \le j \le N - 1, \end{aligned}\right\} \tag{4.13}$$

where

$$H_{n,m,i,j} = J(n, i, 0, \ell_1)\, a_{m,j}^{(\alpha)}(\ell_2) - J(m, j, \alpha, \ell_2)\, b_{n,i}^{(2)}(0, \ell_1), \quad 0 \le i \le N - 2, \quad 0 \le j \le N - 1, \tag{4.14}$$

and

$$\left.\begin{aligned} K_{n,m,i} &= G_{n,\ell_1}^{A,B}(x_i)(G_{m,\ell_2}^{A,B}(0) - G_{m,\ell_2}^{A,B}(\ell_2)), && 0 \le i \le N,\\ L_{n,m,j} &= G_{n,\ell_1}^{A,B}(0)\, G_{m,\ell_2}^{A,B}(t_j), && 0 \le j \le N - 1,\\ M_{n,m,j} &= G_{n,\ell_1}^{A,B}(\ell_1)\, G_{m,\ell_2}^{A,B}(t_j), && 0 \le j \le N - 1,\\ f_i &= f(x_i). \end{aligned}\right\} \tag{4.15}$$

The system (4.12) and (4.13) consists of $(N + 1)^2$ equations in $(N + 1)^2$ unknowns, $c_{0,0}, \dots, c_{0,N}, \dots, c_{N,0}, \dots, c_{N,N}$.

4.2. The collocation approach for treating TFHE

This section provides a detailed explanation of the algorithm (USCCM) that will be used to handle TFHE (4.1) using collocation spectral method as follows:

Suppose that $y(x,t) \in \Omega$ may be approximately represented as (4.4). The Collocation method's main goal is to identify $y_N(x,t)$ such that

$$R_N(x_i, t_j) = 0, \quad 0 \le i \le N-2, \quad 0 \le j \le N-1, \tag{4.16}$$

which gives

$$\sum_{n=0}^{N} \sum_{m=0}^{N} c_{n,m} \hat{H}_{n,m,i,j} = z_{i,j}, \quad 0 \le i \le N-2, \quad 0 \le j \le N-1, \tag{4.17}$$

where

$$\left.\begin{aligned} \hat{H}_{n,m,i,j} &= G^{A,B}_{n,\ell_1}(x_i)\, D^{\alpha} G^{A,B}_{m,\ell_2}(t_j) - G^{A,B}_{m,\ell_2}(t_j)\, D^2 G^{A,B}_{n,\ell_1}(x_i), \quad 0 \le i \le N-2, \quad 0 \le j \le N-1, \\ z_{i,j} &= z(x_i, t_j), \end{aligned}\right\} \tag{4.18}$$

and $x_i,\, t_i\, (0 \le i \le N)$ are choosen to be either the zeros of $G^{A,B}_{N+1,\ell}(x)$ or $x_i,\, t_i = \dfrac{i+1}{N+2}\, (0 \le i \le N)$, in addition to Eq (4.15) which provide us $(N+1)^2$ equations in the $(N+1)^2$ unknowns, $c_{0,0}, \dots, c_{0,N}, \dots, c_{N,0}, \dots, c_{N,N}$.

Remark 4.1. *Attempting to demonstrate how to use the two algorithms—USCTM and USCCM—presented. While the stages for solving the TFHE using Algorithm 1 are expressed using the USCTM notation, Algorithm 2 uses the USCCM notation. The Mathematica application, version 13.1, is used to do the necessary computations.*

Algorithm 1 USCTM Algorithm

Step 1.	Given A, B, ℓ_1, ℓ_2, α and N
Step 2.	Find $G^{A,B}_{n,\ell_1}(x)$, $D^{(2)}_x G^{A,B}_{n,\ell_1}(x)$ and $D^{(\alpha)}_t G^{A,B}_{m,\ell_2}(t)$
Step 3.	Evaluate $R_N(x,t)$ defined in (4.5)
Step 4.	Evaluate the used collocation points x_i and t_i, $i = 0, 1, \dots, N$
Step 5.	Evaluate $H_{n,m,i,j}$, $K_{n,m,i}$, $L_{n,m,j}$, $M_{n,m,j}$, f_j as defined in (4.14) and (4.15)
Step 6.	List the equations system as defined in (4.12) and (4.13)
Step 7.	Join [Output 6]
Step 8.	Solve [Output 7]

5. Investigation of convergence and error analysis

In this part, we extensively examine the suggested expansion's convergence and error analysis (4.4). The following lemmas are necessary to continue with our investigation.

Lemma 5.1. *The polynomials $G^{A,B}_{n,\ell}(x)$ satisfy the following inequality*

$$|G^{A,B}_{n,\ell}(x)| < \rho_n, \quad x \in [0, \ell], \tag{5.1}$$

Algorithm 2 USCCM Algorithm

Step 1.	Given A, B, ℓ_1, ℓ_2, α and N
Step 2.	Find $G^{A,B}_{n,\ell_1}(x)$, $D^{(2)}_x G^{A,B}_{n,\ell_1}(x)$ and $D^{(\alpha)}_t G^{A,B}_{m,\ell_2}(t,\ell_2)$
Step 3.	Evaluate $R_N(x,t)$ defined in (4.5)
Step 4.	Evaluate the used collocation points x_i and t_i, $i = 0, 1, \ldots, N$
Step 5.	Evaluate $\hat{H}_{n,m,i,j}$, $K_{n,m,i}$, $L_{n,m,j}$, $M_{n,m,j}$, f_j as defined in (4.15) and (4.18)
Step 6.	List the equations system as defined in (4.13) and (4.17)
Step 7.	Join [Output 6]
Step 8.	Solve [Output 7]

where

$$\rho_n = \begin{cases} A, & B = A, \\ A(n+1), & B = 2A, \\ r_n, & \textit{otherwise}, \end{cases} \tag{5.2}$$

such that r_n is defined as

$$r_n = \begin{cases} A, & n = 0, \\ B, & n = 1, \\ \rho\, n, & n \geq 2, \end{cases} \tag{5.3}$$

where $\rho = max\{|B - 2A|, B\}$.

Proof. According to Remark 3.1 and formula (3.1), it is easy to prove (5.1). □

Lemma 5.2. *The coefficients $F_{p,m}$ that appear in the inversion formula* (3.13) *satisfy the following inequality*

$$|F_{p,m}| < \frac{\rho\, c_p (2m)!\,\ell^m}{B 2^{2m-1}(m-p)!(m+p)!}. \tag{5.4}$$

Proof. Formula (3.14) leads to the following inequality:

$$|F_{p,m}| \leq \rho\, c_p B^{-1}(2m+1)!4^{1-m}\ell^m S_{p,m}, \tag{5.5}$$

where

$$S_{p,m} = \sum_{k=0}^{\lfloor \frac{m-p}{2} \rfloor} \frac{(2k+p+1)}{(m-p-2k)!(2k+m+p+2)!}.$$

By using computer algebra algorithms, especially Zeilberger's algorithm (see, [47]), $S_{p,m}$ are able to meet the first-order recurrence relation shown below:

$$(m+p+1)S_{p+1,m} - (m-p)S_{p,m} = 0, \quad S_{0,m} = \frac{1}{2(2m+1)(m!)^2}. \tag{5.6}$$

The exact solution to this recurrence relation has the form

$$S_{p,m} = \frac{1}{2(2m+1)(m-p)!(m+p)!}. \tag{5.7}$$

Inserting (5.7) into (5.5) yields (5.4). The lemma's proof is now complete. □

Theorem 5.1. *Let $y(x,t)$ be an infinitely differentiable function at the origin with $\left|\frac{\partial^{i+j} y}{\partial x^i\, \partial t^j}(0,0)\right| < M$. Then it has the following expansion*

$$y(x,t) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} Y_{i,j}\, G^{A,B}_{i,\ell_1}(x)\, G^{A,B}_{j,\ell_2}(t), \tag{5.8}$$

where

$$Y_{i,j} = \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} y_{i+p,j+q}\, F_{i,p+i}\, F_{j,q+j}, \quad y_{i,j} = \frac{1}{i!\, j!} \frac{\partial^{i+j} y}{\partial x^i\, \partial t^j}(0,0).$$

These expansion coefficients satisfy the following inequalities

$$\left.\begin{aligned}
&|Y_{0,0}| \leq \frac{M\rho^2}{A^2} e^{\ell_1} e^{\ell_2},\\
&|Y_{i,0}| < \frac{8\, M\rho^2\, e^{\ell_2} e^{\ell_1}}{A\, B}\, \frac{\ell_1^i}{i!\, 4^i}, \quad i \geq 1,\\
&|Y_{0,j}| < \frac{8\, M\rho^2\, e^{\ell_1} e^{\ell_2}}{A\, B}\, \frac{\ell_2^j}{j!\, 4^j}, \quad j \geq 1,\\
&|Y_{i,j}| < \frac{64\, M\rho^2\, e^{\ell_1} e^{\ell_2}}{B^2}\, \frac{\ell_1^i \ell_2^j}{i!\, j!\, 4^i\, 4^j}, \quad i, j \geq 1.
\end{aligned}\right\} \tag{5.9}$$

The inequalities in (5.9) can be combined to give the following expression

$$|Y_{i,j}| \lesssim \frac{\ell_1^i \ell_2^j}{i!\, j!\, 4^i\, 4^j}, \quad \forall i, j \geq 0, \tag{5.10}$$

where $\lesssim$ means that a generic constant d exists such that $|Y_{i,j}| \leq \frac{d\, \ell_1^i \ell_2^j}{i!\, j!\, 4^i\, 4^j}$. The series in (5.8) converges uniformly to $y(x,t)$.

Proof. First, we expand $y(x,t)$ as

$$y(x,t) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} y_{i,j}\, x^i\, t^j. \tag{5.11}$$

This expansion can be written in the form:

$$y(x,t) = \sum_{i=0}^{\infty} b_i(t) x^i, \tag{5.12}$$

where

$$b_i(t) = \sum_{j=0}^{\infty} y_{i,j}\, t^j. \tag{5.13}$$

Inserting (3.13) into (5.12) enables one to write

$$y(x,t) = \sum_{i=0}^{\infty} b_i(t) \sum_{p=0}^{i} F_{p,i}\, G^{A,B}_{p,\ell_1}(x). \tag{5.14}$$

Expanding the right hand side of (5.14), and rearranging the similar terms, the following expansion is obtained

$$y(x,t)=\sum_{i=0}^{\infty}\left(\sum_{p=0}^{\infty} b_{p+i}(t)\,F_{i,p+i}\right)G_{i,\ell_1}^{A,B}(x). \tag{5.15}$$

Now, we have

$$\sum_{p=0}^{\infty} b_{p+i}(t)\,F_{i,p+i}=\sum_{p=0}^{\infty}\left(\sum_{j=0}^{\infty} y_{p+i,j}\,t^j\right)F_{i,p+i}=\sum_{j=0}^{\infty}\left(\sum_{p=0}^{\infty} y_{p+i,j}\,F_{i,p+i}\right)t^j. \tag{5.16}$$

Inserting (3.13) into (5.16) and following the same procedures enables one to write

$$\sum_{p=0}^{\infty} b_{p+i}(t)\,F_{i,p+i}=\sum_{j=0}^{\infty}\left(\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} y_{p+i,j+q}\,F_{i,p+i}\,F_{j,q+j}\right)G_{j,\ell_2}^{A,B}(t). \tag{5.17}$$

Substituting (5.17) for (5.15) immediately proves (5.8). The first part of Theorem is now proved. Now, we need to prove (5.9). Using Lemma 5.2, one can write

$$\begin{aligned}|Y_{i,j}|&\le M\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\frac{1}{(i+p)!(j+q)!}|F_{i,p+i}||F_{j,q+j}|,\\ &\le\frac{M\rho^2\,c_ic_j\ell_1^i\ell_2^j}{B^2\,2^{2(i+j-1)}}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\frac{1}{(i+p)!(j+q)!}\frac{(2(p+i))!(2(q+j))!\ell_1^p\ell_2^q}{2^{2(p+q)}p!q!(p+2i)!(q+2j)!},\quad i,j\ge 0.\end{aligned} \tag{5.18}$$

Using $(2n)!=2^{2n}n!(1/2)_n$, we obtain

$$|Y_{i,j}|\le\frac{4\,M\rho^2\,c_ic_j\ell_1^i\ell_2^j}{B^2}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\frac{(1/2)_{p+i}(1/2)_{q+j}\ell_1^p\ell_2^q}{p!q!(p+2i)!(q+2j)!},\quad i,j\ge 0. \tag{5.19}$$

Using $\frac{(1/2)_p}{p!}\le 1, p\ge 0$, it is easy to see that

$$|Y_{0,0}|\le\frac{M\rho^2}{A^2}e^{\ell_1}e^{\ell_2}, \tag{5.20}$$

$$|Y_{i,0}|\le\frac{2\,M\rho^2\,\ell_1^i\,e^{\ell_2}}{A\,B}\sum_{p=0}^{\infty}\frac{(1/2)_{p+i}\,\ell_1^p}{p!(p+2i)!},\quad i\ge 1, \tag{5.21}$$

and

$$|Y_{0,j}|\le\frac{2\,M\rho^2\,\ell_2^j\,e^{\ell_1}}{A\,B}\sum_{q=0}^{\infty}\frac{(1/2)_{q+j}\,\ell_2^q}{q!(q+2j)!},\quad j\ge 1. \tag{5.22}$$

Using $\frac{\ell^p}{p!}<e^{\ell},\ p\ge 0$, the three relations (5.19), (5.21) and (5.22) take respectively the forms

$$|Y_{i,j}|<\frac{4\,M\rho^2\,\ell_1^i\ell_2^j e^{\ell_1}e^{\ell_2}}{B^2}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\frac{(1/2)_{p+i}(1/2)_{q+j}}{(p+2i)!(q+2j)!},\quad i,j\ge 1, \tag{5.23}$$

$$|Y_{i,0}| < \frac{2\,M\rho^2\,\ell_1^i e^{\ell_2} e^{\ell_1}}{A\,B}\sum_{p=0}^{\infty}\frac{(1/2)_{p+i}}{(p+2i)!}, \quad i \geq 1, \tag{5.24}$$

and

$$|Y_{0,j}| < \frac{2\,M\rho^2\,\ell_2^j e^{\ell_1} e^{\ell_2}}{A\,B}\sum_{q=0}^{\infty}\frac{(1/2)_{q+j}}{(q+2j)!}, \quad j \geq 1. \tag{5.25}$$

Then by using $(a)_{n+k} = (a)_k(a+k)_n$ and Chu-V Gauss formula [48], one can get

$$\sum_{p=0}^{\infty}\frac{(1/2)_{p+i}}{(p+2i)!} = \frac{(1/2)_i}{(1)_{2i}}\sum_{p=0}^{\infty}\frac{(1/2+i)_p(1)_p}{p!(1+2i)_p} = \frac{(1/2)_i}{(1)_{2i}}\;{}_2F_1\left(\begin{array}{c}1/2+i,\,1\\ 1+2i\end{array}\bigg|1\right) = \frac{\Gamma\left(\frac{1}{2}(2i-1)\right)}{\sqrt{\pi}\,\Gamma(2i)}. \tag{5.26}$$

Again, using $(2n)! = 2^{2n}n!(1/2)_n$, leads to

$$\sum_{p=0}^{\infty}\frac{(1/2)_{p+i}}{(p+2i)!} = \frac{2i\,\Gamma\left(\frac{1}{2}(2i-1)\right)}{\sqrt{\pi}\,(2i)!} = \frac{1}{(i-1)!(2i-1)4^{i-1}} \leq \frac{1}{i!4^{i-1}}, \quad i \geq 1. \tag{5.27}$$

Hence the three relations (5.23)–(5.25) take respectively the forms

$$|Y_{i,j}| < \frac{64\,M\rho^2\,\ell_1^i\ell_2^j e^{\ell_1} e^{\ell_2}}{B^2}\,\frac{1}{4^i\,i!}\frac{1}{4^j\,j!}, \quad i,j \geq 1, \tag{5.28}$$

$$|Y_{i,0}| < \frac{8\,M\rho^2\,\ell_1^i e^{\ell_2} e^{\ell_1}}{A\,B}\,\frac{1}{4^i\,i!}, \quad i \geq 1, \tag{5.29}$$

and

$$|Y_{0,j}| < \frac{8\,M\rho^2\,\ell_2^j e^{\ell_1} e^{\ell_2}}{A\,B}\,\frac{1}{4^j\,j!}, \quad j \geq 1. \tag{5.30}$$

At this point, the second part of the theorem is now proved.
Now, in view of Lemma 5.1, we can see that

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}|Y_{i,j}\,G^{A,B}_{i,\ell_1}(x)\,G^{A,B}_{j,\ell_2}(t)| \leq\; d\,\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{\ell_1^i}{4^i\,i!}\frac{\ell_2^j}{4^j\,j!}\rho_i\rho_j \leq C\,e^{\ell_1/4}e^{\ell_2/4}, \tag{5.31}$$

where C is a constant depending on the two constants A and B. This shows that the series in (5.8) converges uniformly to $y(x,t)$. The theorem's proof is complete. □

Theorem 5.2. *If $y(x,t)$ satisfies the hypothesis of Theorem 5.1, and if*

$$y(x,t) = \sum_{i=0}^{N}\sum_{j=0}^{N}Y_{i,j}\,G^{A,B}_{i,\ell_1}(x)\,G^{A,B}_{j,\ell_2}(t), \tag{5.32}$$

then the following error estimate is satisfied:

$$|y - y_N| \lesssim \frac{1}{4^{N+1}}. \tag{5.33}$$

Proof. The truncation error may be written as:

$$
\begin{aligned}
|y - y_N| &= \left|\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} Y_{i,j}\, G_{i,\ell_1}^{A,B}(x)\, G_{j,\ell_2}^{A,B}(t) - \sum_{i=0}^{N}\sum_{j=0}^{N} Y_{i,j}\, G_{i,\ell_1}^{A,B}(x)\, G_{j,\ell_2}^{A,B}(t)\right| \\
&\le \sum_{i=0}^{N}\sum_{j=N+1}^{\infty} |Y_{i,j}|\, |G_{i,\ell_1}^{A,B}(x)|\, |G_{j,\ell_2}^{A,B}(t)| + \sum_{i=N+1}^{\infty}\sum_{j=0}^{\infty} |Y_{i,j}|\, |G_{i,\ell_1}^{A,B}(x)|\, |G_{j,\ell_2}^{A,B}(t)| \\
&\le d\sum_{i=0}^{N}\sum_{j=N+1}^{\infty} \frac{\ell_1^i \ell_2^j}{i!\, j!\, 4^j\, 4^i}\,\rho_i\,\rho_j + d\sum_{i=N+1}^{\infty}\sum_{j=0}^{\infty} \frac{\ell_1^i \ell_2^j}{i!\, j!\, 4^i\, 4^j}\,\rho_i\,\rho_j, \\
&\le d\,\frac{1}{4^{N+1}}\sum_{i=0}^{N}\sum_{j=N+1}^{\infty} \frac{\ell_1^i \ell_2^j}{i!\, j!\, 4^i}\,\rho_i\,\rho_j + d\,\frac{1}{4^{N+1}}\sum_{i=N+1}^{\infty}\sum_{j=0}^{\infty} \frac{\ell_1^i \ell_2^j}{i!\, j!\, 4^j}\,\rho_i\,\rho_j,
\end{aligned}
\tag{5.34}
$$

we have ρ_i has the forms λ, $\lambda\, i$ or $\lambda(i+1)$, where λ is a constant. So, we have three cases:

Case 1: $\rho_i = \lambda$

$$
\begin{aligned}
|y - y_N| \le\ & d\,\frac{\lambda}{4^{N+1}}\sum_{i=0}^{N}\sum_{j=N+1}^{\infty} \frac{\ell_1^i \ell_2^j}{i!\, j!\, 4^i} + d\,\frac{\lambda}{4^{N+1}}\sum_{i=N+1}^{\infty}\sum_{j=0}^{\infty} \frac{\ell_1^i \ell_2^j}{i!\, j!\, 4^j} \\
&\le d\,\frac{\lambda}{4^{N+1}}\, e^{\ell_1/4} e^{\ell_2} + d\,\frac{\lambda}{4^{N+1}}\, e^{\ell_1} e^{\ell_2/4} \lesssim \frac{1}{4^{N+1}}.
\end{aligned}
\tag{5.35}
$$

Case 2: $\rho_i = \lambda\, i$

$$
\begin{aligned}
|y - y_N| \le\ & d\,\frac{\lambda\,\ell_1}{4^{N+2}}\sum_{i=0}^{N-1}\sum_{j=N+1}^{\infty} \frac{\ell_1^i \ell_2^j}{i!\, j!\, 4^i} + d\,\frac{\lambda\,\ell_2}{4^{N+2}}\sum_{i=N+1}^{\infty}\sum_{j=0}^{\infty} \frac{\ell_1^i \ell_2^j}{i!\, j!\, 4^j} \\
&\le d\,\frac{\lambda\,\ell_1}{4^{N+2}}\, e^{\ell_1/4} e^{\ell_2} + d\,\frac{\lambda\,\ell_2}{4^{N+2}}\, e^{\ell_1} e^{\ell_2/4} \lesssim \frac{1}{4^{N+1}}.
\end{aligned}
\tag{5.36}
$$

Case 3: $\rho_i = \lambda\,(i+1)$ By the same way, it can be proven that

$$
|y - y_N| \lesssim \frac{1}{4^{N+1}}. \tag{5.37}
$$

□

Error stability is further emphasized in the following theorem through an estimation of error propagation.

Theorem 5.3. *For any two successive approximations of* $y(x,t)$*, we get:*

$$
|y_{N+1} - y_N| \lesssim \frac{1}{4^{N+1}}. \tag{5.38}
$$

Proof. In view of (5.33), it is not difficult to obtain (5.38). □

6. Numerical examples

To demonstrate the effectiveness, high accuracy and application of the two suggested algorithms, this part focuses on the presentation of some numerical results followed by comparisons with certain numerical findings from the literature. The error is measured using the maximum absolute error (MAE) in the tests that follow, namely:

$$E_N = \max_{(x,t)\in I} E_N(x,t), \; I = [0,1]\times[0,1], \tag{6.1}$$

where

$$E_N(x,t) = |y(x,t) - y_N(x,t)|, \; (x,t)\in I. \tag{6.2}$$

Example 6.1. *Consider the following fractional initial value problem:*

$$\begin{cases} \dfrac{\partial^{0.5} y(x,t)}{\partial t^{0.5}} = \dfrac{\partial^2 y(x,t)}{\partial x^2} + 2t - \dfrac{2\sqrt{t}(x-1)x}{\sqrt{\pi}}, & 0 < x, t < 1, \\ y(x,0) - y(x,1) = -x(1-x), & 0 < x < 1, \\ y(0,t) = y(1,t) = 0, & 0 < t \le 1. \end{cases} \tag{6.3}$$

If USCTM or USCCM are applied with $N = 2$, then the following nine coefficients are obtained:

$$c_{0,0} = c_{0,1} = -\frac{A-2B}{16A^2B}, \quad c_{2,0} = -\frac{1}{16AB}, \quad c_{2,1} = -\frac{1}{16B^2}, \quad c_{0,2} = c_{1,0} = c_{1,1} = c_{1,2} = c_{2,2} = 0, \tag{6.4}$$

and consequently $y_2(x,t) = x(1-x)\,t$, which is the exact solution.

If USCTM and USCCM are applied to the following three TFHEs (6.5)–(6.7) using some different values of N, then the obtained numerical results are presented in Tables 1–12 and they affirm that compared to other approaches, the suggested methods provide more accurate findings. Additionally, Figures1–10 show that the exact and approximate solutions to the given problems agree very well. They also show how error depends on N and how the solutions to Examples 6.2–6.5 converge when USCTM and USCCM are used. Furthermore, the stability of solutions is demonstrated.

Example 6.2. *Consider the following TFHE:*

$$\begin{cases} \dfrac{\partial^{\alpha} y(x,t)}{\partial t^{\alpha}} = \dfrac{\partial^2 y(x,t)}{\partial x^2} + 2\,t^2\left(\dfrac{t^{-\alpha}}{\Gamma(3-\alpha)} + 2\pi^2\right)\sin(2\pi x), & 0 < x < 1,\, 0 < t \le 1, \\ y(x,0) - y(x,1) = -\sin(2\pi x), & 0 < x < 1, \\ y(0,t) = y(1,t) = 0, & 0 < t \le 1, \end{cases} \tag{6.5}$$

where the exact solution is $y(x,t) = t^2\,\sin(2\pi x)$. Table 1 presents MAE for Example 6.2 for different N, A, B and $\alpha = 0.5$ using the two proposed numerical methods, while Table 2 presents a comparison with some other methods. The results of this table show that our methods are more accurate.

Table 1. Maximum absolute error E_N for Example 6.2 ($\alpha = 0.5$).

A	B	Method	$N=4$	$N=8$	$N=12$	$N=16$	$N=19$	$N=20$
1	1	USCTM	$1.4 \cdot 10^{-2}$	$1.5 \cdot 10^{-5}$	$3.1 \cdot 10^{-9}$	$2.7 \cdot 10^{-13}$	$2.3 \cdot 10^{-15}$	$1.1. 10^{-15}$
		USCCM	$2.8 \cdot 10^{-2}$	$1.3 \cdot 10^{-4}$	$1.2 \cdot 10^{-7}$	$1.3 \cdot 10^{-10}$	$2.9 \cdot 10^{-13}$	$6.9. 10^{-13}$
0.9	0.9	USCTM	$1.5 \cdot 10^{-2}$	$1.7 \cdot 10^{-5}$	$3.5 \cdot 10^{-9}$	$2.4 \cdot 10^{-12}$	$4.1 \cdot 10^{-14}$	$1.2. 10^{-14}$
		USCCM	$2.3 \cdot 10^{-2}$	$1.4 \cdot 10^{-4}$	$1.3 \cdot 10^{-7}$	$2.3 \cdot 10^{-10}$	$6.9 \cdot 10^{-13}$	$5.1. 10^{-14}$
0.9	1.1	USCTM	$2.4 \cdot 10^{-2}$	$3.3 \cdot 10^{-5}$	$8.0 \cdot 10^{-9}$	$6.0 \cdot 10^{-13}$	$4.0 \cdot 10^{-15}$	$2.9. 10^{-15}$
		USCCM	$2.3 \cdot 10^{-2}$	$1.5 \cdot 10^{-4}$	$1.4 \cdot 10^{-7}$	$4.6 \cdot 10^{-10}$	$4.9 \cdot 10^{-13}$	$2.1. 10^{-13}$
0.6	0.8	USCTM	$1.8 \cdot 10^{-2}$	$3.3 \cdot 10^{-5}$	$5.1 \cdot 10^{-9}$	$8.0 \cdot 10^{-13}$	$4.1 \cdot 10^{-15}$	$3.8. 10^{-15}$
		USCCM	$2.4 \cdot 10^{-2}$	$1.1 \cdot 10^{-4}$	$1.5 \cdot 10^{-7}$	$2.3 \cdot 10^{-10}$	$6.9 \cdot 10^{-13}$	$1.5. 10^{-14}$

Table 2. Comparison between different methods of Example 6.2 ($A = 0.6,\ B = 0.8,\ \alpha = 0.5$).

USCTM		USCCM		[46]		[24]($N = M$)		[26]		[25]	
N	MAE	N	MAE	N	MAE	N	MAE	N	MAE	N	MAE
4	$1.8 \cdot 10^{-2}$	4	$2.4 \cdot 10^{-2}$	6	$2.5 \cdot 10^{-2}$	4	$2.3 \cdot 10^{-1}$	4	$3.6 \cdot 10^{-1}$	4	$3.0 \cdot 10^{-2}$
8	$3.3 \cdot 10^{-5}$	8	$1.1 \cdot 10^{-4}$	8	$8.8 \cdot 10^{-4}$	8	$5.4 \cdot 10^{-2}$	6	$5.9 \cdot 10^{-1}$	6	$2.9 \cdot 10^{-3}$
12	$5.1 \cdot 10^{-9}$	12	$1.5 \cdot 10^{-7}$	10	$2.1 \cdot 10^{-5}$	16	$1.4 \cdot 10^{-2}$	8	$5.0 \cdot 10^{-3}$	8	$1.6 \cdot 10^{-4}$
16	$8.0 \cdot 10^{-13}$	16	$2.3 \cdot 10^{-10}$	12	$1.8 \cdot 10^{-6}$	32	$3.8 \cdot 10^{-3}$	10	$2.7 \cdot 10^{-4}$	10	$6.4 \cdot 10^{-6}$
19	$4.1 \cdot 10^{-14}$	19	$6.9 \cdot 10^{-13}$	16	$1.0 \cdot 10^{-7}$	64	$1.2 \cdot 10^{-3}$	12	$9.6 \cdot 10^{-6}$	12	$1.7 \cdot 10^{-7}$
20	$3.8 \cdot 10^{-15}$	20	$1.5 \cdot 10^{-14}$	18	$4.4 \cdot 10^{-7}$	256	$1.4 \cdot 10^{-4}$			14	$3.6 \cdot 10^{-9}$

Example 6.3. *Consider the following TFHE:*

$$\begin{cases} \dfrac{\partial^\alpha y(x,t)}{\partial t^\alpha} = \dfrac{\partial^2 y(x,t)}{\partial x^2} + \dfrac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} t^{\beta-\alpha}(1-x)\sin(x) + t^\beta(2\cos(x) + (1-x)\sin(x)), & 0 < x < 1,\ 0 < t \leq 1, \\ y(x,0) - y(x,1) = (x-1)\sin(x), & 0 < x < 1, \\ y(0,t) = y(1,t) = 0, & 0 < t \leq 1, \end{cases} \tag{6.6}$$

where the exact solution is $y(x,t) = t^\beta\,(1-x)\sin\ x$.

Table 3. Maximum absolute error E_N for Example 6.3 ($\beta = 2,\ A = 1,\ B = 1$).

$\alpha = 0.1$				$\alpha = 0.5$				$\alpha = 0.95$			
N	USCTM	N	USCCM	N	USCTM	N	USCCM	N	USCTM	N	USCCM
4	$5.8 \cdot 10^{-04}$	4	$5.1 \cdot 10^{-04}$	4	$1.5 \cdot 10^{-04}$	5	$5.1 \cdot 10^{-04}$	4	$1.1 \cdot 10^{-04}$	5	$2.0 \cdot 10^{-04}$
8	$1.1 \cdot 10^{-09}$	8	$1.5 \cdot 10^{-08}$	8	$3.1 \cdot 10^{-10}$	8	$1.2 \cdot 10^{-08}$	8	$2.2 \cdot 10^{-10}$	10	$1.5 \cdot 10^{-11}$
11	$2.2 \cdot 10^{-14}$	12	$1.1 \cdot 10^{-14}$	11	$6.2 \cdot 10^{-15}$	13	$1.1 \cdot 10^{-15}$	11	$3.1 \cdot 10^{-15}$	13	$1.2 \cdot 10^{-15}$
15	$6.1 \cdot 10^{-15}$	13	$8.1 \cdot 10^{-16}$	15	$5.2 \cdot 10^{-16}$	14	$7.2 \cdot 10^{-16}$	12	$1.9 \cdot 10^{-16}$	15	$8.1 \cdot 10^{-16}$

Table 4. Maximum absolute error E_N for Example 6.3 ($\beta = 2,\ A = 1,\ B = 2$).

$\alpha = 0.1$				$\alpha = 0.5$				$\alpha = 0.95$			
N	USCTM	N	USCCM	N	USCTM	N	USCCM	N	USCTM	N	USCCM
4	$1.0\,.10^{-3}$	4	$6.1\,.10^{-04}$	4	$2.0\,.10^{-04}$	4	$6.0\,.10^{-04}$	4	$1.5\,.10^{-04}$	4	$6.0\,.10^{-03}$
8	$3.0\,.10^{-9}$	8	$1.6\,.10^{-08}$	8	$8.0\,.10^{-10}$	8	$1.5\,.10^{-08}$	8	$4.0\,.10^{-10}$	8	$1.5\,.10^{-08}$
11	$1.5\,.10^{-14}$	12	$2.2\,.10^{-14}$	11	$1.5\,.10^{-14}$	12	$2.0\,.10^{-14}$	11	$1.5\,.10^{-14}$	11	$1.4\,.10^{-12}$
13	$1.0\,.10^{-14}$	13	$1.3\,.10^{-15}$	12	$2.0\,.10^{-15}$	13	$2.0\,.10^{-15}$	12	$8.0\,.10^{-16}$	12	$2.0\,.10^{-14}$
15	$6.0\,.10^{-15}$	14	$1.4\,.10^{-15}$	15	$6.0\,.10^{-16}$	14	$1.5\,.10^{-15}$	15	$8.3\,.10^{-16}$	13	$1.0\,.10^{-15}$

Table 5. Maximum absolute error E_N for Example 6.3 ($\beta = 2,\ A = 0.5,\ B = 0.6$).

$\alpha = 0.1$				$\alpha = 0.45$				$\alpha = 0.9$			
N	USCTM	N	USCCM	N	USCTM	N	USCCM	N	USCTM	N	USCCM
4	$1.3\,.10^{-4}$	4	$2.3\,.10^{-04}$	4	$2.1\,.10^{-04}$	4	$5.1\,.10^{-04}$	4	$1.6\,.10^{-04}$	5	$1.8\,.10^{-04}$
8	$3.2\,.10^{-9}$	9	$1.2\,.10^{-09}$	8	$8.1\,.10^{-10}$	9	$1.0\,.10^{-09}$	8	$4.0\,.10^{-09}$	9	$1.1\,.10^{-09}$
12	$3.1\,.10^{-14}$	12	$1.5\,.10^{-14}$	12	$2.2\,.10^{-14}$	12	$1.4\,.10^{-14}$	12	$2.4\,.10^{-14}$	12	$1.5\,.10^{-14}$
13	$4.0\,.10^{-15}$	13	$5.5\,.10^{-16}$	13	$7.1\,.10^{-16}$	13	$1.1\,.10^{-16}$	13	$3.1\,.10^{-16}$	13	$1.4\,.10^{-16}$
15	$3.1\,.10^{-16}$	14	$1.2\,.10^{-16}$	14	$5.8\,.10^{-16}$	14	$6.3\,.10^{-16}$	14	$2.3\,.10^{-16}$	14	$4.1\,.10^{-16}$

Table 6. Maximum absolute error E_N for Example 6.3 ($\alpha = 0.5,\ A = 0.9,\ B = 1.9$).

$\beta = 0.1$				$\beta = 0.5$				$\beta = 0.95$			
N	USCTM	N	USCCM	N	USCTM	N	USCCM	N	USCTM	N	USCCM
4	$1.2\,.10^{-2}$	4	$4.3\,.10^{-03}$	4	$1.1\,.10^{-04}$	4	$2.0\,.10^{-04}$	4	$1.4\,.10^{-04}$	4	$1.8\,.10^{-04}$
8	$2.2\,.10^{-8}$	9	$2.5\,.10^{-08}$	8	$2.0\,.10^{-10}$	9	$1.7\,.10^{-09}$	6	$2.2\,.10^{-07}$	6	$1.2\,.10^{-08}$
12	$7.2\,.10^{-13}$	12	$2.2\,.10^{-13}$	12	$3.0\,.10^{-14}$	12	$1.5\,.10^{-15}$	8	$5.0\,.10^{-10}$	8	$3.3\,.10^{-10}$
13	$5.0\,.10^{-14}$	13	$4.4\,.10^{-14}$	13	$5.0\,.10^{-15}$	13	$2.2\,.10^{-15}$	12	$2.0\,.10^{-15}$	12	$1.2\,.10^{-15}$
14	$3.1\,.10^{-14}$	14	$3.2\,.10^{-15}$	15	$4.0\,.10^{-16}$	14	$2.4\,.10^{-16}$	14	$1.7\,.10^{-16}$	14	$2.5\,.10^{-16}$

Table 7. Comparison between different methods of Example 6.3 ($\beta = 2,\ A = 0.7,\ B = 1.5,\ \alpha = 0.95$).

USCTM		USCCM		[26]		[25]	
N	MAE	N	MAE	N	MAE	N	MAE
6	$1.3\,.\,10^{-07}$	7	$7.4\,.\,10^{-07}$	4	$1.7\,.\,10^{-4}$	4	$8.5\,.\,10^{-7}$
8	$2.0\,.\,10^{-10}$	10	$2.1\,.\,10^{-11}$	6	$9.9\,.\,10^{-6}$	6	$1.1\,.\,10^{-9}$
10	$1.1\,.\,10^{-13}$	12	$2.0\,.\,10^{-14}$	8	$7.7\,.\,10^{-6}$	8	$1.8\,.\,10^{-12}$
11	$3.0\,.\,10^{-15}$	13	$3.1\,.\,10^{-15}$	10	$6.0\,.\,10^{-6}$	10	$1.4\,.\,10^{-13}$
12	$1.7\,.\,10^{-16}$	14	$2.9\,.\,10^{-15}$	12	$4.4\,.\,10^{-6}$	12	$2.3\,.\,10^{-12}$

Remark 6.1. *It is known that the exact solution of TFHE has a weak singularity near the initial time point, i.e., the exact solution is nonsmooth near the initial time* $t = 0$ *(see [49]). Table 6 presents the numerical solutions obtained for the TFHE Eq* (6.6) *(for* $\alpha = 0.5$*), whose exact solution is a nonsmooth solution for values of* β, $0 < \beta < 1$. *These results show that our algorithm still provides accurate solutions.*

Example 6.4. *Consider the following TFHE:*

$$\begin{cases} \dfrac{\partial^\alpha y(x,t)}{\partial t^\alpha} = \dfrac{\partial^2 y(x,t)}{\partial x^2} + \dfrac{2}{\Gamma(3-\alpha)} t^{2-\alpha}\, \ln(1+x-x^2) + t^2\, \dfrac{2\,x^2 - 2\,x + 3}{(x^2-x-1)^2}, & 0 < x < 1,\ 0 < t \le 1, \\ y(x,0) - y(x,1) = -\ln(1+x-x^2), & 0 < x < 1, \\ y(0,t) = y(1,t) = 0, & 0 < t \le 1, \end{cases} \tag{6.7}$$

where the exact solution is $y(x,t) = t^2\, \ln(1+x-x^2)$.

Table 8. Maximum absolute error E_N for Example 6.4 ($A = 0.8,\ B = 1.2$).

$\alpha = 0.1$				$\alpha = 0.5$				$\alpha = 0.9$			
N	USCTM	N	USCCM	N	USCTM	N	USCCM	N	USCTM	N	USCCM
4	$4.0\,.10^{-03}$	4	$2.1\,.10^{-3}$	4	$1.0\,.10^{-03}$	4	$6.0\,.10^{-3}$	4	$8.0\,.10^{-04}$	4	$1.3\,.10^{-3}$
8	$1.0\,.10^{-05}$	9	$1.0\,.10^{-5}$	8	$3.0\,.10^{-06}$	9	$2.6\,.10^{-5}$	8	$2.0\,.10^{-06}$	9	$2.0\,.10^{-5}$
11	$4.0\,.10^{-08}$	12	$3.1\,.10^{-7}$	14	$4.0\,.10^{-10}$	12	$5.1\,.10^{-7}$	12	$6.0\,.10^{-09}$	12	$4.1\,.10^{-7}$
15	$1.5\,.10^{-10}$	15	$4.3\,.10^{-9}$	18	$4.2\,.10^{-12}$	16	$4.3\,.10^{-9}$	14	$3.0\,.10^{-10}$	16	$5.3\,.10^{-9}$
18	$6.0\,.10^{-13}$	20	$1.3\,.10^{-11}$	20	$2.1\,.10^{-14}$	20	$2.4\,.10^{-11}$	18	$3.0\,.10^{-13}$	20	$1.1\,.10^{-11}$
22	$6.0\,.10^{-15}$	22	$4.3\,.10^{-13}$	22	$1.5\,.10^{-15}$	22	$4.3\,.10^{-13}$	22	$3.0\,.10^{-15}$	22	$5.3\,.10^{-13}$

Table 9. Maximum absolute error E_N for Example 6.4 ($A = 1.6,\ B = 2.2$).

$\alpha = 0.1$				$\alpha = 0.5$				$\alpha = 0.9$			
N	USCTM	N	USCCM	N	USCTM	N	USCCM	N	USCTM	N	USCCM
4	$4.0\,.10^{-03}$	4	$3.0\,.10^{-3}$	4	$1.0\,.10^{-03}$	4	$1.3\,.10^{-3}$	4	$8.0\,.10^{-04}$	4	$1.5\,.10^{-3}$
8	$7.0\,.10^{-06}$	9	$1.1\,.10^{-6}$	8	$2.0\,.10^{-06}$	9	$2.5\,.10^{-5}$	8	$2.0\,.10^{-06}$	9	$4.0\,.10^{-5}$
11	$2.0\,.10^{-08}$	12	$3.1\,.10^{-7}$	12	$6.0\,.10^{-09}$	12	$5.4\,.10^{-7}$	12	$4.0\,.10^{-09}$	12	$6.0\,.10^{-7}$
15	$1.0\,.10^{-10}$	16	$2.5\,.10^{-9}$	16	$2.0\,.10^{-11}$	16	$4.3\,.10^{-9}$	14	$3.0\,.10^{-10}$	16	$4.0\,.10^{-9}$
19	$4.0\,.10^{-13}$	20	$1.5\,.10^{-11}$	19	$4.0\,.10^{-13}$	20	$4.3\,.10^{-11}$	18	$6.0\,.10^{-13}$	19	$5.3\,.10^{-11}$
21	$2.0\,.10^{-14}$	24	$1.5\,.10^{-13}$	20	$4.0\,.10^{-14}$	24	$2.3\,.10^{-13}$	20	$3.0\,.10^{-14}$	24	$3.5\,.10^{-13}$

Table 10. Comparison between different methods of Example 6.4 ($A = 0.4$, $B = 0.6$, $\alpha = 0.95$).

USCTM		USCCM		[46]		[24](M=16)		[26]		[25]	
N	MAE	N	MAE	N	MAE	N	MAE	N	MAE	N	MAE
4	$8.0 . 10^{-4}$	5	$6.0 . 10^{-03}$	4	$1.1 . 10^{-3}$	4	$7.5 . 10^{-3}$	4	$1.3 . 10^{-03}$	4	$6.8 . 10^{-04}$
8	$2.0 . 10^{-6}$	9	$5.1 . 10^{-05}$	8	$5.9 . 10^{-5}$	8	$2.3 . 10^{-3}$	6	$1.1 . 10^{-04}$	6	$1.8 . 10^{-05}$
12	$6.0 . 10^{-9}$	16	$1.2 . 10^{-09}$	12	$5.1 . 10^{-7}$	16	$1.4 . 10^{-3}$	8	$1.0 . 10^{-05}$	8	$1.3 . 10^{-06}$
16	$1.5 . 10^{-11}$	19	$1.5 . 10^{-10}$	14	$2.8 . 10^{-7}$	32	$9.6 . 10^{-4}$	10	$1.3 . 10^{-05}$	10	$1.2 . 10^{-07}$
20	$4.0 . 10^{-14}$	20	$2.0 . 10^{-11}$	16	$1.7 . 10^{-7}$	64	$7.3 . 10^{-4}$	12	$1.1 . 10^{-05}$	12	$1.2 . 10^{-08}$
22	$1.7 . 10^{-15}$	22	$1.5 . 10^{-13}$	18	$1.7 . 10^{-7}$	256	$5.6 . 10^{-4}$			16	$4.1 . 10^{-09}$

Table 11. Comparison between different methods of Example 6.4 ($A = 1.4$, $B = 1.6$, $\alpha = 0.45$).

USCTM		USCCM		[46]		[24]		[26]	
N	MAE	N	MAE	N	MAE	$M(N = 16)$	MAE	N	MAE
4	$1.0 . 10^{-3}$	4	$8.0 . 10^{-03}$	4	$1.1 . 10^{-3}$	4	$9.5 . 10^{-3}$	4	$1.4 . 10^{-03}$
8	$2.0 . 10^{-6}$	7	$7.1 . 10^{-04}$	8	$6.3 . 10^{-6}$	8	$3.7 . 10^{-3}$	6	$1.2 . 10^{-04}$
12	$4.0 . 10^{-9}$	10	$5.2 . 10^{-06}$	12	$9.0 . 10^{-7}$	16	$2.3 . 10^{-3}$	8	$1.1 . 10^{-05}$
15	$8.0 . 10^{-11}$	12	$6.1 . 10^{-07}$	14	$5.1 . 10^{-7}$	32	$1.9 . 10^{-3}$	10	$4.1 . 10^{-06}$
18	$6.1 . 10^{-13}$	16	$3.2 . 10^{-09}$	16	$3.1 . 10^{-7}$	64	$1.8 . 10^{-3}$	12	$3.3 . 10^{-06}$
20	$2.9 . 10^{-14}$	20	$4.0 . 10^{-12}$	18	$1.9 . 10^{-7}$	256	$1.8 . 10^{-3}$		
23	$1.9 . 10^{-15}$	24	$3.1 . 10^{-13}$						

Example 6.5. *Consider the following TFHE:*

$$\begin{cases} \dfrac{\partial^{0.5} y(x,t)}{\partial t^{0.5}} = \dfrac{\partial^2 y(x,t)}{\partial x^2} + z(x,t), & 0 < x < 1,\ 0 < t \leq 1, \\ y(x,0) - y(x,1) = -e^{x^2}(1 - erf(x)) + (e\,(1 - erf(1)) - 1)x + 1, & 0 < x < 1, \\ y(0,t) = y(1,t) = 0, & 0 < t \leq 1, \end{cases} \tag{6.8}$$

where the function $z(x,t)$ is chosen such that the exact solution is

$$y(x,t) = E_{\frac{1}{2},1}\left(-x\sqrt{t}\right) + x\left(-E_{\frac{1}{2},1}\left(-\sqrt{t}\right)\right) + x - 1, \tag{6.9}$$

where the functions

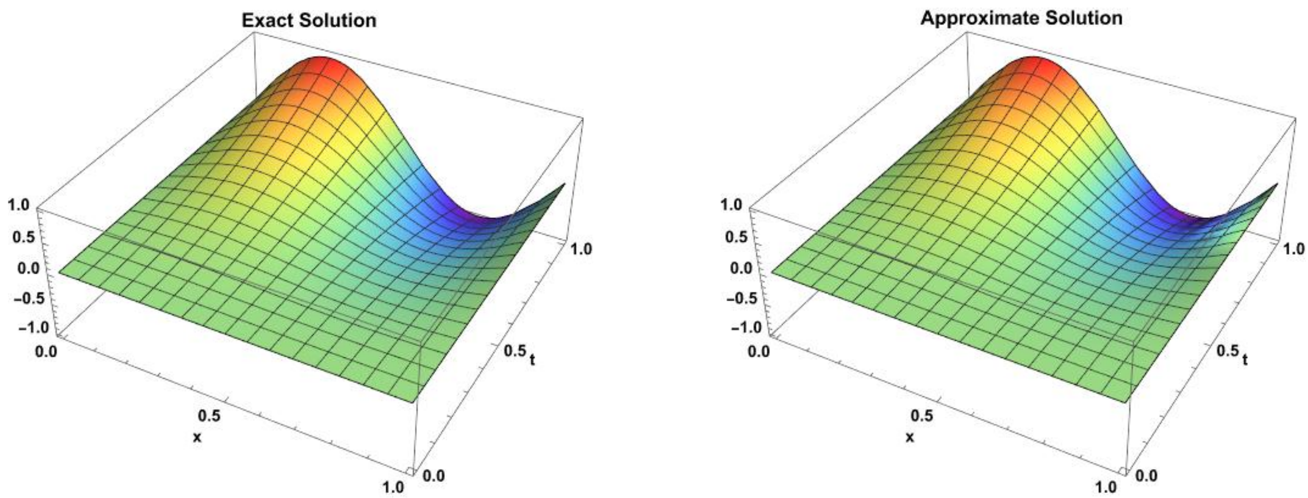
$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt, \quad and \quad E_{\alpha,\beta}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(k\alpha + \beta)}, \tag{6.10}$$

are the Gaussian error function and the generalized Mitta-Leffler function, respectively.

Table 12. Maximum absolute error E_N for Example 6.5.

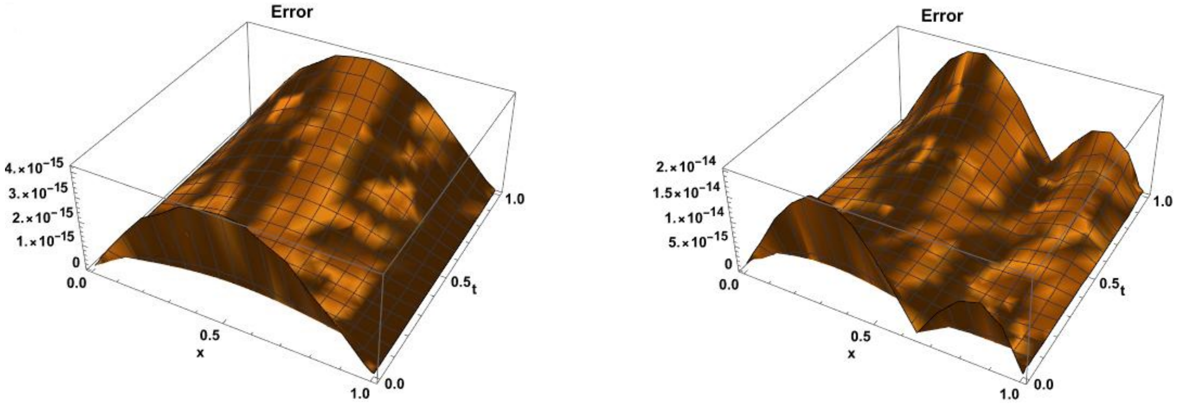
A	B	Method	$N=2$	$N=4$	$N=6$	$N=10$	$N=15$	$N=20$
1	1	USCTM	$2.2\,.\,10^{-2}$	$1.8\,.\,10^{-3}$	$3.4\,.\,10^{-4}$	$3.2\,.\,10^{-5}$	$4.4\,.\,10^{-6}$	$4.0.\,10^{-8}$
		USCCM	$3.1\,.\,10^{-2}$	$2.3\,.\,10^{-3}$	$4.2\,.\,10^{-4}$	$1.5\,.\,10^{-5}$	$2.7\,.\,10^{-6}$	$4.9.\,10^{-7}$
1	2	USCTM	$1.5\,.\,10^{-2}$	$1.7\,.\,10^{-3}$	$3.5\,.\,10^{-4}$	$2.4\,.\,10^{-5}$	$4.1\,.\,10^{-6}$	$1.2.\,10^{-8}$
		USCCM	$3.3\,.\,10^{-2}$	$2.1\,.\,10^{-3}$	$3.3\,.\,10^{-4}$	$2.8\,.\,10^{-5}$	$4.2\,.\,10^{-6}$	$5.5.\,10^{-8}$
0.8	1.8	USCTM	$1.4\,.\,10^{-2}$	$2.1\,.\,10^{-3}$	$5.0\,.\,10^{-4}$	$2.2\,.\,10^{-5}$	$3.1\,.\,10^{-7}$	$2.9.\,10^{-8}$
		USCCM	$2.2\,.\,10^{-2}$	$2.5\,.\,10^{-3}$	$1.7\,.\,10^{-4}$	$1.5\,.\,10^{-5}$	$1.3\,.\,10^{-7}$	$1.9.\,10^{-8}$
0.5	0.6	USCTM	$2.2\,.\,10^{-2}$	$3.5\,.\,10^{-3}$	$5.5\,.\,10^{-4}$	$4.2\,.\,10^{-5}$	$2.2\,.\,10^{-6}$	$3.7.\,10^{-8}$
		USCCM	$1.2\,.\,10^{-2}$	$1.3\,.\,10^{-3}$	$1.7\,.\,10^{-4}$	$2.0\,.\,10^{-5}$	$5.8\,.\,10^{-7}$	$2.5.\,10^{-7}$
1.5	1.5	USCTM	$3.2\,.\,10^{-2}$	$4.6\,.\,10^{-3}$	$2.5\,.\,10^{-4}$	$1.2\,.\,10^{-5}$	$2.7\,.\,10^{-7}$	$4.5.\,10^{-8}$
		USCCM	$3.2\,.\,10^{-2}$	$1.1\,.\,10^{-3}$	$1.4\,.\,10^{-4}$	$1.3\,.\,10^{-5}$	$5.1\,.\,10^{-7}$	$2.6.\,10^{-7}$

Remark 6.2. *The results of Table 1 show that the first and second kinds of Chebyshev approximations are not always the best, along with other approximations for the UCPs. This, of course, clarifies the importance of our generalization to the CPs in this paper.*



(a) Exact solution of Example 6.2 for $\alpha = 0.5$. **(b)** $y_{19}(x,t)$ with $A = 0.6$ and $B = 0.8$ using USCTM.

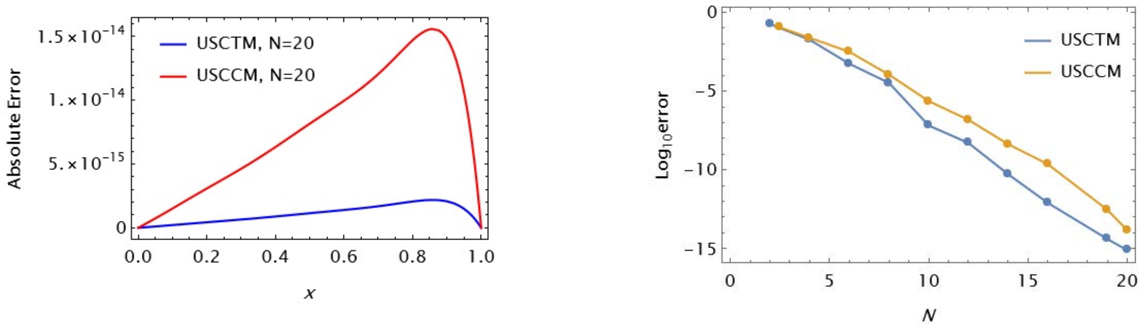
Figure 1. Figures of exact and approximate solutions for Example 6.2.



(a) $E_{20}(x,t)$ using USCTM ($A = 0.6$ and $B = 0.8$).

(b) $E_{20}(x,t)$ using USCCM ($A = 0.6$ and $B = 0.8$).

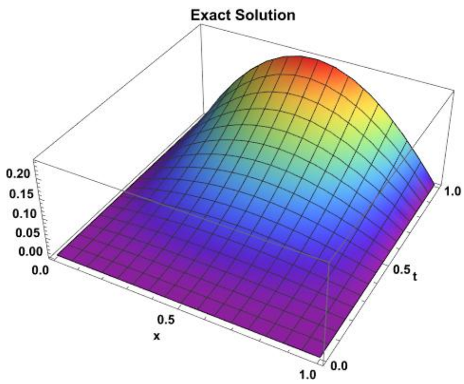
Figure 2. Obtained Errors for Example 6.2 at $\alpha = 0.5$.



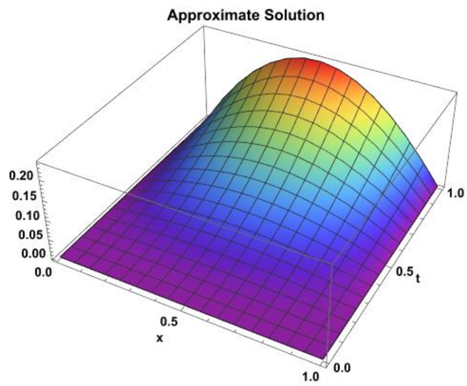
(a) Absolute errors at $t = 0.5$

(b) Graph of $Log_{10}E_N$ against N.

Figure 3. Errors results using USCTM and USCCM ($A = 0.6$ and $B = 0.8$) for Example 6.2.

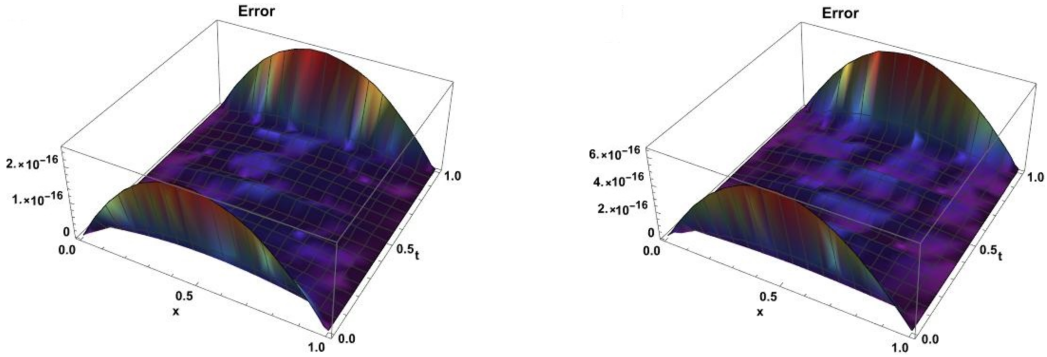


(a) Exact solution.



(b) $y_{12}(x,t)$ using USCTM with $A = 0.7$, $B = 1.5$.

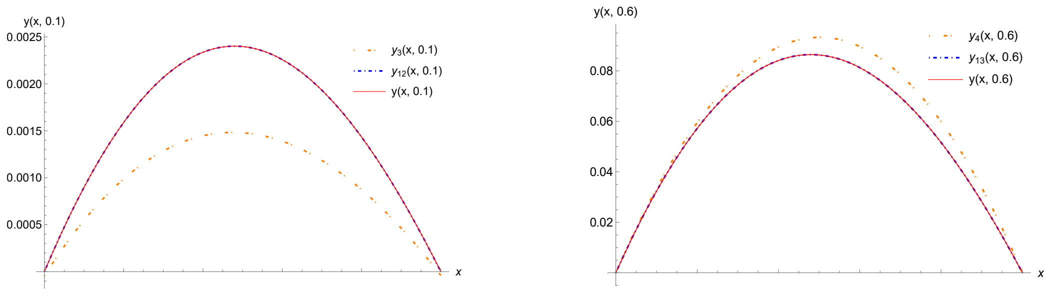
Figure 4. Figures of exact and approximate solutions for Example 6.3 at $\alpha = 0.95$.



(a) $E_{12}(x,t)$ using USCTM ($A = 0.7, B = 1.5$) at $\alpha = 0.95$.

(b) $E_{13}(x,t)$ using USCCM ($A = 0.5$, $B = 0.6$) at $\alpha = 0.1$.

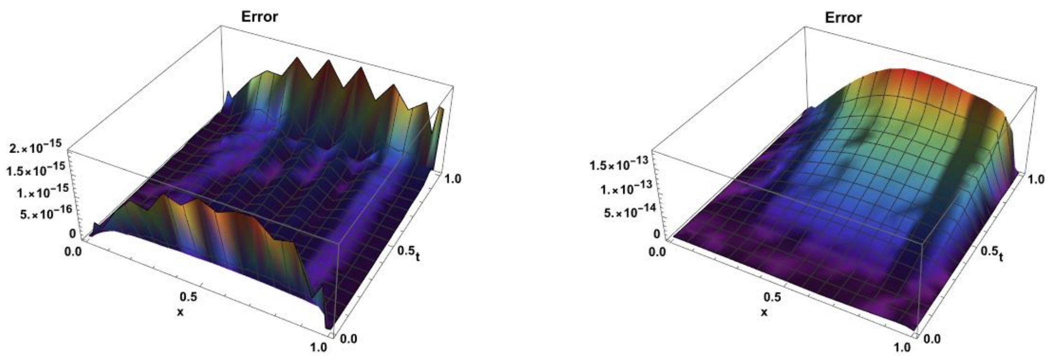
Figure 5. Obtained errors for Example 6.3 using USCTM and USCCM.



(a) Using USCTM ($A = 0.7, B = 1.5$) at $\alpha = 0.95$.

(b) Using USCCM ($A = 0.5, B = 0.6$) at $\alpha = 0.1$.

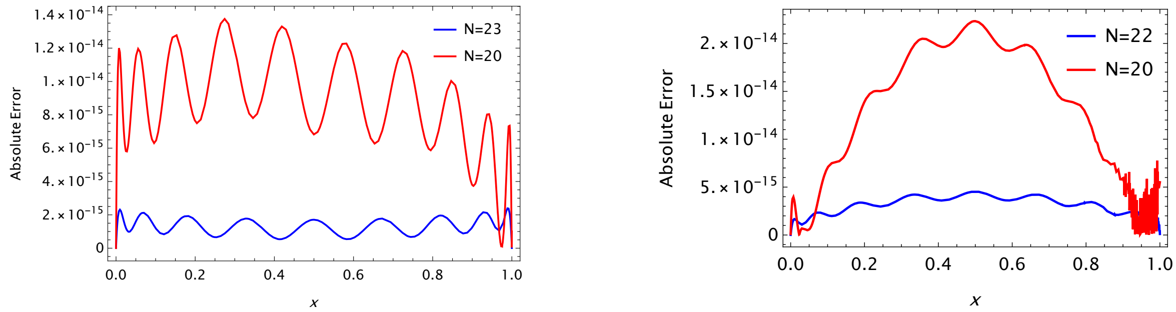
Figure 6. The behavior of exact solution and approximate solution for Example 6.3 using USCTM and USCCM.



(a) $E_{23}(x,t)$ using USCTM ($A = 1.4, B = 1.6$) at $\alpha = 0.45$.

(b) $E_{22}(x,t)$ using USCCM ($A = 0.4, B = 0.6$) at $\alpha = 0.95$.

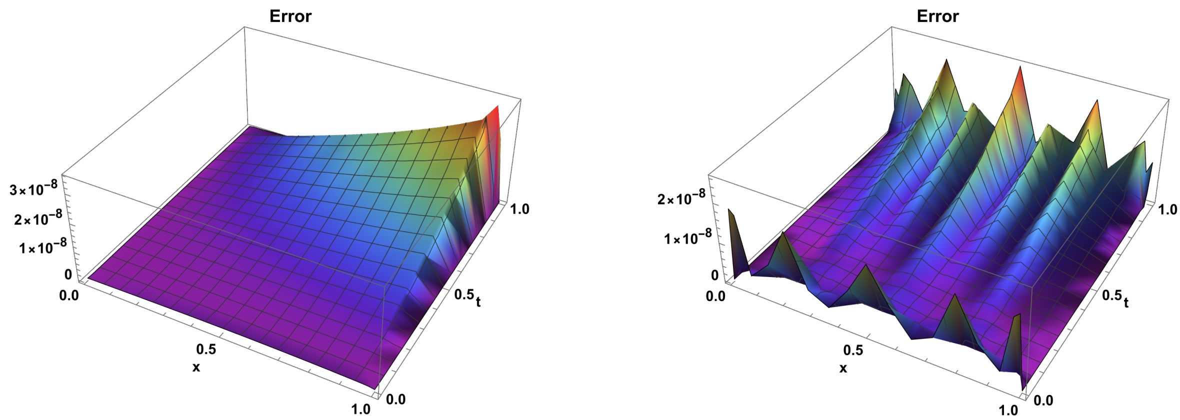
Figure 7. Obtained errors for Example 6.4 using USCTM and USCCM.



(a) $E_{20}(x, 0.3)$ and $E_{22}(x, 0.3)$ using USCTM $(A = 0.4, B = 0.6)$ at $\alpha = 0.5$.

(b) $E_{20}(x, 0.3)$ and $E_{22}(x, 0.3)$ using USCCM $(A = 1.4, B = 1.6)$ at $\alpha = 0.9$.

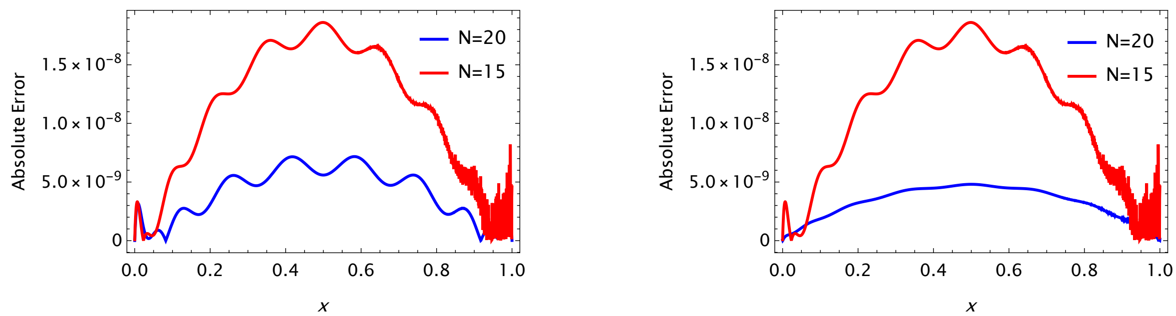
Figure 8. Obtained errors for Example 6.4 using USCTM and USCCM.



(a) $E_{20}(x, t)$ using USCTM $(A = 0.8, B = 1.8)$.

(b) $E_{20}(x, t)$ using USCCM $(A = 0.8, B = 1.8)$.

Figure 9. Obtained errors for Example 6.5 using USCTM and USCCM.



(a) $E_{15}(x, 0.2)$ and $E_{20}(x, 0.2)$ using USCTM $(A = 0.8, B = 1.8)$.

(b) $E_{15}(x, 0.2)$ and $E_{20}(x, 0.2)$ using USCCM $(A = 0.8, B = 1.8)$.

Figure 10. Obtained errors for Example 6.5 using USCTM and USCCM.

7. Conclusions

In order to solve TFHE in non-local conditions, this work developed spectral tau and collocation methods. To choose appropriate sets of basis functions, UCPs, and their shifted polynomials were employed. An approximate solution can be obtained by solving the given system of algebraic equations using an appropriate solver. We emphasize the benefit of using the properties of second-kind CPs, which help us calculate some of the computational formulas. In Section 6, we illustrated the accuracy and usefulness of our methods by comparing them to other methodologies in the literature. To the best of our knowledge, this is the first time that this type of polynomial has been utilized in numerical analysis. It is shown that the first- and second-kinds are not always the best among other Chebyshev approximations. In addition, in future work, we aim to employ these polynomials to treat other types of differential equations.

Use of AI tools declaration

The authors declare that they have not used artificial intelligence tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interests.

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