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Research article

A differential equation approach for solving implicit state-dependent convex sweeping processes in Banach spaces

Messaoud Bounkhel[1,*] and Bushra R. Al-sinan[2]

[1] Department of Mathematics, College of Science, King Saud University, Riyadh 11451, Saudi Arabia

[2] Department of Administrative and Financial Sciences, Nairiyah College, University of Hafr Al-Batin, Hafr Al-Batin 31991, Saudi Arabia; Bushrar@uhb.edu.sa

* **Correspondence:** Email: bounkhel@ksu.edu.sa.

Abstract: In the setting of 2-uniformly convex Banach spaces, we prove the existence of solutions for a variant of the implicit state-dependent convex sweeping processes. Our approach is based on a differential equation associated with the generalized projection operator.

Keywords: uniformly smooth and uniformly convex Banach spaces; differential equations; implicit convex sweeping process; generalized projection; duality mapping
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1. Introduction

In this paper, we aim to study the following variant of implicit state-dependent convex sweeping process in Banach spaces

$$\begin{cases} J(\dot{x}(t)) \in -N_{C(t,x(t))}(BJ[B^*J(\dot{x}(t)) + h(t, x(t))]) + g(t, x(t)) \text{ a.e. } I, \\ x(0) = x_0 \in X, \end{cases} \tag{1.1}$$

where $I := [0, T](T > 0)$, $C : I \times X \to X$ is a set-valued mapping defined from I to a given reflexive Banach space X with nonempty, closed and convex values, $B : X^* \to X$ is a bounded linear operator, $B^* : X^* \to X$ is the adjoint operator of B, $J : X \to X^*$ is the normalized duality mapping, and $g : I \times X \to X^*$, $h : I \times X \to X$ are two given single-valued mappings. Here, $N_S(\cdot)$ stands for the convex normal cone associated to a given closed convex set S. First, we start with the following special cases motivating the study of the proposed problem (1.1).

(1) Assume that X is a Hilbert space. In this case the duality mapping J reduces to the identity Id_X and $X = X^*$. Hence, problem (1.1) becomes

$$\begin{cases} \dot{x}(t) \in -N_{C(t,x(t))}(A\dot{x}(t) + Bh(t, x(t))) + g(t, x(t)) \text{ a.e. on } I, \\ x(0) = x_0 \in X, \end{cases}$$

where $A = BB^*$ is a linear bounded self-adjoint operator on the Hilbert space X. This problem has been proposed and studied recently in [1].

(2) If X is a Banach space and $h = 0$, problem (1.1) becomes

$$\begin{cases} J(\dot{x}(t)) \in -N_{C(t,x(t))}(A\dot{x}(t)) + g(t, x(t)) \text{ a.e. on } I, \\ x(0) = x_0 \in X, \end{cases}$$

where $A = BJB^*J$ is a nonlinear bounded operator from the Banach space X to itself. This implicit state-dependent convex sweeping process problem seems to be new in the Banach space setting. A different variant of implicit sweeping process with A is the identity mapping on X, has been proposed and studied in [2].

(3) If X is a Hilbert space, $h = 0$, A is the identity operator on X, and C is not state-dependent, then the problem (1.1) becomes

$$\begin{cases} \dot{x}(t) \in -N_{C(t)}(\dot{x}(t)) + g(t, x(t)) \text{ a.e. on } I, \\ x(0) = x_0 \in X. \end{cases}$$

This implicit convex sweeping process problem has been studied in [2].

For other types of implicit differential inclusions we refer to [3,4] and their references.

This paper is organized as follows. In Section 2, we recall some definitions and results that will be needed in the paper. In Section 3, we prove our main existence theorem. We end Section 3 with an illustrative application of our abstract results to differential variational inequalities on Banach spaces.

2. Preliminaries

Throughout the paper, we will use X to refer to a Banach space, while X^* will denote its topological dual space. The closed unit balls in X and X^* will be denoted by $\mathbb{B}$ and $\mathbb{B}_*$, respectively. For definitions and properties of q-uniformly convex and p-uniformly smooth Banach spaces, please refer to [5,6]. As example of these two classes of Banach spaces, we state all the spaces $l^p, \mathbf{C}^p, W^{p,m}, L^p$ for $p \in (1, \infty)$. For the proof of the uniform convexity and uniform smoothness of the spaces $l^p, L^p, W^{p,m}$ we refer for instance to Remark 1.6.9 in [5] and for the Schatten trace ideals $\mathbf{C}^p$ we refer to [7].

Let us also revisit the definition of the normalized duality mapping $J : X \rightrightarrows X^*$ which is expressed as follows:

$$J(x) = \{j(x) \in X^* : \langle j(x), x\rangle = \|x\|^2 = \|j(x)\|^2\}.$$

Numerous properties pertaining to the normalized duality mapping J can be found in [5, 8]. Additionally, we would like to revisit the definition of the functional: $V : X^* \times X \to \mathbb{R}$

$$V(x^*, x) = \|x^*\|^2 - 2\langle x^*, x\rangle + \|x\|^2.$$

We can define the generalized projection of $x^* \in X^*$ onto S by means of the functional V using the following expression.

Definition 2.1. Suppose we have a nonempty subset S of X and an element $x^* \in X^*$. We define a generalised projection of x^* onto S (see [9]) as any point $\bar{x} \in S$ that satisfies the following inequality:

$$V(x^*, \bar{x}) = \inf_{x \in S} V(x^*, x),$$

In such a scenario, we refer to $\bar{x}$ as the generalized projection of x^* onto S. The set of all points that satisfy this condition is denoted by $\pi_S(x^*)$.

We refer to the references [2, 9–12] for more properties and applications of the generalized projection π_S on closed convex and nonconvex sets. Let us also recall the definition of convex normal cones:

$$N(S; x) := \{x^* \in X^* : \langle x^*, y - x\rangle \leq 0, \forall y \in S\}.$$

3. Existence of solutions for implicit sweeping process

In this section, we present an existence result for the proposed implicit sweeping process (1.1). Our approach consists in tranforming the differetial inclusion (1.1) into a differential equation. Indeed, we show that the differential inclusion (1.1) is equivalent to the following differential equation:

$$\begin{cases} \dot{x}(t) = (B^*J)^{-1}\Big[J^{-1}\pi_{B^{-1}C(t,x(t))}\Big(B^*g(t, x(t)) + h(t, x(t))\Big) - h(t, x(t))\Big] \text{ a.e. } I, \\ x(0) = x_0 \in X. \end{cases} \tag{3.1}$$

Since the right hand side of this differential equation may have set-values in general, we need the following assumptions on the space X, operator B, and the set-valued mapping C: We check the well definedness of (3.1) under the following assumptions on B, g, h and C. We need J^{-1} and the generalized projection on $B^{-1}C(t, x(t))$ to be single-valued, which can be ensured by assuming that the Banach space X to be strictly convex and that the values $B^{-1}C(t, x(t))$ to be convex. Consequently, our proposed differential equation (3.1) is well defined. To start our study, we need the following assumptions on h, g, B, and C:

($\mathcal{A}_1$) For all $t \in I$ and all $x \in x_0 + r\mathbb{B}$

$$\max\{\|h(t, x)\|, \|g(t, x)\|\} \leq M_1;$$

($\mathcal{A}_2$) For all $t \in I$ and all $x, y \in x_0 + r\mathbb{B}$

$$\max\{\|h(t, x) - h(t, y)\|, \|g(t, x) - g(t, y)\|\} \leq M_2\|x - y\|;$$

($\mathcal{A}_3$) $B : X^* \to X$ is a bounded linear operator;

($\mathcal{A}_4$) For all $t \in I$ and for all $x, y \in X$

$$\mathcal{H}(C(t, x), C(t, y)) := \sup_{z \in X} |d_{C(t,x)}(z) - d_{C(t,y)}(z)| \leq K_1\|x - y\|;$$

($\mathcal{A}_5$) There exists $k \in C(I)$ such that for every $t \in I$, every $l > 0$ and every bounded set $A \subset X$

$$\gamma(C(t, A) \cap l\mathbb{B}) \leq k(t)\gamma(A),$$

where γ is either the Kuratowski or the Hausdorff measure of noncompactness on X;

($\mathcal{A}_6$) For all $x \in X$, the set-valued mapping $(t, x) \mapsto C(t, x)$ is measurable and there exists $\mu \in C(I)$ such that for all $t \in I$ and all $x \in X$

$$d_{C(t,x)}(0) \leq \mu(t)(\|x\| + 1).$$

We state the following important result on generalised projection on closed convex sets in q-uniformly convex Banach spaces. It is presented and proved in Theorem 4.5 in [10].

Proposition 3.1. *Let S be a nonempty closed convex subset of q-uniformly convex Banach space X. Then for any $M > 0$, there exists $L_M > 0$ such that $\forall x^* \in M\mathbb{B}_*$, the generalised projection of x^* on S is singleton and*

$$\|\pi_S(x_1^*) - \pi_S(x_2^*)\| \leq L_M \|x_1^* - x_2^*\|^{\frac{1}{q-1}}, \ \forall x_1^*, x_2^* \in M\mathbb{B}_*.$$

We need some important results that we gather in the following proposition.

Proposition 3.2. *Let X be a Banach space.*

(1) If X is q-uniformly convex, then for any $\alpha > 0$ there exists some constant $K_\alpha > 0$ such that

$$\langle J(x) - J(y); x - y \rangle \geq K_\alpha \|x - y\|^q, \quad \forall x, y \in \alpha\mathbb{B};$$

(2) If X is p-uniformly smooth, then the duality mapping J is Holder continuous with constant $p - 1$ on bounded sets, that is, for any $\alpha > 0$ there exists some constant $K'_\alpha > 0$ such that

$$\|J(x) - J(y)\| \leq K'_\alpha \|x - y\|^{p-1}, \quad \forall x, y \in \alpha\mathbb{B};$$

(3) If X is a reflexive smooth Banach space, S be closed convex set in X, and $\bar{x} \in S$, then

$$\begin{aligned} x^* \in N(S; \bar{x}) \quad &\Leftrightarrow \quad \exists \alpha > 0, \ \text{such that } \bar{x} \in \pi_S(J(\bar{x}) + \alpha x^*), \\ &\Leftrightarrow \quad \forall \alpha > 0, \ \text{such that } \bar{x} \in \pi_S(J(\bar{x}) + \alpha x^*). \end{aligned}$$

We also need to prove the following important result on the generalized projection on closed convex subsets of q-uniformly convex Banach spaces.

Proposition 3.3. *Let S_1 and S_2 be two nonempty closed convex subsets of q-uniformly convex Banach space X. Then for any $x^* \in X^*$ we have*

$$\|\pi_{S_1}(x^*) - \pi_{S_2}(x^*)\| \leq \left[\frac{2\|x^*\| + 2\beta}{K_\beta}\right]^{\frac{1}{q}} \mathcal{H}(S_1, S_2)^{\frac{1}{q}},$$

where $\beta := \max\{\|\pi_{S_1}(x^*)\|, \|\pi_{S_2}(x^*)\|\}$.

Proof. Let x^* be any point in X^*. Denote $\bar{x}_1 := \pi_{S_1}(x^*)$ and $\bar{x}_2 := \pi_{S_2}(x^*)$. Let $x := J^{-1}(x)$. Observe that

$$\langle J(x) - J(\bar{x}_1); y - \bar{x}_1 \rangle \leq 0, \quad \forall y \in S_1$$

and

$$\langle J(x) - J(\bar{x}_2); y - \bar{x}_2 \rangle \leq 0, \quad \forall y \in S_2.$$

Using Part (1) in Proposition 3.2, we have for $\beta = \max\{\|\pi_{S_1}(x^*)\|, \|\pi_{S_2}(x^*)\|\} > 0$

$$\langle J(\bar{x}_2) - J(\bar{x}_1); \bar{x}_2 - \bar{x}_1 \rangle \geq K_\beta \|\bar{x}_2 - \bar{x}_1\|^q. \tag{3.2}$$

If $\mathcal{H}(S_1, S_2) = \infty$, then we are done. Assume that $\mathcal{H}(S_1, S_2) < \infty$. Then there exists $\xi_1 \in S_1$ such that

$$\|\bar{x}_2 - \xi_1\| \leq \mathcal{H}(S_1, S_2).$$

So

$$\begin{aligned}\langle J(x) - J(\bar{x}_1); \bar{x}_2 - \bar{x}_1\rangle &= \langle J(x) - J(\bar{x}_1); \bar{x}_2 - \xi_1\rangle + \langle J(x) - J(\bar{x}_1); \xi_1 - \bar{x}_1\rangle \\ &\leq \|J(x) - J(\bar{x}_1)\|\|\bar{x}_2 - \xi_1\| \leq \|J(x) - J(\bar{x}_1)\|\mathcal{H}(S_1, S_2).\end{aligned}$$

Similarly, we have

$$\langle J(x) - J(\bar{x}_2); \bar{x}_1 - \bar{x}_2\rangle \leq \|J(x) - J(\bar{x}_2)\|\mathcal{H}(S_1, S_2).$$

Therefore, by adding the two above inequalities:

$$\langle J(\bar{x}_2) - J(\bar{x}_1); \bar{x}_2 - \bar{x}_1\rangle \leq [\|J(x) - J(\bar{x}_1)\| + \|J(x) - J(\bar{x}_2)\|]\,\mathcal{H}(S_1, S_2).$$

Hence, by (3.2) we obtain

$$\begin{aligned}K_\beta\|\bar{x}_2 - \bar{x}_1\|^q &\leq [\|J(x) - J(\bar{x}_1)\| + \|J(x) - J(\bar{x}_2)\|]\,\mathcal{H}(S_1, S_2) \\ &\leq [2\|x^*\| + \|\bar{x}_1\| + \|\bar{x}_2\|]\,\mathcal{H}(S_1, S_2),\end{aligned}$$

and so

$$\|\pi_{S_1}(x^*) - \pi_{S_2}(x^*)\| = \|\bar{x}_1 - \bar{x}_2\| \leq \left[\frac{2\|x^*\| + 2\beta}{K_\beta}\right]^{\frac{1}{q}} \mathcal{H}(S_1, S_2)^{\frac{1}{q}}.$$

Thus, completing the proof of Proposition 3.3. □

We start by proving that the Eq (3.1) is equivalent to the proposed variant of implicit sweeping process (1.1).

Proposition 3.4. *Under the assumptions* $(\mathcal{A}_1)$–$(\mathcal{A}_3)$*, we have* x *is a solution of (1.1) if and only if* x *is a solution of (3.1).*

Proof. Let x be a solution of (1.1). Then for a.e. $t \in I$ and for any $y \in C(t, x(t))$ we have by definition of the normal cone of closed convex sets

$$\langle g(t, x(t)) - J(\dot{x}(t)); y - BJ[B^* J(\dot{x}(t)) + h(t, x(t))]\rangle \leq 0.$$

Hence

$$\begin{aligned}&\langle g(t, x(t)) - J(\dot{x}(t)); B[B^{-1}y - J(B^* J(\dot{x}(t)) + h(t, x(t)))]\rangle \\ &= \langle B^* g(t, x(t)) - B^* J(\dot{x}(t)); B^{-1}y - J(B^* J(\dot{x}(t)) + h(t, x(t)))\rangle \\ &\leq 0.\end{aligned}$$

This ensures by definition of convex normal cones that

$$B^* g(t, x(t)) - B^* J(\dot{x}(t)) \in N_{B^{-1}C(t,x(t))}(J[B^* J(\dot{x}(t)) + h(t, x(t))]).$$

Using the characterization of the normal cone of closed convex sets in terms of generalised projection stated in Part (3) in Proposition 3.2, we can write

$$\begin{aligned} J[B^*J(\dot{x}(t)) + h(t,x(t))] &= \pi_{B^{-1}C(t,x(t))}\Big[B^*g(t,x(t)) - B^*J(\dot{x}(t)) \\ &+ J^{-1}J[B^*J(\dot{x}(t)) + h(t,x(t))]\Big] \\ &= \pi_{B^{-1}C(t,x(t))}\Big[B^*g(t,x(t)) + h(t,x(t))\Big]. \end{aligned}$$

This ensures that

$$J(\dot{x}(t)) = (B^*)^{-1}\left[J^{-1}\pi_{B^{-1}C(t,x(t))}\Big(B^*g(t,x(t)) + h(t,x(t))\Big) - h(t,x)\right],$$

that is, x is a solution of (3.1). Reciprocally, let x be a solution of (3.1). Then following the same reasoning as above in the opposite direction, we conclude that x is a solution of (1.1) and so the proof of Proposition 3.4 is complete. □

In order to prove the existence of solution for our main problem (1.1) using the above proposition, we need to prove the existence of solutions for differential equations in Banach spaces.

Theorem 3.5. *Let $f : I \times X \to X$ be a mapping satisfying:*

($\mathcal{H}_1$) $\|f(t,x)\| \leq L_1, \forall (t,x) \in I \times (x_0 + r\mathbb{B})$ for some $r \geq L_1 T$.
($\mathcal{H}_2$) f is uniformly continuous on $I \times (x_0 + r\mathbb{B})$.
($\mathcal{H}_3$) For a.e. $t \in I$, $\forall A \subset x_0 + r\mathbb{B}$ with $\gamma(A) > 0$, we have for some $L_2 > 0$

$$\gamma(f(t,A)) \leq L_2\gamma(A).$$

Then the differential equation

$$\begin{cases} \dot{x}(t) = f(t,x(t)) \text{ a.e. on } I, \\ x(0) = x_0 \in X, \end{cases}$$

has at least one Lipschitz solution.

Proof. Let $N \geq 1$ and $P_N = \{t_0, t_1, \ldots.., t_N\}$ be a partition of I with $t_0 = 0$ and $t_N = T$. We prove by considering on $[t_0, t_1]$ the differential equation with constant right hand side

$$\dot{x}(t) = f(t_0, x_0), \text{ with } x(0) = x_0.$$

This equation has a unique solution $x_N(t)$ on $[t_0, t_1]$ given by

$$x_N(t) = x_0 + f(t_0, x_0)(t - t_0), \quad \forall t \in [t_0, t_1].$$

Set $x_1 := x_N(t_1)$, we iterate by considering on $[t_1, t_2]$ the initial value problem

$$\dot{x}(t) = f(t_1, x_1), \text{ with } x(t_1) = x_1.$$

Similarly, we define $x_2 := x_N(t_2)$, we proceed in this way until a piecewise of mapping x_N has been defined on all the interval I. On the interval (t_i, t_{i+1}) we have

$$\|\dot{x}_N(t)\| = \|f(t_i, x_i)\| \leq L_1.$$

Set $x_i^N := x_N(t_i)$, for all $i = 0, \cdots, N$. So,

$$\begin{aligned} \|x_{i+1}^N - x_0^N\| &\leq \|x_{i+1}^N - x_i^N\| + \|x_i^N - x_0^N\| \\ &\leq \|f(t_i, x_i^N)(t_{i+1} - t_i)\| + \|x_i^N - x_0^N\| \\ &\leq L_1|t_{i+1} - t_i| + \|x_i^N - x_0^N\|. \end{aligned}$$

Hence, by induction we get

$$\begin{aligned} \|x_{i+1}^N - x_0^N\| &\leq L_1|t_{i+1} - t_i| + \Big[L_1|t_i - t_{i-1}| + \|x_{i-1}^N - x_0^N\|\Big] \\ &\leq L_1\Big[|t_{i+1} - t_i| + |t_i - t_{i-1}|\Big] + \|x_{i-1}^N - x_0^N\| \\ &\leq L_1\Big[|t_{i+1} - t_i| + |t_i - t_{i-1}|\Big] + \Big[L_1|t_{i-1} - t_{i-2}| + \|x_{i-2}^N - x_0^N\|\Big] \\ &\vdots \\ &\leq L_1\Big[|t_{i+1} - t_i| + |t_i - t_{i-1}| + ... + |t_2 - t_1|\Big] + \|x_1^N - x_0^N\| \\ &\leq L_1|t_{i+1} - t_0| \leq L_1 T. \end{aligned}$$

Thus,

$$\|x_i^N - x_0^N\| \leq L_1 T \leq r, \text{ for all } i = 1, 2, ..., N.$$

This ensures thet $x_i^N \in x_0 + L_1 T\mathbb{B}$, $\forall\, i \in \{1, 2, ..., N\}$ and hence by convexity of the ball and by definition of x_N on $[0, T]$, we get all the values of x_N are in $x_0 + r\mathbb{B}$. We also have $\|\dot{x}_N(t)\| \leq L_1$ for almost all $t \in I$ and so the sequence of mappings $(x_N)_N$ is equilipschitz with ratio L_1 on I. Set $B(t) = \{x_N(t) : N \geq 1\}$. We wish to prove that $B(t)$ is relatively compact in X, for any $t \in I$. By construction we have

$$\begin{aligned} x_N(t) &= x_N(0) + \int_0^t f(s, x_N(s))ds \\ &= x_0 + \int_0^t f(s, x_N(s))ds \\ &\in x_0 + \int_0^t f(s, B(s))ds, \ \forall t \in I, \quad \forall N \geq 1. \end{aligned}$$

Hence,

$$B(t) \subset x_0 + \int_0^t f(s, B(s))ds.$$

Using the assumption $(\mathcal{H}_3)$ and the properties of the measure of non compactness γ we obtain

$$\begin{aligned} \gamma(B(t)) &\leq \gamma(\{x_0\}) + \gamma(\int_0^t f(s, B(s))ds) \\ &\leq \int_0^t \gamma(f(s, B(s)))ds \leq L_2 \int_0^t \gamma(B(s))ds. \end{aligned}$$

Thus

$$\gamma(B(t)) \leq L_2 \int_0^t \gamma(B(s))ds, \ \forall t \in I.$$

Let $y(t) := \int_0^t \gamma(B(s))ds, \forall t \in I$. Then $y'(t) = \gamma(B(t))$ and

$$y'(t) \leq L_2 y(t), \forall t \in I.$$

Multiplying both sides by $e^{-L_2 t}$ gives

$$y'(t)e^{-L_2 t} \leq L_2 y(t)e^{-L_2 t}$$

and so

$$\frac{d}{dt}(y(t)e^{-L_2 t}) = y'(t)e^{-L_2 t} - L_2 y(t)e^{-L_2 t} \leq 0.$$

Therefore,

$$y(t)e^{-L_2 t} - y(0)e^0 \leq 0$$

and hence

$$y(t) \leq y(0)e^{L_2 t} = 0.$$

Thus,

$$\gamma(B(t)) \leq L_2 y(t) \leq 0.$$

This ensures that $\gamma(B(t)) = 0$, for all $t \in I$, that is, $B(t)$ is relatively compact in X fo all $t \in I$. Consequently, by Arzela-Ascoli theorem we conclude that (x_N) has a subsequence coverging uniformly to some x and $(\dot{x}_N)$ converges weakly in $L^1(I, X)$ to $\dot{x}$. Now, by uniform continuity of f and the uniform convergence of x_N to x on I as $N \to \infty$, it follows that $f(t, x_N(t)) \to f(t, x(t))$ uniformly on I, and so

$$\int_0^t f(s, x_N(s)) \to \int_0^t f(s, x(s))ds.$$

On the other hand we have,

$$x_N(t) = x_0 + \int_0^t f(s, x_N(s))ds$$

for all $t \in I$. Taking $N \to \infty$ gives

$$x(t) = x_0 + \int_0^t f(s, x(s))ds \ \forall t \in I.$$

This ensures that $\dot{x}(t) = f(t, x(t))$ a.e. on I, which completes the proof. □

We need to prove the following technical lemma.

Lemma 3.6. *Assume that $(\mathcal{A}_6)$ is satisfied. Then we have for any $z \in X$*

$$\|\pi_{B^{-1}C(t,x)}(z)\| \leq \|B^{-1}\|\mu(t)(\|x\| + 1) + 2\|z\|, \quad \forall x \in X \text{ and } \forall t \in I. \tag{3.3}$$

Proof. By assumption $(\mathcal{A}_6)$ there exists some element $s_0 \in C(t, x)$ with $\|s_0\| \leq \mu(t)(\|x\| + 1)$. Let $s_0^* := B^{-1}s_0 \in B^{-1}C(t, x)$. Then, by definition of the generalised projection $\pi_{B^{-1}C(t,x)}$ we have

$$V(z; \pi_{B^{-1}C(t,x)}(z)) \leq V(z; s_0^*) \leq (\|z\| + \|s_0^*\|)^2.$$

Thus,

$$\left(\|\pi_{B^{-1}C(t,x)}(z)\| - \|z\|\right)^2 \leq V(z; \pi_{B^{-1}C(t,x)}(z)) \leq (\|z\| + \|s_0^*\|)^2$$

and so

$$\left|\|\pi_{B^{-1}C(t,x)}(z)\| - \|z\|\right| \leq \|z\| + \|B^{-1}\|\mu(t)(\|x\| + 1).$$

Therefore,

$$\begin{aligned}\|\pi_{B^{-1}C(t,x)}(z)\| &= \|\pi_{B^{-1}C(t,x)}(z)\| - \|z\| + \|z\| \\ &\leq \Big|\|\pi_{B^{-1}C(t,x)}(z)\| - \|z\|\Big| + \|z\| \\ &\leq 2\|z\| + \|B^{-1}\|\mu(t)(\|x\| + 1).\end{aligned}$$

This completes the proof. □

In the proof of our next theorem, we need an additional assumption on the dual space X^* in terms of the measure of noncompactness of its normalized duality mapping $J^* = J^{-1}$. We say that X satisfies the assumption $(\mathcal{A})$ provided that for any $l > 0$ there exists some $k_l > 0$ such that for any set $A \subset l\mathbb{B}_*$ in X^* we have

$$\gamma(J^{-1}(A)) \leq k_l \gamma(A). \tag{3.4}$$

Obviously this assumption is satisfied for any Hilbert space with $k_l = 1$ for any $l > 0$. Also, it is satisfied for any 2-uniformly convex spaces (for example L^p spaces with $p \in (1, 2]$). Indeed, if X is 2-uniformly convex spaces, the dual space X^* is 2-uniformly smooth and so by Part (2) in Proposition 3.2, the duality mapping J^{-1} is Lipschitz on bounded sets and so for any $l > 0$ there exists some $K_l > 0$ such that

$$\|J^{-1}(x^*) - J^{-1}(y^*)\| \leq K_l \|x^* - y^*\|, \quad \forall x^*, y^* \in l\mathbb{B}_*.$$

Fix any $\epsilon > 0$. By definition of the measure of non compactness γ there exists a finite covering $\{A_i\}_{i=1}^m$ of A in X^* such that

$$\gamma(A) + \epsilon > \mathrm{diam}(A_i), \forall i = 1, \cdots, m.$$

Define $B_i := J^{-1}(A_i)$, $\forall i = 1, \cdots, m$. Obviously $\{B_i\}_{i=1}^m$ is a finite covering of $J^{-1}(A_i)$ in X. Fix now any two points x, y in B_i we have $x^* := J(x), y^* := J(y) \in A_i$ and so

$$K_l \mathrm{diam}(A_i) \geq K_l \|x^* - y^*\| \geq \|J^{-1}(x^*) - J^{-1}(y^*)\| = \|x - y\|.$$

Therefore, for any x, y in B_i we have

$$K_l(\gamma(A) + \epsilon) > K_l \|x^* - y^*\| \geq \|x - y\|,$$

which ensures

$$K_l(\gamma(A) + \epsilon) \geq \mathrm{diam}(B_i) > \gamma(J^{-1}(A)).$$

Taking $\epsilon \to 0$ gives the desired inequality:

$$K_l \gamma(A) \geq \gamma(J^{-1}(A)).$$

We may pose the following natural questions: Can we characterize the class of Banach spaces that satisfy assumption $(\mathcal{A})$? Unfortunately, at present, there exists no answer or literature addressing this inquiry. Nonetheless, it is worth mentioning that [13, 14] delved into a distinct approach to analyze the measure of noncompactness for duality mappings, and their concepts and methodologies may serve as a basis for tackling the aforementioned question. Based on the aforesaid reasoning, we can deduce that all L^p spaces with $p \in (1, 2]$ satisfy assumption $(\mathcal{A})$. However, the case of $p > 2$ remains unresolved, though we put forth the conjecture that ($\mathcal{A}$) still holds. It is noteworthy that the examination of the same inequality for operators other than the duality mapping has been studied in the reference [15, 16].

Now, we use our previous results in Theorem 3.5 and Lemma 3.6, to state and prove the main result of this paper, that is, the existence of solutions for the proposed implicit convex sweeping processes (1.1).

Theorem 3.7. *Let X be a 2-uniformly convex Banach space, $x_0 \in X$, $r > 0$, $C : I \times X \to X$ be a set-valued mapping with nonempty, closed, and convex values, and let $B : X^* \to X$ be a bounded linear operator, and let $g : I \times X \to X^*$, and $h : I \times X \to X$ be two given mappings. Assume that X satisfies ($\mathcal{A}$) and that ($\mathcal{A}_1$) − ($\mathcal{A}_6$) are also satisfied. Suppose that the following inequalities hold: $T\bar{\mu}\|B^{-1}\|^2 < 1$ and $\frac{3M_1\|B^{-1}\|+2M_1\|B\|\|B^{-1}\|+\|B^{-1}\|^2\bar{\mu}(1+\|x_0\|)}{T^{-1}-\bar{\mu}\|B^{-1}\|^2} < r$, with $\bar{\mu} := \max_{t\in I} \mu(t)$. Then there exists a mapping $x : I \to X$ satisfying* (1.1).

Proof. Let us consider the mapping

$$f(t, x) := J^{-1}\Big[(B^*)^{-1}J^{-1}\pi_{B^{-1}C(t,x)}\Big(B^*g(t, x) + h(t, x)\Big) - (B^*)^{-1}h(t, x)\Big]. \tag{3.5}$$

We are going to show that all hypothesis ($\mathcal{H}_1$)–($\mathcal{H}_3$) are satisfied for the mapping f defined in (3.5). Let $t \in I$ and let $x \in x_0 + r\mathbb{B}$. Then $x \in M\mathbb{B}$ with $M := \|x_0\| + r$. Set

$$z := B^*g(t, x) + h(t, x)$$

and

$$y^* := (B^*)^{-1}J^{-1}\pi_{B^{-1}C(t,x)}(z) - (B^*)^{-1}h(t, x).$$

Under our assumptions ($\mathcal{A}_1$)–($\mathcal{A}_6$) we have

$$\|z\| \le \|B^*\|\|g(t, x)\| + \|h(t, x)\| \le M_1(\|B\| + 1) =: M_3$$

and

$$\begin{aligned} \|y^*\| &\le \|B^{-1}\|\|\pi_{B^{-1}C(t,x)}(z)\| + \|B^{-1}\|\|h(t, x)\| \\ &\le \|B^{-1}\|(M_1 + \|\pi_{B^{-1}C(t,x)}(z)\|). \end{aligned}$$

On the other hand we have by Lemma 3.6

$$\begin{aligned} \|\pi_{B^{-1}C(t,x)}(z)\| &\le 2\|z\| + \|B^{-1}\|\mu(t)(\|x\| + 1) \\ &\le 2M_3 + \|B^{-1}\|\bar{\mu}(M + 1) =: R. \end{aligned}$$

Therefore, we obtain

$$\|y^*\| \le \|B^{-1}\|(M_1 + R).$$

This ensures that ($\mathcal{H}_1$) is satisfied with $L_1 := \|B^{-1}\|(M_1 + R)$. Also, we note that f is uniformly continuous on $I \times (x_0 + r\mathbb{B})$ because J^{-1} and $\pi_{B^{-1}C(t,x)}$ are holder continuous from Propositions 3.2 and 3.3 and also by using ($\mathcal{A}_4$). On the other hand there exists some $K_{L_1} > 0$ such that for a.e. $t \in [0, T]$ and every $A \subset x_0 + r\mathbb{B}$,

$$\gamma(f(t, A)) \le \|(B^*)^{-1}\|(K_{L_1}\|B^{-1}\|\bar{\mu} + M_2)\gamma(A).$$

Indeed, since the space X satisfies the assumption $(\mathcal{A})$ we have for some $K_{L_1} > 0$ such that

$$\gamma(J^{-1}(D)) \leq K_{L_1}\gamma(D), \text{ for any subset } D \text{ in } L_1\mathbb{B}_*.$$

Fix now any $A \subset x_0 + r\mathbb{B}$. Set

$$D := (B^*)^{-1}J^{-1}\pi_{B^{-1}C(t,A)}\Big[B^*g(t,A) + h(t,A)\Big] - (B^*)^{-1}h(t,A).$$

Then, D is a subset of $L_1\mathbb{B}_*$ and so by the previous inequality we obtain

$$\begin{aligned}
\gamma(f(t,A)) &= \gamma\Big(J^{-1}\Big[(B^*)^{-1}J^{-1}\pi_{B^{-1}C(t,A)}[B^*g(t,A) + h(t,A)] - (B^*)^{-1}h(t,A)\Big]\Big)\\
&\leq \gamma(J^{-1}(D))\\
&\leq K_{L_1}\gamma(D)\\
&\leq K_{L_1}\gamma\Big((B^*)^{-1}J^{-1}\pi_{B^{-1}C(t,A)}\Big[B^*g(t,A) + h(t,A)\Big] - (B^*)^{-1}h(t,A)\Big).
\end{aligned}$$

Thus, using the properties of γ and our assumptions we get

$$\begin{aligned}
\gamma(f(t,A)) &\leq K_{L_1}\|B^{-1}\|\Big[\gamma(J^{-1}\pi_{B^{-1}C(t,A)}(B^*g(t,A) + h(t,A))) + \gamma(h(t,A))\Big]\\
&\leq \|B^{-1}\|K_{L_1}\Big[K_{L_1}\gamma(\pi_{B^{-1}C(t,A)}(B^*g(t,A) + h(t,A))) + \gamma(h(t,A))\Big]\\
&\leq \|B^{-1}\|K_{L_1}\Big[K_{L_1}\gamma\Big(B^{-1}C(t,A) \cap R\mathbb{B}\Big) + \gamma(h(t,A))\Big]\\
&\leq \|B^{-1}\|K_{L_1}\Big[K_{L_1}\|B^{-1}\|\gamma\Big(C(t,A) \cap R\|B\|\mathbb{B}\Big) + 2M_2\gamma(A))\Big]\\
&\leq \|B^{-1}\|K_{L_1}\Big[K_{L_1}\|B^{-1}\|k(t)\gamma(A) + 2M_2\gamma(A))\Big]\\
&\leq \|B^{-1}\|K_{L_1}\Big[K_{L_1}\|B^{-1}\|\max_{t\in I} k(t) + 2M_2\Big]\gamma(A)\\
&\leq \Big[\|B^{-1}\|^2K_{L_1}^2\bar{k} + 2M_2\|B^{-1}\|K_{L_1}\Big]\gamma(A), \text{ where } \bar{k} := \max_{t\in I} k(t).
\end{aligned}$$

Thus, the assumption $(\mathcal{H}_3)$ is satisfied with $L_2 := \|B^{-1}\|^2K_{L_1}^2\bar{k} + 2M_2\|B^{-1}\|K_{L_1}$. Now, all the assumptions $(\mathcal{H}_1)$–$(\mathcal{H}_3)$ of Theorem 3.5 are fulfilled but we need to verify the additional assumption $L_1T < r$. Indeed, using the inequalities

$$T\bar{\mu}\|B^{-1}\|^2 < 1,$$

$$\frac{3M_1\|B^{-1}\| + 2M_1\|B\|\|B^{-1}\| + \|B^{-1}\|^2\bar{\mu}(1 + \|x_0\|)}{T^{-1} - \bar{\mu}\|B^{-1}\|^2} < r,$$

we deduce that

$$3M_1\|B^{-1}\| + 2M_1\|B\|\|B^{-1}\| + \|B^{-1}\|^2\bar{\mu}(1 + \|x_0\|) < rT^{-1} - r\bar{\mu}\|B^{-1}\|^2$$

and so

$$TL_1 = T\Big[3M_1\|B^{-1}\| + 2M_1\|B\|\|B^{-1}\| + \|B^{-1}\|^2\bar{\mu}(1 + \|x_0\|) + r\bar{\mu}\|B^{-1}\|^2\Big] < r.$$

Now, we can apply Theorem 3.5 to get a Lispchitz solution of the Eq (3.1) which is, in fact by using Proposition 3.4, our desired solution of (1.1) and hence we achieve the poof of Theorem 3.7. □

Now, we present an illustrative example showing the applicability of our abstract results in Banach spaces.

Example 3.8 (Differential Variational Inequalities (DVI))**.** *Let $X := L^p(0,T;\mathbb{R})$, with $p \in (1,2]$, $W : [0,T] \times X \to S$ be a Lipschitz non increasing function w.r.t. the second variable with Lipschitz ratio $k > 0$, S convex compact subset of $L^p(0,T;\mathbb{R})$ and we define the set-valued mapping $C : [0,T] \times X \rightrightarrows X$ as follows: $C(t,x) := S - W(t,x)$. Consider the following differential variational inequality: Find $x : [0,T] \to X$ such that $x(0) = x_0 \in X$ and*

$$\begin{cases} \langle J(\dot{x}(t)) - g(t,x(t)), v - BJ[B^* J(\dot{x}(t)) + h(t,x(t))]\rangle \geq 0, & \textit{(DVI)} \\ \textit{for all } v \in C(t,x), \textit{ and for a.e. on } [0,T]. \end{cases}$$

Here, $h : I \times X \to X$, $g : I \times X \to X^*$ are bounded Lipschitz functions and $B : X^* \to X$ is a bounded linear operator. Let us prove the existence of solutions for DVI by using our abstract results proved in Theorem 3.7. First, we rewrite DVI in the form of (1.1). Clearly C has closed convex values and so using the definition of normal cones for convex sets, DVI is equivalent to

$$J(\dot{x}(t)) - g(t,x(t)) \quad \in \quad -N(C(t,x); BJ[B^* J(\dot{x}(t)) + h(t,x(t))])$$

and hence DVI is equivalent to

$$J(\dot{x}(t)) \quad \in \quad -N(C(t,x); BJ[B^* J(\dot{x}(t)) + h(t,x(t))]) + g(t,x(t)).$$

Clearly ($\mathcal{A}_1$) and ($\mathcal{A}_2$) are satisfied. Also, we observe that for any $x, y \in X$ and for any $z \in C(t,y)$

$$\begin{aligned} d_{C(t,x)}(z) &= \inf_{u \in C(t,x)} \|u - z\| = \inf_{s \in S} \|s - W(t,x) - z\| = d_S(z + W(t,x)) \\ &= d_S(s_z - W(t,y) + W(t,x)), \end{aligned}$$

where $s_z \in S$ with $z = s_z - W(t,y)$. Thus, we obtain

$$d_{C(t,x)}(z) \quad \leq \quad \|W(t,x) - W(t,y)\| \leq k\|x - y\|, \quad \forall z \in C(t,y).$$

Similarly, we have

$$d_{C(t,y)}(u) \quad \leq \quad \|W(t,y) - W(t,x)\| \leq k\|x - y\|, \quad \forall u \in C(t,x).$$

Therefore,

$$\mathcal{H}(C(t,x), C(t,y)) = \max\{ \sup_{u \in C(t,x)} d_{C(t,y)}(u), \sup_{z \in C(t,y)} d_{C(t,x)}(z)\} \leq k\|x - y\|.$$

This ensures that ($\mathcal{A}_4$) is satisfied. On the other hand, since S is compact, we have $\gamma(C(t,A) \cap r\mathbb{B}) = 0$, and ($\mathcal{A}_5$) is obviously satisfied. Also, since by definition we have $W(t,x) \in S$, $\forall (t,x) \in I \times X$, we get $0 \in S - W(t,x) = C(t,x)$ for every $t \in I$ and $x \in X$, and hence the assumption ($\mathcal{A}_6$) is satisfied. Therefore, by Theorem 3.7 there exists a solution for DVI.

4. Conclusions

In summary, our study delves into the realm of 2-uniformly convex Banach spaces, successfully establishing the existence of solutions for a specific adaptation of implicit state-dependent convex sweeping processes. The core of our methodology revolves around a meticulously crafted differential equation strongly linked to the generalized projection operator. Through the meticulous examination in this study, we significantly contribute to enhancing our comprehension of convex sweeping processes in Banach spaces. Our subsequent objective is not only to deepen our understanding of these processes but also to pave the way for extending these pivotal existence results into the realm of nonconvex settings. Additionally, we present an illustrative example showing the applicability of our abstract results in Banach spaces.

Use of AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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