AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 9(1): 2063–2083.
DOI:10.3934/math.2024103
Received: 01 October 2023
Revised: 08 November 2023
Accepted: 09 November 2023
Published: 20 December 2023

Research article

On a new nonlinear convex structure

Andreea Bejenaru[1,*] **and Mihai Postolache**[2,3,4]

[1] Department of Mathematics and Informatics, National University of Science and Technology Politehnica Bucharest, 060042 Bucharest, Romania

[2] The Key Laboratory of Intelligent Information and Big Data Processing of NingXia Province, and Health Big Data Research Institute, North Minzu University, 750021 Yinchuan, China

[3] Department of Mathematics and Informatics, National University of Science and Technology Politehnica Bucharest, 060042 Bucharest, Romania

[4] Gh. Mihoc-C. Iacob Institute of Mathematical Statistics and Applied Mathematics, Romanian Academy, 050711 Bucharest, Romania

* **Correspondence:** Email: andreea.bejenaru@upb.ro.

Abstract: In this work we start from near vector spaces, which we endow with some additional properties that allow convex analysis. The seminormed structure used here will also be improved by adding properties such as the null condition and null equality, thus resulting in a new type of space, which is still weaker than the conventional Banach structures: pre-convex regular near-Banach space. On the newly defined structure, we introduce the concept of uniform convexity and analyze several resulting properties. The major outcomes prove a remarkable resemblance to the classical properties resulting from uniform convexity on hyperbolic metric spaces or modular function spaces, including the famous Browder-Göhde fixed point theorem.

Keywords: near-Banach spaces; uniform convexity; type function; asymptotic center; nonexpansive mapping
Mathematics Subject Classification: 46B20, 47H09, 52A01

1. Introduction

Uniform convexity was first defined for Banach spaces in [1]. This property has a strong geometric significance, stating that the midpoint of two given distinct points in the unit ball always falls inside the ball, or, equivalently, if the midpoint tends to the boundary, the considered points approach each other. In fixed point theory, the amazing contribution of this geometric property is emphasized by the famous Browder-Göhde Theorem (see [2, 3]): Each nonexpansive self-mapping of a bounded closed convex subset in a uniformly convex Banach space has a fixed point, and the set of fixed points is closed and convex. In connection to this theorem, see Sections 2 and 5 (uniform convexity and fixed points in uniformly convex spaces, respectively) in [4]. Another interesting consequence of uniform convexity was revealed by Edelstein ([5, 6]) who defined in connection with bounded sequences the concept of asymptotic center, proving that, in uniformly convex Banach spaces, the asymptotic center is a singleton. A comprehensive approach to asymptotic centers is given in Section 4 of [4].

One of the first significant extensions of uniform convexity was done in [7] to hyperbolic metric spaces. In connection to this, see also Section 4 of the early paper [8]. Obviously, normed linear spaces are hyperbolic spaces. As nonlinear examples, one can consider the Hadamard manifolds, the Hilbert open unit ball equipped with the hyperbolic metric and the CAT(0) spaces (see Chapter 2 in [4]). In [9], remarkable consequences of metric uniform convexity were stated:

- property (R) of each uniformly convex metric space (which is equivalent to reflexivity for Banach spaces), stating that the intersection of each nonincreasing sequence of nonempty closed, bounded and convex subsets is nonempty;
- the existence and uniqueness of best approximants in closed and convex subsets of uniformly convex metric spaces;
- a metric version of parallelogram identity;
- a very useful technical lemma stating sufficient conditions for the distance limit between two sequences to be zero;
- using the type-function of an bounded sequence, an analogue of Edelstein's result was proven, stating that all minimizing sequences are convergent toward the same, unique, limit-point.

The next relevant step regarding uniform convexity was the extension of this concept to modular function spaces [10], resulting in properties similar to those described above. In addition, the existence of fixed points for pointwise asymptotically nonexpansive self-mappings was proved.

Following the ideas applied in [9] and [10] for hyperbolic metric spaces and modular function spaces, respectively, we wish to properly define uniform convexity on regular near-Banach spaces. The concepts of near vector space, near (pseudo, semi) norm, as well as a number of versions of near-Banach spaces are extremely recent and provide weaker types of normed structures, where the classical distributivity laws from conventional vector spaces and the existence of additive inverse elements are completely omitted.

Several versions of near-Banach spaces were introduced by Wu in [11]. Starting from a significantly weakened vector structure (so-called near vector space), in which the laws of distributivity, as well as the existence of the additive inverse element are completely excluded, and the existence of the zero element is optional (being substituted by the so-called null set), Wu managed to define several types of norms resulting near pseudo-seminormed, near seminormed, near pseudo-normed or near normed spaces. In connection to these norms, he defined appropriate concepts of convergence and completeness, obtaining several types of near-Banach structures. Sometimes other properties as null condition or null equality were assumed and these led to some important consequences. Despite the limitations imposed by the weak underlying structure, Wu managed to state and prove some adequate versions of Banach's contraction principle. Moreover, three important examples were provided, proving that the newly considered structures are not trivial: The space of all bounded and closed intervals in $\mathbb{R}$, the space of fuzzy numbers in $\mathbb{R}$ and the hyperspace of all subsets of a conventional vector space.

However, looking closely at these highly general vector structures, we notice that they lack the basic properties that allow us to approach convexity problems. Still, the examples listed above have some features that make possible the definition of convexity: although scalar multiplication does not generally act like it does for conventional vector spaces, it obeys the laws of distributivity when dealing with scalars in the interval $[0, 1]$. In this paper we shall assume this additional property and we shall name it pre-convexity. Moreover, the norm function we choose to work with will be the near seminorm from [11] endowed with two additional properties: null condition and null equality. We will name this stronger structure regular near norm. So, the underlying structure to be studied here is a pre-convex regular near-Banach space. We will adapt uniform convexity to this type of space, studying several properties arising from this concept. The major outcomes are similar to those already existing for hyperbolic metric spaces and modular function spaces. In addition, we prove that Browder-Göhde Theorem holds true for uniformly convex regular near-Banach spaces. Some examples are provided to underline the utility of our study.

2. Preliminaries on regular near normed spaces

In this section, we recall the definition of a near vector space and other related concepts from [11]. Starting from a near seminorm, and adding to it properties such as the null condition and null equality, we define the regular near norm on a near vector space, as well as corresponding concepts concerning convergence analysis: Cauchy sequence, convergence, completeness and regular near-Banach space.

Definition 1 ([11])**.** A universal set over $\mathbb{R}$ is any nonempty set U endowed with two operations, an internal one called vector addition and conventionally marked $\oplus$ and an external one acting as scalar multiplication with real numbers.

Remark 1. The minus sign could be also used in connection to universal sets, if considered as follows:

$$-x = (-1)x.$$

We notice that it marks just a simple notation, without any connection to opposite elements. In general, for universal sets, the concept of additive inverse element is not even defined.

Moreover, for any two elements $x, y \in U$, the subtraction $x \ominus y$ could be also conventionally defined, as follows:

$$x \ominus y = x \oplus (-y) = x \oplus (-1)y.$$

Definition 2 ([11])**.** Let U be a universal set over $\mathbb{R}$. The set

$$\Psi = \{x \ominus x \ : \ x \in U\}$$

is called the null set of U.

Definition 3 ([11])**.** Let U be a universal set over $\mathbb{R}$. We say that U is a near vector space over $\mathbb{R}$ if the following additional conditions are satisfied:

(i) $1x = x, \ \forall x \in U$;

(ii) if $x = y$, then $\begin{cases} x \oplus z = y \oplus z, \ \forall z \in U; \\ \alpha x = \alpha y, \ \forall \alpha \in \mathbb{R}; \end{cases}$

(iii) $x \oplus y = y \oplus x, \ \forall x, y \in U$;
(iv) $x \oplus (y \oplus z) = x \oplus y \oplus z, \ \forall x, y, z \in U$.

Definition 4 ([11])**.** Let U be a near vector space over $\mathbb{R}$, with the null set Ψ. Two elements x, y are called almost identical, and denoted $x \stackrel{\Psi}{=} y$, if they satisfy any of the conditions below:

(i) $x = y$;
(ii) $\exists\, \psi \in \Psi$ such that $x = y \oplus \psi$ or $y = x \oplus \psi$;
(iii) $\exists\, \psi_1, \psi_2 \in \Psi$ such that $x \oplus \psi_1 = y \oplus \psi_2$.

In [11], a near seminorm was defined by homogeneity and subadditivity. In addition, properties as null condition and null equality were often used to provide stronger features in near seminormed spaces. In this work we will put together all these conditions, thus defining the concept of regular near norm.

Definition 5. Let U be a near vector space over $\mathbb{R}$ with the null set Ψ. A function $\|\cdot\|\colon U \to \mathbb{R}_+$ is called regular near norm on U, if the following conditions hold true:

(i) the null condition: $\|x\| = 0$ if and only if $x \in \Psi$;
(ii) the null equality: $\|x \oplus \psi\| = \|x\|, \ \forall x \in U, \ \forall \psi \in \Psi$;
(iii) homogeneity: $\|\alpha x\| = |\alpha| \cdot \|x\|, \ \forall \alpha \in \mathbb{R}, \ \forall x \in U$;
(iv) subadditivity: $\|x \oplus y\| \leq \|x\| + \|y\|, \ \forall x, y \in U$.

Example 1. Let us consider the set U of all bounded functions on $\mathbb{R}$. We define the addition of two bounded functions in the usual way and the scalar multiplication as follows:

$$(\alpha f)(x) = \begin{cases} \alpha f(x), & \text{if } x \geq 0, \\ \alpha^2 f(\alpha^2 x), & \text{if } x < 0. \end{cases}$$

It is not difficult to see that all the properties in Definition 3 are satisfied, making U a near vector space over $\mathbb{R}$. We also notice that, since $(\alpha + \beta) f \neq \alpha f + \beta f$, U is not a conventional vector space. Moreover, one can identify the null set Ψ in all the bounded functions that are identically null on $[0, \infty)$.

For a given function $f \in U$, we define $\|f\| = \sup\limits_{x \geq 0} |f(x)|$. We thus obviously obtain a regular near norm on U.

3. Properties of regular near normed spaces

The next proposition lists some regularity properties of the null set. They will play a significant role in the development of our study. In [11], Wu demonstrated that these properties also work in a less restrictive framework than the regular near norm studied here (for this wider setting, please see Remark 3.3 and Proposition 3.9 in the cited paper).

Proposition 1. Let $\|\cdot\|$ be a regular near norm on a near vector space U, with null set Ψ. The following properties hold true:

(i) The null set is closed under internal addition and scalar multiplication (see also Remark 3.3 in [11]);

(ii) $x \stackrel{\Psi}{=} y \Leftrightarrow x \ominus y \in \Psi \Leftrightarrow \|x \ominus y\| = 0$ (Proposition 3.9 in [11]);
(iii) If $x \stackrel{\Psi}{=} y$, then $\|x\| = \|y\|$ (Proposition 3.9 in [11]);

Proof.

(i) Let $\alpha \in \mathbb{R}$ and $\psi, \phi \in \Psi$. Then, from the null condition it follows that $\|\psi\| = \|\phi\| = 0$ and

$$\|\alpha\psi\| = |\alpha| \cdot \|\psi\| = 0; \;\; 0 \leq \|\psi + \phi\| \leq \|\psi\| + \|\phi\| = 0.$$

Using again the null condition, we conclude that $\alpha\psi$, $\psi + \phi \in \Psi$.

(ii) The second equivalence results directly from the null condition. Let us suppose now that $x \stackrel{\Psi}{=} y$. Then, one of the three conditions in Definition 4 must be satisfied. We run the proof for the third condition, the other cases being similar. So, there exist ψ_1, $\psi_2 \in \Psi$, such that $x \oplus \psi_1 = y \oplus \psi_2$. Then,

$$x \oplus \psi_1 \ominus y = y \oplus \psi_2 \ominus y,$$

that is

$$x \ominus y \oplus \psi_1 = y \ominus y \oplus \psi_2 = \psi \oplus \psi_2 \in \Psi,$$

where ψ denotes $y \ominus y$. According to the null condition, $\|x \ominus y \oplus \psi_1\| = 0$. On the other side, from null equality, $\|x \ominus y \oplus \psi_1\| = \|x \ominus y\|$, which proves that $\|x \ominus y\| = 0$, equivalent to $x \ominus y \in \Psi$.
For the converse statement, let us assume that $x \ominus y \in \Psi$. Then, $x \ominus y = \psi$, hence $x \ominus y \oplus y = \psi \oplus y$. Since $y \ominus y \in \Psi$, then $x \stackrel{\Psi}{=} y$, according to Definition 4.

(iii) Let $x \stackrel{\Psi}{=} y$. Same as before, we consider ψ_1, $\psi_2 \in \Psi$, such that $x \oplus \psi_1 = y \oplus \psi_2$. From null equality it follows that

$$\|x\| = \|x \oplus \psi_1\| = \|y \oplus \psi_2\| = \|y\|\,.$$

□

Further properties concerning regular near normed spaces are listed below. They provide conclusions about some partial distributivity rules that could be applied despite the weak vector structure of the underlying near vector space.

Proposition 2. Let $\|\cdot\|$ be a regular near norm on a near vector space U, with null set Ψ. The following properties hold true:

(i) $-(x \ominus y) \stackrel{\Psi}{=} -x \oplus y = y \ominus x$ and $-(x \oplus y) \stackrel{\Psi}{=} -x \oplus (-y) = -x \ominus y$, although equalities are not necessarily satisfied. Further on, the symmetric condition $\|x \ominus y\| = \|y \ominus x\|$ (see Definition 3.5 and Proposition 3.6 in [11]) is satisfied;

(ii) $z \ominus (x \oplus y) \stackrel{\Psi}{=} z \ominus x \ominus y$ and $z \ominus (x \ominus y) \stackrel{\Psi}{=} z \ominus x \oplus y$. Consequently, opening parentheses inside the norm is allowed (see Proposition 3.7 in [11]):

$$\|z \ominus (x \oplus y)\| = \|z \ominus x \ominus y\|; \quad \|z \ominus (x \ominus y)\| = \|z \ominus x \oplus y\|.$$

(iii) $-(-x) \stackrel{\Psi}{=} x$, although $-(-x)$ is not necessarily equal to x.

(iv) The triangle inequality $\|x \ominus y\| \leq \|x \ominus z\| + \|z \ominus y\|$ is satisfied (see also Proposition 3.8 in [11]);

Proof.

(i) From the definition of the null set Ψ, one has $-(x \ominus y) \oplus (x \ominus y) = (x \ominus y) \ominus (x \ominus y) = \psi \in \Psi$. We add in both sides $-x$ and y and obtain

$$-(x \ominus y) \oplus (x \ominus y) \ominus x \oplus y = \psi \ominus x \oplus y,$$

that is, after commuting some elements in the left side,

$$-(x \ominus y) \oplus x \ominus x \ominus y \oplus y = \psi \ominus x \oplus y.$$

By denoting $x \ominus x = \psi_1 \in \Psi$ and $y \ominus y = \psi_2 \in \Psi$, we find

$$-(x \ominus y) \oplus \psi_1 \oplus \psi_2 = -x \oplus y \oplus \psi,$$

which provides immediately $-(x \ominus y) \stackrel{\Psi}{=} -x \oplus y = y \ominus x$. Applying the regular norm, we also find

$$\|x \ominus y\| = \|-(x \ominus y)\| = \|y \ominus x\| .$$

The second relationship can be proved similarly.

(ii) These properties follow immediately after adding z in both sides of the relations proved bei the previous item.

(iii) By the definition of the null set, one has $-(-x) \oplus (-x) = -x \ominus (-x) = \psi \in \Psi$. Then, we add x in both sides and obtain

$$-(-x) \oplus (-x) \oplus x = \psi \oplus x,$$

that is

$$-(-x) \oplus \psi_1 = x \oplus \psi,$$

where ψ_1 denotes $(-x) \oplus x = x \ominus x$. This proves the claimed identity.

(iv) According to Definition 4, $x \ominus y \stackrel{\Psi}{=} x \ominus y + \psi$, for every $\psi \in \Psi$. In particular, if $z \in U$, then $z \ominus z \in \Psi$, so

$$x \ominus y \stackrel{\Psi}{=} x \ominus y \oplus (z \ominus z) = (x \ominus z) \oplus (z \ominus y).$$

From Proposition 1(iii) and subadditivity of the regular near norm, we find

$$\|x \ominus y\| = \|(x \ominus z) \oplus (z \ominus y)\| \leq \|x \ominus z\| + \|z \ominus y\| .$$

□

Due to the properties listed above, we are now able to define concepts related to metric convergence for regular near normed spaces.

Definition 6. Let $(U, \|\cdot\|)$ be a regular near normed space.

- A sequence $\{x_n\}$ in U is said to be convergent to $x \in U$ if $\lim_{n\to\infty} \|x_n \ominus x\| = 0$.
- The sequence $\{x_n\}$ in U is called Cauchy if $\lim_{m,n\to\infty} \|x_n \ominus x_m\| = 0$.
- The regular near normed space $(U, \|\cdot\|)$ is called complete (or regular near-Banach space) if every Cauchy sequence is also convergent.
- A subset $C \subset U$ is called bounded if $\|x \ominus y\| \leq M, \ \forall x, y \in C$, for a given constant $M > 0$.

- A subset $C \subset U$ is called closed if for each sequence $\{x_n\} \subset C$ satisfying $\lim_{n\to\infty} \|x_n \ominus x\| = 0$ for some $x \in U$, one has $x \in C$.

In [11], several alternative definitions for convergence, Cauchy sequences and completeness were provided for more general cases. However, since in a regular near normed space the symmetric condition is satisfied (Proposition 2(i)), the definitions above are well posed.

Remark 2. The limit of a sequence $\{x_n\}$ is not necessarily unique. However, the sequence $\{x_n\}$ converges simultaneously to distinct limit points $x, y \in U$ if and only if $x \stackrel{\Psi}{=} y$. For details, see [11], Proposition 4.3.

4. Uniform convexity on regular near-Banach spaces

In order to operate with convexity related properties, we previously ask for a certain precondition to be fulfilled. More precisely, we ask that the space behave as a conventional vector space, at least for scalars from interval $[0, 1]$.

Definition 7. A near vector space U is endowed with pre-convexity property, if

(i) $\alpha x \oplus \beta x = (\alpha + \beta)x,\ \forall \alpha, \beta \in [0, 1], \alpha + \beta \le 1,\ \forall x \in U$;
(ii) $\alpha(\beta x) = (\alpha\beta)x,\ \forall \alpha, \beta \in [0, 1],\ \forall x \in U$;
(iii) $\alpha x \oplus \alpha y = \alpha(x \oplus y),\ \forall \alpha \in [0, 1],\ \forall x, y \in U$.

Lemma 1. Let $(U, \|\cdot\|)$ be a pre-convex regular near normed space. Then,

$$\|[(1-\alpha)x \oplus \alpha y] \ominus z\| \le (1-\alpha)\|x \ominus z\| + \alpha\|y \ominus z\|,\ \forall x, y, z \in U,\ \forall \alpha \in [0, 1].$$

Proof. Indeed, since $z = (1-\alpha)z \oplus \alpha z$ one has

$$\begin{aligned}
\|[(1-\alpha)x \oplus \alpha y] \ominus z\| &= \|[(1-\alpha)x \oplus \alpha y] \ominus [(1-\alpha)z \oplus \alpha z]\| \\
&= \|(1-\alpha)x \oplus \alpha y \ominus (1-\alpha)z \ominus \alpha z\| \text{ (from Proposition 2(ii))} \\
&= \|(1-\alpha)(x \ominus z) \oplus \alpha(y \ominus z)\| \text{ (from pre-convexity)} \\
&\le \|(1-\alpha)(x \ominus z)\| + \|\alpha(y \ominus z)\| \text{ (from norm subadditivity)} \\
&= (1-\alpha)\|x \ominus z\| + \alpha\|y \ominus z\| \text{ (from norm homogeneity),}
\end{aligned}$$

hence the proof. □

Definition 8. Let $(U, \|\cdot\|)$ be a regular near-Banach space. For every $\epsilon > 0$ and $r > 0$ we denote

$$D(r, \epsilon) = \{(x, y) : x, y \in U,\ \|x\|, \|y\| \le r,\ \|x \ominus y\| \ge \epsilon r\}$$

and

$$\delta(r, \epsilon) = \begin{cases} \inf\left\{1 - \frac{1}{r}\left\|\frac{1}{2}x \oplus \frac{1}{2}y\right\| : (x, y) \in D(\epsilon, r)\right\}, \text{ if } D(r, \epsilon) \ne \emptyset; \\ \\ 1, \text{ if } D(r, \epsilon) = \emptyset. \end{cases}$$

Then U is said to be uniformly convex, if U is endowed with pre-convexity property and for each $s \ge 0$ and $\epsilon > 0$, there exists $\eta(s, \epsilon) > 0$ such that

$$\delta(r, \epsilon) > \eta(s, \epsilon),\ \forall r > s.$$

Lemma 2. Let $(U, \|\cdot\|)$ be a uniformly convex regular near-Banach space and $t \in (0, 1)$. For $\epsilon > 0$ and $r > 0$ consider

$$\delta^t(r, \epsilon) = \begin{cases} \inf\left\{1 - \frac{1}{r}\|(1-t)x \oplus ty\| : (x, y) \in D(r, \epsilon)\right\}, \text{ if } D(r, \epsilon) \neq \emptyset; \\ \\ 1, \text{ if } D(r, \epsilon) = \emptyset. \end{cases}$$

If U is uniformly convex, then for each $s \geq 0$ and $\epsilon > 0$, there exists $\eta^t(s, \epsilon) > 0$ such that

$$\delta^t(r, \epsilon) > \eta^t(s, \epsilon), \ \forall r > s.$$

Proof. Obviously, the property is satisfied for $t = \frac{1}{2}$ according to the definition of uniform convexity. Since $\delta^t(r, \epsilon) = \delta^{1-t}(r, \epsilon)$ it is enough to prove the result for $t > \frac{1}{2}$. For $x, y \in D(r, \epsilon)$ and $t \in (0, 1)$, $t > \frac{1}{2}$, consider $u = (1 - (2t - 1))x \oplus (2t - 1)y$ and $v = y$. Then, $2t - 1 \in (0, 1)$ and

$$\begin{aligned} \|u\| &= \|(1 - (2t - 1))x \oplus (2t - 1)y\| \\ &\leq (1 - (2t - 1))\|x\| + (2t - 1)\|y\| \\ &\leq (1 - (2t - 1))r + (2t - 1)r = r. \end{aligned}$$

Obviously, one also has $\|v\| \leq r$. On the other side,

$$\begin{aligned} \|u - v\| &= \|[(1 - (2t - 1))x \oplus (2t - 1)y] \ominus y]\| \\ &= \|2(1 - t)(x \ominus y)\| \text{ (from pre-convexity)} \\ &= 2(1 - t)\|x \ominus y\| \geq 2(1 - t)r\epsilon. \end{aligned}$$

Therefore, $(u, v) \in D(r, 2(1 - t)\epsilon)$. Moreover,

$$1 - \frac{1}{r}\|(1 - t)x \oplus ty\| = 1 - \frac{1}{r}\left\|\frac{1}{2}u \oplus \frac{1}{2}v\right\|,$$

and consequently,

$$\delta^t(r, \epsilon) \geq \delta(r, 2(1 - t)\epsilon).$$

Since U is uniformly convex, for each $s \geq 0$ and $\epsilon > 0$, there exists $\eta^t(s, \epsilon) = \eta(s, 2(1 - t)\epsilon) > 0$ such that

$$\delta^t(r, \epsilon) \geq \delta(r, 2(1 - t)\epsilon) > \eta(s, 2(1 - t)\epsilon) = \eta^t(s, \epsilon) > 0, \ \forall r > s,$$

so the proof is completed. □

The next technical Lemma is expected to play an important role in fixed point theory, and is the expression of similar results from uniformly convex Banach spaces, uniformly convex metric spaces or uniformly convex modular spaces.

Lemma 3. Let $(U, \|\cdot\|)$ be a uniformly convex regular near-Banach space and $\{t_n\} \subset (0,1)$ be bounded away from 0 and 1 (i.e. $0 < p \leq t_n \leq q < 1$, for some $p \leq q \in (0,1)$). If there exists $r \geq 0$ such that

$$\limsup_{n\to\infty} \|x_n\| \leq r, \ \limsup_{n\to\infty} \|y_n\| \leq r$$

and

$$\lim_{n\to\infty} \|(1-t_n)x_n \oplus t_n y_n\| = r,$$

then

$$\lim_{n\to\infty} \|x_n \ominus y_n\| = 0.$$

Proof. If $r = 0$, the proof is trivial, since $\|x_n \ominus y_n\| \leq \|x_n\| + \|y_n\|$. So, we assume from start that $r > 0$. Consider also $\epsilon > 0$. First of all, for each pair $(x, y) \in D(r, \epsilon)$, consider the function $f \colon [0,1] \to \mathbb{R}$, $f_{x,y}(t) = \|(1-t)x \oplus ty\|$. We start by showing that $f_{x,y}$ is a convex function. Indeed, based on pre-convexity, one has

$$\begin{aligned}
f_{x,y}((1-\alpha)s + \alpha t) &= \|(1 - [(1-\alpha)s + \alpha t])x \oplus [(1-\alpha)s + \alpha t]y\| \\
&= \|[(1-\alpha)(1-s) + \alpha(1-t)]x \oplus [(1-\alpha)s + \alpha t]y\| \\
&= \|(1-\alpha)[(1-s)x \oplus sy] \oplus \alpha[(1-t)x \oplus ty]\| \\
&\leq (1-\alpha)\|(1-s)x \oplus sy\| + \alpha\|(1-t)x \oplus ty\| \\
&= (1-\alpha)f_{x,y}(s) + \alpha f_{x,y}(t), \ \forall \alpha, t, s \in [0,1].
\end{aligned}$$

Then $t \to \lambda_{r,\epsilon}(t) = \sup\{f_{x,y}(t) : (x,y) \in D(r,\epsilon)\}$ is also convex on $[0,1]$ and hence it is continuous. Since $\delta^t(r,\epsilon) = 1 - \frac{1}{r}\lambda_{r,\epsilon}(t)$, then $t \to \delta^t(r,\epsilon)$ is continuous as well, for each pair (r,ϵ).

In order to prove the statement of the Lemma, let us assume first the contrary. Then, let $\gamma > 0$. Passing to a subsequence if necessary, we could assume that

$$\|x_n\| \leq r + \gamma; \ \|y_n\| \leq r + \gamma; \ \|x_n \ominus y_n\| \geq (r+\gamma)\epsilon.$$

Moreover, we could also assume that $\{t_n\}$ is convergent toward a point $t^0 \in [p,q]$ (otherwise, it contains a convergent subsequence). Then

$$\lim_{n\to\infty} \delta^{t_n}(r+\gamma,\epsilon) = \delta^{t_0}(r+\gamma,\epsilon).$$

On the other side,

$$\lim_{n\to\infty} \delta^{t_n}(r+\gamma,\epsilon) \leq \lim_{n\to\infty}\left[1 - \frac{1}{r+\gamma}\|(1-t_n)x_n \oplus t_n y_n\|\right] = \frac{\gamma}{r+\gamma},$$

so

$$\delta^{t_0}(r+\gamma,\epsilon) \leq \frac{\gamma}{r+\gamma}. \tag{4.1}$$

Using the uniform convexity and Lemma 2, we could state that for each $\epsilon > 0$, there exists $\eta^{t_0}(r,\epsilon) > 0$ such that

$$\delta^{t_0}(r+\gamma,\epsilon) > \eta^{t_0}(r,\epsilon), \ \forall \gamma > 0. \tag{4.2}$$

From Eqs (4.1) and (4.2), we find

$$\frac{\gamma}{r+\gamma} > \eta^{t_0}(r,\epsilon),\ \ \forall\gamma > 0.$$

Letting $\gamma \to 0$, we find $\eta^{t_0}(r,\epsilon) \le 0$, which is a contradiction. □

Lemma 4. Let $(U, \|\cdot\|)$ be a uniformly convex regular near-Banach space. For every $\epsilon \ge 0$ and $r > 0$ we consider the set $D(r,\epsilon)$ as in Definition 8 and, if $D(r,\epsilon) \neq \emptyset$, we define the function

$$\lambda(\epsilon, r) = \inf_{(x,y)\in D(r,\epsilon)} \left\{ \frac{1}{2}\|x\|^2 + \frac{1}{2}\|y\|^2 - \left\| \frac{1}{2}x \oplus \frac{1}{2}y \right\|^2 \right\}.$$

Then, for some given $r > 0$, $\lambda(\epsilon, r) = 0$ if and only if $\epsilon = 0$.

Proof. It is not difficult to see that the regular near norm, as well as the square functions are convex, so

$$\left\| \frac{1}{2}x \oplus \frac{1}{2}y \right\|^2 \le \frac{1}{2}\|x\|^2 + \frac{1}{2}\|y\|^2$$

hence $\lambda(\epsilon, r) \ge 0,\ \forall\epsilon \ge 0,\ \forall r > 0$.

We prove first the direct statement. Let us assume that $\lambda(\epsilon, r) = 0$. The definition of infimum ensures the existence of some sequences $\{x_l\}$ and $\{y_l\}$ in $D(r,\epsilon)$ such that

$$\lim_{l\to\infty} \left(\frac{1}{2}\|x_l\|^2 + \frac{1}{2}\|y_l\|^2 - \left\| \frac{1}{2}x_l \oplus \frac{1}{2}y_l \right\|^2 \right) = \lambda(\epsilon, r) = 0. \tag{4.3}$$

Consider the equality in $\mathbb{R}_+$:

$$\left(\frac{\|x_l\| - \|y_l\|}{2} \right)^2 = \frac{\|x_l\|^2}{2} + \frac{\|y_l\|^2}{2} - \left(\frac{\|x_l\| + \|y_l\|}{2} \right)^2.$$

Since

$$0 \le \left\| \frac{1}{2}x_l \oplus \frac{1}{2}y_l \right\| \le \frac{\|x_l\| + \|y_l\|}{2},$$

we find

$$\left(\frac{\|x_l\| - \|y_l\|}{2} \right)^2 \le \frac{\|x_l\|^2}{2} + \frac{\|y_l\|^2}{2} - \left\| \frac{1}{2}x_l \oplus \frac{1}{2}y_l \right\|^2.$$

Taking $l \to \infty$, one has $\lim\limits_{l\to\infty}(\|x_l\| - \|y_l\|) = 0$. Since $\|x_l\|, \|x_l\| \le r$, by going if necessary to a subsequence, we could assume that $\lim\limits_{l\to\infty}\|x_l\| = z$. Then $\lim\limits_{l\to\infty}\|y_l\| = z$. Moreover, from equation (4.3) we also obtain

$$\lim_{l\to\infty} \left\| \frac{1}{2}x_l \oplus \frac{1}{2}y_l \right\| = z.$$

Applying now Lemma 3, we obtain $\lim\limits_{l\to\infty}\|x_l - y_l\| = 0$. On the other hand, since $(x_l, y_l) \in D(r,\epsilon)$, one has $\|x_l - y_l\| \ge \epsilon r,\ \forall l$. So

$$\epsilon r \le \lim_{l\to\infty}\|x_l - y_l\| = 0,$$

hence $\epsilon = 0$.

For the converse statement, we prove that $\lambda(0, r) = 0$. Let $x, y \in U$ such that $\|x\| \leq r$ and $y \overset{\Psi}{=} x$. Then $\|y\| = \|x\| \leq r$ and $\|x \ominus y\| = 0$, so (x, y) belongs to $D(0, r)$. Moreover, from Lemma 1

$$\left\|\frac{1}{2}x \oplus \frac{1}{2}y \ominus x\right\| \leq \frac{1}{2}\|x \ominus x\| + \frac{1}{2}\|x \ominus y\| = 0,$$

so

$$\frac{1}{2}x \oplus \frac{1}{2}y \overset{\Psi}{=} x \overset{\Psi}{=} y,$$

leading to

$$\left\|\frac{1}{2}x \oplus \frac{1}{2}y\right\| = \|x\| = \|y\|$$

and so $\lambda(0, r) = 0$. □

Remark 3. Let us also notice that, for given $x, y \in U$, with $\|x\| \leq r$ and $\|y\| \leq r$, taking $\epsilon = \frac{\|x \ominus y\|}{r}$ leads to the inequality

$$\lambda\left(\frac{\|x \ominus y\|}{r}, r\right) \leq \frac{1}{2}\|x\|^2 + \frac{1}{2}\|y\|^2 - \left\|\frac{1}{2}x \oplus \frac{1}{2}y\right\|^2.$$

Theorem 1 (Existence of best approximants)**.** Let $(U, \|\cdot\|)$ be a uniformly convex regular near-Banach space, $C \subset U$ a nonempty convex and closed subset and $x \in U$ such that $d(x, C) = \inf_{y\in C}\|x \ominus y\| < \infty$. Then, there exists a best approximant x_C of x in C, i.e.,

$$\|x \ominus x_C\| = \inf_{y\in C}\|x \ominus y\|.$$

Moreover, if x_C and x'_C are two best approximants, then $x_C \overset{\Psi}{=} x'_C$.

Proof. We shall assume from the start that $d = \inf_{y\in C}\|x \ominus y\| > 0$. Then, there exists a sequence $\{x_n\} \in C$ such that $\lim_{n\to\infty}\|x_n \ominus x\| = d$. Consequently, for each $\sigma > 0$, there is $N_\sigma > 0$ such that $\|x_n \ominus x\| \leq d + \sigma$, $\forall n \geq N_\sigma$. Without loosing the generality, we could assume $\sigma < 1$. We prove next that $\{x_n\}$, $n \geq N_\sigma$ is Cauchy. If we assume the contrary, there exists $\epsilon_0 > 0$ and two subsequences x_{k_n} and x_{m_n}, $k_n, m_n \geq n \geq N_\sigma$ such that

$$\left\|x_{k_n} \ominus x_{m_n}\right\| \geq \epsilon_0 = \frac{\epsilon_0}{d + \sigma}(d + \sigma).$$

Since

$$\left\|x_{k_n} \ominus x\right\| \leq d + \sigma; \ \left\|x_{m_n} \ominus x\right\| \leq d + \sigma,$$

it follows that $(x_{k_n} \ominus x, x_{m_n} \ominus x) \in D(d + \sigma, \frac{\epsilon_0}{d + \sigma})$.

The definition of the modulus of convexity provides the inequality

$$\left\|\frac{1}{2}x_{k_n} \oplus \frac{1}{2}x_{m_n} \ominus x\right\| \leq (d + \sigma)\left(1 - \delta\left(d + \sigma, \frac{\epsilon_0}{d + \sigma}\right)\right).$$

Since the function $\epsilon \to \delta(r, \epsilon)$ is nondecreasing, we find

$$\delta\left(d + \sigma, \frac{\epsilon_0}{d + \sigma}\right) \geq \delta\left(d + \sigma, \frac{\epsilon_0}{d + 1}\right).$$

Moreover, from uniform convexity it follows that there exists $\eta\left(d, \frac{\epsilon_0}{d+1}\right) > 0$ such that

$$\delta\left(d+\sigma, \frac{\epsilon_0}{d+1}\right) > \eta\left(d, \frac{\epsilon_0}{d+1}\right) > 0.$$

In conclusion,

$$\left\|\frac{1}{2}x_{k_n} \oplus \frac{1}{2}x_{m_n} \ominus x\right\| \leq (d+\sigma)\left(1-\eta\left(d, \frac{\epsilon_0}{d+1}\right)\right),\ \forall n \geq N_\sigma.$$

So,

$$\lim_{n\to\infty}\left\|\frac{1}{2}x_{k_n} \oplus \frac{1}{2}x_{m_n} \ominus x\right\| \leq (d+\sigma)\left(1-\eta\left(d, \frac{\epsilon_0}{d+1}\right)\right).$$

On the other side, C is convex, so $\frac{1}{2}x_{k_n} \oplus \frac{1}{2}x_{m_n} \in C$ and $\left\|\frac{1}{2}x_{k_n} \oplus \frac{1}{2}x_{m_n} \ominus x\right\| \geq d$. therefore,

$$d \leq \lim_{n\to\infty}\left\|\frac{1}{2}x_{k_n} \oplus \frac{1}{2}x_{m_n} \ominus x\right\| \leq (d+\sigma)\left(1-\eta\left(d, \frac{\epsilon_0}{d+1}\right)\right).$$

Taking $\sigma \to 0$, we find $d \leq d\left(1-\eta\left(d, \frac{\epsilon_0}{d+1}\right)\right) < d$, which is absurd, unless our assumption about the sequence $\{x_n\}$ not being Cauchy is false. By the completeness of U, $\{x_n\}$ should have a limit point, let us say x_C. This limit point is obviously an element of C, since this set was assumed closed. Moreover, x_C is a best approximant of x in C because

$$d \leq \|x \ominus x_C\| \leq \|x \ominus x_n\| + \|x_n - x_C\|,$$

and after passing to the limit we find $\|x \ominus x_C\| = d$.

In order to complete our proof, we consider two best approximants x_C, x'_C of x in C. If $x_C \overset{\Psi}{\neq} x'_C$, it follows that there exists $\epsilon > 0$, such that

$$\left\|x_C \ominus x'_C\right\| = \epsilon = \frac{\epsilon}{d}d.$$

From the relationships

$$\|x \ominus x_C\| = d;\ \left\|x - x'_C\right\| = d,$$

and, from the definition for modulus of convexity, it follows that

$$d \leq \left\|\frac{1}{2}x_C \oplus \frac{1}{2}x'_C \ominus x\right\| \leq d\left(1-\delta\left(d, \frac{\epsilon}{d}\right)\right) < d,$$

which is absurd. Therefore, $x_C \overset{\Psi}{=} x'_C$. □

Definition 9. We say that a regular near-Banach space $(U, \|\cdot\|)$ has property (R) if every nonincreasing sequence $\{C_n\}$ of nonempty, bounded, closed, convex subsets of U has a nonempty intersection.

Theorem 2. If $(U, \|\cdot\|)$ is a uniformly convex regular near-Banach space, then U has property (R).

Proof. Let $\{C_n\}$ be a nonincreasing sequence of nonempty, bounded, closed, convex subsets of U and $x \in U$ such that $\lim_{n\to\infty} d(x, C_n) < \infty$. From Theorem 1, there exists $x_n \in C_n$, such that

$$\|x_n \ominus x\| = \inf_{y\in C_n} \|x \ominus y\| = d(x, C_n).$$

Since $C_{n+1} \subset C_n$, one has $d(x, C_{n+1}) \geq d(x, C_n)$, so $\{d(x, C_n)\}$ is a nondecreasing sequence in $\mathbb{R}$. Let $d \geq 0$ be its limit. It follows that

$$\lim_{n\to\infty} \|x_n \ominus x\| = d.$$

Case 1. If $d = 0$ it follows from the monotonicity of $\{d(x, C_n)\}$ that $d(x, C_n) = 0,\ \forall n$. Therefore, there exists $\{y_k^{(n)}\} \subset C_n$ such that $\lim_{k\to\infty} \left\|y_k^{(n)} \ominus x\right\| = 0$. So, the sequence $\{y_k^{(n)}\}$ converges to x. The set C_n is closed, therefore $x \in C_n$. This relationship is independent of n, so $x \in \cap C_n$.
Case 2. We assume now that $d > 0$. We will prove that $\{x_n\}$ is Cauchy. If we assume the contrary, then the same arguments we used in the proof of Theorem 1 lead to the inequality

$$\lim_{n\to\infty} \left\|\frac{1}{2}x_{k_n} \oplus \frac{1}{2}x_{m_n} \ominus x\right\| \leq (d + \sigma)\left(1 - \eta\left(d, \frac{\epsilon_0}{d+1}\right)\right),$$

for some subsequences x_{k_n} and x_{m_n}. Let $p_n = \min(k_n, m_n)$. Then $x_{k_n}, x_{m_n} \in C_{p_n}$, which is a convex subset. Therefore, $\frac{1}{2}x_{k_n} \oplus \frac{1}{2}x_{m_n} \in C_{p_n}$, and $\left\|\frac{1}{2}x_{k_n} \oplus \frac{1}{2}x_{m_n} \ominus x\right\| \geq d(x, C_{p_n})$. Taking $n \to \infty$ leads to

$$d = \lim_{n\to\infty} d(x, C_{p_n}) \leq \lim_{n\to\infty} \left\|\frac{1}{2}x_{k_n} \oplus \frac{1}{2}x_{m_n} \ominus x\right\| \leq (d + \sigma)\left(1 - \eta\left(d, \frac{\epsilon_0}{d+1}\right)\right),$$

and after taking $\sigma \to 0$, to $d \leq d\left(1 - \eta\left(d, \frac{\epsilon_0}{d+1}\right)\right) < d$, which is not true. So $\{x_n\}$ is a Cauchy sequence and hence convergent. Let x be one of its limits. We notice that, for each $n \in \mathbb{N}$, $x_k \in C_n$, $\forall k \geq n$. Since every set C_n is closed it follows that $x \in C_n$, for every $n \in \mathbb{N}$, so $x \in \cap C_n$.

□

5. Two examples

Example 2 ([11])**.** Let $\mathcal{I}$ be the set of all bounded and closed intervals on $\mathbb{R}$. With interval addition $[a, b] \oplus [c, d] = [a + c, b + d]$ and with scalar multiplication: $\alpha[a, b] = [\alpha a, \alpha b]$ when $\alpha \geq 0$ and $\alpha[a, b] = [\alpha b, \alpha a]$ when $\alpha < 0$, thus it was proven in [11] that $\mathcal{I}$ is a near vector space, with null set $\Psi = \{[-a, a], : a \geq 0\}$ (see Example 2.1, Example 2.3 in [11]). Moreover, together with the norm-function $\|[a, b]\| = |a + b|$, $\mathcal{I}$ is a complete near-normed space, satisfying both the null condition and null equality, so it is a regular near-Banach space (see Example 3.4 and Example 4.14 in [11]).

We shall prove next that $\mathcal{I}$ is uniformly convex. The pre-convexity is an immediate consequence of the fact that multiplication of intervals with positive scalars acts similar as for $\mathbb{R}^2$, so distributivity laws when positive scalars are involved run naturally. Let us consider now $\epsilon, r > 0$ and two intervals $[a, b]$ and $[c, d]$ such that

$$\|[a, b]\| \leq r,\ \ \|[c, d]\| \leq r,\ \ \|[a, b] \ominus [c, d]\| \geq \epsilon r.$$

Explicitly, this means

$$|a + b| \leq r,\ \ |c + d| \leq r,\ \ |(a + b) - (c + d)| \geq \epsilon r,$$

or

$$\left|\frac{a+b}{r}\right| \leq 1, \ \left|\frac{c+d}{r}\right| \leq 1, \ \left|\frac{a+b}{r} - \frac{c+d}{r}\right| \geq \epsilon.$$

Since $\mathbb{R}$ is known to be uniformly convex, there is $\delta(\epsilon) > 0$, such that

$$\left|\frac{1}{2}\left(\frac{a+b}{r} + \frac{c+d}{r}\right)\right| \leq 1 - \delta(\epsilon),$$

that is

$$\left|\frac{(a+b)+(c+d)}{2}\right| \leq r[1-\delta(\epsilon)].$$

On the other side,

$$\left\|\frac{1}{2}[a,b] \oplus \frac{1}{2}[c,d]\right\| = \left|\frac{(a+b)+(c+d)}{2}\right|.$$

Hence

$$\left\|\frac{1}{2}[a,b] \oplus \frac{1}{2}[c,d]\right\| \leq r[1-\delta(\epsilon)],$$

or

$$1 - \frac{1}{r}\left\|\frac{1}{2}[a,b] \oplus \frac{1}{2}[c,d]\right\| \geq \delta(\epsilon).$$

For $\epsilon > 0$ and $s \geq 0$, let us denote $\delta(s,\epsilon) = \frac{1}{2}\delta(\epsilon)$. Then, for each $r > s$, one has

$$\delta(r,\epsilon) \geq \delta(\epsilon) > \frac{1}{2}\delta(\epsilon) = \delta(s,\epsilon),$$

ending the proof.

Example 3. We consider the upper half plane $U = \{(x,y) \in \mathbb{R}^2 \ : \ y \geq 0\}$. On this set we shall define vector addition and scalar multiplication as follows:

- $(x_1,y_1) \oplus (x_2,y_2) = (x_1+x_2, y_1+y_2)$;
- $\alpha(x,y) = (\alpha x, |\alpha| y)$.

It is quite easy to notice that U together with these two operations is a near vector space. Moreover, $(x_1,y_1) \ominus (x_2,y_2) = (x_1 - x_2, y_1 + y_2)$ and, in particular, $(x,y) \ominus (x,y) = (0,2y)$, therefore the null set is precisely the nonnegative y-axis,

$$\Psi = \{(0,y) \ : \ y \geq 0\}.$$

Moreover,

$$(x_1,y_1) \overset{\Psi}{=} (x_2,y_2) \text{ if and only if } x_1 = x_2,$$

so two elements are almost identical when their projections onto the x-axis are equal.

For an arbitrary element $(x,y) \in U$, we define

$$\|(x,y)\| = |x|.$$

We prove next that $\|\cdot\|$ is a regular near norm.

- To start, we test the null condition. $\|(x, y)\| = |x| = 0$ if and only if $x = 0$, which is equivalent to (x, y) laying on the nonnegative y-axis, that is, the null set Ψ.
- For testing the null equality, let $\psi = (0, z) \in \Psi$. Then

$$\|(x, y) \oplus \psi\| = \|(x, y + z)\| = |x| = \|(x, y)\|\,,\ \ \forall (x, y) \in U.$$

- The homogeneity and subadditivity follow immediately from similar properties on $\mathbb{R}$.

The next step of our analysis is related to completeness. Let $\{(x_n, y_n)\}$ be a Cauchy sequence in U. Then

$$0 = \lim_{m,n\to\infty} \|(x_n, y_n) \ominus (x_m, y_m)\| = \lim_{m,n\to\infty} \|(x_n - x_m, y_n + y_m)\| = \lim_{m,n\to\infty} |x_n - x_m|.$$

It follows that $\{x_n\}$ is a Cauchy sequence in $\mathbb{R}$, so it is convergent. Let $x^* \in \mathbb{R}$ be its limit, i.e., $\lim_{n\to\infty} |x_n - x^*| = 0$ and consider an element of type $(x^*, y) \in U$, with $y \geq 0$. Then

$$\lim_{n\to\infty} \|(x_n, y_n) \ominus (x^*, y)\| = \lim_{n\to\infty} \|(x_n - x^*, y_n + y)\| = \lim_{n\to\infty} |x_n - x^*| = 0,$$

so the sequence (x_n, y_n) is convergent to (x^*, y), $\forall y \geq 0$. This also emphasizes the general fact we have already stated, that the limit is not necessarily unique, but every two limits are almost identical.

Finally, we shall check the uniform convexity. The properties from pre-convexity are clearly satisfied since, for positive scalars, the scalar multiplication acts as classical multiplication on $\mathbb{R}^2$. The uniform convexity is an immediate consequence of the relationships:

$$\|(x_1, y_1)\| = |x_1|,\ \ \|(x_2, y_2)\| = |x_2|,\ \ \|(x_1, y_1) \ominus (x_2, y_2)\| = |x_1 - x_2|,$$

and

$$\left\|\frac{1}{2}(x_1, y_1) \oplus \frac{1}{2}(x_1, y_1)\right\| = \left\|\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)\right\| = \left|\frac{x_1 + x_2}{2}\right|,$$

as well as of uniform convexity of $(\mathbb{R}, |\cdot|)$.

6. The type function of a bounded sequence

Definition 10. Let $(U, \|\cdot\|)$ be a regular near-Banach space, C a nonempty bounded subset and $\{x_n\}$ a sequence in C.

(i) The function $\tau\colon C \to [0, \infty]$, $\tau(x) = \limsup_{n\to\infty} \|x \ominus x_n\|$ is called the type function of $\{x_n\}$ in C.
(ii) The value $r(C) = \inf_{x\in C} \tau(x)$ is called the asymptotic radius of $\{x_n\}$ relative to C.
(iii) A sequence $\{c_n\}$ in C is called a minimizing sequence of τ if $\lim_{n\to\infty} \tau(c_n) = r(C)$.
(iv) The set $A(C) = \{x \in C\ :\ \tau(x) = r(C)\}$ is called the asymptotic center of $\{x_n\}$ in C.

Proposition 3. If $(U, \|\cdot\|)$ is a regular near-Banach space endowed with the pre-convexity property, C a nonempty bounded convex subset and $\{x_n\}$ a sequence in C, then the type function of $\{x_n\}$ is convex.

Proof. Let $x, y \in C$ and $\alpha \in [0, 1]$. Then,

$$\tau((1 - \alpha)x \oplus \alpha y) \ \ = \ \ \limsup_{n\to\infty} \|(1 - \alpha)x \oplus \alpha y \ominus x_n\|$$

$$\begin{aligned} &\leq \limsup_{n\to\infty}[(1-\alpha)\|x\ominus x_n\|+\alpha\|y\ominus x_n\|] \text{ (from Lemma 1)} \\ &\leq (1-\alpha)\limsup_{n\to\infty}\|x\ominus x_n\|+\alpha\limsup_{n\to\infty}\|y\ominus x_n\| \\ &= (1-\alpha)\tau(x)+\alpha\tau(y), \end{aligned}$$

hence the conclusion follows. □

Proposition 4. Let $(U, \|\cdot\|)$ be a uniformly convex regular near-Banach space and $C \subset U$ a closed, bounded, convex subset. Let τ be the type function of a sequence $\{x_n\}$ on C. Then

(i) Every minimizing sequence $\{c_n\}$ of τ is convergent and all its limits belong to the asymptotic center $A(C)$. Consequently, the asymptotic center is nonempty.
(ii) Any convex combination $(1-\lambda)c_k + \lambda d_k$, $\lambda \in (0,1)$ of minimizing sequences is a minimizing sequence as well.
(iii) For every two minimizing sequences, $\{c_n\}$ and $\{d_n\}$ hold true $\lim_{n\to\infty}\|c_n - d_n\| = 0$.
(iv) The asymptotic center is not necessarily a singleton. However, if $c, d \in A(C)$, then $c \overset{\Psi}{=} d$ (all the asymptotic centers are almost identical).

Proof. (i) Let $\{c_n\} \subset C$ be a minimizing sequence of τ (the definition of infimum ensures the existence of at least such sequence). Since C is bounded, there exists $M > 0$ such that $\|x-y\| \leq M$, for every $x, y \in C$. Let k, m be some fixed indices and denote $c_{k,m} = \frac{1}{2}c_k \oplus \frac{1}{2}c_m$. In particular,

$$\|c_k - x_n\|, \|c_m - x_n\| \leq M, \ \forall k, m, n \in \mathbb{N}. \tag{6.1}$$

First of all, let us note from Proposition 2(ii) that

$$(c_k \ominus x_n) \ominus (c_m \ominus x_n) = c_k \ominus x_n \ominus c_m \oplus x_n = c_k \ominus c_m \oplus x_n \ominus x_n \overset{\Psi}{=} c_k \ominus c_m,$$

so $\|(c_k \ominus x_n) \ominus (c_m \ominus x_n)\| = \|c_k \ominus c_m\|$.

Then, using the inequalities (6.1) and Remark 3, one has

$$\begin{aligned} 0 &\leq \lambda\left(\frac{\|c_k\ominus c_m\|}{M}, M\right) \\ &= \lambda\left(\frac{\|(c_k\ominus x_n)\ominus(c_m\ominus x_n)\|}{M}, M\right) \\ &\leq \frac{1}{2}\|c_k\ominus x_n\|^2+\frac{1}{2}\|c_m\ominus x_n\|^2-\left\|\frac{1}{2}(c_k\ominus x_n)\oplus\frac{1}{2}(c_m\ominus x_n)\right\|^2 \\ &= \frac{1}{2}\|c_k\ominus x_n\|^2+\frac{1}{2}\|c_m\ominus x_n\|^2-\left\|c_{k,m}\ominus x_n\right\|^2, \ \forall n\in\mathbb{N}. \end{aligned} \tag{6.2}$$

It is well known that every sequence in $\mathbb{R}$ has a subsequence convergent to its limit superior. Therefore, there exists a subsequence $\{x_{l_n}\}$ such that

$$\lim_{n\to\infty}\left\|c_{k,m}\ominus x_{l_n}\right\| = \limsup_{n\to\infty}\left\|c_{k,m}\ominus x_n\right\| = \tau(c_{k,m}) \geq r(C). \tag{6.3}$$

Also, going to a subsequence leads to the inequalities

$$\begin{aligned}\limsup_{n\to\infty}\left\|c_k\ominus x_{l_n}\right\| &\leq \limsup_{n\to\infty}\|c_k\ominus x_n\| = \tau(c_k);\\ \limsup_{n\to\infty}\left\|c_m\ominus x_{l_n}\right\| &\leq \limsup_{n\to\infty}\|c_m\ominus x_n\| = \tau(c_m).\end{aligned} \tag{6.4}$$

Writing Eq (6.2) for l_n and taking lim sup for $n\to\infty$, based on the relationships (6.3) and (6.4), we obtain

$$0\leq\lambda\left(\frac{\|c_k\ominus c_m\|}{M},M\right)\leq\frac{1}{2}\tau(c_k)^2+\frac{1}{2}\tau(c_m)^2-(r(C))^2,$$

and this inequality holds true for each k,m. Since $\{c_n\}$ is a minimizing sequence, we find

$$\lim_{k,m\to\infty}\lambda\left(\frac{\|c_k\ominus c_m\|}{M},M\right)=0. \tag{6.5}$$

We wish to prove first that $\{c_n\}$ is Cauchy. If we assume the contrary, then for some $\epsilon_0>0$, there exists two subsequences satisfying

$$\left\|c_{k_n}\ominus c_{m_n}\right\|\geq\epsilon_0 M.$$

Since $D\left(M,\frac{\left\|c_{k_n}\ominus c_{m_n}\right\|}{M}\right)\subset D(M,\epsilon_0)$ it follows that

$$\lambda\left(\frac{\left\|c_{k_n}\ominus c_{m_n}\right\|}{M},M\right)\geq\lambda\left(\epsilon_0,M\right)\geq 0.$$

It follows immediately from equation (6.5) that $\lambda\left(\epsilon_0,M\right)=0$. This, together with Lemma 4 leads to $\epsilon_0=0$, which is a contradiction. Therefore, $\{c_n\}$ is Cauchy sequence, and hence convergent. Let $c\in U$ be one of its limits. Since C is closed, $c\in C$. Moreover, we can prove that c is an asymptotic center.

Indeed,

$$\begin{aligned}r(C)\leq\tau(c) &= \limsup_{n\to\infty}\|c-x_n\|\leq\|c-c_k\|+\limsup_{n\to\infty}\|c_k-x_n\|\\ &\leq \|c-c_k\|+\tau(c_k),\ \ \forall k\in\mathbb{N}.\end{aligned}$$

Taking $k\to\infty$, we find

$$r(C)\leq\tau(c)\leq r(C),$$

so $c\in A(C)$.

(ii) Let $\{c_k\},\{d_k\}$ be two minimizing sequences for τ and $e_k=\lambda c_k+(1-\lambda)\,d_k$, $\lambda\in(0,1)$, $k\geq 0$. From Lemma 1, for any $x\in C$, we have

$$\|e_k-x\|\leq\lambda\|c_k-x\|+(1-\lambda)\|d_k-x\|,k\geq 0,$$

which implies

$$\limsup_{n\to\infty}\|e_k-x_n\|\leq\lambda\limsup_{n\to\infty}\|c_k-x_n\|+(1-\lambda)\limsup_{n\to\infty}\|d_k-x_n\|,$$

that is,

$$\tau(e_k) \le \lambda\tau(c_k) + (1-\lambda)\tau(d_k).$$

Passing to the limit and keeping in mind that $\{c_k\}$ and $\{d_k\}$ are minimizing sequences, we obtain

$$r(C) \le \lim_{k\to\infty} \tau(e_k) \le \lambda r(C) + (1-\lambda) r(C) = r(C),$$

which gives the conclusion.

(iii) Let $\{c_n\}, \{d_n\}$ be two minimizing sequences. Using similar arguments as in point i), one could write

$$0 \le \lambda\left(\frac{\|c_n \ominus d_n\|}{M}, M\right) \le \frac{1}{2}\tau(c_n)^2 + \frac{1}{2}\tau(d_n)^2 - (r(C))^2,$$

and this inequality holds true for each n. Since $\{c_n\}, \{d_n\}$ are minimizing sequences, we find

$$\lim_{n\to\infty} \lambda\left(\frac{\|c_n \ominus d_n\|}{M}, M\right) = 0. \tag{6.6}$$

Let us assume that $\lim_{n\to\infty} \|c_n - d_n\| \ne 0$. Then, for some $\epsilon_0 > 0$, there exists two subsequences satisfying

$$\left\|c_{k_n} \ominus d_{k_n}\right\| \ge \epsilon_0 M.$$

This leads to inequality

$$\lambda\left(\frac{\left\|c_{k_n} \ominus d_{k_n}\right\|}{M}, M\right) \ge \lambda(\epsilon_0, M) \ge 0.$$

It follows immediately from equation (6.6) that $\lambda(\epsilon_0, M) = 0$. This, together with Lemma 4 leads to $\epsilon_0 = 0$, which is a contradiction. Therefore, $\lim_{n\to\infty} \|c_n - d_n\| = 0$.

(iv) Let c, d be two asymptotic points. Then, the sequences $c_n \equiv c$ and $d_n \equiv d$ are minimizing sequences. It follows from *iii*) that

$$\|c - d\| = 0,$$

so $c \overset{\Psi}{=} d$. □

7. Nonexpansive mappings on regular near-normed spaces

Definition 11. Let $(U, \|\cdot\|)$ be a regular near-normed space and $C \subset U$ a nonempty subset. A mapping $T\colon C \to C$ is called nonexpansive if

$$\|Tx \ominus Ty\| \le \|x \ominus y\|, \ \forall x, y \in C.$$

A point $x \in C$ is called a near fixed point of T if $Tx \overset{\Psi}{=} x$. We shall denote by $F(T)$ the set of near fixed points.

Theorem 3 (Existence of fixed points for nonexpansive mappings)**.** Let $(U, \|\cdot\|)$ be a uniformly convex regular near-Banach space and $C \subset U$ be a closed, bounded, convex subset. Let $T\colon C \to C$ be a nonexpansive mapping. Then,

(i) T has a fixed point.

(ii) The set of fixed points $F(T)$ is closed.

Proof. (i) For an arbitrarily fixed element $x_0 \in C$, consider the Picard iteration $x_n = T^n(x_0)$, $n \geq 1$. Let τ be the type function of $\{x_n\}$ and $A(C)$, $r(C)$ the corresponding asymptotic center and asymptotic radius, respectively. By Proposition 4(i) the asymptotic center $A(C)$ is nonempty. Let $p \in A(C)$, so $\tau(p) = r(C)$. Then,

$$\begin{aligned} r(C) \leq \tau(Tp) &= \limsup_{n\to\infty} \|Tp \ominus x_n\| \\ &= \limsup_{n\to\infty} \|Tp \ominus Tx_{n-1}\| \\ &\leq \limsup_{n\to\infty} \|p \ominus x_{n-1}\| \\ &= \tau(p) = r(C). \end{aligned}$$

It follows that Tp belongs to $A(C)$. Using Proposition 4(iv), we may conclude that $Tp \stackrel{\Psi}{=} p$, so p is a near fixed point for T.

(ii) Let $\{p_n\} \subset F(T)$ be a convergent sequence of fixed points and let $p \in C$ (C is closed subset) be a limit point. Then, $Tp_n \stackrel{\Psi}{=} p_n$, resulting that $\|Tp_n - p_n\| = 0$. It follows

$$\begin{aligned} \|Tp \ominus p\| &\leq \|Tp \ominus Tp_n\| + \|Tp_n \ominus p_n\| + \|p_n \ominus p\| \\ &\leq \|p \ominus p_n\| + 0 + \|p_n \ominus p\| \\ &= 2\,\|p \ominus p_n\|\,. \end{aligned}$$

Taking $n \to \infty$, we find $\|Tp \ominus p\| = 0$, so $Tp \stackrel{\Psi}{=} p$. Therefore, $p \in F(T)$. □

Example 4. On the regular near-Banach space defined in Example 3, we consider the mapping

$$T\colon U \to U,\ T(x,y) = \left(e^{-x^2} - 1, x^2 + y\right).$$

Then

$$\begin{aligned} T(x_1,y_1) \ominus T(x_2,y_2) &= \left(e^{-x_1^2} - 1, x_1^2 + y_1\right) \ominus \left(e^{-x_2^2} - 1, x_2^2 + y_2\right) \\ &= \left(e^{-x_1^2} - e^{-x_2^2}, x_1^2 + y_1 + x_2^2 + y_2\right). \end{aligned}$$

So

$$\|T(x_1,y_1) \ominus T(x_2,y_2)\| = \left|e^{-x_1^2} - e^{-x_2^2}\right| \leq |x_1 - x_2| = \|(x_1,y_1) \ominus (x_2,y_2)\|,$$

proving that T is nonexpansive.

To search the fixed points of T, one has to solve the equation $T(x,y) \stackrel{\Psi}{=} (x,y)$. This is equivalent to $\|T(x,y) \ominus (x,y)\| = 0$, which leads to $\left|e^{-x^2} - 1 - x\right| = 0$. This equation in $\mathbb{R}$ has the unique solution $x_0 = 0$, so the fixed points of T are all the pairs $(0,y)$, $y \geq 0$. In this case, the set of fixed points $F(T)$ is precisely the null set Ψ.

Example 5. On the regular near-Banach space $\mathcal{I}$ defined in Example 2, we consider the mapping

$$T\colon \mathcal{I} \to \mathcal{I},\ T([a,b]) = \left[\frac{\sin(a+b) + a - b}{2}, \frac{\sin(a+b) + b - a}{2}\right].$$

Then,

$$\|T([a,b]) \ominus T([c,d])\| = |\sin(a+b) - \sin(c+d)| \leq |(a+b)-(c+d)| = \|[a,b] \ominus [c,d]\|,$$

proving that T is nonexpansive.

In order to find the fixed points of T, we have to solve the equation $\|T([a,b]) \ominus [a,b]\| = 0$, that is

$$\left\| \left[\frac{\sin(a+b)+a-b}{2}, \frac{\sin(a+b)+b-a}{2} \right] \ominus [a,b] \right\| = 0,$$

which leads to

$$\left| \frac{\sin(a+b)+a-b}{2} - b + \frac{\sin(a+b)+b-a}{2} - a \right| = 0,$$

or

$$\sin(a+b) = (a+b).$$

If x_0 is the unique positive solution of equation $\sin x = x$, then $a+b = \pm x_0$. Therefore the fixed points of T are all the intervals $[a,b]$, such that $\|[a,b]\| = |a+b| = x_0$.

8. Conclusions and further development

Through this paper we managed to prove that uniform convexity could be extended to regular near-Banach spaces, producing similar properties as for hyperbolic metric spaces or modular function spaces: property (R), existence of best approximants, existence of fixed points for nonexpansive mappings, as well as the existence of asymptotic centers. Moreover, it is not difficult to prove that the binary relation indicating that two elements are almost identical is an equivalence relation. These will allow us to say that the limit class of a convergent sequence is unique, and that the asymptotic center of a bounded sequence is a singleton up to an equivalence class.

As further development, we consider first the possibility of analyzing the normal structure property together with an adequate version of Kirk's fixed point theorem on pre-convex regular near-Banach spaces. Second, a study regarding numerical reckoning of fixed points for (generalized) nonexpansive mappings using various types of iteration procedures could also provide significant contribution to the study of regular near-Banach structures. Finally, we must point out that, out of the three examples of near vector spaces provided by Wu [11], we have proved uniform convexity here only for the space of closed and bounded intervals in $\mathbb{R}$. The hyperspace is not suitable for our analysis because it does not satisfy the necessary null condition. However, the set of fuzzy numbers in $\mathbb{R}$ is a pre-convex regular near-Banach space. Still, the presence of the supremum in the definition of the regular near norm function makes it difficult the verify the uniform convexity. Therefore, the question of whether this space is uniformly convex or not remains an open problem.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This work was supported by a grant from the National Program for Research of the National Association of Technical Universities - GNAC ARUT 2023.

Conflict of interest

The authors declare that there is no conflict of interest in this paper.

References

1. J. A. Clarkson, Uniformly convex spaces, *Trans. Am. Math. Soc.*, **40** (1936), 396–414. https://doi.org/10.2307/1989630
2. F. E. Browder, Nonexpansive nonlinear operators in a Banach space, *Proc. Natl. Acad. Sci.*, **54** (1965), 1041–1044. https://doi.org/10.1073/pnas.54.4.1041
3. D. Göhde, Zum Prinzip der kontraktiven Abbildung, *Math. Nachr.*, **30** (1965), 251–258. https://doi.org/10.1002/mana.19650300312
4. K. Goebel, S. Reich, *Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings*, Marcel Dekker, New York and Basel, 1984.
5. M. Edelstein, The construction of an asymptotic center with a fixed-point property, *Bull. Amer. Math. Soc.*, **78** (1972), 206–208. https://doi.org/10.1090/S0002-9904-1972-12918-5
6. M. Edelstein, Fixed point theorems in uniformly convex Banach spaces, *Proc. Amer. Math. Soc.*, **44** (1974), 369–374. https://doi.org/10.2307/2040439
7. K. Goebel, T. Sekowski, A. Stachura, Uniform convexity of the hyperbolic metric and fixed points of holomorphic mappings in the Hilbert ball, *Nonlinear Anal.-Theor.,* **4** (1980), 1011–1021. https://doi.org/10.1016/0362-546X(80)90012-7
8. S. Reich, I. Shafrir, Nonexpansive iterations in hyperbolic spaces, *Nonlinear Anal.*, **15** (1990), 537–558. https://doi.org/10.1016/0362-546X(90)90058-O
9. M. A. Khamsi, A. R. Khan, Inequalities in metric spaces with applications, *Nonlinear Anal.-Theor.,* **74** (2011), 4036–4045. https://doi.org/10.1016/j.na.2011.03.034
10. M. A. Khamsi, W. M. Kozlowski, S. Reich, Fixed point theory in modular function spaces, *Nonlinear Anal.*, **14** (1990), 935–953. https://doi.org/10.1016/0362-546X(90)90111-S
11. H. C. Wu, Near fixed point theorems in near-Banach spaces, *AIMS Mathematics*, **8** (2023), 1269–1303. https://doi.org/10.3934/math.2023064