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Research article

Some identities involving derangement polynomials and r-Bell polynomials

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Abstract: In this paper two kinds of identities involving derangement polynomials and r-Bell polynomials were established. The identities of the first kind extended the identity on derangement numbers and Bell numbers due to Clarke and Sved and its generalizations due to Du and Fonseca. The identities of the second kind extended some of the results on derangement polynomials and Bell polynomials due to Kim et al.

Keywords: derangement polynomials; r-Bell polynomials; Stirling numbers; identities
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1. Introduction

A permutation of a set of elements ranking from one to n is called derangement if none of the elements is left at its original place. The number of derangements of a set of n elements is denoted by D_n throughout this paper. The first few derangement numbers are $D_1 = 0, D_2 = 1, D_3 = 2, D_4 = 9, D_5 = 44$, with $D_0 = 1$ being defined by convention and the familiar inclusion-exclusion principle giving a closed formula as follows [5, 16]

$$D_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!} = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} k!, \tag{1.1}$$

which can be also obtained by applying the generating function of the derangement numbers [6, 16]

$$\frac{e^{-t}}{1-t} = \sum_{n=0}^{\infty} D_n \frac{t^n}{n!}. \tag{1.2}$$

The derangement numbers satisfy the following recurrence relations

$$D_n = (n-1)(D_{n-1} + D_{n-2}),\ n \geq 2 \tag{1.3}$$

and

$$D_n = nD_{n-1} + (-1)^n,\ n \geq 1, \tag{1.4}$$

with $D_0 = 1$ and $D_1 = 0$. The above two recursion formulae allow us fast evaluation of D_n. Moreover, Qi, Wang and Guo [15] established a new recurrence relation

$$D_n = \sum_{k=0}^{n-2} \binom{n}{k} (n-k-1) D_k,\ n \geq 2. \tag{1.5}$$

A short proof was given by Fonseca [9].

Let B_n denote the nth Bell number, defined as the number of partitions of a set of cardinality n (with $B_0 = 1$). The Bell number B_n can be represented by the sum of Stirling numbers of the second kind, as follows

$$B_n = \sum_{k=0}^{n} S(n,k), \tag{1.6}$$

where $S(n,k)$ are the Stirling numbers of the second kind. The Bell numbers obey the recurrence

$$B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k \tag{1.7}$$

and satisfy the generating function

$$e^{e^t-1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}. \tag{1.8}$$

Both Bell numbers and derangement numbers are important tools in the study of special sequences and combinatorics. In [5], an interesting connection between the derangement numbers and the Bell numbers was established by the probabilistic method:

$$\sum_{k=0}^{n} \binom{n}{k} k^s D_k = n! \sum_{k=0}^{s} \binom{s}{k} (-1)^k n^{s-k} B_k,\ n \geq s \geq 0. \tag{1.9}$$

Recently, Du and Fonseca [7] provided a general identity for the derangement numbers and the Bell numbers which includes (1.1), (1.5) and (1.9) as particular cases. They also provided a combinatorial interpretation and established a general determinantal representation in terms of a Hessenberg matrix. For more interesting identities involving the derangement numbers, the Bell numbers and their generalizations, one is referred to [13] and the references therein.

The aim of this short note is to establish two kinds of identities involving derangement polynomials and r-Bell polynomials. The identities of the first kind are mainly inspired by the work in [5, 7], and they are the extensions of (1.9). The identities of the second kind generalize some of the results on derangement polynomials and Bell polynomials in [12]. The definitions of the derangement polynomials, the r-Bell polynomials and some necessary properties will be presented in the next section.

2. Preliminaries

We begin by recalling the definition of the derangement polynomials. The derangement polynomials $D_n(x)$ are defined by

$$\frac{e^{-t}}{1-t}e^{xt} = \sum_{n=0}^{\infty} D_n(x)\frac{t^n}{n!}, \tag{2.1}$$

which have been considerably investigated in [8, 10–12]. They are natural extensions of the derangement numbers because $D_n(x) = D_n$ when $x = 0$. The derangement polynomials $D_n(x)$ obey the recursive relation

$$D_n(x) = nD_{n-1}(x) + (x-1)^n. \tag{2.2}$$

As a direct consequence of (2.1), two closed formulae for the derangement polynomials are

$$D_n(x) = \sum_{k=0}^{n} \binom{n}{k} D_k x^{n-k} \tag{2.3}$$

and

$$D_n(x) = \sum_{k=0}^{n} \binom{n}{k} k!(x-1)^{n-k}, \tag{2.4}$$

respectively. The r-Stirling number denoted by $S_r(n,k)$ enumerates the partitions of a set of n elements into k nonempty, disjoint subsets such that the first r elements are in distinct subsets. A systematic treatment on the r-Stirling numbers was given in [1] and a different approach was described in [2, 3]. The r-Stirling numbers have the 'horizontal' generating function

$$(x+r)^n = \sum_{k=0}^{n} S_r(n+r, k+r)(x)_k, \tag{2.5}$$

where the falling factorial of a given real number x is $(x)_k = x(x-1)\cdots(x-k+1)$. From (2.5), the explicit expressions of $S_r(n+r, k+r)$ are given by

$$S_r(n+r, k+r) = \frac{1}{k!}\sum_{j=0}^{k} (-1)^{k-j}\binom{k}{j}(j+r)^n. \tag{2.6}$$

In particular, when $r = 0$, we have $S_r(n+r, k+r) = S(n,k)$, where $S(n,k)$ are the Stirling numbers of the second kind. In [14], r-Bell numbers and r-Bell polynomials are defined by

$$B_{n,r} = \sum_{k=0}^{n} S_r(n+r, k+r) \tag{2.7}$$

and

$$B_{n,r}(x) = \sum_{k=0}^{n} S_r(n+r, k+r)x^k, \tag{2.8}$$

respectively. The exponential generating function for the r-Bell polynomials is

$$\sum_{n=0}^{\infty} B_{n,r}(x)\frac{t^n}{n!} = e^{x(e^t-1)+rt}. \tag{2.9}$$

It is clear that the r-Bell polynomials $B_{n,r}(x)$ reduce to the well-known Bell polynomials $B_n(x)$ [6] when $r = 0$. For more about r-Bell polynomials, especially from an algebraic perspective, one can refer to reference [4], in which partial r-Bell polynomials in three combinatorial Hopf algebras are introduced.

3. Main results

In this section, two kinds of identities involving the derangement polynomials and the r-Bell polynomials are established. The following lemma plays an important role.

Lemma 3.1. *If both r and s are nonnegative integers, then we have*

$$\sum_{k=0}^{n}\binom{n}{k}(n-k+r)^s x^{n-k} D_k(1-x) = n!\left\{B_{s,r}(x) - \sum_{j=n+1}^{s} S_r(s+r, j+r)x^j\right\}. \tag{3.1}$$

Proof. From the explicit expression of the derangement polynomials $D_n(x)$ (2.4), we have

$$\begin{aligned}
\sum_{k=0}^{n}\binom{n}{k}(n-k+r)^s x^{n-k} D_k(1-x) &= \sum_{k=0}^{n}\binom{n}{k}(n-k+r)^s x^{n-k}\sum_{j=0}^{k}\binom{k}{j} j!(-x)^{k-j} \\
&= \sum_{j=0}^{n} j! x^{n-j}\sum_{k=j}^{n}(-1)^{k-j}\binom{n}{k}\binom{k}{j}(n-k+r)^s \\
&= \sum_{j=0}^{n} j!\binom{n}{j} x^{n-j}\sum_{k=j}^{n}(-1)^{k-j}\binom{n-j}{k-j}(n-k+r)^s,
\end{aligned}$$

since

$$\binom{n}{k}\binom{k}{j} = \binom{n}{j}\binom{n-j}{k-j}.$$

We replace k by $k+j$ and obtain

$$\begin{aligned}
&\sum_{j=0}^{n} j!\binom{n}{j} x^{n-j}\sum_{k=j}^{n}(-1)^{k-j}\binom{n-j}{k-j}(n-k+r)^s \\
&= \sum_{j=0}^{n} j!\binom{n}{j} x^{n-j}\sum_{k=0}^{n-j}(-1)^{k}\binom{n-j}{k}(n-j-k+r)^s \\
&= \sum_{j=0}^{n} j!\binom{n}{j} x^{n-j}\sum_{k=0}^{n-j}(-1)^{n-j-k}\binom{n-j}{k}(k+r)^s.
\end{aligned}$$

By (2.6), we have

$$\begin{aligned}
&\sum_{j=0}^{n} j!\binom{n}{j}x^{n-j}\sum_{k=0}^{n-j}(-1)^{n-j-k}\binom{n-j}{k}(k+r)^s\\
=&\sum_{j=0}^{n} j!\binom{n}{j}x^{n-j}(n-j)!S_r(s+r,n-j+r)\\
=&n!\sum_{j=0}^{n} S_r(s+r,j+r)x^j.
\end{aligned}$$

From (2.8), it is equivalent to the righthand side of (3.1).

Remark 3.1. *For $j > i \geq 0$, $S_r(i, j) = 0$. Thus, when $n \geq s$, it is natural that* (3.1) *reduces to*

$$\sum_{k=0}^{n}\binom{n}{k}(n-k+r)^s x^{n-k}D_k(1-x) = n!B_{s,r}(x). \tag{3.2}$$

By Lemma 3.1, we have the following theorem.

Theorem 3.1. *Let $f(x) = a_s x^s + a_{s-1}x^{s-1} + \cdots + a_0$ be a polynomial of degree s. If r is a nonnegative integer, then*

$$\sum_{k=0}^{n}\binom{n}{k}f(n-k+r)x^{n-k}D_k(1-x) = n!\left\{\sum_{i=0}^{s} a_iB_{i,r}(x) - \sum_{n+1\leq j\leq i\leq s} a_iS_r(i+r,j+r)x^j\right\}. \tag{3.3}$$

Proof. Using (3.1), the theorem is easily obtained by linearity.

Remark 3.2. *In fact, here we introduce truncated r-Bell polynomials:*

$$B^{(n)}_{s,r}(x) = \sum_{k=0}^{n} S_r(s+r,k+r)x^k,$$

with the exponential generating function

$$\sum_{s=0}^{\infty} B^{(n)}_{s,r}(x)\frac{t^s}{s!} = e^{x\left(\sum_{i=1}^{n}\frac{t^i}{i!}\right)+rt}.$$

For $x = 1$, we obtain numbers that have a simple combinatorial interpretation: They are partitions of a set of size s into $k \leq n$ disjoint subsets such that the first r elements are in distinct sets. For a fixed n, these polynomials form a generating set of the space of polynomials of degree at most n. As the referee pointed out, we can use this notation to give a simpler form to the statement of Lemma 3.1, *which becomes*

$$B^{(n)}_{s,r}(x) = \frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}(n-k+r)^s x^{n-k}D_k(1-x).$$

Theorem 3.1 *can then be rewritten as follows:*

$$\sum_{i=0}^{s} a_i B_{s,r}^{(n)}(x) = \frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k} f(n-k+r)x^{n-k}D_k(1-x).$$

We therefore have an equality between two linear combinations of families of polynomials. Suppose that $f(x) = \sum_{k=0}^{\infty} a_k x^k$ *has an infinite radius of convergence. Passing to the limit, the equality becomes*

$$\sum_{i=0}^{\infty} a_i B_{s,r}^{(n)}(x) = \frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k} f(n-k+r)x^{n-k}D_k(1-x).$$

The identity relates the values of a function and the coefficients of its Taylor expansion.

Remark 3.3. *The referee also pointed out that using the truncated r-Bell polynomials and generating series, the proof of Lemma* 3.1 *can be simplified. Now, we present his (her) proof. Lemma* 3.1 *is equivalent to*

$$\sum_{n=0}^{\infty} B_{s,r}^{(n)}(x)t^n = \sum_{n=0}^{\infty}(n+r)^s\frac{(xt)^n}{n!}\sum_{n=0}^{\infty} D_n(1-x)\frac{t^n}{n!}.$$

The left hand side is

$$\sum_{n=0}^{\infty} B_{s,r}^{(n)}(x)t^n = \sum_{n=0}^{\infty}\sum_{k=0}^{n} S_r(s+r,k+r)x^k t^n = \sum_{k=0}^{\infty} x^k S_r(s+r,k+r)\frac{t^k}{1-t} = \frac{B_{s,r}(xt)}{1-t}.$$

By Dobinski's formula for r-Bell polynomials [14]

$$B_{n,r}(x) = e^{-x}\sum_{k=0}^{\infty}\frac{(k+r)^n}{k!}x^k,$$

we obtain the righthand side

$$\sum_{n=0}^{\infty}(n+r)^s\frac{(xt)^n}{n!}\sum_{n=0}^{\infty} D_n(1-x)\frac{t^n}{n!} = B_{s,r}(xt)e^{xt}\frac{e^{-xt}}{1-t}.$$

The two expressions are equal and Lemma 3.1 *is proved.*

When $r = 0$, we establish the relationship between the derangement polynomials and the Bell polynomials.

Corollary 3.1. *Let* $f(x) = a_s x^s + a_{s-1}x^{s-1} + \cdots + a_0$ *be a polynomial of degree s, then*

$$\sum_{k=0}^{n}\binom{n}{k} f(n-k)x^{n-k}D_k(1-x) = n!\left\{\sum_{i=0}^{s} a_i B_i(x) - \sum_{n+1\le j\le i\le s} a_i S(i,j)x^j\right\}. \tag{3.4}$$

Remark 3.4. *Theorem* 4 *in [7] describes a relationship between the derangement numbers and the Bell numbers, while Corollary* 3.1 *in our paper is extended to the case of the derangement polynomials and the Bell polynomials. When* $x = 1$, (3.4) *reduces to*

$$\sum_{k=0}^{n}\binom{n}{k}f(n-k)D_k = n!\left\{\sum_{i=0}^{s}a_iB_i - \sum_{n+1\le j\le i\le s}a_iS(i,j)\right\}. \tag{3.5}$$

In (3.4)*, when* $f(x) = 1$*, we get*

$$\sum_{k=0}^{n}\binom{n}{k}x^{n-k}D_k(1-x) = n!; \tag{3.6}$$

equivalently, by the binomial inversion formula we have

$$D_n(x) = \sum_{k=0}^{n}\binom{n}{k}k!(x-1)^{n-k}. \tag{3.7}$$

In fact, if we rewrite (2.1) *as*

$$\frac{1}{1-t} = e^{xt}\sum_{n=0}^{\infty}D_n(1-x)\frac{t^n}{n!},$$

we can easily find that (3.6) is a direct consequence of the above identity. When $f(x) = x - 1$*, we get*

$$\sum_{k=0}^{n}\binom{n}{k}(n-k-1)x^{n-k}D_k(1-x) = n!\{B_1(x) - B_0(x)\} = n!(x-1),\ n\ge 2. \tag{3.8}$$

It can be rewritten as

$$D_n(x) = \sum_{k=0}^{n-2}\binom{n}{k}(n-k-1)(1-x)^{n-k}D_k(x) + n!x,\ n\ge 2. \tag{3.9}$$

When $x = 0$*,* (3.9) *reduces to* (1.5)*.*

Lemma 3.2. *If both r and s are nonnegative integers, then we have*

$$\begin{aligned}&\sum_{k=0}^{n}\binom{n}{k}(k-r)^s x^{n-k}D_k(1-x)\\ &= n!\left\{\sum_{j=0}^{s}(-1)^j\binom{s}{j}n^{s-j}B_{j,r}(x) - \sum_{n+1\le i\le j\le s}(-1)^j\binom{s}{j}n^{s-j}S_r(j+r,i+r)x^i\right\}.\end{aligned} \tag{3.10}$$

Proof. We have

$$\sum_{k=0}^{n}\binom{n}{k}(k-r)^s x^{n-k}D_k(1-x) = \sum_{k=0}^{n}\binom{n}{k}(n-(n-k+r))^s x^{n-k}D_k(1-x)$$

$$= \sum_{k=0}^{n}\binom{n}{k}x^{n-k}D_k(1-x)\sum_{j=0}^{s}(-1)^j\binom{s}{j}n^{s-j}(n-k+r)^j$$
$$= \sum_{j=0}^{s}(-1)^j\binom{s}{j}n^{s-j}\sum_{k=0}^{n}\binom{n}{k}(n-k+r)^jx^{n-k}D_k(1-x).$$

By Lemma 3.1, we immediately obtain Lemma 3.2.

Remark 3.5. *When $n \geq s$,* (3.10) *reduces to*

$$\sum_{k=0}^{n}\binom{n}{k}(k-r)^sx^{n-k}D_k(1-x) = n!\sum_{j=0}^{s}(-1)^j\binom{s}{j}n^{s-j}B_{j,r}(x), \tag{3.11}$$

which is a generalized identity of (1.9).

By Lemma 3.2 and simple calculations, we have the following theorem.

Theorem 3.2. *Let $f(x) = a_sx^s + a_{s-1}x^{s-1} + \cdots + a_0$ be a polynomial of degree s. If r is a nonnegative integer, then*

$$\sum_{k=0}^{n}\binom{n}{k}f(k-r)x^{n-k}D_k(1-x)$$
$$= n!\left\{\sum_{0\leq j\leq i\leq s}(-1)^ja_i\binom{i}{j}n^{i-j}B_{j,r}(x) - \sum_{n+1\leq t\leq j\leq i\leq s}(-1)^ja_i\binom{i}{j}n^{i-j}S_r(j+r,t+r)x^t\right\}. \tag{3.12}$$

Remark 3.6. *As discussed in Remark* 3.2*, we can have the same approach for Theorem* 3.2 *with the formula*

$$\sum_{i=0}^{s}a_i\sum_{j=0}^{i}(-1)^j\binom{i}{j}n^{i-j}B_{j,r}^{(n)}(x) = \frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}f(k-r)x^{n-k}D_k(1-x).$$

In particular, suppose that $f(x) = \sum_{k=0}^{\infty}a_kx^k$ has an infinite radius of convergence. Passing to the limit we obtain

$$\sum_{i=0}^{\infty}a_i\sum_{j=0}^{i}(-1)^j\binom{i}{j}n^{i-j}B_{j,r}^{(n)}(x) = \frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}f(k-r)x^{n-k}D_k(1-x).$$

The identity also relates the values of a function and the coefficients of its Taylor expansion.

In particular, when $r = 0$, we have

Corollary 3.2. *Let $f(x) = a_sx^s + a_{s-1}x^{s-1} + \cdots + a_0$ be a polynomial of degree s, then*

$$\sum_{k=0}^{n}\binom{n}{k}f(k)x^{n-k}D_k(1-x)$$
$$= n!\left\{\sum_{0\leq j\leq i\leq s}(-1)^ja_i\binom{i}{j}n^{i-j}B_j(x) - \sum_{n+1\leq t\leq j\leq i\leq s}(-1)^ja_i\binom{i}{j}n^{i-j}S(j,t)x^t\right\}. \tag{3.13}$$

If we take $x = 1$ in Corollary 3.2, we recover Corollary 5 in [7].

In the following theorems, we establish the identities of the second kind involving the derangement polynomials and the r-Bell polynomials.

Theorem 3.3. *If both r and s are nonnegative integers, then we have*

$$B_{n,r}(x) = \sum_{j=0}^{n} \sum_{k=0}^{j} \binom{n}{j} (r+1)^{n-j} (-1)^k S(j,k) D_k(1-x). \tag{3.14}$$

Proof. If we let $u = 1 - e^t$, then

$$e^{x(e^t-1)+rt} = \frac{e^{-xu}}{1-u} \cdot e^{(r+1)t} = \sum_{k=0}^{\infty} D_k(1-x) \frac{u^k}{k!} \sum_{k=0}^{\infty} (r+1)^k \frac{t^k}{k!}.$$

Since

$$\frac{u^k}{k!} = (-1)^k \frac{(e^t-1)^k}{k!} = (-1)^k \sum_{n \geq k} S(n,k) \frac{t^n}{n!},$$

we have

$$\begin{aligned} e^{x(e^t-1)+rt} &= \sum_{p=0}^{\infty} \frac{t^p}{p!} \sum_{k=0}^{p} (-1)^k S(p,k) D_k(1-x) \cdot \sum_{q=0}^{\infty} (r+1)^q \frac{t^q}{q!} \\ &= \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{j=0}^{n} \binom{n}{j} (r+1)^{n-j} \sum_{k=0}^{j} (-1)^k S(j,k) D_k(1-x). \end{aligned}$$

From (2.9), we arrive at

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{j=0}^{n} \binom{n}{j} (r+1)^{n-j} \sum_{k=0}^{j} (-1)^k S(j,k) D_k(1-x) = \sum_{n=0}^{\infty} B_{n,r}(x) \frac{t^n}{n!}.$$

By comparing the coefficients of $t^n/n!$, we get

$$B_{n,r}(x) = \sum_{j=0}^{n} \binom{n}{j} (r+1)^{n-j} \sum_{k=0}^{j} (-1)^k S(j,k) D_k(1-x),$$

which leads to Theorem 3.3.

Taking $x = 1$ in (3.14), we obtain the explicit expression of the r-Bell numbers $B_{n,r}$ in terms of the derangement numbers D_k.

Corollary 3.3. *If both r and s are nonnegative integers, then we have*

$$B_{n,r} = \sum_{j=0}^{n} \sum_{k=0}^{j} \binom{n}{j} (r+1)^{n-j} (-1)^k S(j,k) D_k. \tag{3.15}$$

When $r = 0$ in (3.14), we have the relationship between the derangement polynomials and the Bell polynomials.

Corollary 3.4. *[12] For $n \geq 0$, we have*

$$B_n(x) = \sum_{j=0}^{n} \sum_{k=0}^{j} \binom{n}{j} (-1)^k S(j,k) D_k(1-x). \tag{3.16}$$

Remark 3.7. *By the binomial inversion formula, we can rewrite* (3.14) *as*

$$\sum_{k=0}^{n} (-1)^k S(n,k) D_k(1-x) = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} (r+1)^{n-j} B_{j,r}(x). \tag{3.17}$$

By (3.17), we obtain the explicit expression of the derangement polynomials $D_n(x)$ in terms of the r-Bell polynomials.

Theorem 3.4. *If both r and s are nonnegative integers, then we have*

$$D_n(x) = \sum_{k=0}^{n} \sum_{j=0}^{k} (-1)^{n-k-j} s(n,k) \binom{k}{j} (r+1)^{k-j} B_{j,r}(1-x), \tag{3.18}$$

where $s(n,k)$ are the Stirling numbers of the first kind.

Proof. Let $T_n = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} (r+1)^{n-j} B_{j,r}(x)$ in (3.17). By the orthogonal relationship between two kinds of Stirling numbers [6],

$$\sum_{j=k}^{n} s(n,j) S(j,k) = \delta_{n,k},$$

where $\delta_{n,k}$ is the Kronecker symbol defined by $\delta_{n,k} = 1$ if $n = k$ and $\delta_{n,k} = 0$ otherwise, and one can obtain

$$D_n(1-x) = (-1)^n \sum_{k=0}^{n} s(n,k) T_k.$$

Replacing x by $1 - x$, we arrive at (3.18).

Taking $x = 1$ in (3.14), we obtain the explicit expression of the derangement numbers D_n in terms of the r-Bell numbers $B_{j,r}$.

Corollary 3.5. *For $n \geq 0$, we have*

$$D_n = \sum_{k=0}^{n} \sum_{j=0}^{k} (-1)^{n-k-j} s(n,k) \binom{k}{j} (r+1)^{k-j} B_{j,r}, \tag{3.19}$$

where $s(n,k)$ are the Stirling numbers of the first kind.

Taking $r = 0$ in (3.14), we have the following corollary.

Corollary 3.6. *[12] For $n \geq 0$, we have*

$$D_n(x) = \sum_{k=0}^{n} \sum_{j=0}^{k} (-1)^{n-k-j} s(n,k) \binom{k}{j} B_j(1-x), \tag{3.20}$$

where $s(n,k)$ are the Stirling numbers of the first kind.

Remark 3.8. *In [12], Kim et al. obtained Corollaries 3.4 and 3.6 by using the generating function method. Their method provides us with a good idea to prove Theorems 3.3 and 3.4. In the proof of Theorem 3.4, we use the binomial inversion formula and the orthogonal relationship of two kinds of Stirling numbers, which seems more direct.*

Remark 3.9. *Theorems 3.3 and 3.4 have a common feature, which is that the lefthand member of the equality is a combinatorial number, while the righthand member of the equality is a somewhat complicated double sum. For algorithmic motivation, these results show that either derangement polynomials or r-Bell polynomials can be calculated using the other one. Furthermore, these results may well have combinatorial consequences, as double sums involve combinatorial numbers, binomial coefficients, signs and so on. It would therefore be interesting to look for an interpretation of these two identities from the point of view of combinatorial objects. We will continue our research in the following paper.*

4. Conclusions

We have obtained two kinds of identities involving derangement polynomials and r-Bell polynomials. The identities of the first kind presented a relation between two linear combinations of families of polynomials. They also related the values of a function and the coefficients of its Taylor expansion. The identities of the second kind extended some of the results on derangement polynomials and Bell polynomials due to Kim et al and had a common feature, which is that the lefthand member of the equality is a combinatorial number, while the righthand member of the equality is a somewhat complicated double sum.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interests.

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