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Research article

Classification of the symmetry Lie algebras for six-dimensional co-dimension two Abelian nilradical Lie algebras

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Abstract: In this paper, we consider the symmetry algebra of the geodesic equations of the canonical connection on a Lie group. We mainly consider the solvable indecomposable six-dimensional Lie algebras with co-dimension two abelian nilradical that have an abelian complement. In dimension six, there are nineteen such algebras, namely, $A_{6,1}$–$A_{6,19}$ in Turkowski's list. For each algebra, we give the geodesic equations, a basis for the symmetry Lie algebra in terms of vector fields, and finally we identify the symmetry Lie algebra from standard lists.

Keywords: Lie symmetry; Lie group; canonical connection; geodesic system
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1. Introduction

Symmetry methods for differential equations were introduced by Lie towards the end of the nineteenth century. Lie was looking for transformations that would change a differential equation to a simpler version, so that it might, for example, become separable and then easy to solve. Lie noticed that the ordinary differential equations (ODE's) are invariant under certain transformations, which later came to be known as Lie groups. The use of symmetry methods has become an increasingly important part of the study of differential equations. For example, we can obtain solutions of differential equations if we know their symmetries; we can also use symmetries to reduce the order of the ODE, and determine whether or not the ODE or partial differential equations (PDE) can be linearized. We can also classify differential equations based on their symmetry Lie algebras. For more information on the history of symmetry Lie algebras and their applications, we refer the reader to [1–4]. Another very accessible reference is [5].

In this paper, we consider special systems of second order ordinary differential equations, known as geodesic equations. The geodesic equation is a second order differential equation, where the independent variable is t, which represents time, and the equation itself represents the motion of a particle moving in a curved space. In Riemannian geometry, a geodesic curve gives the shortest path between two points in that space. In general relativity, geodesic equations describe the motion in spacetime. In the case of a differentiable manifold with a connection, geodesic curves provide a generalization of straight lines in Euclidean space. On a smooth manifold, where (x^i) are a system of local coordinates, and Γ^i_{jk} are the connection components or Christoffel symbols, the geodesic equation is given by

$$\frac{d^2x}{dt^2} + \Gamma^i_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt} = 0. \tag{1.1}$$

A significant amount of work has been done on analyzing the symmetry Lie algebras of the geodesic equations of the canonical connection on Lie groups. See [6–9] for further details about this connection. In [8], the main geometrical properties of this connection have been listed and proofs supplied. Ghanam and Thompson have considered the problem in dimensions two, three and four [10]. They have also considered the problem for six-dimensional nilpotent Lie algebras [11]. They have followed the list of algebras classified by Patera et al. [12] for indecomposable Lie algebras in dimension up to six. In recent articles, Almusawa et al. have considered the problem of classifying the symmetry Lie algebra of the geodesic equations for indecomposable nilpotent Lie algebras in dimension five [13], and they considered the problem for indecomposable five-dimensional solvable Lie algebras [14]. Finally, they considered the problem for a general n-dimensional Lie algebra with co-dimension one, and general results have been obtained [15]. In [16], the symmetry Lie algebras of the geodesic equations of the canonical connection on Lie groups whose Lie algebras have a co-dimension two abelian nilradical in dimensions four and five, were identified.

In this paper, we continue our investigation of the Lie symmetries of the geodesic system of the canonical connection on a Lie group. The present article extends the investigation to indecomposable Lie algebras in dimensions six, that have a co-dimension two abelian nilradical, together with an abelian complement. The six-dimensional solvable real Lie algebras were classified into isomorphism classes by Turkowski [17] and comprise forty cases, some of which contain up to four parameters. Of these forty classes, the first nineteen, denoted by $A_{6,1}$–$A_{6,19}$, have co-dimension two abelian nilradical and abelian complement. Since most of these algebras have parameters, we need to consider sub-cases based on the values of these parameters: Symmetry may be broken in the sense that, for exceptional values of the parameters, the symmetry algebra may have a higher dimension.

The outline of the paper is as follows: In Section 2, we give the definition of the of the canonical connection ∇ on Lie groups and we review the main properties of ∇. We also show how the definition is used to calculate the geodesic equations of the connection. In Section 3, we review the symmetries of differential equations and the Lie invariance condition. We also show how the Lie invariance condition leads to a system of partial differential equations. The solution of that PDE system gives the Lie symmetry of the geodesic equations. In Section 4, for each algebra $A_{6,1}$–$A_{6,19}$ in Turkowski's list, we give the geodesic equations, a basis for the symmetry algebra in terms of vector fields is given and finally we identify and describe the symmetry algebra, which in all cases is a solvable Lie algebra, in terms of its nilradical and its complement. Regarding our notation, we use $\rtimes$ for the semi-direct product and $\oplus$ for the direct sum of algebras.

2. The canonical Lie group connection

On left invariant vector fields, X and Y the canonical symmetric connection ∇ on a Lie group G is defined by

$$\nabla_X Y = \frac{1}{2}\,[X, Y], \tag{2.1}$$

and then extended to arbitrary vector fields using linearity and the Leibnitz rule. ∇ is left-invariant. One could just as well use right-invariant vector fields to define ∇, but one must check that ∇ is well defined, a fact that we will prove next.

Proposition 1. *In the definition of ∇ we can equally assume that X and Y are right-invariant vector fields and hence ∇ is also left-invariant and hence bi-invariant. Moreover ∇ is symmetric, that is, its torsion is zero.*

Proof. The fact that ∇ is symmetric is obvious from Eq (2.1). Now we choose a fixed basis in the tangent space at the identity $T_I G$. We shall denote its left and right invariant extensions by $\{X_1, X_2, \cdots, X_n\}$ and $\{Y_1, Y_2, \cdots, Y_n\}$, respectively. Then there must exist a non-singular matrix A of functions on G such that $Y_i = a_i^j X_j$. We shall suppose that

$$[X_i, X_j] = C_{ij}^k X_k. \tag{2.2}$$

Changing from the left-invariant basis to the right gives

$$C_{ij}^k a_k^p = a_i^k a_j^m C_{km}^p. \tag{2.3}$$

Next, we use the fact that left and right vector fields commute to deduce that

$$a_j^k C_{ik}^m + X_i a_j^m = 0, \tag{2.4}$$

where the second term in (2.4) denotes directional derivative. We note that necessarily

$$[Y_i, Y_j] = -C_{ij}^k Y_k. \tag{2.5}$$

Now we compute

$$\nabla_{Y_i} Y_j + \frac{1}{2} C_{ij}^k Y_k = \frac{1}{2} a_i^k a_j^m C_{km}^p + a_i^k (X_k a_i^p) + \frac{1}{2} C_{ij}^k a_k^p. \tag{2.6}$$

Next we use (2.4) to replace the second term on the right hand side of (2.6) so as to obtain

$$\nabla_{Y_i} Y_j + \frac{1}{2} C_{ij}^k Y_k = \frac{1}{2} a_i^k a_j^m C_{km}^p - a_i^k a_j^m C_{km}^p + \frac{1}{2} C_{ij}^k a_k^p. \tag{2.7}$$

However, the right hand side of (2.7) is seen to be zero by virtue of (2.3). Thus

$$\nabla_X Y = \frac{1}{2}[X, Y], \tag{2.8}$$

whenever X and Y are right invariant vector fields. □

An alternative proof of Proposition 1 uses the inversion map ψ defined by, for $S \in G$,

$$\psi(S) = S^{-1}. \tag{2.9}$$

As such, one checks that ψ_{*_I} maps a left-invariant vector field evaluated at I to minus its right-invariant counterpart evaluated at I. Then ψ_{*_I} is an isomorphism and there is no change of sign in the structure constants, as compared with Eq (2.5). Since there are two minus signs in Eq (2.1) the same condition Eq (2.1) applies also to right-invariant vector fields.

Proposition 2. *(i) An element in the center of g engenders a bi-invariant vector field.*

(ii) A vector field in the center of g is parallel.

(iii) A bi-invariant differential k-form θ is closed and so defines an element of the cohomology group $H^k(M, \mathbb{R})$.

Proof. (i) Suppose that $Z \in T_I G$ is in the center of g and let $\exp(tZ)$ be the associated one-parameter subgroup of G so that Z corresponds to the equivalence class of curves $[\exp(tZ)]$ based at I. Let $S \in G$; then $L_{S*}Z$ corresponds to the equivalence class of curves $[S \exp(tZ)]$ based at S. Since Z is in the center of g then $\exp(tZ)$ will be in the center of G and hence

$$[S \exp(tZ)] = [\exp(tZ)S].$$

It follows that any element in the center of g engenders a bi-invariant vector field.

(ii) Obvious from Eq (2.1).

(iii) A proof can be found in [18]. Spivak shows that

$$\psi^*(\theta) = (-1)^k \theta,$$

whereas $d\theta$, which is also bi-invariant, changes by

$$\psi^*(d\theta) = (-1)^{k+1} d\theta.$$

It follows that $d\theta = 0$. □

Proposition 3. *(i) The curvature tensor, which is also bi-invariant, on vector fields X, Y, Z is given by*

$$R(X, Y)Z = \frac{1}{4}\,[[X, Y], Z]. \tag{2.10}$$

(ii) The connection ∇ is flat if and only if the Lie algebra g of G is two-step nilpotent.

(iii) The tensor R is parallel in the sense that $\nabla_W R(X, Y)Z = 0$, where W is a fourth right invariant vector field, so that G is in a sense a symmetric space.

(iv) The Ricci tensor R_{ij} of ∇ is given by

$$R_{ij} = \frac{1}{4}\, C^l_{jm} C^m_{il} \tag{2.11}$$

and is symmetric and bi-invariant and is obtained by translating to the left or right one quarter of the Killing form. It engenders a bi-invariant pseudo-Riemannian metric if and only if the Lie algebra g is semi-simple.

Proof. (i) Is obvious and applies to arbitrary vector fields since it is a tensorial object.

(ii) Is obvious.

(iii) This fact follows from a series of implications:

$$\begin{aligned}
&4\nabla_W R(X,Y)Z + 4R(\nabla_W X, Y)Z + 4R(X, \nabla_W Y)Z + 4R(X,Y)\nabla_W Z = \nabla_W[[X,Y],Z],\\
&4\nabla_W R(X,Y)Z + 2R([W,X],Y)Z + 2R(X,[W,Y])Z + 2R(X,Y)[W,Z] - \tfrac{1}{2}[W,[[X,Y],Z] = 0,\\
&4\nabla_W R(X,Y)Z + \tfrac{1}{2}[[W,X],Y],Z] + \tfrac{1}{2}[X,[W,Y]],Z] + \tfrac{1}{2}[[X,Y],[W,Z]] - \tfrac{1}{2}[W,[[X,Y],Z] = 0,\\
&4\nabla_W R(X,Y)Z + \tfrac{1}{2}[[W,X],Y],Z] + \tfrac{1}{2}[X,[W,Y]],Z] - \tfrac{1}{2}[Z,[[X,Y],W] = 0,\\
&\nabla_W R(X,Y)Z = 0.
\end{aligned} \tag{2.12}$$

(iv) The formula Eq (2.11) is obvious from Eqs (2.1) and (2.10). The last remark follows from Cartan's criterion. □

Proposition 4. *(i) Any left or right-invariant vector field is geodesic.*

(ii) Any geodesic curve emanating from the identity is a one-parameter subgroup.

(iii) An arbitrary geodesic curve is a translation, to the left or right, of a one-parameter subgroup.

Proof. (i) Is obvious because of the skew-symmetry in Eq (2.1).

(ii) By definition the curve $t \mapsto [S\, \exp(tX)]$ integrates a geodesic field X.

(iii) If the geodesic curve at $t = 0$ starts at S, translate the curve to I by multiplying on the left or right by S^{-1} and apply (ii). □

Proposition 5. *(i) A left or right-invariant vector field is a symmetry, a.k.a. affine collineation, of* ∇.

(ii) Any left or right-invariant one-form engenders a first integral of the geodesic system of ∇.

Proof. (i) The following condition for vector fields X and Y says that vector field W is a symmetry or, affine collineation, of a symmetric linear connection:

$$\nabla_X \nabla_Y W - \nabla_{\nabla_X Y} W - R(W,X)Y = 0. \tag{2.13}$$

In the case at hand of the canonical connection, this condition just reduces to the Jacobi identity when W, X and Y are all left or right-invariant.

(ii) A one-form α is a *Killing one-form*, if the following condition holds:

$$\langle \nabla_X \alpha, Y\rangle + \langle X, \nabla_Y \alpha\rangle = 0. \tag{2.14}$$

In the case of the canonical connection, if X and Y are right-invariant and α is right-invariant then Eq (2.1) gives

$$\langle X, \nabla_Y \alpha\rangle = \frac{1}{2}\langle [X,Y], \alpha\rangle. \tag{2.15}$$

Clearly, (2.15) implies (2.14) so that every left or right-invariant one-form engenders a first integral of the geodesics: if the one-form is given in a coordinate system as $\alpha_i dx^i$ on G, the first integral is $\alpha_i u^i$ viewed as a function on the tangent bundle TG that is linear in the fibers. □

Proposition 6. *Any left or right-invariant one-form α is closed if and only if $\langle [g, g], \alpha\rangle = 0$, that is, α annihilates the derived algebra of g.*

Proof. Consider the identity

$$d\alpha(X, Y) = X\langle Y, \alpha\rangle - Y\langle X, \alpha\rangle - \langle [X, Y], \alpha\rangle. \tag{2.16}$$

If α is left-invariant and we take X and Y left-invariant, then the first and second terms in Eq (2.16) are zero. Now the conclusion of the proposition is obvious. The proof for right-invariant one-forms is similar. □

Proposition 7. *Consider the following conditions for a one-form α on G:*

(i) α is bi-invariant.

(ii) α is right-invariant and closed.

(iii) α is left-invariant and closed.

(iv) α is parallel.

Then we have the following implications: (i)–(iii) are equivalent and any one of them implies (iv).

Proof. The fact that (i) implies (ii) and (iii) follows from Proposition 2 part (iii). Now, suppose that (iii) holds and let X and Y be right and left-invariant vector fields, respectively. Then, consider again the identity

$$d\alpha(X, Y) = X\langle Y, \alpha\rangle - Y\langle X, \alpha\rangle - \langle [X, Y], \alpha\rangle. \tag{2.17}$$

Assuming that α is closed, then either because $[X, Y] = 0$ or by using Proposition 6, we find that Eq (2.17) reduces to

$$X\langle Y, \alpha\rangle = Y\langle X, \alpha\rangle. \tag{2.18}$$

Now, the left hand side of Eq (2.18) is zero, since Y and α are left-invariant. Hence, $\langle X, \alpha\rangle$ is constant, which implies that α is right-invariant and hence bi-invariant. Thus, (iii) implies (i). The proof that (ii) implies (i) is similar. Finally, supposing that (ii) or (iii) holds we show that (iv) holds. Then as with any symmetric connection, the closure condition may be written, for arbitrary vector fields X and Y, as

$$\langle \nabla_X \alpha, Y\rangle - \langle X, \nabla_Y \alpha\rangle = 0. \tag{2.19}$$

Clearly Eq (2.14) and Eq (2.19)imply that α is parallel. So a closed, invariant one-form is parallel. □

Of course, it may well be the case that there are no bi-invariant one-forms on G, for example if G is semi-simple so that $[g, g] = g$. However, there must be at least one such one-form if G is solvable and at least two if G is nilpotent.

If we choose a basis of dimension dim g-dim $[g, g]$ for the bi-invariant one-forms on G, it may be used to obtain a partial coordinate system on G, since each such form is closed. Such a partial coordinate system is significant in terms of the geodesic system, in that it gives rise to second order differential equations that resemble the system in Euclidean space.

Proposition 8. *Each of the bi-invariant one-forms on G projects to a one-form on the quotient space $G/[G,G]$, assuming that the commutator subgroup $[G,G]$ is closed topologically in G. Furthermore, the canonical connection ∇ on G projects to a flat connection on $G/[G,G]$ and the induced system of one-forms on $G/[G,G]$ comprises a "flat" coordinate system.*

Proof. The fact that a bi-invariant one-form on G projects to a one-form on $G/[G,G]$ follows because each such form annihilates the vertical distribution of the principal right $[G,G]$-bundle $G \to G/[G,G]$ and furthermore the equivariance, or Lie-derivative condition along the fibers, is trivially satisfied since the one-form is closed. The fact that ∇ projects to $G/[G,G]$ follows because $[G,G] \triangleleft G$, as was noted in [19]. □

3. Lie invariance condition and symmetries of the geodesic equations

In this section, we explain the algorithm of finding the Lie symmetry of the geodesic equations. Consider the system of the geodesic equations given by

$$\frac{d^2x}{dt^2} = f^i(t, x^i), \quad i = 1, 2, \cdots, 6, \tag{3.1}$$

where in in this case

$$(x^1, x^2, x^3, x^4, x^5, x^6) = (p, q, x, y, z, w)$$

and t is the independant variable and x^i's are the dependant variables. We now consider a symmetry vector field Γ of the form:

$$\Gamma = T\frac{\partial}{\partial t} + P\frac{\partial}{\partial p} + Q\frac{\partial}{\partial q} + X\frac{\partial}{\partial x} + Y\frac{\partial}{\partial y} + Z\frac{\partial}{\partial z} + W\frac{\partial}{\partial w}, \tag{3.2}$$

where T, P, Q, X, Y, Z and W are unknown functions of (t, p, q, x, y, z, w). The first prolongation Γ^1 and second prolongation Γ^2 of Γ are give by

$$\Gamma^1 = \Gamma + P_t\frac{\partial}{\partial \dot{p}} + Q_t\frac{\partial}{\partial \dot{q}} + X_t\frac{\partial}{\partial \dot{x}} + Y_t\frac{\partial}{\partial \dot{y}} + Z_t\frac{\partial}{\partial \dot{z}} + W_t\frac{\partial}{\partial \dot{w}}, \tag{3.3}$$

$$\Gamma^2 = \Gamma^1 + P_{tt}\frac{\partial}{\partial \ddot{p}} + Q_{tt}\frac{\partial}{\partial \ddot{q}} + X_{tt}\frac{\partial}{\partial \ddot{x}} + Y_{tt}\frac{\partial}{\partial \ddot{y}} + Z_{tt}\frac{\partial}{\partial \ddot{z}} + W_{tt}\frac{\partial}{\partial \ddot{w}}, \tag{3.4}$$

where

$$\begin{aligned} P_t &= D_t(P) - \dot{p}D_t(T), & P_{tt} &= D_t(P_t) - \ddot{p}D_t(T), \\ Q_t &= D_t(Q) - \dot{q}D_t(T), & Q_{tt} &= D_t(Q_t) - \ddot{q}D_t(T), \\ X_t &= D_t(X) - \dot{x}D_t(T), & X_{tt} &= D_t(X_t) - \ddot{x}D_t(T), \\ Y_t &= D_t(Y) - \dot{y}D_t(T), & Y_{tt} &= D_t(Y_t) - \ddot{y}D_t(T), \\ Z_t &= D_t(Z) - \dot{z}D_t(T), & Z_{tt} &= D_t(Z_t) - \ddot{z}D_t(T), \\ W_t &= D_t(W) - \dot{w}D_t(T), & W_{tt} &= D_t(W_t) - \ddot{w}D_t(T), \end{aligned} \tag{3.5}$$

where D_t is given by

$$D_t = \frac{\partial}{\partial t} + \dot{p}\frac{\partial}{\partial p} + \dot{q}\frac{\partial}{\partial q} + \dot{x}\frac{\partial}{\partial x} + \dot{y}\frac{\partial}{\partial y} + \dot{z}\frac{\partial}{\partial z} + \dot{w}\frac{\partial}{\partial w} + \ddot{p}\frac{\partial}{\partial \dot{p}} + \ddot{q}\frac{\partial}{\partial \dot{q}} + \ddot{x}\frac{\partial}{\partial \dot{x}} + \ddot{y}\frac{\partial}{\partial \dot{y}} + \ddot{z}\frac{\partial}{\partial \dot{z}} + \ddot{w}\frac{\partial}{\partial \dot{w}}. \tag{3.6}$$

Finally, Γ is said to be a Lie symmetry of the system the geodesic equations if

$$\Gamma^2(\Delta_i^{(2)})|_{\Delta_i^{(2)}=0} = 0, \tag{3.7}$$

where

$$\Delta_i^{(2)} = \frac{d^2x}{dt^2} - f^i(t, x^i), \qquad i = 1, 2, \cdots, 6. \tag{3.8}$$

Equation (3.7) is called the Lie invariance condition. We equate the coefficients of the linearly independent derivation terms to zero and this yields to an overdetermined system of PDEs.

4. Classification of the symmetry Lie algebras

In this section, we consider the nineteen six-dimensional Lie algebra with co-dimension two abelian nilradical. For each Lie algebra, we will list the non-zero brackets, the system of the geodesic equations and the symmetry vector fields. Finally, we analyze the symmetry Lie algebra in terms of its nilradical and identify it.

4.1. Algebra $A_{6,1}^{abcd}$ ($abcd : ab \neq 0$, $c^2 + d^2 \neq 0$)

The non-zero brackets for the algebra $A_{6,1}^{abcd}$ are given by

$$\begin{aligned} &[e_1, e_3] = ae_3, \quad [e_1, e_4] = ce_4, \quad [e_1, e_6] = e_6, \\ &[e_2, e_3] = be_3, \quad [e_2, e_4] = de_4, \quad [e_2, e_5] = e_5. \end{aligned} \tag{4.1}$$

The geodesic equations are given by

$$\ddot{p} = \dot{p}\dot{w}, \quad \ddot{q} = \dot{q}\dot{z}, \quad \ddot{x} = \dot{x}(d\dot{z} + c\dot{w}), \quad \ddot{y} = \dot{y}(b\dot{z} + a\dot{w}), \quad \ddot{z} = 0, \; \ddot{w} = 0. \tag{4.2}$$

For the general case $A_{6,1}^{a\neq0,b\neq0,c\neq0,d\neq0}$, the symmetry Lie algebra is spanned by

$$\begin{aligned} &e_1 = D_q, \quad e_2 = D_t, \quad e_3 = D_x, \quad e_4 = D_y, \quad e_5 = D_p, \quad e_6 = wD_t, \quad e_7 = zD_t, \\ &e_8 = e^z D_q, \quad e_9 = e^w D_p, \quad e_{10} = e^{cw+dz} D_x, \quad e_{11} = e^{aw+bz} D_y, \quad e_{12} = D_w, \\ &e_{13} = D_z, \quad e_{14} = tD_t, \quad e_{15} = xD_x, \quad e_{16} = qD_q, \quad e_{17} = yD_y. \quad e_{18} = pD_p. \end{aligned} \tag{4.3}$$

The non-zero brackets of the symmetry algebra are given by

$$\begin{aligned} &[e_1, e_{16}] = e_1, \quad [e_2, e_{14}] = e_2, \quad [e_3, e_{15}] = e_3, \quad [e_4, e_{17}] = e_4, \quad [e_5, e_{18}] = e_5, \\ &[e_6, e_{12}] = -e_2, \quad [e_6, e_{14}] = e_6, \quad [e_7, e_{13}] = -e_2, \quad [e_7, e_{14}] = e_7, \quad [e_8, e_{13}] = -e_8, \\ &[e_8, e_{16}] = e_8, \quad [e_9, e_{12}] = -e_9, \quad [e_9, e_{18}] = e_9, \quad [e_{10}, e_{15}] = e_{10}, \quad [e_{11}, e_{17}] = e_{11}, \\ &[e_{10}, e_{13}] = -de_{10}, \quad [e_{11}, e_{12}] = -ae_{11}, \quad [e_{11}, e_{13}] = -be_{11}, \quad [e_{10}, e_{12}] = -ce_{10}. \end{aligned} \tag{4.4}$$

In this case, based on the Lie invariance condition, we have to consider eight subcases based on the values of the parameters to see if taking certain values will generate new solutions to the system of PDE. The cases we consider are: $A_{6,1}^{a=1,b\neq0}$, $A_{6,1}^{a=c,a\neq0}$, $A_{6,1}^{b=1,a\neq0}$, $A_{6,1}^{b=d,b\neq0}$, $A_{6,1}^{c=0,d\neq0}$, $A_{6,1}^{c=1}$, $A_{6,1}^{d=0,c\neq0}$ and $A_{6,1}^{d=1}$. In the generic and sub-cases, we find that the structure of the symmetry Lie algberas are the same. We summarize the results in the following proposition.

Proposition 9. *The symmetry Lie algebra is an eighteen-dimensional solvable Lie algebra which is the semidirect product of eleven-dimensional abelian nilradical spanned by e_1–e_{11} and a seven-dimensional abelian complement spanned by e_{12}–e_{18}. Hence, it can be described as $\mathbb{R}^{11} \rtimes \mathbb{R}^7$.*

4.2. Algebra $A_{6,2}^{abc}$ ($a^2 + b^2 \neq 0$)

The non-zero brackets for the algebra $A_{6,2}^{abc}$ are given by

$$[e_1, e_4] = e_4, \quad [e_1, e_5] = e_6, \quad [e_2, e_5] = e_5, \quad [e_2, e_6] = e_6,$$
$$[e_2, e_3] = be_3, \quad [e_2, e_4] = ce_4, \quad [e_1, e_3] = ae_3. \tag{4.5}$$

The geodesic equations are given by

$$\ddot{p} = \dot{p}\dot{z}, \quad \ddot{q} = \dot{p}\dot{w} + \dot{q}\dot{z}, \quad \ddot{x} = \dot{x}(c\dot{z} + \dot{w}), \quad \ddot{y} = \dot{y}(b\dot{z} + a\dot{w}), \quad \ddot{z} = 0, \quad \ddot{w} = 0. \tag{4.6}$$

For the general case $A_{6,2}^{a\neq 0, b\neq 0, c\neq 0}$, the symmetry Lie algebra is spanned by

$$e_1 = e^z D_q, \quad e_2 = D_q, \quad e_3 = e^z D_p + we^z D_q, \quad e_4 = D_p, \quad e_5 = pD_q, \quad e_6 = D_t,$$
$$e_7 = D_x, \quad e_8 = D_y, \quad e_9 = xD_x, \quad e_{10} = xD_t, \quad e_{11} = zD_t, \quad e_{12} w^{w+cz} D_x,$$
$$e_{13} = e^{aw+bz} D_y, \quad e_{14} = D_w, \quad e_{15} = D_z, \quad e_{16} = tD_t, \quad e_{17} = yD_y, \quad e_{18} = pD_p = qD_q. \tag{4.7}$$

The non-zero brackets of the symmetry algebra are given by

$$[e_1, e_{15}] = e_1, \quad [e_2, e_{18}] = e_2, \quad [e_3, e_{16}] = e_3, \quad [e_4, e_{17}] = e_4, \quad [e_8, e_{10}] = -e_9,$$
$$[e_5, e_{18}] = e_5, \quad [e_6, e_{13}] = -e_1, \quad [e_6, e_{15}] = e_6, \quad [e_7, e_{14}] = -e_1, \quad [e_7, e_{15}] = e_7,$$
$$[e_5, e_8] = e_2, \quad [e_9, e_{14}] = -e_9, \quad [e_9, e_{18}] = e_9, \quad [e_{12}, e_{17}] = e_{12}, \quad [e_{11}, e_{16}] = e_{11},$$
$$[e_{10}, e_{18}] = e_{10}, \quad [e_{11}, e_{13}] = -e_{11}, \quad [e_{10}, e_{13}] = -e_9, \quad [e_{10}, e_{14}] = -e_{10},$$
$$[e_{12}, e_{13}] = -ae_{12}, \quad [e_{12}, e_{14}] = -be_{12}, \quad [e_{11}, e_{14}] = -ce_{11}. \tag{4.8}$$

In this case, based on the Lie invariance condition, we have to consider seven sub-cases based on the values of the parameters to see if taking certain values will generate new solutions to the system of PDE. The cases we consider are: $A_{6,2}^{a=0,b\neq 0}$, $A_{6,2}^{a=1}$, $A_{6,2}^{b=0,a\neq 0}$, $A_{6,2}^{b=1}$, $A_{6,2}^{a\neq 0,b=c}$, $A_{6,2}^{c=0}$ and $A_{6,2}^{c=1}$. In the generic and sub-cases, we find that the structure of the symmetry Lie algebras are the same. We summarize the results in the following proposition.

Proposition 10. *The symmetry Lie algebra is an eighteen-dimensional solvable Lie algebra which is the semidirect product of twelve-dimensional decomposable nilradical spanned by e_1–e_{12} and a six-dimensional abelian complement spanned by e_{12}–e_{18}. In fact, the nilradical is a direct sum of $A_{5,1}$ in Winternitz list and $\mathbb{R}^7$. Hence, symmetry algebra can be can be described as $(A_{5,1} \oplus \mathbb{R}^7) \rtimes \mathbb{R}^6$ where the non-zero brackets of $A_{5,1}$ are given by*

$$[e_3, e_5] = e_1, [e_4, e_5] = e_2. \tag{4.9}$$

4.3. Algebra $A_{6,3}^{a}$

The non-zero brackets for the algebra $A_{6,3}^{a}$ are given by

$$[e_1, e_3] = e_3, \quad [e_1, e_4] = e_4, \quad [e_1, e_5] = e_6, \quad [e_2, e_5] = e_5,$$

$$[e_2, e_3] = ae_3 + e_4, \quad [e_2, e_4] = ae_4, \quad [e_2, e_6] = e_6. \tag{4.10}$$

The geodesic equations are given by

$$\ddot{p} = \dot{p}\dot{z} + \dot{q}\dot{w}, \quad \ddot{q} = \dot{q}\dot{z}, \quad \ddot{x} = \dot{x}(a\dot{z} + \dot{w}) + \dot{y}\dot{z}, \quad \ddot{y} = \dot{y}(a\dot{z} + \dot{w}), \quad \ddot{z} = 0, \ \ddot{w} = 0. \tag{4.11}$$

For the general case $A_{6,3}^{a\neq0}$, the symmetry Lie algebra is spanned by

$$\begin{gathered} e_1 = D_t, \quad e_2 = D_x, \quad e_3 = D_p, \quad e_4 = D_y, \quad e_5 = D_q, \quad e_6 = wD_t, \quad e_7 = zD_t, \\ e_8 = yD_x, \quad e_9 = qD_p, \quad e_{10} = e^z D_p, \quad e_{11} = we^z D_p + e^z D_q, \quad e_{12} = \frac{e^w e^{az} D_x}{a}, \\ e_{13} = \frac{(az-1)e^{az+w} D_x}{a} + e^w e^{az} D_y, \quad e_{14} = D_w, \quad e_{15} = D_z, \\ e_{16} = tD_t, \quad e_{17} = xD_x + yD_y, \quad e_{18} = pD_p + qD_q. \end{gathered} \tag{4.12}$$

The non-zero brackets of the symmetry algebra are given by

$$\begin{gathered} [e_1, e_{16}] = e_1, \quad [e_2, e_{17}] = e_2, \quad [e_3, e_{18}] = e_3, \quad [e_{13}, e_{15}] = -ae_{12} - ae_{13}, \\ [e_5, e_9] = e_3, \quad [e_5, e_{18}] = e_5, \quad [e_6, e_{14}] = -e_1, \quad [e_7, e_{15}] = -e_1, \\ [e_7, e_{16}] = e_7, \quad [e_9, e_{11}] = -e_{10}, \quad [e_{10}, e_{15}] = -e_{10}, \quad [e_{10}, e_{18}] = e_{10}, \\ [e_{11}, e_{14}] = -e_{10}, \quad [e_{11}, e_{15}] = -e_{11}, \quad [e_{11}, e_{18}] = e_{11}, \quad [e_4, e_{17}] = e_4, \\ [e_4, e_8] = e_2, \quad [e_6, e_{16}] = e_6, \quad [e_8, e_{13}] = -ae_{12}, \quad [e_{12}, e_{14}] = -e_{12}, \\ [e_{12}, e_{17}] = e_{12}, \quad [e_{13}, e_{14}] = -e_{13}, \quad [e_{12}, e_{15}] = -ae_{12}, \quad [e_{13}, e_{17}] = e_{13}. \end{gathered} \tag{4.13}$$

In this case, based on the Lie invariance condition, we have to consider two sub-cases based on the values of the parameters to see if taking certain values will generate new solutions to the system of PDE. The cases we consider are $A_{6,3}^{a=0}$ and $A_{6,3}^{a=1}$. In the generic and sub-cases, we find that the structure of the symmetry Lie algebras are the same. We summarize the results in the following proposition.

Proposition 11. *The symmetry Lie algebra is an eighteen-dimensional solvable Lie algebra with thirteen-dimensional nilradical spanned by e_1–e_{13} and five-dimensional abelian complement spanned by e_{14}–e_{18}. In fact, the nilradical is a direct sum of two copies of $A_{5,1}$ and $\mathbb{R}^3$. Hence, the symmetry algebra is $(A_{5,1} \oplus A_{5,1} \oplus \mathbb{R}^3) \rtimes \mathbb{R}^5$, where $A_{5,1}$ is given by Eq (4.9).*

4.4. Algebra $A_{6,4}^{ab}$ $(a \neq 0)$

The non-zero brackets for the algebra $A_{6,4}^{ab}$ are given by

$$\begin{gathered} [e_1, e_3] = e_3, \quad [e_1, e_4] = e_4, \quad [e_1, e_5] = e_6, \quad [e_2, e_3] = e_4, \\ [e_2, e_4] = -e_3, \quad [e_2, e_5] = ae_5 + be_6, \quad [e_2, e_6] = ae_6. \end{gathered} \tag{4.14}$$

The geodesic equations are given by

$$\ddot{p} = \dot{z}(a\dot{p} + b\dot{q}) + \dot{q}\dot{w}, \quad \ddot{q} = a\dot{q}\dot{z}, \quad \ddot{x} = \dot{x}\dot{w} - \dot{y}\dot{z}, \quad \ddot{y} = \dot{x}\dot{z} + \dot{y}\dot{w}, \quad \ddot{z} = 0, \quad \ddot{w} = 0. \tag{4.15}$$

For the general case $A_{6,4}^{a\neq0,b\neq0}$, the symmetry Lie algebra is spanned by

$$e_1 = D_x, \quad e_2 = D_y, \quad e_3 = D_p, \quad e_4 = D_q, \quad e_5 = D_t, \quad e_6 = qD_p,$$

$$e_7 = wD_t, \quad e_8 = zD_t, \quad e_9 = \frac{e^{az}D_p}{a}, \quad e_{10} = e^w \cos(z)D_x + e^w \sin(z)D_y,$$

$$e_{11} = e^w \sin(z)D_x - e^w \cos(z)D_y, \quad e_{12} = \frac{((bz+w)a-b)e^{az}D_p}{a} + e^{az}D_q, \quad e_{13} = D_w,$$

$$e_{14} = D_z, \quad e_{15} = tD_t, \quad e_{16} = pD_p + qD_q, \quad e_{17} = xD_x + yD_y, \quad e_{18} = yD_x - xD_y. \tag{4.16}$$

The non-zero brackets of the symmetry algebra are given by

$$[e_1, e_{17}] = e_1, \quad [e_1, e_{18}] = -e_2, \quad [e_2, e_{17}] = e_2, \quad [e_2, e_{18}] = e_1, \quad [e_3, e_{16}] = e_3,$$

$$[e_4, e_6] = e_3, \quad [e_4, e_{16}] = e_4, \quad [e_{15}, e_{15}] = e_5, \quad [e_{10}, e_{17}] = e_{10}, \quad [e_7, e_{13}] = -e_5,$$

$$[e_7, e_{15}] = e_7, \quad [e_8, e_{14}] = -e_5, \quad [e_8, e_{15}] = e_8, \quad [e_{10}, e_{14}] = e_{11}, \quad [e_9, e_{16}] = e_9,$$

$$[e_{11}, e_{13}] = -e_{11}, \quad [e_{11}, e_{14}] = -e_{10}, \quad [e_{11}, e_{17}] = e_{11}, \quad [e_{11}, e_{18}] = -e_{10},$$

$$[e_{10}, e_{18}] = e_{11}, \quad [e_{10}, e_{13}] = -e_{10}, \quad [e_6, e_{12}] = -ae_9, \quad [e_9, e_{14}] = -ae_9,$$

$$[e_{12}, e_{13}] = -ae_9, \quad [e_{12}, e_{14}] = -abe_9 - ae_{12}, \quad [e_{12}, e_{16}] = e_{12}. \tag{4.17}$$

In this case, based on the Lie invariance condition, we have to consider one sub-case based on the values of the parameters to see if taking certain values will generate new solutions to the system of PDE. The only case we consider is $A_{6,4}^{b=0}$. In the generic and sub-cases, we find that the structure of the symmetry Lie algebras are the same. We summarize the results in the following proposition.

Proposition 12. *The symmetry Lie algebra is an eighteen-dimensional solvable Lie algebra which is the semidirect product of twelve-dimensional decomposable nilradical spanned by e_1–e_{12} and a six-dimensional abelian complement spanned by e_{12}–e_{18}. In fact, the nilradical is a direct sum of $A_{5,1}$ in Winternitz list and $\mathbb{R}^7$. Hence, symmetry algebra can be can be described as $(A_{5,1} \oplus \mathbb{R}^7) \rtimes \mathbb{R}^6$, where $A_{5,1}$ is given by Eq (4.9).*

4.5. Algebra $A_{6,5}^{ab}$ $(ab \neq 0)$

The non-zero brackets for the algebra $A_{6,5}^{ab}$ are given by

$$[e_1, e_3] = ae_3, \quad [e_1, e_5] = e_5 + e_6, \quad [e_1, e_6] = e_6, \quad [e_2, e_3] = be_3, \quad [e_2, e_4] = e_4. \tag{4.18}$$

The geodesic equations are given by

$$\ddot{p} = \dot{w}(\dot{p} + \dot{q}), \quad \ddot{q} = \dot{q}\dot{w}, \quad \ddot{x} = \dot{x}(b\dot{z} + a\dot{w}), \quad \ddot{y} = \dot{y}\dot{z}, \quad \ddot{z} = 0, \quad \ddot{w} = 0. \tag{4.19}$$

For the general case $A_{6,5}^{a\neq0,b\neq0}$, the symmetry Lie algebra is spanned by

$$e_1 = D_t, \quad e_2 = D_p, \quad e_3 = D_x, \quad e_4 = D_y, \quad e_5 = D_q, \quad e_6 = wD_t, \quad e_7 = zD_t,$$

$$e_8 = qD_p, \quad e_9 = e^w D_p, \quad e_{10} = e^z D_y, \quad e_{11} = (w-1)e^w D_p + e^w D_q, \quad e_{12} = e^{aw}e^{bz}D_x,$$

$$e_{13} = D_z, \quad e_{14} = D_w, \quad e_{15} = tD_t, \quad e_{16} = xD_x, \quad e_{17} = yD_y, \quad e_{18} = pD_p + qD_q. \tag{4.20}$$

The non-zero brackets of the symmetry algebra are given by

$$[e_3, e_{16}] = e_3, \quad [e_4, e_{17}] = e_4, \quad [e_5, e_8] = e_2, \quad [e_{11}, e_{14}] = -e_{11} - e_9,$$

$$[e_5, e_{18}] = e_5, \quad [e_6, e_{14}] = -e_1, \quad [e_6, e_{15}] = e_6, \quad [e_7, e_{13}] = -e_1,$$
$$[e_8, e_{11}] = -e_9, \quad [e_9, e_{14}] = -e_9, \quad [e_9, e_{18}] = e_9, \quad [e_{10}, e_{13}] = -e_{10},$$
$$[e_{10}, e_{17}] = e_{10}, \quad [e_7, e_{15}] = e_7, \quad [e_{12}, e_{13}] = -be_{12}, \quad [e_{11}, e_{18}] = e_{11},$$
$$[e_1, e_{15}] = e_1, \quad [e_2, e_{18}] = e_2, \quad [e_{12}, e_{14}] = -ae_{12}, \quad [e_{12}, e_{16}] = e_{12}. \tag{4.21}$$

In this case, based on the Lie invariance condition, we have to consider two sub-cases based on the values of the parameters to see if taking certain values will generate new solutions to the system of PDE. The cases we consider are $A_{6,5}^{a=1,b\neq0}$ and $A_{6,5}^{a\neq0,b=1}$. In the generic and sub-cases, we find that the structure of the symmetry Lie algebras are the same. We summarize the results in the following proposition.

Proposition 13. *The symmetry Lie algebra is an eighteen-dimensional solvable Lie algebra which is the semidirect product of twelve-dimensional decomposable nilradical spanned by e_1–e_{12} and a six-dimensional abelian complement spanned by e_{12}–e_{18}. In fact, the nilradical is a direct sum of $A_{5,1}$ in Winternitz list and $\mathbb{R}^7$. Hence, symmetry algebra can be can be described as $(A_{5,1} \oplus \mathbb{R}^7) \rtimes \mathbb{R}^6$, where $A_{5,1}$ is given by Eq (4.9).*

4.6. Algebra $A_{6,6}^{ab}$ $(a^2 + b^2 \neq 0)$

The non-zero brackets for the algebra $A_{6,6}^{ab}$ are given by

$$[e_1, e_3] = ae_3, \quad [e_1, e_4] = ae_4, \quad [e_2, e_4] = e_4, \quad [e_1, e_6] = e_6,$$
$$[e_1, e_5] = e_5 + e_6, \quad [e_2, e_3] = e_3 + e_4, \quad [e_2, e_5] = be_6. \tag{4.22}$$

The geodesic equations are given by

$$\ddot{p} = \dot{w}(\dot{p} + \dot{q}) + b\dot{q}\dot{z}, \quad \ddot{q} = \dot{q}\dot{w}, \quad \ddot{x} = \dot{x}(\dot{z} + a\dot{w}) + \dot{y}\dot{z}, \quad \ddot{y} = \dot{y}(\dot{z} + a\dot{w}), \quad \ddot{z} = 0, \ \ddot{w} = 0. \tag{4.23}$$

For the general case $A_{6,6}^{a\neq0,b\neq0}$, the symmetry Lie algebra is spanned by

$$e_1 = D_t, \quad e_2 = D_x, \quad e_3 = D_p, \quad e_4 = D_y, \quad e_5 = D_q, \quad e_6 = wD_t,$$
$$e_7 = zD_t, \quad e_8 = yD_x, \quad e_9 = qD_p, \quad e_{10} = e^w D_p, \quad e_{11} = e^z e^{aw} D_x,$$
$$e_{12} = (bz + w - 1)e^w D_p + e^w D_q, \quad e_{13} = (z-1)e^{aw+z} D_x + e^z e^{aw} D_y,$$
$$e_{14} = D_z, \quad e_{15} = D_w, \quad e_{16} = tD_t, \quad e_{17} = xD_x + yD_y, \quad e_{18} = pD_p + qD_q. \tag{4.24}$$

The non-zero brackets of the symmetry algebra are given by

$$[e_1, e_{16}] = e_1, \quad [e_2, e_{17}] = e_2, \quad [e_3, e_{18}] = e_3, \quad [e_4, e_8] = e_2, \quad [e_4, e_{17}] = e_4,$$
$$[e_5, e_{18}] = e_5, \quad [e_6, e_{15}] = -e_1, \quad [e_6, e_{16}] = e_6, \quad [e_5, e_9] = e_3, \quad [e_7, e_{16}] = e_7,$$
$$[e_{19}, e_{12}] = -e_{10}, \quad [e_{10}, e_{15}] = -e_{10}, \quad [e_{10}, e_{18}] = e_{10}, \quad [e_{11}, e_{15}] = -ae_{11},$$
$$[e_{11}, e_{14}] = -e_{11}, \quad [e_{11}, e_{17}] = e_{11}, \quad [e_8, e_{13}] = -e_{11}, \quad [e_{12}, e_{18}] = e_{12},$$
$$[e_{12}, e_{15}] = -e_{10} - e_{12}, \quad [e_7, e_{14}] = -e_1, \quad [e_{12}, e_{14}] = -be_{10},$$
$$[e_{13}, e_{14}] = -e_{11} - e_{13}, \quad [e_{13}, e_{15}] = -ae_{13}, \quad [e_{13}, e_{17}] = e_{13}. \tag{4.25}$$

In this case, based on the Lie invariance condition, we have to consider three sub-cases based on the values of the parameters to see if taking certain values will generate new solutions to the system of PDE. The cases we consider are $A_{6,6}^{a=0,b\neq 0}$, $A_{6,6}^{a=1}$ and $A_{6,6}^{a\neq 0,b=0}$. In the generic and sub-cases, we find that the structure of the symmetry Lie algebras are the same. We summarize the results in the following proposition.

Proposition 14. *The symmetry Lie algebra is an eighteen-dimensional solvable Lie algebra with thirteen-dimensional nilradical spanned by e_1–e_{13} and five-dimensional abelian complement spanned by e_{14}–e_{18}. In fact, the nilradical is a direct sum of two copies of $A_{5,1}$ and $\mathbb{R}^3$. Hence, the symmetry algebra is $(A_{5,1} \oplus A_{5,1} \oplus \mathbb{R}^3) \rtimes \mathbb{R}^5$, where $A_{5,1}$ is given by Eq (4.9).*

4.7. Algebra $A_{6,7}^{abc}$ $(a^2 + b^2 \neq 0)$

The non-zero brackets for the algebra $A_{6,7}^{abc}$ are given by

$$[e_1, e_3] = ae_3, \quad [e_1, e_4] = ae_4, \quad [e_2, e_5] = be_6, \quad [e_1, e_6] = e_6,$$
$$[e_1, e_5] = e_5 + e_6, \quad [e_2, e_3] = ce_3 + e_4, \quad [e_2, e_4] = -e_3 + ce_4. \tag{4.26}$$

The geodesic equations are given by

$$\ddot{p} = \dot{p}(c\dot{z} + a\dot{w}) + \dot{q}\dot{z}, \;\; \ddot{q} = \dot{z}(-\dot{p} + c\dot{q}) + a\dot{q}\dot{w}, \;\; \ddot{x} = \dot{x}\dot{w}, \;\; \ddot{y} = \dot{x}(b\dot{z} + \dot{w}) + \dot{y}\dot{w}, \;\; \ddot{z} = 0, \; \ddot{w} = 0. \tag{4.27}$$

For the general case $A_{6,7}^{a\neq 0,b\neq 0,c\neq 0}$, the symmetry Lie algebra is spanned by

$$e_1 = D_q, \quad e_2 = D_p, \quad e_3 = D_y, \quad e_4 = D_x, \quad e_5 = D_t,$$
$$e_6 = xD_y, \quad e_7 = wD_t, \quad e_8 = zD_t, \quad e_9 = e^w D_y,$$
$$e_{10} = e^w D_x + (bz + w - 1)e^w D_y, \quad e_{11} = \sin(z)e^{aw+cz} D_p + e^{cz}\cos(z)e^{aw} D_q,$$
$$e_{12} = -\cos(z)e^{aw+cz} D_p + e^{cz}\sin(z)e^{aw} D_q, \quad e_{13} = D_z, \quad e_{14} = D_w, \quad e_{15} = tD_t,$$
$$e_{16} = xD_x + yD_y, \quad e_{17} = pD_p + qD_q, \quad e_{18} = -qD_p + pD_q. \tag{4.28}$$

The non-zero brackets of the symmetry algebra are given by

$$[e_1, e_{17}] = e_1, \quad [e_1, e_{18}] = -e_2, \quad [e_2, e_{17}] = e_2, \quad [e_9, e_{14}] = -e_9, \quad [e_3, e_{16}] = e_3,$$
$$[e_4, e_{16}] = e_4, \quad [e_5, e_{15}] = e_5, \quad [e_6, e_{10}] = -e_9, \quad [e_7, e_{14}] = -e_5, \quad [e_7, e_{15}] = e_7,$$
$$[e_8, e_{15}] = e_8, \quad [e_2, e_{18}] = e_1, \quad [e_9, e_{16}] = e_9, \quad [e_{11}, e_{17}] = e_{11}, \quad [e_{10}, e_{16}] = e_{10},$$
$$[e_{11}, e_{14}] = -ae_{11}, \quad [e_{10}, e_{13}] = -be_9, \quad [e_{12}, e_{18}] = -e_{11}, \quad [e_{11}, e_{18}] = e_{12},$$
$$[e_{11}, e_{13}] = -ce_{11} + e_{12}, \quad [e_{12}, e_{17}] = e_{12}, \quad [e_4, e_6] = e_3, \quad [e_8, e_{13}] = -e_5,$$
$$[e_{12}, e_{13}] = -ce_{12} - e_{11}, \quad [e_{12}, e_{14}] = -ae_{12}, \quad [e_{10}, e_{14}] = -e_{10} - e_9. \tag{4.29}$$

In this case, based on the Lie invariance condition, we have to consider four sub-cases based on the values of the parameters to see if taking certain values will generate new solutions to the system of PDE. The cases we consider are $A_{6,7}^{a=0,b\neq 0}$, $A_{6,7}^{a=1}$, $A_{6,7}^{a\neq 0,b=0}$ and $A_{6,7}^{c=0}$. In the generic and sub-cases, we find that the structure of the symmetry Lie algebras are the same. We summarize the results in the following proposition.

Proposition 15. *The symmetry Lie algebra is an eighteen-dimensional solvable Lie algebra which is the semidirect product of twelve-dimensional decomposable nilradical spanned by e_1–e_{12} and a six-dimensional abelian complement spanned by e_{12}–e_{18}. In fact, the nilradical is a direct sum of $A_{5,1}$ in Winternitz list and $\mathbb{R}^7$. Hence, symmetry algebra can be can be described as $(A_{5,1} \oplus \mathbb{R}^7) \rtimes \mathbb{R}^6$, where $A_{5,1}$ is given by Eq (4.9).*

4.8. Algebra $A_{6,8}$

The non-zero brackets for the algebra $A_{6,8}$ are given by

$$[e_1, e_3] = e_3, \quad [e_1, e_4] = e_6, \quad [e_2, e_5] = e_5 + e_6, \quad [e_2, e_6] = e_6, \quad [e_2, e_4] = e_4. \tag{4.30}$$

The geodesic equations are given by

$$\ddot{p} = \dot{z}(\dot{p} + \dot{y}) + \dot{q}\dot{w}, \quad \ddot{q} = \dot{q}\dot{z}, \quad \ddot{x} = \dot{x}\dot{w}, \quad \ddot{y} = \dot{y}\dot{z}, \quad \ddot{z} = 0, \quad \ddot{w} = 0. \tag{4.31}$$

The symmetry Lie algebra is spanned by

$$\begin{gathered} e_1 = D_t, \quad e_2 = D_p, \quad e_3 = D_x, \quad e_4 = D_q, \quad e_5 = D_y, \quad e_6 = wD_t, \\ e_7 = zD_t, \quad e_8 = qD_p, \quad e_9 = yD_p, \quad e_{10} = e^z D_p, \quad e_{11} = e^w D_x, \\ e_{12} = we^z D_p + e^z D_q, \quad e_{13} = (z-1)e^z D_p + e^z D_y, \quad e_{14} = D_z, \\ e_{15} = tD_t, \quad e_{16} = D_w, \quad e_{17} = xD_x, \quad e_{18} = pD_p + qD_q + yD_y. \end{gathered} \tag{4.32}$$

The non-zero brackets of the symmetry algebra are given by

$$\begin{gathered} [e_1, e_{15}] = e_1, \quad [e_2, e_{18}] = e_2, \quad [e_3, e_{17}] = e_3, \quad [e_4, e_8] = e_2, \quad [e_4, e_{18}] = e_4, \\ [e_5, e_9] = e_2, \quad [e_5, e_{18}] = e_5, \quad [e_6, e_{15}] = e_6, \quad [e_6, e_{16}] = -e_1, \quad [e_7, e_{15}] = e_7, \\ [e_7, e_{14}] = -e_1, \quad [e_8, e_{12}] = -e_{10}, \quad [e_9, e_{13}] = -e_{10}, \quad [e_{10}, e_{14}] = -e_{10}, \\ [e_{10}, e_{18}] = e_{10}, \quad [e_{11}, e_{16}] = -e_{11}, \quad [e_{11}, e_{17}] = e_{11}, \quad [e_{12}, e_{14}] = -e_{12}, \\ [e_{12}, e_{16}] = -e_{10}, \quad [e_{12}, e_{18}] = e_{12}, \quad [e_{13}, e_{14}] = -e_{10} - e_{13}, \quad [e_{13}, e_{18}] = e_{13}. \end{gathered} \tag{4.33}$$

Proposition 16. *The symmetry Lie algebra is an eighteen-dimensional solvable Lie algebra with thirteen-dimensional nilradical spanned by e_1–e_{13}, which is a direct sum of an eight-dimensional indecomposable solvable Lie algebra $B_{8(a=0)}$ and a five-dimensional abelian Lie algebra. The complement of the nilradical is a another five-dimensional abelian Lie algebra spanned by e_{14}–e_{18}. Therefore, the symmetry algebra can be identified as: $(B_{8(a=0)} \oplus \mathbb{R}^5) \rtimes \mathbb{R}^5$, where the non-zero brackets of B_{8a} are given by Eq (4.38).*

4.9. Algebra $A_{6,9}^{a}$

The non-zero brackets for the algebra $A_{6,9}^{a}$ are given by

$$[e_1, e_3] = e_3, \quad [e_1, e_4] = e_6, \quad [e_2, e_4] = e_4 + e_5, \quad [e_2, e_5] = e_5 + ae_6, \quad [e_2, e_6] = e_6. \tag{4.34}$$

The geodesic equations are given by

$$\ddot{p} = \dot{z}(\dot{p} + a\dot{y}) + \dot{q}\dot{w}, \;\; \ddot{q} = \dot{q}\dot{z}, \;\; \ddot{x} = \dot{x}\dot{w}, \;\; \ddot{y} = \dot{z}(\dot{q} + \dot{y}), \;\; \ddot{z} = 0, \;\; \ddot{w} = 0. \tag{4.35}$$

For the general case $A_{6,9}^{a\neq 0}$, the symmetry Lie algebra is spanned by

$$\begin{aligned}
&e_1 = D_t, \quad e_2 = D_y, \quad e_3 = D_p, \quad e_4 = D_x, \quad e_5 = D_q, \quad e_6 = wD_t,\\
&e_7 = zD_t, \quad e_8 = qD_p, \quad e_9 = e^z D_p, \quad e_{10} = e^w D_x, \quad e_{11} = ayD_p + qD_y,\\
&e_{12} = (z-1)ae^z D_p + e^z D_y, \quad e_{13} = [\frac{(z^2 - 2z + 2)a}{2} + w]e^z D_p + e^z D_q + (z-1)e^z D_y,\\
&e_{14} = D_w, \quad e_{15} = D_z, \quad e_{16} = tD_t, \quad e_{17} = xD_x, \quad e_{18} = pD_p + qD_q + yD_y.
\end{aligned} \tag{4.36}$$

The non-zero brackets of the symmetry algebra are given by

$$\begin{aligned}
&[e_1, e_{16}] = e_1, \quad [e_2, e_{18}] = e_2, \quad [e_3, e_{18}] = e_3, \quad [e_4, e_{17}] = e_4,\\
&[e_5, e_{11}] = e_2, \quad [e_5, e_{18}] = e_5, \quad [e_6, e_{14}] = -e_1, \quad [e_6, e_{16}] = e_6,\\
&[e_7, e_{15}] = -e_1, \quad [e_8, e_{13}] = -e_9, \quad [e_9, e_{15}] = -e_9, \quad [e_9, e_{18}] = e_9,\\
&[e_{10}, e_{14}] = -e_{10}, \quad [e_{11}, e_{12}] = -ae_9, \quad [e_2, e_{11}] = ae_3, \quad [e_{11}, e_{13}] = -e_{12},\\
&[e_{12}, e_{18}] = e_{12}, \quad [e_5, e_8] = e_3, \quad [e_{12}, e_{15}] = -ae_9 - e_{12}, \quad [e_7, e_{16}] = e_7,\\
&[e_{10}, e_{17}] = e_{10}, \quad [e_{13}, e_{14}] = -e_9, \quad [e_{13}, e_{15}] = -e_{12} - e_{13}, \quad [e_{13}, e_{18}] = e_{13}.
\end{aligned} \tag{4.37}$$

In this case, based on the Lie invariance condition, we have to consider one sub-case based on the values of the parameters to see if taking certain values will generate new solutions to the system of PDE. The only case we consider is $A_{6,9}^{a=0}$. In the generic and sub-cases, we find that the structure of the symmetry Lie algebras are the same. We summarize the results in the following proposition.

Proposition 17. *The symmetry Lie algebra is an eighteen-dimensional solvable Lie algebra with thirteen-dimensional nilradical spanned by* e_1–e_{13}, *which is a direct sum of an eight-dimensional indecomposable solvable Lie algebra* B_8 *and a five-dimensional abelian Lie algebra. The complement of the nilradical is a another five-dimensional abelian Lie algebra spanned by* e_{14}–e_{18}. *Therefore, the symmetry algebra can be identified as* $(B_8 \oplus \mathbb{R}^5) \rtimes \mathbb{R}^5$, *where* B_8 *is the following solvable indecomposable eight-dimensional Lie algebra given by the non-zero brackets*

$$[e_3, \; e_4] = e_2, \;\; [e_3, \; e_6] = e_1, \;\; [e_4, \; e_8] = -e_5, \;\; [e_6, \; e_7] = -ae_5, \;\; [e_6, \; e_8] = -e_7. \tag{4.38}$$

4.10. Algebra $A_{6,10}^{ab}$

The non-zero brackets for the algebra $A_{6,10}^{ab}$ are given by

$$[e_1, e_3] = ae_3, \quad [e_1, e_4] = e_4 + be_6, \quad [e_1, e_5] = e_5,$$
$$[e_1, e_6] = e_6, \quad [e_2, e_3] = e_3, \quad [e_2, e_4] = e_5, \quad [e_2, e_5] = e_6. \tag{4.39}$$

The geodesic equations are given by

$$\ddot{p} = \dot{w}(\dot{p} + b\dot{x}) + \dot{q}\dot{z}, \ \ \ddot{q} = \dot{q}\dot{w} + \dot{x}\dot{z}, \ \ \ddot{x} = \dot{x}\dot{w}, \ \ \ddot{y} = \dot{y}(\dot{z} + a\dot{w}), \ \ \ddot{z} = 0, \ \ddot{w} = 0. \tag{4.40}$$

For the general case $A_{6,10}^{a\neq 0, b\neq 0}$, the symmetry Lie algebra is spanned by

$$e_1 = D_t, \quad e_2 = D_q, \quad e_3 = D_p, \quad e_4 = D_y, \quad e_5 = D_x, \quad e_6 = wD_t,$$
$$e_7 = zD_t, \quad e_8 = xD_p, \quad e_9 = qD_p + xD_q, \quad e_{10} = e^w D_p, \quad e_{11} = ze^w D_p + e^w D_q,$$
$$e_{12} = e^{aw} e^z D_y, \quad e_{13} = [\frac{(2w-2)b}{2} + \frac{z^2}{2}]e^w D_p + ze^w D_q + e^w D_x,$$
$$e_{14} = D_z, \quad e_{15} = D_w, \quad e_{16} = tD_t, \quad e_{17} = yD_y, \quad e_{18} = pD_p + qD_q + xD_x. \tag{4.41}$$

The non-zero brackets of the symmetry algebra are given by

$$[e_1, e_{16}] = e_1, \quad [e_2, e_9] = e_3, \quad [e_2, e_{18}] = e_2, \quad [e_3, e_{18}] = e_3, \quad [e_4, e_{17}] = e_4,$$
$$[e_5, e_8] = e_3, \quad [e_5, e_9] = e_2, \quad [e_5, e_{18}] = e_5, \quad [e_6, e_{15}] = -e_1, \quad [e_6, e_{16}] = e_6,$$
$$[e_7, e_{14}] = -e_1, \quad [e_7, e_{16}] = e_7, \quad [e_8, e_{13}] = -e_{10}, \quad [e_{13}, e_{15}] = -be_{10} - e_{13},$$
$$[e_9, e_{13}] = -e_{11}, \quad [e_{10}, e_{15}] = -e_{10}, \quad [e_{10}, e_{18}] = e_{10}, \quad [e_{11}, e_{14}] = -e_{10},$$
$$[e_{11}, e_{18}] = e_{11}, \quad [e_{11}, e_{15}] = -e_{11}, \quad [e_{12}, e_{14}] = -e_{12}, \quad [e_{12}, e_{15}] = -ae_{12},$$
$$[e_{12}, e_{17}] = e_{12}, \quad [e_{13}, e_{14}] = -e_{11}, \quad [e_9, e_{11}] = -e_{10}, \quad [e_{13}, e_{18}] = e_{13}. \tag{4.42}$$

In this case, based on the Lie invariance condition, we have to consider three sub-cases based on the values of the parameters to see if taking certain values will generate new solutions to the system of PDE. The cases we consider are $A_{6,10}^{a=0}$, $A_{6,10}^{a=1}$ and $A_{6,10}^{b=0}$. In the generic and sub-cases, we find that the structure of the symmetry Lie algebras are the same. We summarize the results in the following proposition.

Proposition 18. *The symmetry Lie algebra is an eighteen-dimensional solvable Lie algebra with thirteen-dimensional nilradical spanned by e_1–e_{13}, which is a direct sum of an eight-dimensional indecomposable solvable Lie algebra $B_{8(a=1)}$ and a five-dimensional abelian Lie algebra. The complement of the nilradical is a another five-dimensional abelian Lie algebra spanned by e_{14}–e_{18}. Therefore, the symmetry algebra can be identified as $(B_{8(a=1)} \oplus \mathbb{R}^5) \rtimes \mathbb{R}^5$, where the non-zero brackets of B_{8a} are given by Eq (4.38).*

4.11. Algebra $A^a_{6,11}$

The non-zero brackets for the algebra $A^a_{6,11}$ are given by

$$[e_1, e_3] = e_4, \quad [e_2, e_4] = e_4, \quad [e_1, e_6] = e_6, \quad [e_2, e_3] = e_3,$$
$$[e_1, e_5] = e_5 + e_6, \quad [e_2, e_5] = ae_5, \quad [e_2, e_6] = ae_6. \tag{4.43}$$

The geodesic equations are given by

$$\ddot{p} = \dot{p}(a\dot{z} + \dot{w}) + \dot{q}\dot{w}, \;\; \ddot{q} = \dot{q}(a\dot{z} + \dot{w}), \;\; \ddot{x} = \dot{x}\dot{z}, \;\; \ddot{y} = \dot{x}\dot{w} + \dot{z}\dot{y}, \;\; \ddot{z} = 0, \; \ddot{w} = 0. \tag{4.44}$$

For the general case $A^{a\neq 0}_{6,11}$, the symmetry Lie algebra is spanned by

$$e_1 = D_t, \quad e_2 = D_p, \quad e_3 = D_y, \quad e_4 = D_q, \quad e_5 = D_x, \quad e_6 = wD_t,$$
$$e_7 = zD_t, \quad e_8 = qD_p, \quad e_9 = xD_y, \quad e_{10} = e^z D_y, \quad e_{11} = e^z D_x + we^z D_y,$$
$$e_{12} = e^w e^{az} D_p, \quad e_{13} = we^w e^{az} D_p + e^w e^{az} D_q, \quad e_{14} = D_w, \quad e_{15} = D_z,$$
$$e_{16} = tD_t, \quad e_{17} = pD_p + qD_q, \quad e_{18} = xD_x + yD_y. \tag{4.45}$$

The non-zero brackets of the symmetry algebra are given by

$$[e_1, e_{16}] = e_1, \quad [e_2, e_{17}] = e_2, \quad [e_3, e_{18}] = e_3, \quad [e_4, e_8] = e_2, \quad [e_4, e_{17}] = e_4,$$
$$[e_6, e_{14}] = -e_1, \quad [e_6, e_{16}] = e_6, \quad [e_{13}, e_{15}] = -ae_{13}, \quad [e_7, e_{15}] = -e_1,$$
$$[e_8, e_{13}] = -e_{12}, \quad [e_9, e_{11}] = -e_{10}, \quad [e_{10}, e_{15}] = -e_{10}, \quad [e_{10}, e_{18}] = e_{10},$$
$$[e_{11}, e_{15}] = -e_{11}, \quad [e_{11}, e_{18}] = e_{11}, \quad [e_{12}, e_{14}] = -e_{12}, \quad [e_5, e_9] = e_3,$$
$$[e_{12}, e_{15}] = -ae_{12}, \quad [e_{11}, e_{14}] = -e_{10}, \quad [e_{13}, e_{14}] = -e_{12} - e_{13},$$
$$[e_{12}, e_{17}] = e_{12}, \quad [e_5, e_{18}] = e_5, \quad [e_7, e_{16}] = e_7, \quad [e_{13}, e_{17}] = e_{13}. \tag{4.46}$$

In this case, based on the Lie invariance condition, we have to consider two sub-cases based on the values of the parameters to see if taking certain values will generate new solutions to the system of PDE. The cases we consider are $A^{a=0}_{6,11}$ and $A^{a=1}_{6,11}$. In the generic and sub-cases, we find that the structure of the symmetry Lie algebras are the same. We summarize the results in the following proposition.

Proposition 19. *The symmetry Lie algebra is an eighteen-dimensional solvable Lie algebra with thirteen-dimensional nilradical spanned by e_1–e_{13} and five-dimensional abelian complement spanned by e_{14}–e_{18}. In fact, the nilradical is a direct sum of two copies of $A_{5,1}$ and $\mathbb{R}^3$. Hence, the symmetry algebra is $(A_{5,1} \oplus A_{5,1} \oplus \mathbb{R}^3) \rtimes \mathbb{R}^5$, where $A_{5,1}$ is given by Eq (4.9).*

4.12. Algebra $A_{6,12}^{ab}$

The non-zero brackets for the algebra $A_{6,12}^{ab}$ are given by

$$[e_1,e_4]=e_4,\quad [e_1,e_5]=e_5+e_6,\quad [e_1,e_6]=e_6,\quad [e_2,e_3]=ae_4+e_5-be_6,$$
$$[e_1,e_3]=e_3+e_4,\quad [e_2,e_4]=e_6,\quad [e_2,e_5]=-e_3+be_4+ae_6,\quad [e_2,e_6]=-e_4. \tag{4.47}$$

The geodesic equations are given by

$$\ddot{p}=\dot{z}(\dot{p}+\dot{x})+\dot{w}(a\dot{x}+b\dot{y}-\dot{q}),\quad \ddot{q}=\dot{z}(\dot{q}+\dot{y})+\dot{w}(a\dot{y}-b\dot{x}+\dot{p}),$$
$$\ddot{x}=\dot{x}\dot{z}-\dot{y}\dot{w},\quad \ddot{y}=\dot{y}\dot{z}+\dot{x}\dot{w},\quad \ddot{z}=0,\quad \ddot{w}=0. \tag{4.48}$$

For the general case $A_{6,12}^{a\neq0,b\neq0}$, the symmetry Lie algebra is spanned by

$$e_1=D_y,\quad e_2=D_p,\quad e_3=D_q,\quad e_4=D_t,\quad e_5=D_x,$$
$$e_6=wD_t,\quad e_7=zD_t,\quad e_8=xD_p+yD_q,\quad e_9=yD_p-xD_q,$$
$$e_{10}=\cos(w)e^zD_p+\sin(w)e^zD_q,\quad e_{11}=\sin(w)e^zD_p-\cos(w)e^zD_q,$$
$$e_{12}=((aw+b+z-1)\ \cos(w)+bw\ \sin(w))\ e^zD_p+((aw+b+z-1)\sin(w)$$
$$-w\ \cos(w)b)\ e^zD_q+\cos(w)\ e^zD_x+\sin(w)\ e^zD_y,$$
$$e_{13}=((-bw+a)\ \cos(w)+\sin(w)(aw+z-1))\ e^zD_p+(-\cos(w)(aw+z-1)$$
$$-(bw-a)\sin(w))\ e^zD_q+\sin(w)\ e^zD_x-\cos(w)\ e^zD_y,$$
$$e_{14}=tD_t,\quad e_{15}=D_z,\quad e_{16}=D_w,$$
$$e_{17}=pD_p+qD_q+xD_x+yD_y,\quad e_{18}=qD_p-pD_q+yD_x-xD_y. \tag{4.49}$$

The non-zero brackets of the symmetry algebra are given by

$$[e_1,e_8]=e_3,\quad [e_1,e_9]=e_2,\quad [e_1,e_{17}]=e_1,\quad [e_{13},e_{15}]=-e_{11}-e_{13},$$
$$[e_{13},e_{16}]=2be_{10}-e_{12},\quad [e_3,e_{17}]=e_3,\quad [e_3,e_{18}]=e_2,\quad [e_4,e_{14}]=e_4,$$
$$[e_5,e_9]=-e_3,\quad [e_5,e_{17}]=e_5,\quad [e_5,e_{18}]=-e_1,\quad [e_6,e_{14}]=e_6,$$
$$[e_7,e_{14}]=e_7,\quad [e_7,e_{15}]=-e_4,\quad [e_8,e_{12}]=-e_{10},\quad [e_8,e_{13}]=-e_{11},$$
$$[e_9,e_{12}]=-e_{11},\quad [e_9,e_{13}]=e_{10},\quad [e_{10},e_{15}]=-e_{10},\quad [e_{10},e_{16}]=e_{11},$$
$$[e_{10},e_{17}]=e_{10},\quad [e_{10},e_{18}]=e_{11},\quad [e_{11},e_{15}]=-e_{11},\quad [e_{11},e_{16}]=-e_{10},$$
$$[e_{11},e_{17}]=e_{11},\quad [e_{11},e_{18}]=-e_{10},\quad [e_5,e_8]=e_2,\quad [e_{12},e_{17}]=e_{12},$$
$$[e_{13},e_{17}]=e_{13},\quad [e_{12},e_{18}]=-ae_{10}+be_{11}+e_{13},\quad [e_1,e_{18}]=e_5,$$
$$[e_{12},e_{16}]=-2ae_{10}+e_{13},\quad [e_2,e_{17}]=e_2,\quad [e_{12},e_{15}]=-e_{10}-e_{12},$$
$$[e_2,e_{18}]=-e_3,\quad [e_6,e_{16}]=-e_4,\quad [e_{13},e_{18}]=ae_{11}+be_{10}-e_{12}. \tag{4.50}$$

In this case, based on the Lie invariance condition, we have to consider two sub-cases based on the values of the parameters to see if taking certain values will generate new solutions to the system of PDE. The cases we consider are $A_{6,12}^{a=0}$ and $A_{6,12}^{b=0}$. In the generic and sub-cases, we find that the structure of the symmetry Lie algebras are the same. We summarize the results in the following proposition.

Proposition 20. *The symmetry Lie algebra is an eighteen-dimensional solvable Lie algebra with thirteen-dimensional nilradical spanned by e_1–e_{13} and five-dimensional abelian complement spanned by e_{14}–e_{18}. In fact, the nilradical is a direct sum of ten-dimensional solvable Lie algebra C_{10} and $\mathbb{R}^3$. Hence, the symmetry algebra is $(C_{10} \oplus \mathbb{R}^3) \rtimes \mathbb{R}^5$, where C_{10} is an indecoposable ten-dimensional solvable Lie algebra given by the non-zero brackets*

$$\begin{aligned} &[e_1, e_5] = e_3, \quad [e_1, e_6] = e_2, \quad [e_4, e_5] = e_2, \quad [e_4, e_6] = -e_3, \\ &[e_5, e_9] = -e_7, \quad [e_5, e_{10}] = -e_8, \quad [e_6, e_9] = -e_8, \quad [e_6, e_{10}] = e_7. \end{aligned} \tag{4.51}$$

4.13. Algebra $A^{abcd}_{6,13}$ ($abcd : a^2 + b^2 \neq 0$, $c^2 + d^2 \neq 0$)

The non-zero brackets for the algebra $A^{abcd}_{6,13}$ are given by

$$\begin{aligned} &[e_1, e_3] = ae_3, \quad [e_1, e_4] = ce_4, \quad [e_1, e_5] = e_6, \quad [e_1, e_6] = -e_5, \\ &[e_2, e_3] = be_3, \quad [e_2, e_4] = de_4, \quad [e_2, e_6] = e_6, \quad [e_2, e_5] = e_5. \end{aligned} \tag{4.52}$$

The geodesic equations are given by

$$\ddot{p} = \dot{p}\dot{z} - \dot{q}\dot{w}, \;\; \ddot{q} = \dot{p}\dot{w} + \dot{q}\dot{z}, \;\; \ddot{x} = \dot{x}(d\dot{z} + c\dot{w}), \;\; \ddot{y} = \dot{y}(b\dot{z} + a\dot{w}), \;\; \ddot{z} = 0, \; \ddot{w} = 0. \tag{4.53}$$

For the general case $A^{a\neq0,b\neq0,c\neq0,d\neq0}_{6,13}$, the symmetry Lie algebra is spanned by

$$\begin{aligned} &e_1 = D_t, \quad e_2 = D_p, \quad e_3 = D_q, \quad e_4 = D_x, \quad e_5 = D_y, \quad e_6 = wD_t, \quad e_7 = zD_t, \\ &e_8 = e^{cw}e^{dz}D_x, \quad e_9 = e^{aw}e^{bz}D_y, \quad e_{10} = \cos(w)e^zD_p + \sin(w)e^zD_q, \\ &e_{11} = \sin(w)e^zD_p - \cos(w)e^zD_q, \quad e_{12} = D_w, \quad e_{13} = D_z, \quad e_{14} = tD_t, \\ &e_{15} = xD_x, \quad e_{16} = yD_y, \quad e_{17} = pD_p + qD_q \quad e_{18} = qD_p - pD_q. \end{aligned} \tag{4.54}$$

The non-zero brackets of the symmetry algebra are given by

$$\begin{aligned} &[e_1, e_{14}] = e_1, \quad [e_2, e_{17}] = e_2, \quad [e_2, e_{18}] = -e_3, \quad [e_3, e_{17}] = e_3, \quad [e_3, e_{18}] = e_2, \\ &[e_4, e_{15}] = e_4, \quad [e_6, e_{12}] = -e_1, \quad [e_9, e_{12}] = -ae_9, \quad [e_6, e_{14}] = e_6, \\ &[e_7, e_{14}] = e_7, \quad [e_8, e_{12}] = -ce_8, \quad [e_8, e_{13}] = -de_8, \quad [e_8, e_{15}] = e_8, \\ &[e_9, e_{16}] = e_9, \quad [e_{10}, e_{12}] = e_{11}, \quad [e_{10}, e_{13}] = -e_{10}, \quad [e_{10}, e_{17}] = e_{10}, \\ &[e_{11}, e_{13}] = -e_{11}, \quad [e_9, e_{13}] = -be_9, \quad [e_{10}, e_{18}] = e_{11}, \quad [e_7, e_{13}] = -e_1, \\ &[e_5, e_{16}] = e_5, \quad [e_{11}, e_{12}] = -e_{10}, \quad [e_{11}, e_{17}] = e_{11}, \quad [e_{11}, e_{18}] = -e_{10}. \end{aligned} \tag{4.55}$$

In this case, based on the Lie invariance condition, we have to consider eight sub-cases based on the values of the parameters to see if taking certain values will generate new solutions to the system of PDE. The cases we consider are: $A^{a=0,b\neq0}_{6,13}$, $A^{a=c,a\neq0}_{6,13}$, $A^{b=0,a\neq0}_{6,13}$, $A^{b=1}_{6,13}$, $A^{b=d,b\neq0}_{6,13}$, $A^{c=0,d\neq0}_{6,13}$, $A^{d=1}_{6,13}$ and $A^{d=0,c\neq0}_{6,13}$. In the generic and sub-cases, we find that the structure of the symmetry Lie algebras are the same. We summarize the results in the following proposition.

Proposition 21. *The symmetry Lie algebra is an eighteen-dimensional solvable Lie algebra which is the semidirect product of eleven-dimensional abelian nilradical spanned by e_1–e_{11} and a seven-dimensional abelian complement spanned by e_{12}–e_{18}. Hence, it can be described as $\mathbb{R}^{11} \rtimes \mathbb{R}^7$.*

4.14. Algebra $A_{6,14}^{abc}$ $(ab \neq 0)$

The non-zero brackets for the algebra $A_{6,14}^{abc}$ are given by

$$[e_1, e_3] = ae_3, \quad [e_1, e_5] = ce_5 + e_6, \quad [e_1, e_6] = e_5 + ce_6, \quad [e_2, e_3] = be_3, \quad [e_2, e_4] = e_4. \tag{4.56}$$

The geodesic equations are given by

$$\ddot{p} = \dot{w}(c\dot{p} + \dot{q}), \quad \ddot{q} = \dot{w}(-\dot{p} + c\dot{q}), \quad \ddot{x} = \dot{x}\dot{z}, \quad \ddot{y} = \dot{y}(b\dot{z} + a\dot{w}), \quad \ddot{z} = 0, \quad \ddot{w} = 0. \tag{4.57}$$

For the general case $A_{6,14}^{a\neq0,b\neq0,c\neq0}$, the symmetry Lie algebra is spanned by

$$\begin{gathered} e_1 = D_t, \quad e_2 = D_q, \quad e_3 = D_p, \quad e_4 = D_y, \quad e_5 = D_x, \quad e_6 = wD_t, \\ e_7 = zD_t, \quad e_8 = e^z D_x, \quad e_9 = e^{aw}e^{bz}D_y, \quad e_{10} = e^{cw}\sin(w)D_p + e^{cw}\cos(w)D_q, \\ e_{11} = -e^{cw}\cos(w)D_p + e^{cw}\sin(w)D_q, \quad e_{12} = D_z, \quad e_{13} = D_w, \quad e_{14} = tD_t, \\ e_{15} = yD_y, \quad e_{16} = xD_x, \quad e_{17} = pD_p + qD_q, \quad e_{18} = -qD_p + pD_q. \end{gathered} \tag{4.58}$$

The non-zero brackets of the symmetry algebra are given by

$$\begin{gathered} [e_1, e_{14}] = e_1, \quad [e_2, e_{17}] = e_2, \quad [e_7, e_{14}] = e_7, \quad [e_3, e_{17}] = e_3, \quad [e_3, e_{18}] = e_2, \\ [e_4, e_{15}] = e_4, \quad [e_5, e_{16}] = e_5, \quad [e_6, e_{13}] = -e_1, \quad [e_6, e_{14}] = e_6, \quad [e_8, e_{16}] = e_8, \\ [e_{11}, e_{13}] = -ce_{11} - e_{10}, \quad [e_{11}, e_{17}] = e_{11}, \quad [e_{11}, e_{18}] = -e_{10}, \quad [e_7, e_{12}] = -e_1, \\ [e_9, e_{15}] = e_9, \quad [e_{10}, e_{13}] = -ce_{10} + e_{11}, \quad [e_{10}, e_{17}] = e_{10}, \quad [e_{10}, e_{18}] = e_{11}, \\ [e_2, e_{18}] = -e_3, \quad [e_8, e_{12}] = -e_8, \quad [e_9, e_{13}] = -ae_9, \quad [e_9, e_{12}] = -be_9. \end{gathered} \tag{4.59}$$

In this case, based on the Lie invariance condition, we have to consider two sub-cases based on the values of the parameters to see if taking certain values will generate new solutions to the system of PDE. The cases we consider are: $A_{6,14}^{b=1,a\neq0}$ and $A_{6,14}^{a=c,a\neq0}$. In the generic and sub-cases, we find that the structure of the symmetry Lie algebras are the same. We summarize the results in the following proposition.

Proposition 22. *The symmetry Lie algebra is an eighteen-dimensional solvable Lie algebra which is the semidirect product of eleven-dimensional abelian nilradical spanned by e_1–e_{11} and a seven-dimensional abelian complement spanned by e_{12}–e_{18}. Hence, it can be described as $\mathbb{R}^{11} \rtimes \mathbb{R}^7$.*

4.15. Algebra $A_{6,15}^{abcd}$ $(abcd : b \neq 0)$

The non-zero brackets for the algebra $A_{6,15}^{abcd}$ are given by

$$\begin{gathered} [e_1, e_3] = e_3, \quad [e_1, e_4] = e_4, \quad [e_1, e_5] = ae_5 + be_6, \quad [e_1, e_6] = -be_5 + ae_6, \\ [e_2, e_3] = ce_3 + e_4, \quad [e_2, e_4] = -e_3 + ce_4, \quad [e_2, e_6] = de_6, \quad [e_2, e_5] = de_5. \end{gathered} \tag{4.60}$$

The geodesic equations are given by

$$\ddot{p} = \dot{p}(c\dot{z} + \dot{w}) - \dot{q}\dot{z}, \quad \ddot{q} = \dot{z}(\dot{p} + c\dot{q}) + \dot{q}\dot{w}, \quad \ddot{z} = 0,$$

$$\ddot{x} = \dot{x}(d\dot{z} + a\dot{w}) - b\dot{y}\dot{w}, \quad \ddot{y} = b\dot{x}\dot{w} + \dot{y}(d\dot{z} + q\dot{w}), \quad \ddot{w} = 0. \tag{4.61}$$

For the general case $A_{6,15}^{a\neq0,b\neq0,c\neq0,d\neq0}$, the symmetry Lie algebra is spanned by

$$e_1 = D_p, \quad e_2 = D_t, \quad e_3 = D_w, \quad e_4 = D_z, \quad e_5 = D_x, \quad e_6 = D_y,$$
$$e_7 = wD_t, \quad e_8 = zD_t, \quad e_9 = tD_t, \quad e_{10} = xD_x + yD_y. \tag{4.62}$$

The non-zero brackets of the symmetry algebra are given by

$$[e_2, e_9] = e_2, \quad [e_3, e_7] = e_2, \quad [e_4, e_8] = e_2, \quad [e_5, e_{10}] = e_5,$$
$$[e_6, e_{10}] = e_6, \quad [e_7, e_9] = e_7, \quad [e_8, e_9] = e_8. \tag{4.63}$$

In this case, based on the Lie invariance condition, we have to consider five sub-cases based on the values of the parameters to see if taking certain values will generate new solutions to the system of PDE. The cases we consider are $A_{6,15}^{a=0}$, $A_{6,15}^{a=1}$, $A_{6,15}^{c=0}$, $A_{6,15}^{d=0}$ and $A_{6,15}^{c=d}$. In the generic and sub-cases, we find that the structure of the symmetry Lie algebras are the same. We summarize the results in the following proposition.

Proposition 23. *The symmetry Lie algebra is a ten-dimensional solvable Lie algebra with eight -dimensional not abelian nilradical spanned by e_1–e_8 and two-dimensional abelian complement spanned by e_9 and e_{10}. In fact, The nilradical is a direct sum of $A_{5,4}$ and $\mathbb{R}^3$. Hence, the symmetry Lie algebra can be identified as $(A_{5,4} \oplus \mathbb{R}^3) \rtimes \mathbb{R}^2$, where the non-zero brackets of $A_{5,4}$ are given by*

$$[e_2, e_4] = e_1, \ [e_3, e_5] = e_1. \tag{4.64}$$

4.16. Algebra $A_{6,16}^{ab}$

The non-zero brackets for the algebra $A_{6,16}^{ab}$ are given by

$$[e_1, e_3] = e_4, \quad [e_2, e_4] = e_4, \quad [e_1, e_5] = ae_5 + e_6, \quad [e_2, e_3] = e_3,$$
$$[e_1, e_6] = -e_5 + ae_6, \quad [e_2, e_5] = be_5, \quad [e_2, e_6] = be_6. \tag{4.65}$$

The geodesic equations are given by

$$\ddot{p} = \dot{p}(b\dot{z} + a\dot{w}) - \dot{q}\dot{w}, \quad \ddot{q} = \dot{p}\dot{w} + \dot{q}(b\dot{z} + a\dot{w}), \quad \ddot{x} = \dot{x}\dot{z}, \quad \ddot{y} = \dot{y}\dot{z}, \quad \ddot{z} = 0, \quad \ddot{w} = 0. \tag{4.66}$$

For the general case $A_{6,16}^{a\neq0,b\neq0}$, the symmetry Lie algebra is spanned by

$$e_1 = D_t, \quad e_2 = D_p, \quad e_3 = D_q, \quad e_4 = D_x, \quad e_5 = D_y, \quad e_6 = wD_t,$$
$$e_7 = zD_t, \quad e_8 = e^z D_x, \quad e_9 = e^z D_y, \quad e_{10} = e^{bz}\cos(w)e^{aw}D_p + \sin(w)e^{aw+bz}D_q,$$
$$e_{11} = e^{bz}\sin(w)e^{aw}D_p - \cos(w)e^{aw+bz}D_q, \quad e_{12} = D_z, \quad e_{13} = tD_t, \quad e_{14} = D_w,$$
$$e_{15} = xD_x, \quad e_{16} = yD_x, \quad e_{17} = xD_y, \quad e_{18} = yD_y, \quad e_{19} = pD_p + qD_q, \quad e_{20} = qD_p - pD_q. \tag{4.67}$$

The non-zero brackets of the symmetry algebra are given by

$$[e_1, e_{13}] = e_1, \quad [e_2, e_{19}] = e_2, \quad [e_3, e_{19}] = e_3, \quad [e_{10}, e_{14}] = -ae_{10} + e_{11},$$

$$[e_4, e_{15}] = e_4, \quad [e_4, e_{17}] = e_5, \quad [e_5, e_{16}] = e_4, \quad [e_{11}, e_{14}] = -ae_{11} - e_{10},$$
$$[e_6, e_{14}] = -e_1, \quad [e_7, e_{13}] = e_7, \quad [e_8, e_{15}] = e_8, \quad [e_{16}, e_{17}] = -e_{15} + e_{18},$$
$$[e_{15}, e_{17}] = e_{17}, \quad [e_{16}, e_{18}] = -e_{16}, \quad [e_7, e_{12}] = -e_1, \quad [e_{17}, e_{18}] = e_{17},$$
$$[e_2, e_{20}] = -e_3, \quad [e_{10}, e_{19}] = e_{10}, \quad [e_{10}, e_{20}] = e_{11}, \quad [e_{11}, e_{12}] = -be_{11},$$
$$[e_{11}, e_{19}] = e_{11}, \quad [e_5, e_{18}] = e_5, \quad [e_{11}, e_{20}] = -e_{10}, \quad [e_{10}, e_{12}] = -be_{10},$$
$$[e_3, e_{20}] = e_2, \quad [e_6, e_{13}] = e_6, \quad [e_8, e_{12}] = -e_8, \quad [e_{15}, e_{16}] = -e_{16},$$
$$[e_8, e_{17}] = e_9, \quad [e_9, e_{12}] = -e_9, \quad [e_9, e_{16}] = e_8, \quad [e_9, e_{18}] = e_9. \tag{4.68}$$

In this case, based on the Lie invariance condition, we have to consider three sub-cases based on the values of the parameters to see if taking certain values will generate new solutions to the system of PDE. The cases we consider are $A^{a=0}_{6,16}$, $A^{b=0}_{6,16}$ and $A^{b=1}_{6,16}$. In the generic and sub-cases, we find that the structure of the symmetry Lie algebras are the same. We summarize the results in the following proposition.

Proposition 24. *The symmetry Lie algebra is a twenty-dimensional semi direct product of seventeen solvable Lie algebra and three -dimensional semi-simple $sl(2,\mathbb{R})$. Furthermore, the symmetry Lie algebra has eleven dimensional abelian nilradical and nine-dimensional complement. Therefore, the symmetry Lie algebra can be identified as $(\mathbb{R}^{11} \rtimes \mathbb{R}^6) \rtimes sl(2,\mathbb{R})$.*

4.17. Algebra $A^a_{6,17}$

The non-zero brackets for the algebra $A^a_{6,17}$ are given by

$$[e_1, e_3] = ae_3 + e_4, \quad [e_1, e_4] = ae_4, \quad [e_1, e_5] = e_6,$$
$$[e_1, e_6] = -e_5, \quad [e_2, e_5] = e_5, \quad [e_2, e_6] = e_6. \tag{4.69}$$

The geodesic equations are given by

$$\ddot{p} = \dot{p}\dot{z} - \dot{q}\dot{w}, \quad \ddot{q} = \dot{p}\dot{w} + \dot{q}\dot{z}, \quad \ddot{x} = \dot{w}(a\dot{x} + \dot{y}), \quad \ddot{y} = a\dot{y}\dot{w}, \quad \ddot{z} = 0, \quad \ddot{w} = 0. \tag{4.70}$$

Based on the system of PDE's obtained from the Lie Invarince Condition we consider the following subcases of certain values of the parameters.

4.17.1. Case 1: $A^{a\neq 0}_{6,17}$

The symmetry Lie algebra is spanned by

$$e_1 = D_x, \quad e_2 = D_p, \quad e_3 = D_q, \quad e_4 = D_y, \quad e_5 = D_t, \quad e_6 = yD_x,$$
$$e_7 = wD_t, \quad e_8 = zD_t, \quad e_9 = \frac{e^{aw}D_x}{a}, \quad e_{10} = \cos(w)e^zD_p + \sin(w)e^zD_q,$$
$$e_{11} = \sin(w)e^zD_p - \cos(w)e^zD_q, \quad e_{12} = \frac{(aw-1)e^{aw}D_x}{a} + e^{aw}D_y, \quad e_{13} = D_z,$$
$$e_{14} = D_w, \quad e_{15} = tD_t, \quad e_{16} = xD_x + yD_y, \quad e_{17} = pD_p + qD_q, \quad e_{18} = qD_p - pD_q. \tag{4.71}$$

The non-zero brackets of the symmetry algebra are given by

$$\begin{gathered}[e_1,e_{16}]=e_1,\quad [e_2,e_{17}]=e_2,\quad [e_3,e_{18}]=-e_3,\quad [e_3,e_{17}]=e_3,\quad [e_3,e_{18}]=e_2,\\ [e_4,e_{16}]=e_4,\quad [e_5,e_{15}]=e_5,\quad [e_6,e_{12}]=-ae_9,\quad [e_7,e_{14}]=-e_5,\quad [e_7,e_{15}]=e_7,\\ [e_8,e_{13}]=-e_5,\quad [e_8,e_{15}]=e_8,\quad [e_9,e_{14}]=-ae_9,\quad [e_{10},e_{13}]=-e_{10},\\ [e_{10},e_{14}]=e_{11},\quad [e_{10},e_{17}]=e_{10},\quad [e_{10},e_{18}]=e_{11},\quad [e_{11},e_{13}]=-e_{11},\\ [e_{11},e_{17}]=e_{11},\quad [e_4,e_6]=e_1,\quad [e_9,e_{16}]=e_9,\quad [e_{11},e_{14}]=-e_{10},\\ [e_{11},e_{18}]=-e_{10},\quad [e_{12},e_{14}]=-ae_{12}-ae_9,\quad [e_{12},e_{16}]=e_{12}. \end{gathered} \tag{4.72}$$

Proposition 25. *The symmetry Lie algebra is an eighteen-dimensional solvable Lie algebra which is the semidirect product of twelve-dimensional decomposable nilradical spanned by e_1–e_{12} and a six-dimensional abelian complement spanned by e_{12}–e_{18}. In fact, the nilradical is a direct sum of $A_{5,1}$ in Winternitz list and $\mathbb{R}^7$. Hence, symmetry algebra can be can be described as $(A_{5,1}\oplus\mathbb{R}^7)\rtimes\mathbb{R}^6$, where $A_{5,1}$ is given by Eq (4.9).*

4.17.2. Case 2: $A_{6,17}^{a=0}$

The symmetry Lie algebra is spanned by

$$\begin{gathered} e_1=D_t,\quad e_2=D_x,\quad e_3=D_p,\quad e_4=D_q,\quad e_5=D_y,\quad e_6=wD_t,\quad e_7=zD_t,\\ e_8=zD_x,\quad e_9=wD_x,\quad e_{10}=\frac{1}{2}\,w^2D_x+wD_y,\quad e_{11}=\frac{1}{2}\,wzD_x+zD_y,\\ e_{12}=\cos(w)e^zD_p+\sin(w)e^zD_q,\quad e_{13}=\sin(w)e^zD_p-\cos(w)e^zD_q,\\ e_{14}=tD_x,\quad e_{15}=D_z,\quad e_{16}=D_w,\quad e_{17}=tD_t,\quad e_{18}=yD_t,\quad e_{19}=yD_x,\\ e_{20}=pD_p+qD_q,\quad e_{21}=qD_p-pD_q,\quad e_{22}=\frac{1}{2}\,twD_x+tD_y,\quad e_{23}=(wy-2x)D_t,\\ e_{24}=\frac{1}{2}\,wyD_x+yD_y,\quad e_{25}=(x-\frac{wy}{2})D_x,\quad e_{26}=(\frac{1}{2}\,yw^2-wx)D_x+(wy-2x)D_y. \end{gathered} \tag{4.73}$$

The non-zero brackets of the symmetry algebra are given by

$$\begin{gathered}[e_1,e_{14}]=e_2,\quad [e_1,e_{17}]=e_1,\quad [e_1,e_{22}]=e_5+\frac{e_9}{2},\quad [e_2,e_{23}]=-2e_1,\\ [e_2,e_{25}]=e_2,\quad [e_2,e_{26}]=-2e_5-e_9,\quad [e_3,e_{20}]=e_3,\quad [e_3,e_{21}]=-e_4,\\ [e_4,e_{20}]=e_4,\quad [e_4,e_{21}]=e_3,\quad [e_5,e_{18}]=e_1,\quad [e_5,e_{19}]=e_2,\quad [e_5,e_{23}]=e_6,\\ [e_5,e_{24}]=e_5+\frac{e_9}{2},\quad [e_5,e_{25}]=\frac{e_9}{2},\quad [e_5,e_{26}]=e_{10},\quad [e_{10},e_{16}]=-e_5-e_9,\\ [e_6,e_{16}]=-e_1,\quad [e_6,e_{17}]=e_6,\quad [e_6,e_{22}]=e_{10},\quad [e_7,e_{14}]=e_8,\\ [e_7,e_{15}]=-e_1,\quad [e_7,e_{17}]=e_7,\quad [e_7,e_{22}]=e_{11},\quad [e_8,e_{15}]=-e_2,\\ [e_8,e_{23}]=-2e_7,\quad [e_8,e_{25}]=e_8,\quad [e_8,e_{26}]=-2e_{11},\quad [e_9,e_{16}]=-e_2,\\ [e_9,e_{23}]=-2e_6,\quad [e_9,e_{25}]=e_9,\quad [e_9,e_{26}]=-2e_{10},\quad [e_6,e_{14}]=e_9, \end{gathered}$$

$$[e_{10}, e_{18}] = e_6, \quad [e_{10}, e_{19}] = e_9, \quad [e_{10}, e_{24}] = e_{10}, \quad [e_{11}, e_{15}] = -e_5 - \frac{e_9}{2},$$
$$[e_{11}, e_{16}] = -\frac{e_8}{2}, \quad [e_{11}, e_{18}] = e_7, \quad [e_{11}, e_{19}] = e_8, \quad [e_{11}, e_{24}] = e_{11},$$
$$[e_{12}, e_{15}] = -e_{12}, \quad [e_{12}, e_{16}] = e_{13}, \quad [e_{12}, e_{20}] = e_{12}, \quad [e_{12}, e_{21}] = e_{13},$$
$$[e_{13}, e_{15}] = -e_{13}, \quad [e_{13}, e_{16}] = -e_{12}, \quad [e_{13}, e_{20}] = e_{13}, \quad [e_{13}, e_{21}] = -e_{12},$$
$$[e_{14}, e_{25}] = e_{14}, \quad [e_{14}, e_{26}] = -2e_{22}, \quad [e_{16}, e_{22}] = \frac{e_{14}}{2}, \quad [e_{17}, e_{18}] = -e_{18},$$
$$[e_{16}, e_{26}] = e_{24} - e_{25}, \quad [e_{14}, e_{18}] = -e_{19}, \quad [e_{14}, e_{23}] = -2e_{17} + 2e_{25},$$
$$[e_{16}, e_{23}] = e_{18}, \quad [e_{16}, e_{24}] = \frac{e_{19}}{2}, \quad [e_{19}, e_{25}] = e_{19}, \quad [e_{17}, e_{23}] = -e_{23},$$
$$[e_{16}, e_{25}] = -\frac{e_{19}}{2}, \quad [e_{14}, e_{17}] = -e_{14}, \quad [e_{17}, e_{22}] = e_{22}, \quad [e_{19}, e_{22}] = -e_{14},$$
$$[e_{19}, e_{23}] = -2e_{18}, \quad [e_{18}, e_{24}] = -e_{18}, \quad [e_{18}, e_{26}] = -e_{23}, \quad [e_{22}, e_{23}] = -e_{26},$$
$$[e_{18}, e_{22}] = -e_{17} + e_{24}, \quad [e_{19}, e_{24}] = -e_{19}, \quad [e_{19}, e_{26}] = -2e_{24} + 2e_{25},$$
$$[e_{22}, e_{24}] = e_{22}, \quad [e_{23}, e_{25}] = -e_{23}, \quad [e_{24}, e_{26}] = -e_{26}, \quad [e_{25}, e_{26}] = e_{26}. \tag{4.74}$$

Proposition 26. *The symmetry Lie algebra is a twenty six-dimensional semi direct product of eighteen solvable Lie algebra and eight-dimensional semi-simple $sl(3, \mathbb{R})$. Furthermore, the symmetry Lie algebra has thirteen-dimensional complement. Therefore, the symmetry algebra can be identified as $(\mathbb{R}^{13} \rtimes \mathbb{R}^5) \rtimes sl(3, \mathbb{R})$.*

4.18. Algebra $A_{6,18}^{abc}$ $(b \neq 0)$

The non-zero brackets for the algebra $A_{6,18}^{abc}$ are given by

$$[e_1, e_3] = e_4, \quad [e_1, e_4] = -e_3, \quad [e_1, e_5] = ae_5 + be_6, \quad [e_2, e_3] = e_3,$$
$$[e_1, e_6] = -be_5 + ae_6, \quad [e_2, e_4] = e_4, \quad [e_2, e_5] = ce_5, \quad [e_2, e_6] = ce_6. \tag{4.75}$$

The geodesic equations are given by

$$\ddot{p} = \dot{p}\dot{z} + \dot{q}\dot{w}, \;\; \ddot{q} = -\dot{p}\dot{w} + \dot{q}\dot{z}, \;\; \ddot{x} = \dot{x}(c\dot{z} - a\dot{w}) + b\dot{y}\dot{w}, \;\; \ddot{y} = -b\dot{x}\dot{w} + \dot{y}(c\dot{z} - a\dot{w}), \;\; \ddot{z} = 0, \;\; \ddot{w} = 0. \tag{4.76}$$

For the general case $A_{6,18}^{a\neq 0, b\neq 0, c\neq 0}$, the symmetry Lie algebra is spanned by

$$e_1 = D_y, \quad e_2 = D_q, \quad e_3 = D_p, \quad e_4 = D_t, \quad e_5 = D_x, \quad e_6 = wD_t, \quad e_7 = zD_t,$$
$$e_8 = \sin(w)e^z D_p + \cos(w)e^z D_q, \quad e_9 = -\cos(w)e^z D_p + \sin(w)e^z D_q,$$
$$e_{10} = e^{cz}\sin(bw)e^{-aw}D_x + \cos(bw)e^{-aw+cz}D_y, \quad e_{11} = e^{cz}\cos(bw)e^{-aw}D_x - \sin(bw)e^{-aw+cz}D_y,$$
$$e_{12} = D_z, \quad e_{13} = tD_t, \quad e_{14} = D_w, \quad e_{15} = pD_p + qD_q,$$
$$e_{16} = xD_x + yD_y, \quad e_{17} = -qD_p + pD_q, \quad e_{18} = yD_x - xD_y. \tag{4.77}$$

The non-zero brackets of the symmetry algebra are given by

$$[e_1, e_{16}] = e_1, \quad [e_1, e_{18}] = e_5, \quad [e_2, e_{15}] = e_2, \quad [e_2, e_{17}] = -e_3, \quad [e_{10}, e_{18}] = e_{11},$$

$$[e_3, e_{17}] = e_2, \quad [e_4, e_{13}] = e_4, \quad [e_5, e_{16}] = e_5, \quad [e_5, e_{18}] = -e_1,$$
$$[e_6, e_{14}] = -e_4, \quad [e_7, e_{12}] = -e_4, \quad [e_7, e_{13}] = e_7, \quad [e_8, e_{12}] = -e_8,$$
$$[e_9, e_{12}] = -e_9, \quad [e_9, e_{14}] = -e_8, \quad [e_9, e_{17}] = -e_8, \quad [e_{10}, e_{12}] = -ce_{10},$$
$$[e_9, e_{15}] = e_9, \quad [e_8, e_{14}] = e_9, \quad [e_{11}, e_{18}] = -e_{10}, \quad [e_{11}, e_{12}] = -ce_{11},$$
$$[e_8, e_{15}] = e_8, \quad [e_8, e_{17}] = e_9, \quad [e_{10}, e_{14}] = ae_{10} - be_{11}, \quad [e_{10}, e_{16}] = e_{10},$$
$$[e_3, e_{15}] = e_3, \quad [e_{11}, e_{14}] = ae_{11} + be_{10}, \quad [e_{10}, e_{16}] = e_{11}, \quad [e_6, e_{13}] = e_6. \tag{4.78}$$

In this case, based on the Lie invariance condition, we have to consider three sub-cases based on the values of the parameters to see if taking certain values will generate new solutions to the system of PDE. The cases we consider are $A_{6,18}^{a=0}$, $A_{6,18}^{c=0}$ and $A_{6,18}^{c=1}$. In the generic and sub-cases, we find that the structure of the symmetry Lie algebras are the same. We summarize the results in the following proposition.

Proposition 27. *The symmetry Lie algebra is an eighteen-dimensional solvable Lie algebra which is the semidirect product of eleven-dimensional abelian nilradical spanned by e_1–e_{11} and a seven-dimensional abelian complement spanned by e_{12}–e_{18}. Hence, it can be described as $\mathbb{R}^{11} \rtimes \mathbb{R}^7$.*

4.19. Algebra $A_{6,19}$

The non-zero brackets for the algebra $A_{6,19}$ are given by

$$[e_1, e_3] = e_4 + e_5, \quad [e_1, e_5] = e_6, \quad [e_1, e_6] = -e_5, \quad [e_2, e_3] = e_3,$$
$$[e_1, e_4] = -e_3 + e_6, \quad [e_2, e_4] = e_4, \quad [e_2, e_5] = e_5, \quad [e_2, e_6] = e_6. \tag{4.79}$$

The geodesic equations are given by

$$\ddot{p} = \dot{p}\dot{z} + \dot{w}(\dot{x} - \dot{q}), \quad \ddot{q} = \dot{w}(\dot{y} - \dot{p}) + \dot{q}\dot{z}, \quad \ddot{x} = \dot{x}\dot{z} - \dot{y}\dot{w}, \quad \ddot{y} = \dot{y}\dot{z} + \dot{x}\dot{w}, \quad \ddot{z} = 0, \quad \ddot{w} = 0. \tag{4.80}$$

The symmetry Lie algebra is spanned by

$$e_1 = D_y, \quad e_2 = D_p, \quad e_3 = D_q, \quad e_4 = D_t, \quad e_5 = D_x, \quad e_6 = wD_t,$$
$$e_7 = zD_t, \quad e_8 = e^z e^w D_p - e^{z+w} D_q, \quad e_9 = e^z e^{-w} D_p + e^{z-w} D_q,$$
$$e_{10} = \sin(w) e^z D_p + \cos(w) e^z D_x + \sin(w) e^z D_y,$$
$$e_{11} = -\cos(w) e^z D_p + \sin(w) e^z D_x - \cos(w) e^z D_y, \quad e_{12} = D_z,$$
$$e_{13} = tD_t, \quad e_{14} = D_w, \quad e_{15} = pD_p + qD_q + xD_x + yD_y,$$
$$e_{16} = qD_p + (-y + p)D_q, \quad e_{17} = -xD_p + yD_x - xD_y, \quad e_{18} = (y - p)D_p - qD_q. \tag{4.81}$$

The non-zero brackets of the symmetry algebra are given by

$$[e_1, e_{15}] = e_1, \quad [e_1, e_{16}] = -e_3, \quad [e_1, e_{17}] = e_5, \quad [e_1, e_{18}] = e_2, \quad [e_2, e_{15}] = e_2,$$
$$[e_2, e_{16}] = e_3, \quad [e_2, e_{18}] = -e_2, \quad [e_3, e_{15}] = e_3, \quad [e_3, e_{16}] = e_2, \quad [e_3, e_{18}] = -e_3,$$
$$[e_4, e_{13}] = e_4, \quad [e_5, e_{15}] = e_5, \quad [e_5, e_{17}] = -e_1 - e_2, \quad [e_6, e_{13}] = e_6, \quad [e_6, e_{14}] = -e_4,$$

$$[e_7, e_{12}] = -e_4, \quad [e_7, e_{13}] = e_7, \quad [e_8, e_{12}] = -e_8, \quad [e_8, e_{14}] = e_8, \quad [e_8, e_{15}] = e_8,$$

$$[e_8, e_{16}] = e_8, \quad [e_8, e_{18}] = -e_8, \quad [e_9, e_{12}] = -e_9, \quad [e_9, e_{14}] = -e_9, \quad [e_9, e_{15}] = e_9,$$

$$[e_9, e_{16}] = -e_9, \quad [e_9, e_{18}] = -e_9, \quad [e_{10}, e_{12}] = -e_{10}, \quad [e_{10}, e_{14}] = e_{11}, \quad [e_{10}, e_{15}] = e_{10},$$

$$[e_{10}, e_{17}] = e_{11}, \quad [e_{11}, e_{12}] = -e_{11}, \quad [e_{11}, e_{14}] = -e_{10}, \quad [e_{11}, e_{15}] = e_{11}, \quad [e_{11}, e_{17}] = -e_{10}. \quad (4.82)$$

Proposition 28. *The symmetry Lie algebra is an eighteen-dimensional solvable Lie algebra which is the semidirect product of eleven-dimensional abelian nilradical spanned by e_1–e_{11} and a seven-dimensional abelian complement spanned by e_{12}–e_{18}. Hence, it can be described as $\mathbb{R}^{11} \rtimes \mathbb{R}^7$.*

5. Conclusions and future work

n this work, we have investigated the symmetry Lie algebra of the geodesic equations of the canonical connection on a Lie group. More precisely, we have considered six-dimensional indecomposable solvable Lie algebras with co-dimension two abelian nilradical and abelian complement. In dimension six, there are nineteen such algebras, namely, $A_{6,1}$–$A_{6,19}$ in [17]. In each case, we list the non-zero brackets of the Lie algebra, the geodesic equations and a basis for the symmetry Lie algebra in terms of vector fields. We also analyze the nilradical of the symmetry Lie algebra. In every case, we identify the symmetry Lie algebra, and a summary of our results is given in Table 1.

Table 1. Six-dimensional Lie algebras and identification of the symmetry algebra.

Six-dimensional Lie algebras	Dimension	Identification
$A_{6,1}^{a\neq0,b\neq0,c\neq0,d\neq0}$	18	$\mathbb{R}^{11} \rtimes \mathbb{R}^7$
$A_{6,2}^{a\neq0,b\neq0,c\neq0}$	18	$(A_{5,1} \oplus \mathbb{R}^7) \rtimes \mathbb{R}^6$
$A_{6,3}^{a\neq0}$	18	$(A_{5,1} \oplus A_{5,1} \oplus \mathbb{R}^3) \rtimes \mathbb{R}^5$
$A_{6,4}^{a\neq0,b\neq0}$	18	$(A_{5,1} \oplus \mathbb{R}^7) \rtimes \mathbb{R}^6$
$A_{6,5}^{a\neq0,b\neq0}$	18	$(A_{5,1} \oplus \mathbb{R}^7) \rtimes \mathbb{R}^6$
$A_{6,6}^{a\neq0,b\neq0}$	18	$(A_{5,1} \oplus A_{5,1} \oplus \mathbb{R}^3) \rtimes \mathbb{R}^5$
$A_{6,7}^{a\neq0,b\neq0,c\neq0}$	18	$(A_{5,1} \oplus \mathbb{R}^7) \rtimes \mathbb{R}^6$
$A_{6,8}$	18	$(B_{8(a=0)} \oplus \mathbb{R}^5) \rtimes \mathbb{R}^5$
$A_{6,9}^{a\neq0}$	18	$(B_8 \oplus \mathbb{R}^5) \rtimes \mathbb{R}^5$
$A_{6,10}^{a\neq0,b\neq0}$	18	$(B_{8(a=1)} \oplus \mathbb{R}^5) \rtimes \mathbb{R}^5$
$A_{6,11}^{a\neq0}$	18	$(A_{5,1} \oplus A_{5,1} \oplus \mathbb{R}^3) \rtimes \mathbb{R}^5$
$A_{6,12}^{a\neq0,b\neq0}$	18	$(C_{10} \oplus \mathbb{R}^3) \rtimes \mathbb{R}^5$
$A_{6,13}^{a\neq0,b\neq0,c\neq0,d\neq0}$	18	$\mathbb{R}^{11} \rtimes \mathbb{R}^7$
$A_{6,14}^{a\neq0,b\neq0,c\neq0}$	18	$\mathbb{R}^{11} \rtimes \mathbb{R}^7$
$A_{6,15}^{a\neq0,b\neq0,c\neq0,d\neq0}$	10	$(A_{5,4} \oplus \mathbb{R}^3) \rtimes \mathbb{R}^2$
$A_{6,16}^{a\neq0,b\neq0}$	20	$(\mathbb{R}^{11} \rtimes \mathbb{R}^6) \rtimes sl(2,\mathbb{R})$
$A_{6,17}^{a\neq0}$	18	$(A_{5,1} \oplus \mathbb{R}^7) \rtimes \mathbb{R}^6$
$A_{6,17}^{a=0}$	26	$(\mathbb{R}^{13} \rtimes \mathbb{R}^5) \rtimes sl(3,\mathbb{R})$
$A_{6,18}^{a\neq0,b\neq0,c\neq0}$	18	$\mathbb{R}^{11} \rtimes \mathbb{R}^7$
$A_{6,19}$	18	$\mathbb{R}^{11} \rtimes \mathbb{R}^7$

In future work, we intend to consider the problem in dimension n. For an n dimensional solvable Lie algebra with co-dimension two abelian nilrdical and abelian complement, we plan to construct the system of geodesic equations in general, the Lie invariance conditions, and try to integrate, to the extent possible, the system of partial differential equations and obtain general results.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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