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*Research article*

# Paired contractive mappings and fixed point results

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**Abstract:** In this study, we explored a novel type of contraction, known as paired contraction (PC), to establish fixed points in metric spaces. It has been demonstrated that mappings possessing the PC property are continuous. We have also provided proofs for the existence of fixed points for such mappings with the classical Banach fixed point theorem emerging as a corollary. Furthermore, we presented examples of mappings that do not satisfy the standard contraction condition, but do exhibit the PC property.

**Keywords:** metric space; Banach contraction; paired contraction (PC); fixed point
**Mathematics Subject Classification:** 47H09, 47H10

## 1. Introduction

The contraction mapping principle, initially formulated by S. Banach in his 1920 dissertation and subsequently published in 1922 (as referenced in [1]), marked a pivotal development in mathematics. While the concept of employing successive approximations had been previously explored by notable figures like P. L. Chebyshev, E. Picard, R. Caccioppoli and others in various specific contexts, it was S. Banach who first articulated this fundamental concept in a precise abstract form that could be applied across a diverse range of applications. After a century, the enthusiasm of mathematicians worldwide for fixed-point theorems remains substantial. This is affirmed by the emergence of numerous articles dedicated to the fixed point theory and its applications in recent decades (see [2]).

Over the years, the Banach contraction principle has undergone various generalizations. In the work of W. Kirk [3], it is pointed out that, in addition to Banach's original fixed point theorem, there exists three classical fixed point theorems that serve as benchmarks for examining metric extensions. These include Nadler's widely recognized set-valued extension of Banach's theorem [4], the extension of Banach's theorem to nonexpansive mappings [5] and Caristi's theorem [6]. Additionally, it is possible to categorize at least two types of generalizations of these theorems. In the first scenario, the strict contractive nature of the mapping is relaxed, as evidenced by works of Boyd and Wong on nonlinear contractions in [7], Suzuki on asymptotic contractions in [8] and Meir Keeler in [9], and more details can be gleaned from [10–16]. In the second scenario, the topological requirements are eased, as exemplified in publications like [17–25].

Motivated by the results mentioned above in this paper, we introduce a novel form of mapping contraction, referred to as paired contraction (PC), with the objective of establishing new fixed points results within the metric spaces. It has been demonstrated that mappings satisfying the PC property are continuous. Although the proof for the main theorem of this work relies on the concepts found in the proof of Banach's classical theorem, the fundamental difference lies in the fact that PCs are defined based on three points in space instead of two (Banach contraction). In addition, we need a condition to ensure that the mapping does not possess periodic points with a prime period of two. The ordinary contraction mappings constitute a significant subclass among these mappings, and this readily enables us to derive the classical Banach theorem as a simple corollary. An illustrative example has been created that satisfies PC conditions but does not meet the criteria for being a contraction mapping, providing support for our result.

## 2. Paired contraction and fixed point results

The key findings of our manuscript are presented in this section. We start with the definition of a well-known metric space.

**Definition 2.1.** *A metric space is a nonempty set $X$ equipped with a distance function $d : X \times X \to [0, \infty)$ that holds:*

*(i)* $d(x, y) \geq 0$ *and* $d(x, y) = 0$ *if, and only if,* $x = y$*;*
*(ii)* $d(x, y) = d(y, x)$*;*
*(iii)* $d(x, z) \leq d(x, y) + d(y, z)$*,*

*for all $x, y, z \in X$. The pair $(X, d)$ is referred to as a metric space.*

**Definition 2.2.** *In the context of a metric space $(X, d)$ with a cardinality of at least three, denoted as $|X| \geq 3$, we will say that mapping $T : X \to X$ will have PC on $X$ if there exists a real number $\alpha \in [0, 1)$ such that the following inequality holds:*

$$d(Tx, Ty) + d(Ty, Tz) \leq \alpha(d(x, y) + d(y, z)) \tag{2.1}$$

*for all $x, y, z \in X$.*

**Remark 2.1.** *It's important to note that the condition necessitating that $x, y, z \in X$ be pairwise distinct is essential. Without this condition, the definition would be equivalent to that of a contraction mapping.*

**Proposition 2.1.** *Mappings possessing the paired contraction property are continuous.*

*Proof.* Consider a metric space $(X, d)$ with a cardinality of at least three, and let $T : X \to X$ be a mapping that has PC on $X$. If $x_0$ is an isolated point in $X$, then it's evident that $T$ is continuous at $x_0$.

Now, let's consider the case where $x_0$ is an accumulation point. We want to demonstrate that for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $d(Tx_0, Tx) < \varepsilon$ whenever $d(x_0, x) < \delta$. Since $x_0$ is an accumulation point, for any $\delta > 0$, there exists a point $y \in X$ such that $d(x_0, y) < \delta$. By (2.1), we have:

$$\begin{aligned} d(Tx_0, Tx) &\leq d(Tx_0, Tx) + d(Tx, Ty) \\ &\leq \alpha(d(x_0, x) + d(x, y)) \\ &\leq \alpha(2d(x_0, x) + d(x_0, y)) \ \ (\textit{using triangle inequality}). \\ &\leq 3\alpha\delta. \end{aligned}$$

Setting $\delta = \varepsilon/(3\alpha)$, we obtain the desired inequality. □

**Remark 2.2.** *In the context of a metric space X, consider a mapping T. A point $x \in X$ is termed a periodic point of period n if $T^n(x) = x$. The smallest positive integer n for which $T^n(x) = x$ is referred to as the prime period of x, as defined in [23].*

**Theorem 2.1.** *Let $(X, d)$ be a complete metric space with cardinality of at least three. Consider a mapping $T : X \to X$ with the PC property on X, then it can be concluded that:*

*(i) T has a fixed point if, and only if, T does not have periodic points of prime period two.*
*(ii) The number of fixed points is at most two.*

*Proof.* Let $T$ be a mapping with no periodic points of prime period two. We aim to demonstrate that there exists a fixed point for $T$. Suppose we have an initial point $x_0 \in X$ such that $Tx_0 = x_1$, $Tx_1 = x_2$ and so on, forming a sequence $x_0, x_1, x_2, \ldots$.

Assuming that none of the points $x_i$ are fixed points of the mapping $T$ for every $i = 0, 1, ...,$ we can prove that all $x_i$ are distinct. Since $x_i$ is not a fixed point, we have $x_i \neq Tx_i = x_{i+1}$. Moreover, due to the absence of periodic points of prime period two, we can conclude that $x_{i+2} = T(T(x_i)) \neq x_i$. Additionally, given our supposition that $x_{i+1}$ is not a fixed point, we can also deduce that $x_{i+1} \neq x_{i+2} = Tx_{i+1}$. Consequently, $x_i$, $x_{i+1}$, and $x_{i+2}$ are all distinct from each other. Now,

$$\begin{aligned} d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) =& d(Tx_{n-1}, Tx_n) + d(Tx_n, Tx_{n+1}) \\ \leq& \alpha(d(x_{n-1}, x_n) + d(x_n, x_{n+1})) \\ \leq& \alpha^2(d(x_{n-2}, x_{n-1}) + d(x_{n-1}, x_n)) \\ & \ldots\ldots\ldots\ldots \\ \leq& \alpha^n(d(x_0, x_1) + d(x_1, x_2)). \end{aligned}$$

If we consider $K_0 = d(x_0, x_1) + d(x_1, x_2)$, $K_1 = d(x_1, x_2) + d(x_2, x_3)$,......, $K_n = d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})$, then we have

$$K_n \leq \alpha K_{n-1} \leq \alpha^2 K_{n-1} \leq \ldots\ldots \leq \alpha^n K_0. \tag{2.2}$$

Assume that there exists a minimal natural number $j \geq 3$ for which $x_j = x_i$ with $i$ satisfying $0 \leq i \leq j - 2$. In this case, it follows that $x_{j+1} = x_{i+1}$ and $x_{j+2} = x_{i+2}$. Hence,

$$K_i = d(x_i, x_{i+1}) + d(x_{i+1}, x_{i+2}) = d(x_j, x_{j+1}) + d(x_{j+1}, x_{j+2}) = K_j.$$

This leads to a contradiction with Eq (2.2). Therefore, there cannot exist such values of $i$ and $j$.

Now, let's demonstrate that the sequence $\{x_n\}$ is a Cauchy sequence. Based on the previous arguments, it is evident that:

$$d(x_n, x_{n+1}) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) = K_n \leq \alpha^n K_0. \tag{2.3}$$

By applying the triangle inequality and considering Eq (2.3), we can establish the following:

$$\begin{aligned} d(x_n, x_{n+l}) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + .... + d(x_{n+l-1}, x_{n+l}) \\ &\leq \alpha^n K_0 + \alpha^{n+1} K_0 + .... + \alpha^{n+l-1} K_0 \\ &= \alpha^n (1 + \alpha + .... + \alpha^{l-1}) K_0 \\ &= \alpha^n \left( \frac{1 - \alpha^l}{1 - \alpha} \right) K_0. \end{aligned}$$

Under the assumption that $\alpha \in [0, 1)$, we can observe that $d(x_n, x_{n+l}) < \alpha^n \frac{1}{1-\alpha} K_0$. Consequently, as we let $n \to \infty$, we find that $d(x_n, x_{n+l})$ tends to zero for any positive value of $l$. This demonstrates that $\{x_n\}$ is a Cauchy sequence.

Utilizing the completeness of the metric space $(X, d)$, we can conclude that the sequence has a limit, denoted as $x^* \in X$. To prove that $x^*$ is a fixed point of $T$, let's employ the triangle inequality and Eq (2.1). This yields:

$$\begin{aligned} d(x^*, Tx^*) &\leq d(x^*, x_{n+1}) + d(x_{n+1}, Tx^*) \\ &= d(x^*, x_{n+1}) + d(Tx_n, Tx^*) \\ &\leq d(x^*, x_n) + d(Tx_{n-1}, Tx_n) + d(Tx_n, Tx^*) \\ &\leq d(x^*, x_n) + \alpha(d(x_{n-1}, x_n) + d(x_n, x^*)). \end{aligned}$$

Letting $n \to \infty$, we obtain $d(x^*, Tx^*) = 0$, that is, $Tx^* = x^*$.

Assume that there exists three pairwise distinct fixed points, denoted as $x$, $y$ and $z$. This implies that $T(x) = x$, $T(y) = y$ and $T(z) = z$. However, this contradicts Eq (2.1).

On the other hand, if we assume that $T$ possesses a fixed point $x^*$ and let $x$ be a periodic point of prime period two, where $Ty = T(Tx) = x$, we can derive the following:

$$d(Tx, Tx^*) + d(Tx^*, Ty) = d(y, x^*) + d(x^*, x),$$

which is a contradiction. □

Here is an example of a mapping $T$ that has PC property and exactly two fixed points:

**Example 2.1.** *Let $X = \{1, 2, 3\}$ and define the distance function $d$ as follows: $d(1, 1) = d(2, 2) = d(3, 3) = 0$, $d(1, 2) = d(2, 1) = \frac{1}{4}$, $d(2, 3) = d(2, 3) = \frac{1}{3}$ and $d(1, 3) = d(3, 1) = \frac{1}{2}$. The mapping $T : X \to X$ is defined by $T(1) = 1$, $T(2) = 2$ and $T(3) = 1$.*

*It can be easily verified that Eq (2.1) holds for this example and, furthermore, $T$ does not possess periodic points of prime period two.*

**Remark 2.3.** *It is not a contraction mapping as $d(T1, T2) = d(1, 2)$.*

Here is an example of a mapping $T$ that has PC property and does not have any fixed points:

**Example 2.2.** *Let* $X = \{1, 2, 3\}$ *and define the distance function d as follows:* $d(1, 1) = d(2, 2) = d(3, 3) = 0$*,* $d(1, 2) = d(2, 1) = 1$*,* $d(2, 3) = d(2, 3) = 1$ *and* $d(1, 3) = d(3, 1) = 1$*. The mapping* $T : X \to X$ *is defined by* $T(1) = 2$*,* $T(2) = 1$*, and* $T(3) = 1$*.*

*In this example, both elements* 1 *and* 2 *are periodic points of prime period two, and there are no fixed points in* $T$*.*

**Theorem 2.2.** *In the context of Theorem 2.1, if the fixed point* $x^*$ *is a limit of the Picard iteration sequence* $x_n = Tx_{n-1}$ *with an initial point* $x_1 = Tx_0$ *for any* $x_0 \in X$*, then* $x^*$ *is the unique fixed point.*

*Proof.* Indeed, assuming the existence of another fixed point $x^{**}$ for $T$ where $x^{**} \neq x^*$, we can deduce from the previous explanation in the proof of Theorem 2.1 that $x_n \neq x^{**}$ for all $n \in \mathbb{N}$. Consequently, it becomes evident that $x^*$, $x^{**}$ and $x_n$ are pairwise distinct for all natural numbers $n$. By employing the triangle inequality and taking into account Eq (2.1), we can derive the following:

$$\begin{aligned} d(x^*, x^{**}) &\leq d(x^*, Tx_n) + d(Tx_n, x^{**}) \\ &= d(Tx^*, Tx_n) + d(Tx_n, Tx^{**}) \\ &\leq \alpha(d(x^*, x_n) + d(x_n, x^{**})). \end{aligned}$$

As we let $n$ approach infinity and consider $\alpha \in [0, 1)$, we arrive at $d(x^*, x^{**}) = 0$, which implies $x^* = x^{**}$. This demonstrates that $x^*$ is indeed the unique fixed point of $T$ in $X$. □

**Definition 2.3.** *In a metric space* $(X, d)$*, a mapping* $T : X \to X$ *is considered a contraction mapping on X if there exists a constant* $\alpha \in [0, 1)$*, such that the inequality*

$$d(Tx, Ty) \leq \alpha d(x, y) \tag{2.4}$$

*holds for all x and y in X.*

**Corollary 2.1.** *(Banach fixed-point theorem) In a nonempty, complete metric space* $(X, d)$ *with a contraction mapping* $T : X \to X$*, there exists only one fixed point for* $T$*.*

*Proof.* For cases where the cardinality of $X$ is $|X| = 1$ or $|X| = 2$, the proof is straightforward. However, when $|X| \geq 3$, the following argument applies:

Suppose there exists an element $x \in X$ such that $T(Tx) = x$. Consequently, we have $d(x, Tx) = d(Tx, x) = d(Tx, T(Tx))$, which contradicts the contraction property (2.4). Thus, $T$ cannot possess periodic points of prime period two.

Now, let $x, y, z \in X$ be pairwise distinct. By applying (2.4), we deduce that $d(T(x), T(y)) \leq \alpha d(x, y)$ and $d(T(y), T(z)) \leq \alpha d(y, z)$. We have

$$d(T(x), T(y) + d(T(y), T(z) \leq \alpha(d(x, y) + d(y, z)).$$

This implies that $T$ is a mapping with the PC property on $X$.

By virtue of Theorem 2.1, we can conclude that the mapping $T$ must have a fixed point. The uniqueness of the fixed point can be demonstrated using a standard method. □

**Note:** In the context of a given metric space $X$, an accumulation point of $X$ is defined as a point $x \in X$ for which every open ball centered at $x$ contains an infinite number of points from $X$.

**Proposition 2.2.** *In a metric space* $(X, d)$ *where* $|X| \geq 3$, *if* $T : X \to X$ *is a mapping with the PC property on* $X$ *and* $x$ *is an accumulation point of* $X$, *then inequality* (2.4) *holds for all points* $y \in X$.

*Proof.* If $x \in X$ is an accumulation point and $y \in X$, it's evident that (2.4) holds when $y = x$. Now, consider the case when $y \neq x$. As $x$ is an accumulation point, there exists a sequence $z_n \to x$ such that $z_n \neq x$, $z_n \neq y$ and all $z_n$ are distinct. Consequently, by (2.1), we have the inequality:

$$d(Tx, Tz_n) + d(Tz_n, Ty) \leq \alpha(d(x, z_n) + d(z_n, y)),$$

which satisfies for any $n \in \mathbb{N}$. Since $d(x, z_n) \to 0$ and considering the continuity of every metric, we can deduce that $d(y, z_n) \to d(x, y)$. Also, since $T$ is continuous, it follows that $d(Tx, Tz_n) \to 0$ and, consequently, $d(Tz_n, Ty) \to d(Tx, Ty)$. Then, by allowing $n \to \infty$, we obtain:

$$d(Tx, Ty) \leq \alpha d(x, y),$$

which is equivalent to (2.4). □

**Corollary 2.2.** *In a metric space* $(X, d)$ *with* $|X| \geq 3$, *where* $T : X \to X$ *is a mapping with PC property, if all points in* $X$ *are accumulation points, then it follows that* $T$ *is a contraction mapping.*

**Example 2.3.** *Let* $X = \{x_0, x_1, x_2, .......\} \subset \mathbb{R}$, *where*

$$x_i = \begin{cases} \frac{4}{2^m}, & \text{when } i \text{ is even and } i = 2m; \\ \frac{3}{2^m}, & \text{when } i \text{ is odd and } i = 2m+1 \end{cases}, \ m = 0, 1, 2.....$$

*and let* $d$ *be the Euclidean metric on* $X$, *then* $(X, d)$ *is a complete metric space.*

*Define a mapping* $T : X \to X$ *as* $Tx = x_{i+1}$, *for all* $x = x_i \in \{x_0, x_1, x_2, ......\}$. *Now, we will show that* $T$ *is not a contraction mapping, but it is a PC mapping satisfying condition (2.1).*

*It can be easily verified that for any three distinct points from the space* $X$, *one of them lies between two others (see Figure 1). Now,*

$$d(Tx_{2m}, Tx_{2m+1}) = d(x_{2m+1}, x_{2m+2}) = d(\frac{3}{2^m}, \frac{4}{2^{m+1}}) = |\frac{3}{2^m} - \frac{4}{2^{m+1}}| = \frac{1}{2^m} \tag{2.5}$$

*and*

$$d(x_{2m}, x_{2m+1}) = d(\frac{4}{2^m}, \frac{3}{2^m}) = |\frac{4}{2^m} - \frac{3}{2^m}| = \frac{1}{2^m}. \tag{2.6}$$

*Hence, it is clear that* $d(Tm_{2m}, Tx_{2m+1}) = d(x_{2m}, x_{2m+1})$ *for all* $m = 0, 1, 2, .....$, *and using (2.4), we see that* $T$ *is not a contraction mapping.*

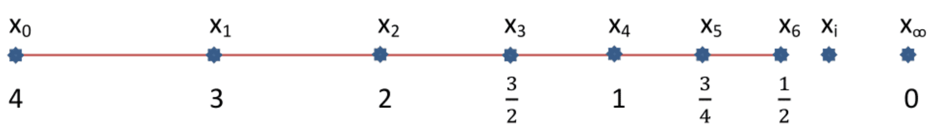


**Figure 1.** Configuration of the points of the metric space $(X, d)$.

*Let us take three distinct points* $x_i, x_j, x_k \in X$. *The following terminologies we will used thoroughly in the proof, which can be easily verified from the configuration of the metric space (Figure 1).*

*(a) For all* $i = 0, 1, 2, 3, .....$

$$d(x_i, x_{i+1}) = \begin{cases} d(\frac{4}{2^m}, \frac{3}{2^m}), & i = 2m; \\ d(\frac{3}{2^m}, \frac{4}{2^{m+1}}), & i = 2m+1, \end{cases}$$
$$= \begin{cases} |\frac{4}{2^m} - \frac{3}{2^m}|, & i = 2m; \\ |\frac{3}{2^m} - \frac{4}{2^{m+1}}|, & i = 2m+1, \end{cases}$$
$$= \frac{1}{2^m} = \frac{1}{2^{[\frac{i}{2}]}}, \tag{2.7}$$

*where [-] is greatest integer function.*

*(b) For all* $0 \leq i < j$, *where* $i, j \in \{0, 1, 2, 3, ......\}$,

$$d(x_{i+1}, x_{j+1}) = d(x_i, x_j) - \left( \frac{1}{2^{[\frac{i}{2}]}} - \frac{1}{2^{[\frac{j}{2}]}} \right). \tag{2.8}$$

*(c) For all* $0 \leq i < j$, *where* $i, j \in \{0, 1, 2, 3, ......\}$,

$$d(x_i, x_j) \leqslant d(x_i, x_\infty) \leqslant 4d(x_i, x_{i+1}). \tag{2.9}$$

*(d) For all* $i, j \in \{0, 1, 2, 3, ......\}$,

$$\left. \begin{array}{lll} i < j & \Longrightarrow & \frac{1}{2^{[\frac{j}{2}]}} \leqslant \frac{1}{2^{[\frac{i}{2}]}}, \\ i + 1 < j & \Longrightarrow & \frac{1}{2^{[\frac{j}{2}]}} \leqslant \frac{1}{2\times 2^{[\frac{i}{2}]}} \end{array} \right\}. \tag{2.10}$$

*Now to prove contractive condition (2.1), let us consider* $i < k$. *On taking account (2.7), (2.8), (2.9) and (2.10), we will prove the contractivity (PC) in the following cases:*

***Case 1.*** $0 \leq i < j < k$ *for all* $i, j, k \in 0, 1, 2, 3, .....$

*Figure 2 illustrates the arrangement of the points* $x_i$, $x_j$ *and* $x_k$, *also establishing relationships among their relative distances in Case 1. Now, we have*

$$\begin{aligned} R_{ijk} &= \frac{d(Tx_i, Tx_j) + d(Tx_j, Tx_k)}{d(x_i, x_j) + d(x_j, x_k)} = \frac{d(Tx_i, Tx_k)}{d(x_i, x_k)} = \frac{d(x_{i+1}, x_{k+1})}{d(x_i, x_k)} \\ &= \frac{d(x_i, x_k) - (\frac{1}{2^{[\frac{i}{2}]}} - \frac{1}{2^{[\frac{k}{2}]}})}{d(x_i, x_k)} = 1 - \frac{\frac{1}{2^{[\frac{i}{2}]}} - \frac{1}{2^{[\frac{k}{2}]}}}{d(x_i, x_k)} \leq 1 - \frac{\frac{1}{2^{[\frac{i}{2}]}} - \frac{1}{2\times 2^{[\frac{i}{2}]}}}{d(x_i, x_\infty)} \leq 1 - \frac{\frac{1}{2\times 2^{[\frac{i}{2}]}}}{4d(x_i, x_{i+1})} \\ &= 1 - \frac{\frac{1}{2\times 2^{[\frac{i}{2}]}}}{4 \times 2^{[\frac{i}{2}]}} = 1 - \frac{1}{8} = \frac{7}{8}. \end{aligned}$$

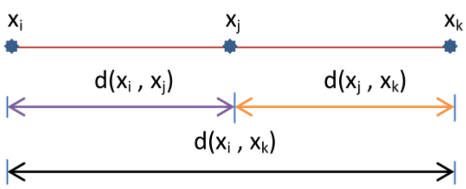


**Figure 2.** $x_i$, $x_j$ and $x_k$ in Case 1.

***Case 2.*** $0 \leq j < i < k$ *for all* $i, j, k \in 0, 1, 2, 3, .....$

*Figure 3 illustrates the arrangement of the points* $x_i$*,* $x_j$ *and* $x_k$*, and their relative distances in Case 2. We have*

$$
\begin{aligned}
R_{jik} &= \frac{d(Tx_i, Tx_j) + d(Tx_j, Tx_k)}{d(x_i, x_j) + d(x_j, x_k)} = \frac{d(Tx_j, Tx_i) + d(Tx_j, Tx_k)}{d(x_j, x_i) + d(x_j, x_k)} \\
&= \frac{d(x_j, x_i) - (\frac{1}{2^{[\frac{j}{2}]}} - \frac{1}{2^{[\frac{i}{2}]}}) + d(x_j, x_k) - (\frac{1}{2^{[\frac{j}{2}]}} - \frac{1}{2^{[\frac{k}{2}]}})}{d(x_j, x_i) + d(x_j, x_k)} \qquad [as\ j < i\ and\ j + 1 < k] \\
&= 1 - \frac{\frac{2}{2^{[\frac{j}{2}]}} - \frac{1}{2^{[\frac{i}{2}]}} - \frac{1}{2^{[\frac{k}{2}]}}}{d(x_j, x_i) + d(x_j, x_k)} \leq 1 - \frac{\frac{2}{2^{[\frac{j}{2}]}} - \frac{1}{2^{[\frac{j}{2}]}} - \frac{1}{2 \times 2^{[\frac{j}{2}]}}}{2d(x_j, x_\infty)} \\
&\leq 1 - \frac{\frac{1}{2 \times 2^{[\frac{j}{2}]}}}{2 \times 4d(x_j, x_{j+1})} \leq 1 - \frac{\frac{1}{2 \times 2^{[\frac{j}{2}]}}}{2 \times 4 \times \frac{1}{2^{[\frac{j}{2}]}}} = 1 - \frac{1}{16} = \frac{15}{16}.
\end{aligned}
$$

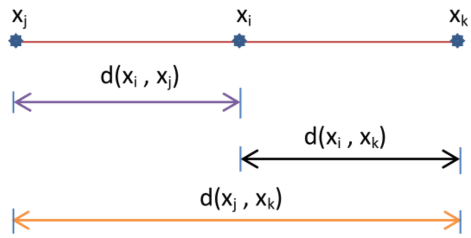


**Figure 3.** $x_i$, $x_j$ and $x_k$ in Case 2.

*Notice that similar reasoning can follow the third case* $0 \leq i < k < j$ *because of symmetry in the expression.*

*Hence, inequality (2.1) holds for every three pairwise distinct points from the space* $X$ *with the coefficient* $\alpha = \frac{15}{16} = \max\{\frac{3}{4}, \frac{15}{16}\}$*. Within this example, zero is the unique fixed point of the mapping* $T$*.*

## 3. Conclusions

In conclusion, our research has introduced and investigated the concept of PC in the context of metric space theory. We have established that mappings with PC, in the absence of periodic points of prime period two, exhibit a maximum of two fixed points. Furthermore, we have uncovered that mappings with PC, with a periodic point of prime period two, may not have a fixed point. Moreover, our study has unveiled an interesting relationship between contraction mappings and PC, demonstrating that all contraction mappings inherently exhibit PC. However, the converse may not universally hold, as evidenced by the examples provided. This insight expands our understanding of contraction mapping theory and its relationship with the classical Banach contraction principle. The concept of PC represents a valuable addition to the realm of metric space theory, offering new perspectives and possibilities. These findings create a strong foundation for exploration in various fields of science and engineering.

## Use of AI tools declaration

The authors declare that they do not have used Artificial Intelligence (AI) tools in the creation of this article.

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## Conflict of interest

The authors declare that there is no conflict of interest.

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