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Research article

Hypersurfaces in a Euclidean space with a Killing vector field

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Abstract: An odd-dimensional sphere admits a killing vector field, induced by the transform of the unit normal by the complex structure of the ambiant Euclidean space. In this paper, we studied orientable hypersurfaces in a Euclidean space that admits a unit Killing vector field and finds two characterizations of odd-dimensional spheres. In the first result, we showed that a complete and simply connected hypersurface of Euclidean space $\mathbb{R}^{n+1}$, $n > 1$ admits a unit Killing vector field ξ that leaves the shape operator S invariant and has sectional curvatures of plane sections containing ξ positive which satisfies $S(\xi) = \alpha\xi$, α mean curvature if, and only if, $n = 2m - 1$, α is constant and the hypersurface is isometric to the sphere $S^{2m-1}(\alpha^2)$. Similarly, we found another characterization of the unit sphere $S^2(\alpha^2)$ using the smooth function $\sigma = g(S(\xi), \xi)$ on the hypersurface.

Keywords: Euclidean space; hypersurface; Killing vector field
Mathematics Subject Classification: 53A50, 53C20

1. Introduction

The study of differential geometry started with the study of curves and surfaces in the Euclidean space $\mathbb{R}^3$ with basic notions such as curvature, torsion, Frenet-Serret frame, first and second fundamental forms, Gauss curvature and mean curvature. With the advancements, it shifted to studying hypersurfaces in higher dimensional Euclidean space $\mathbb{R}^{n+1}$, $n > 1$, with tools such as unit normal N to hypersurface M and the shape operator S, the equations of Gauss, namely, [5]:

$$D_X Y = \nabla_X Y + g(S(X), Y)N \tag{1.1}$$

and

$$D_X N = -S(X), \quad X, Y \in \mathfrak{X}(M), \tag{1.2}$$

where D_X and ∇_X are covariant derivative operators on $\mathbb{R}^{n+1}$ and hypersurface M, respectively, and g is the Riemannian metric induced on M by the Euclidean metric $\langle , \rangle$ on $\mathbb{R}^{n+1}$. The mean curvature α of the hypersurface M is given by $\alpha = \frac{1}{n} trace(S)$, and we have the Gauss and Codazzi equations for the hypersurface M, namely, for all $X, Y, Z \in \mathfrak{X}(M)$ (see [5])

$$R(X, Y)Z = g(S(Y), Z)S(X) - g(S(X), Z)S(Y), \tag{1.3}$$

$$(\nabla_X S)(Y) = (\nabla_Y S)(X), \quad X, Y \in \mathfrak{X}(M), \tag{1.4}$$

where $R(X, Y)Z$ is the curvature tensor of M and $(\nabla_X S)(Y) = \nabla_X S Y - S(\nabla_X Y)$.

The Ricci tensor Ric of the hypersurface M is given by [5]:

$$Ric(X, Y) = n\alpha \left(g(S(X), Y) - g(S(X), S(Y)) \right). \tag{1.5}$$

In the following sections, we will use the notation $R(X, Y; Z, W)$ to refer to the value obtained by applying the metric g to $R(X, Y) Z$ and W.

A hypersurface M of the Euclidean space $\mathbb{R}^{n+1}$ is said to be totally umbilical if the shape operator is $S = \lambda I$, and for $n > 1$, it follows that λ is a constant. It is known that a complete and connected totally umbilical hypersurface M of the Euclidean space $\mathbb{R}^{n+1}$ is isometric to the sphere $S^n(\lambda^2)$ of constant curvature λ^2 [5].

An interesting global result on a compact hypersurface M states that there exists a point $p \in M$ such that all sectional curvatures of M at p are positive [5].

Given a compact hypersurface M of $\mathbb{R}^{n+1}$, the support function $\rho = \langle \psi, N \rangle$ where $\psi : M \longrightarrow \mathbb{R}^{n+1}$ is the immersion and satisfies the Minkowski's formula

$$\int_M (1 + \rho\alpha) = 0, \tag{1.6}$$

where α is the mean curvature of the hypersurface M.

Recall that a hypersurface M of the Euclidean space is said to be a minimal hypersurface if $\alpha = 0$. As a result of Minkowski's formula, it follows that there is no compact minimal hypersurface in a Euclidean space $\mathbb{R}^{n+1}$.

One of the interesting goals in differential geometry of compact hypersurfaces is to find the conditions under which the hypersurface of $\mathbb{R}^{n+1}$ is isometric to the sphere $S^n(c)$ of the constant curvature c.

In [6], it is shown that if the scalar curvature τ of a compact hypersurface M in the Euclidean space $\mathbb{R}^{n+1}$ satisfies $\tau \leq \lambda_1(n-1)$, then M is isometric to $S^n(c)$. Here, λ_1 stands for the first eigenvalue of the Laplace operator. For similar results on compact hypersurfaces in $\mathbb{R}^{n+1}$, we refer to [1, 7–9].

Consider the odd-dimensional sphere $S^{2n-1}(c)$ as a hypersurface in the complex Euclidean space $\mathbb{C}^n$ with natural embedding $\Psi : S^{2n-1}(c) \longrightarrow \mathbb{C}^n$, with $\Psi(x) = x$, then it has shape operator $S = -\sqrt{c} I$ and unit normal $N = \sqrt{c}\Psi$.

Due to the presence of complex structure J on $\mathbb{C}^n$, we get a unit vector field ξ defined on $S^{2n-1}(c)$ by

$$\xi = -JN,$$

which is a Killing vector field on the sphere $S^{2n-1}(c)$, that is, it satisfies

$$\mathcal{L}_\xi g = 0,$$

where $\mathcal{L}_\xi$ is the Lie-derivative with respect to ξ.

In this paper, we are interested in studying compact hypersurfaces in the Euclidean space $\mathbb{R}^{n+1}$, which admit a Killing vector field ξ and analyze the impact of the presence of the Killing vector field on the geometry of hypersurfaces. It is well known that the presence of a Killing vector field on a Riemannian manifold contravenes its topology as well as geometry [2–4, 9–14]. In that, if the length of the Killing vector field is a constant, the influence on the topology and geometry of the Riemannian manifold on which they exist becomes severe. For example, on an even-dimensional Riemannian manifold of positive curvature, there does not exist a nonzero Killing vector field of constant length. It is in this context that even-dimensional spheres $S^{2n}(c)$ do not possess unit Killing vector fields. In [13], it is shown that the fundamental group of a Riemannian manifold admitting a Killing vector field contains a cyclic subgroup of constant index.

Recall that on a compact hypersurface M, each smooth vector field ξ is generated by the global flow on M. Let $\{\phi_t\}$ be the flow of the Killing vector field ξ on the compact hypersurface M of the Euclidean space $\mathbb{R}^{n+1}$. We say that a (1,1)-tensor field T on the hypersurface M is invariant under the killing vector field ξ if

$$\phi_t^*(T) = T \circ d\phi_t,$$

which is equivalent to

$$\mathcal{L}_\xi T = 0. \tag{1.7}$$

Recall that a Killing vector field is said to be a nontrivial Killing vector field if it is not a parallel vector field.

Our first result in this paper is the following.

Theorem 1. *A complete and simply connected hypersurface M of the Euclidean $\mathbb{R}^{n+1}$, $n > 1$ with mean curvature α and shape operator S admits a nontrivial unit Killing vector ξ, such that the sectional curvature of plane sections containing ξ are positive, the shape operator S is invariant under ξ and $S(\xi) = \alpha\xi$ holds if, and only if, $n = 2m - 1$, α is constant and M is isometric to the sphere $S^{2m-1}(\alpha^2)$.*

For a hypersurface M that admits a unit Killing vector field ξ, we have a smooth function $\sigma : M \longrightarrow \mathbb{R}$, defined by

$$\sigma = g((S(\xi), \xi)),$$

and we also get a vector field U on the hypersurface M associated to ξ, defined by

$$U = S(\xi) - \sigma\xi, \tag{1.8}$$

and we call U the associated vector field. It follows that U is orthogonal to ξ.

Finally, we prove the following with constrained sectional curvature $R(S(\xi), \xi; \xi, S(\xi))$ of the hypersurface M.

Theorem 2. *A nontrivial unit Killing vector field ξ on a compact and connected hypersurface M of $\mathbb{R}^{n+1}$, $n > 1$ with mean curvature α leaves the shape operator S invariant, and the function $\sigma = g(S(\xi), \xi) \neq 0$ satisfies*

$$\int_M R(S(\xi), \xi; \xi, S(\xi)) \geq \int_M (n\sigma\alpha\|S(\xi)\|^2 - n\sigma^2\alpha^2)$$

if, and only if, $n = 2m - 1$, α is a constant and M is isometric to $S^{2m-1}(\alpha^2)$.

2. Preliminaries

A smooth vector field ξ on an n-dimensional Riemannian manifold (N^n, g) is said to be a Killing vector field if

$$\mathcal{L}_\xi g = 0. \tag{2.1}$$

In [9], it is shown that for a Killing vector field ξ on (N^n, g), there exists skew-symmetric operator F on (N^n, g), that satisfies

$$\nabla_X \xi = F(X) \tag{2.2}$$

and that

$$(\nabla_X F)(Y) = R(X, \xi)Y, \quad X, Y \in \mathfrak{X}(N^n) \tag{2.3}$$

holds.

Moreover, if ξ is a unit Killing vector field, then it follows that it annihilates F; that is,

$$F(\xi) = 0. \tag{2.4}$$

Using Eqs (2.2)–(2.4), we have

$$R(X, \xi)\xi = (\nabla_X F)(\xi) = -F(\nabla_X \xi) = -F^2(X);$$

that is,

$$R(X, \xi)\xi = -F^2(X), \quad X \in \mathfrak{X}(M), \tag{2.5}$$

and on taking the inner product with X in the above equation, we get the following expression

$$R(X, \xi; \xi, X) = \|F(X)\|^2, \quad X \in \mathfrak{X}(M). \tag{2.6}$$

Note that here, for a unit X that is orthogonal to ξ, $R(X, \xi; \xi, X)$ stands for the sectional curvature of the plane section spanned by ξ and X.

Let M be an orientable hypersurface of the Euclidean space $\mathbb{R}^{n+1}$ with unit normal N and the shape operator S. We denote the induced metric on M by g and the Riemannian connection with respect to g by ∇. Suppose the hypersurace admits a unit Killing vector field ξ.

We shall say the shape operator S is invariant under ξ if

$$\mathcal{L}_\xi S = 0, \tag{2.7}$$

which is equivalent to

$$(\nabla_\xi S)(X) = F(SX) - S(FX)), \quad X \in \mathfrak{X}(M). \tag{2.8}$$

Just like it was previously given, with a unit Killing vector field ξ on the hypersurface M, we can define a smooth function $\sigma : M \longrightarrow R$ by

$$\sigma = g(S(\xi), \xi)$$

and a smooth vector field $U \in \mathfrak{X}(F)$ by

$$U = S(\xi) - \sigma\xi, \tag{2.9}$$

which is called the associated vector field.

It follows that the vector field U is orthogonal to ξ. Note that, according to Codazzi's Eq (1.4) for hypersurface M and Eq (2.8), we confirm

$$(\nabla_X S)(\xi) = F(SX) - S(FX), \quad X \in \mathfrak{X}(M). \tag{2.10}$$

Taking derivative in (2.9) with respect to $X \in \mathfrak{X}(M)$ and using (2.2), we have that

$$\nabla_X U = (\nabla_X S)(\xi) + S(FX) - X(\sigma)\xi - \sigma FX,$$

which in view of Eq (2.10), implies

$$\nabla_X U = F(SX) - X(\sigma)\xi - \sigma FX. \tag{2.11}$$

3. Proof of Theorem 1

Suppose M is a complete and simply connected hypersurface of the Euclidean space $\mathbb{R}^{n+1}$, which admits a unit Killing vector field ξ with shape operator S as invariant under ξ. A sectional curvature of the plane sections containing ξ are positive and the shape operator satisfies

$$S(\xi) = \alpha\xi, \tag{3.1}$$

where $\alpha = \frac{1}{n} trS$ is the mean curvature of M.

Differentiating Eq (3.1) with respect to $X \in \mathfrak{X}(M)$ and using Eq (2.2) yields

$$(\nabla_X S)(\xi) + S(FX) = X(\alpha)\xi + \alpha FX.$$

Using Eq (2.10) in the above equation brings

$$F(SX) = X(\alpha)\xi + \alpha FX, \quad X \in \mathfrak{X}(M);$$

that is,

$$F(SX - \alpha X) = X(\alpha)\xi, \quad X \in \mathfrak{X}(M).$$

Operating F in the above equation and using equation (2.4), yields

$$F^2(SX - \alpha X) = 0, \quad X \in \mathfrak{X}(M).$$

The above equation, in view of Eq (2.5) implies

$$R(SX - \alpha X, \xi)\xi = 0.$$

Taking the inner product in the above equation, with $SX - \alpha X$, we get

$$R(SX - \alpha X, \xi; \xi, SX - \alpha X) = 0, \quad X \in \mathfrak{X}(M). \tag{3.2}$$

Note that for any $X \in \mathfrak{X}(M)$, in view of Eq (3.1), we have

$$\begin{aligned} g(SX - \alpha X, \xi) &= g(SX, \xi) - \alpha g(X, \alpha) \\ &= g(X, S\xi) - \alpha g(X, \xi) \\ &= 0; \end{aligned}$$

that is, $SX - \alpha X$ is orthogonal to ξ. Thus, by Eq (3.2), it follows that the sectional curvatures of the plane sections spanned by $SX - \alpha X$ and ξ are zero, which is contrary to the hypothesis that sectional curvatures of plane sections containing ξ are positive. Hence, we conclude

$$SX - \alpha X = 0, \quad X \in \mathfrak{X}(M);$$

that is,

$$S(X) = \alpha X, \quad X \in \mathfrak{X}(M). \tag{3.3}$$

Note that the mean curvature α satisfies

$$n\alpha = \sum_{j=1}^{n} g(Se_j, e_j) \tag{3.4}$$

for a local orthonormal frame $\{e_1, \ldots, e_n\}$ of the hypersurface M.

Differentiating (3.4) with respect to $X \in \mathfrak{X}(M)$ gives

$$\begin{aligned} nX(\alpha) &= \sum_{j=1}^{n} [g(\nabla_X S e_j, e_j) + g(Se_j, D_X e_j)] \\ &= \sum_{j=1}^{n} [g((\nabla_X S)(e_j), e_j) + 2g(Se_j, D_X e_j)], \end{aligned}$$

and using Eq (1.4) gives

$$nX(\alpha) = \sum_{j=1}^{n} [g((\nabla_{e_j} S)(X), e_j) + 2g(Se_j, D_X e_j)]. \tag{3.5}$$

Note that

$$\nabla_X e_j = \sum_{i=1}^{n} \omega_j^i(X) e_i,$$

where (ω_j^i) are connection forms satisfying

$$\omega_j^i + \omega_i^j = 0. \tag{3.6}$$

We take

$$S(e_j) = \sum_k \lambda_j^k e_k,$$

where (λ_j^k) is a symmetric matrix. Thus,

$$\sum_{j=1}^{n} g(Se_j, \nabla_X ej) = \sum_{ji} \lambda_j^i \omega_j^i(X) = 0,$$

owing to the fact that (λ_j^k) is a symmetric whereas $(\omega_j^i(X))$ is skew-symmetric.

Hence,

$$nX(\alpha) = \sum_{j=1}^{n} g((\nabla_{e_j} S)(X), e_j),$$

and as S is symmetric operator, we have

$$nX(\alpha) = \sum_{j=1}^{n} g(X, (\nabla_{e_j} S)(e_j)), \quad X \in \mathfrak{X}(M).$$

From this, we see that the gradient of the mean curvature α satisfies

$$n\nabla\alpha = \sum_{j=1}^{n} (\nabla_{e_j} S)(e_j). \tag{3.7}$$

Now, differentiating Eq (3.3) with respect to $X \in \mathfrak{X}(M)$ yields

$$\nabla_X SX = X(\alpha)X + \alpha\nabla_X X,$$

and

$$S(\nabla_X X) = \alpha\nabla_X X$$

gives

$$(\nabla_X S)(X) = X(\alpha)X.$$

Taking a local orthonormal frame $\{e_1, \ldots e_n\}$ on the hypersurface M, we get

$$\sum_{j=1}^{n} (\nabla_{e_j} S)(e_j) = \sum_{j=1}^{n} e_j(\alpha)e_j = \nabla\alpha,$$

and combining above the equation with Eq (3.7) yields

$$n\nabla\alpha = \nabla\alpha.$$

However, $n > 1$ in the hypothesis implies

$$\nabla\alpha = 0;$$

that is, the mean curvature α is a constant. Using Eqs (1.3) and (3.3), we see that the curvature tensor of the hypersurface satisfies

$$R(X, Y)Z = \alpha^2\{g(Y, Z)X - g(X, Z)Y\}, \quad X, Y, Z \in \mathfrak{X}(M);$$

that is, M is a space of constant curvature α^2. Note that $\alpha^2 > 0$, as the sectional curvature of the plane sections containing ξ are positive. Hence, with M being complete and a simply connected Riemannian manifold of positive constant curvature α^2, it is isometric to the sphere $S^n(\alpha^2)$.

Note that n cannot be even as a Killing vector field ξ on an even-dimensional Riemannian manifold of positive sectional curvature that has a zero [5]; this is contrary to the assumption that ξ is a unit Killing vector field. Hence, n is odd; that is, $n = 2m - 1$ and M is isometric to the sphere $S^{2m-1}(\alpha^2)$. The converse is trivial.

Note that the condition in the statement of Theorem 1 that the sectional curvatures of plane sections containing ξ are positive is essential. For instance, if a complete and simply connected hypersurface has sectional curvatures of plane sections containing ξ as nonpositive, then by virtue of Eq (2.6), it will imply that ξ is a parallel; that is, it is a trivial Killing vector field contrary to the requirement that ξ is a nontrivial.

4. Proof of Theorem 2

Suppose the compact and connected hypersurface M of the Euclidean space $\mathbb{R}^{n+1}$, $n > 1$ with mean curvature α admits a unit Killing vector field ξ, that the shape operator S is invariant under ξ and the function $\sigma = g(S\xi, \xi) \neq 0$ satisfies

$$\int_M R(S\xi, \xi; \xi, S\xi) \geq \int_M (n\alpha\sigma\|S\xi\|^2 - n\alpha^2\sigma^2). \tag{4.1}$$

For $X \in \mathfrak{X}(M)$, by using Eq (2.2), we have that

$$X(\sigma) = g((\nabla_X S)(\xi) + SFX, \xi) + g(S\xi, FX),$$

which, in view of Eq (2.8), gives

$$X(\sigma) = g(FSX, \xi) + g(S\xi, FX).$$

Using Eq (2.4) in the above equation, we get the gradient of σ as

$$\nabla\sigma = -F(S\xi). \tag{4.2}$$

Differentiating the above equation with respect to $X \in \mathfrak{X}(M)$ and using Eq (2.2), we get

$$\nabla_X \nabla\sigma = -[(\nabla_X F)(S\xi) + F((\nabla_X S)(\xi) + FS(X))].$$

Using Eqs (2.3) and (2.10), we conclude

$$\nabla_X \nabla\sigma = -R(X,\xi)S\xi - F(F(SX) - S(FX)) - FS(FX);$$

that is,

$$\nabla_X \nabla\sigma = -R(X,\xi)S\xi - F^2(SX), \quad X \in \mathfrak{X}(F).$$

Now, employing Eq (2.5) in the above equation, we reach

$$\nabla_X \nabla\sigma = -R(X,\xi)S\xi + R(SX,\xi)\xi,$$

which in view of Eq (1.3), leads to

$$\nabla_X \nabla\sigma = -[\|S\xi\|^2 SX - g(SX,S\xi)S\xi] + \sigma S^2 X - g(SX,S\xi)S\xi;$$

that is,

$$\nabla_X \nabla\sigma = -\|S\xi\|^2 SX + \sigma S^2 X. \tag{4.3}$$

Now, choosing a local orthonormal frame $\{e_1, \ldots, e_n\}$ on the hypersurface M to compute $div(\nabla\sigma)$, by using Eq (4.3) we have

$$\Delta\sigma = div(\nabla\sigma) = \sum_{j=1}^{n} g(\nabla_{e_j}\nabla\sigma, e_j) = -n\alpha\|S\xi\|^2 + \sigma\|S\|^2.$$

Thus, we conclude

$$\sigma\Delta\sigma = -n\sigma\alpha\|S\xi\|^2 + \sigma^2\|S\|^2.$$

Integrating the above equation by parts leads to

$$-\int_M \|\nabla\sigma\|^2 = \int_M (\sigma^2\|S\|^2 - n\sigma\alpha\|S\xi\|^2);$$

that is,

$$\int_M \sigma^2(\|S\|^2 - n\alpha^2) = \int_M (n\sigma\alpha\|S\xi\|^2 - \|\nabla\sigma\|^2 - n\sigma^2\alpha^2). \tag{4.4}$$

Now, Eqs (2.6) and (4.2) give

$$\|\nabla\sigma\|^2 = \|F(S\xi)\|^2 = R(S\xi,\xi;\xi,S\xi),$$

which changes Eq (4.4) to

$$\int_M \sigma^2(\|S\|^2 - n\alpha^2) = \int_M (n\sigma\alpha\|S\xi\|^2 - n\sigma^2\alpha^2) - \int_M R(S\xi,\xi;\xi,S\xi).$$

Now, employing the inequality in the above equation yields

$$\int_M \sigma^2(\|S\|^2 - n\alpha^2) \leq 0. \tag{4.5}$$

Note that, according to Schwartz's inequality $\|S\|^2 \geq n\alpha^2$, the integrand in the integral of inequality (4.5) is nonnegative. Hence, we get

$$\sigma^2(\|S\|^2 - n\alpha^2) = 0, \tag{4.6}$$

and since $\sigma \neq 0$ on the connected M, Eq (4.6) implies $\|S\|^2 = n\alpha^2$. However, $\|S\|^2 = n\alpha^2$ is the equality in the Schwartz's inequality $\|S\|^2 \geq n\alpha^2$, which holds if, and only if, $S = \alpha I$. Following the proof of Theorem 1, we get M as isometric to $S^{2m-1}(\alpha^2)$.

Conversely, suppose that M is isometric to $S^{2m-1}(\alpha^2)$, then as seen in the introduction, we see there is a unit Killing vector field ξ on $S^{2m-1}(\alpha^2)$. Moreover, the shape operator $S = \alpha I$ is invariant under ξ and the function $\sigma = g(S\xi, \xi) = \alpha$.

Thus, $\int_M R(S\xi, \xi; \xi, S\xi) = 0$ and

$$\int_M (n\sigma\alpha\|S\xi\|^2 - m\sigma^2\alpha^2) = \int_M (n\alpha^4 - n\alpha^4) = 0.$$

Consequently,

$$\int_M R(S\xi, \xi; \xi, S\xi) = \int_M (n\sigma\alpha\|S\xi\|^2 - n\sigma^2\alpha^2)$$

holds. This finishes the proof.

We would like to emphasize that the condition $\sigma \neq 0$ is essential in the statement of Theorem 2 to reach the conclusion. For instance, if we consider $\sigma = g(S\xi, \xi) = 0$ on the compact and connected hypersurface, then Eq (4.3) would imply $S\xi = 0$ and it will not allow the hypersurface to be isometric to a sphere.

5. Conclusions

There are two important vector fields on a Riemannian manifold (N, g), namely, a Killing vector field and a conformal vector field, and they have importance in the geometry of a Riemannian manifold in which they live, as well as in physics, especially the theory of relativity. In this paper, we have used a unit Killing vector field ξ on a hypersurface M of the Euclidean space R^{m+1} under the restriction that the shape operator S of the hypersurface is invariant under ξ, and we obtained two characterizations of the odd-dimensional spheres. In these results, we used the restrictions on sectional curvatures of the plane sections containing the unit Killing vector field ξ and the shape operator S to reach the conclusions. There could be a natural question as to what the restriction on the Ricci curvature $Ric(\xi, \xi)$ should be of the orientable hypersurface of the Euclidean space R^{m+1} admitting a Killing vector field ξ, which leaves the shape operator S invariant so that the hypersurface is isometric to an odd-dimensional sphere.

The next important vector field on a Riemannian manifold (N, g) is the conformal vector field. A vector field ζ on (N, g) is said to be a conformal vector field if

$$\mathcal{L}_\zeta g = 2\rho g, \tag{5.1}$$

where $\mathcal{L}_\zeta g$ is the Lie derivative of g, with respect to ζ, and ρ is a smooth function called the conformal factor [3,10]. It is known that all spheres $S^m(c)$ admit many conformal vector fields. Therefore, it is natural to study hypersurfaces of the Euclidean space R^{m+1} admitting a conformal vector field ζ. Naturally, one would like to confront with the question: Under what conditions does an orientable hypersurface M of the Euclidean space R^{m+1} admitting a conformal vector field ζ is isometric to the sphere $S^m(c)$?

Given a unit Killing vector field ξ on an orientable hypersurface M of the Euclidean space R^{n+1}, we have seen that there is a vector field U on M given by Eq (2.9), which is orthogonal to ξ and called the associated vector field to ξ. In addition, if the shape operator S is invariant under ξ, then the associated vector field U satisfies Eq (2.11). Note that in Theorem 1, we assumed the associated vector field $U = 0$. However, it will be an interesting task to explore the geometry of an orientable hypersurface M with unit Killing vector field ξ, with respect to which the shape operator S is invariant under ξ and has a nonzero associated vector field U, by imposing some geometric conditions on U.

These three questions raised above shall be our focus of attention in future studies of an orientable hypersurface of the Euclidean space R^{m+1}.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of Interest

The authors declare that there is no conflict of interest.

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