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Research article

Asymptotic behavior of Levin-Nohel nonlinear difference system with several delays

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Abstract: In this manuscript, we considered a system of difference equations with delays and we established sufficient conditions to guarantee stability, asymptotic stability and exponential stability. In each type of stability, we created an appropriate space that guarantees us the existence of a fixed point that achieves the required stability.

Keywords: delay difference system; fixed points; fundamental matrix; asymptotic stability; exponential stability
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1. Introduction

In the last century, difference equations have been applied to solve many problems in statistics, science and engineering. Difference equations are used to approximate ordinary and partial differential equations due to the development of computing machinery. In addition to approximating these equations, they provide a powerful method for analyzing mechanical, electrical, and other systems with repeated identical sections. Using difference equations greatly facilitates the study of insulator strings, electric-wave filters, magnetic amplifiers, multistage amplifiers, continuous beams of equal span, and acoustical filters (see [2, 7, 14, 15]). Among the important equations that the researchers highlighted and which we will also study in our paper is the Levin-Nohel equation (see [1, 4–6, 8, 11, 13, 16]).

Many researchers resort to using the fixed point theorems and the Lyapunov function to study the qualitative properties of difference systems with and without delays because these methods give impressive results and support the conformity of the conditions of the studied phenomenon with the reality (see [3–6, 8–12]).

Let $\mathbb{Z}$ the set of integers. In this paper we denote $\mathbb{Z}_a = \{a, a+1, ...\}$ and $\mathbb{Z}_a^b = \{a, a+1, ..., b-1, b\}$ for $a, b \in \mathbb{Z}$.

Khelil in [1] obtained results for asymptotic stability of the following difference equation

$$\Delta u(m) + \sum_{s=m-q(m)}^{m-1} a(m,s)\, u(s)\, ds + b(m)\, u(m - p(m)) = 0, \quad m \in \mathbb{Z}_{m_0}, \tag{1.1}$$

with initial condition given by

$$u(m) = \phi(m), \quad m \in \mathbb{Z}_{\theta(m_0)}^{m_0},$$

such that

$$\theta(m_0) = \min\left(\inf_{s \geq m_0} \{s - p(s)\}, \inf_{s \geq m_0} \{s - q(s)\} \right).$$

In [11], we considered the linear Levin-Nohel integro-differential system

$$u'(\zeta) + \int_{\zeta - q(\zeta)}^{\zeta} C(\zeta, s)\, u(s)\, ds + B(\zeta)\, u(\zeta - p(\zeta)) = 0, \ \zeta \geq \zeta_0, \tag{1.2}$$

with initial condition given by

$$u(\zeta) = \psi(\zeta) \ \text{ for } \zeta \in [\theta(\zeta_0), \zeta_0],$$

and we studied the asymptotic stability.

Using the above works as motivation, we present in this paper the nonlinear Levin-Nohel difference system

$$\Delta u(m) + A(m)\, u(m - p(m)) + \sum_{s=m-q(m)}^{m-1} C(m,s)\, g(u(s)) = 0, \quad m \geq m_0, \tag{1.3}$$

with initial condition given by

$$u(m) = \omega(m) \ \text{ for } m \in \mathbb{Z}_{\theta(m_0)}^{m_0},$$

where $\Delta u(m) = u(m+1) - u(m)$ is the forward difference operator for any sequence $\{u(m),\ u(m_0) = u_0,\ m \in \mathbb{N}\}$ and $p(m), q(m) : \mathbb{Z}_{m_0} \to \mathbb{N}$, such that $m - p(m), m - q(m) \to \infty$ when the $m \to \infty$, $g : \mathbb{R}^N \to \mathbb{R}^N$ real sequence satisfies $g(0) = 0$. The $N \times N$ matrices $C : \mathbb{Z}_{m_0} \times \mathbb{Z}_{\theta(m_0)} \to \mathbb{R}^{N^2}$ and $A : \mathbb{Z}_{m_0} \to \mathbb{R}^{N^2}$ are bounded with real sequences as its elements.

The asymptotic behavior of the above system has never been investigated by applying Banach's fixed point theorem. There is known literature on the Levin-Nohel integro-differential systems, but the specific system (1.3) may not be examined yet, So we use the fixed point theorem of Banach to show stability, asymptotic stability and the exponential stability of solutions for the system (1.3).

2. Preliminaries

Let A be an $N \times N$ matrix valued sequence and consider the homogeneous linear system

$$\Delta u(m) = A(m)\, u(m). \tag{2.1}$$

Definition 1. *The state transition matrix $t \to Q(t,q)$ for the homogeneous linear system (2.1) on the open interval J is the family of fundamental matrix solutions parameterized by $q \in J$ satisfying $Q(q,q) = I$, where I is the $N \times N$ identity matrix.*

Throughout this manuscript, we assume that the matrix $I + A(m)$ is nonsingular and we define the forward operator E by $Eu(m) = u(m+1)$. Furthermore, the fundamental matrix solution $Q(m)$ of the unperturbed linear system (2.1) satisfies:

(a) $\det Q(m) \neq 0$.
(b) $Q(m+1) = (I + A(m)) Q(m)$ and $Q^{-1}(m+1) = Q^{-1}(m)(I + A(m))^{-1}$.

In Lemma 1, we convert system (1.3) to a new convenient system to facilitate the application of the fixed point techniques.

Lemma 1. *If $u(m) : \mathbb{Z}_{m_0} \to \mathbb{R}^m$ is the solution of (1.3), then system (1.3) is equivalent to*

$$\begin{aligned} u(m) &= Q(m,m_0)\,\omega(m_0) \\ &\quad + \sum_{s=m_0}^{m-1} Q(m,s)\,B(s)\left(A(s)\left((u(s) + u(s-p(s)))\right) + \sum_{z=s-q(s)}^{s-1} C(s,z)\,g(u(z))\right), \end{aligned} \tag{2.2}$$

where

$$B(m) := A(m)(I + A(m))^{-1} - I, \forall m \in \mathbb{Z}_{\theta(m_0)}.$$

Proof. First, we can write system (1.3) as the form

$$\begin{aligned} \Delta u(m) &= A(m)\,u(m) - A(m)\,(u(m) + u(m-p(m))) \\ &\quad - \sum_{s=m-q(m)}^{m-1} C(m,s)\,g(u(s)). \end{aligned}$$

Let u be a solution of (1.3) and $Q(m,m_0)$ be a fundamental matrix of (2.1). Since

$$Q(m,m_0)\,Q^{-1}(m,m_0) = I,$$

it follows that

$$\begin{aligned} 0 &= \Delta\left[Q(m,m_0)\,Q^{-1}(m,m_0)\right] \\ &= A(m)\,Q(m,m_0)\,EQ^{-1}(m,m_0) + Q(m,m_0)\,\Delta Q^{-1}(m,m_0) \\ &= A(m)\,Q(m,m_0)\,Q^{-1}(m,m_0)(I + A(m))^{-1} + Q(m,m_0)\,\Delta Q^{-1}(m,m_0). \end{aligned}$$

This implies

$$\Delta Q^{-1}(m,m_0) = -Q^{-1}(m,m_0)\,A(m)(I + A(m))^{-1}.$$

On the other hand,

$$\begin{aligned} &\Delta\left[Q^{-1}(m,m_0)\,u(m)\right] \\ &= \Delta Q^{-1}(m,m_0)\,Eu(m) + Q^{-1}(m,m_0)\,\Delta u(m) \end{aligned}$$

$$= -Q^{-1}(m,m_0)A(m)(I+A(m))^{-1}$$
$$\times\left[(I+A(m))u(m)-A(m)(u(m)+u(m-p(m)))-\sum_{s=m-q(m)}^{m-1}C(m,s)g(u(s))ds\right]$$
$$+Q^{-1}(m,m_0)\left[A(m)u(m)-A(m)(u(m)+u(m-p(m)))-\sum_{s=m-q(m)}^{m-1}C(m,s)g(u(s))ds\right],$$

then

$$\Delta\left[Q^{-1}(m,m_0)u(m)\right]=Q^{-1}(m,m_0)\left(A(m)(I+A(m))^{-1}-I\right)$$
$$\times\left[A(m)(u(m)+u(m-p(m)))+\sum_{s=m-q(m)}^{m-1}C(m,s)g(u(s))ds\right].$$

A summation of the above equation from m_0 to $m-1$ gives (2.2). It is easy to obtain the converse implication, and the proof is complete. □

3. Main results

Let $(\mathcal{S},\|\cdot\|)$ be the Banach space of bounded sequences $u: m\in\mathbb{Z}_{m_0}\to\mathbb{R}^N$ with the maximum norm.

$$\|u(\cdot)\|=\sup_{m\in\mathbb{Z}_{m_0}}|u(m)|,$$

where $|\cdot|$ is the infinity norm for $u\in\mathbb{R}^N$. We define the norm of $A(m):=\left[a_{ij}(m)\right]$ by

$$\|A\|:=\sup_{m\in\mathbb{Z}_{m_0}}|A(m)|,$$

where

$$|A(m)|=\max_{1\le i\le N}\sum_{j=1}^{N}\left|a_{ij}(m)\right|.$$

In this paper, we assume that there exists a constant $L_g>0$ such that for $u,v\in\mathbb{R}^N$

$$\|g(u)-g(v)\|\le L_g\|u-v\|. \tag{3.1}$$

Definition 2. *We say that the zero solution of (1.3) is Lyapunov stable if for any $\epsilon>0$ and $m_0\in\mathbb{Z}$ there exists $\delta>0$ such that $|\omega(m)|\le\delta$ for $m\in\mathbb{Z}_{\theta(m_0)}^{m_0}$, which implies $|u(m,m_0,u_0)|\le\epsilon$ for $m\in\mathbb{Z}_{m_0}$.*

Theorem 1. *Assume there exists $M>0$ and $\gamma\in(0,1)$ such that for $m\in\mathbb{Z}_{m_0}$,*

$$|Q(m,m_0)|\le M \tag{3.2}$$

$$\sum_{s=m_0}^{m-1}|Q(m,s)||B(s)|\left(2|A(s)|+L_g\sum_{z=s-q(s)}^{s-1}|C(s,z)|\right)\le\gamma, \tag{3.3}$$

then the zero solution of (1.3) is stable.

Proof. Let $\epsilon > 0$ and choose $\delta > 0$ such that for $|u(m)| \leq \delta, \forall m \in \mathbb{Z}_{\theta(m_0)}^{m_0}$, we have

$$\delta M + \gamma\epsilon \leq \epsilon.$$

Define

$$\Gamma_\epsilon = \left\{ u \in \mathcal{S} : |u(m)| \leq \delta, \forall m \in \mathbb{Z}_{\theta(m_0)}^{m_0} \text{ and } |u(m)| \leq \epsilon, \forall m \in \mathbb{Z}_{m_0} \right\},$$

then $(\Gamma_\epsilon, \|\cdot\|)$ is a complete metric space with the maximum norm.

We define the operator $F : \Gamma_\epsilon \to \mathcal{S}$ due to Lemma 1 by

$$\begin{aligned}(Fu)(m) &= Q(m, m_0)\,\omega(m_0) + \sum_{s=m_0}^{m-1} Q(m, s)\, B(s) \\ &\quad \times \left[A(s)\,(u(s) + u(s - p(s))) + \sum_{z=s-q(s)}^{s-1} C(s, z)\, g(u(z)) \right], \end{aligned} \tag{3.4}$$

for $m \in \mathbb{Z}_{m_0}$.

We first prove that F maps Γ_ϵ into Γ_ϵ. So, by (3.1)–(3.3)

$$\begin{aligned}|(Fu)(m)| &\leq |Q(m, m_0)|\,|\omega(m_0)| + \sum_{s=m_0}^{m-1} |Q(m, s)|\,|B(s)| \\ &\quad \times \left[|A(s)|\,(|u(s)| + |u(s - p(s))|) + \sum_{z=s-q(s)}^{s-1} |C(s, z)|\,|g(u(z))| \right] \\ &\leq M\delta + \sum_{s=m_0}^{m-1} |Q(m, s)|\,|B(s)| \left(2\,|A(s)| + L_g \sum_{z=s-q(s)}^{s-1} |C(s, z)| \right) \|u\| \\ &\leq M\delta + \gamma\epsilon \leq \epsilon.\end{aligned}$$

We next prove that F is a contraction.

Let $u, v \in \Gamma_\epsilon$, then

$$\begin{aligned}|(Fu)(m) - (Fv)(m)| &\leq \sum_{s=m_0}^{m-1} |Q(m, s)|\,|B(s)| \\ &\quad \times \left(2\,|A(s)|\,\|u - v\| + L_g \sum_{z=s-q(s)}^{s-1} |C(s, z)|\,\|u - v\| \right) \\ &\leq \gamma\,\|u - v\|.\end{aligned}$$

Hence,

$$\|Fu - Fv\| \leq \gamma\,\|u - v\|,$$

since $\gamma \in (0, 1)$, then F is a contraction.

Thus, by the fixed point of Banach, F has a unique fixed point u in Γ_ϵ, which is a solution of (1.3) with $u(m_0) = u_0$ and $|u(m)| = |u(m, m_0, u_0)| \leq \epsilon$ for $m \in \mathbb{Z}_{m_0}$. This proves that the zero solution of (1.3) is stable. □

Definition 3. *We say that the zero solution of (1.3) is asymptotically stable if it is stable and if for any integer* $m_0 \geq 0$ *there exists* $\delta > 0$*, such that* $|\omega(m)| \leq \delta$ *for* $m \in [\theta(m_0), m_0]$*, which implies* $|u(m, m_0, u_0)| \to 0$ *as* $m \to \infty$*.*

Theorem 2. *If (3.1)–(3.3) and*

$$Q(m, m_0) \to 0, \text{ as } m \to \infty, \tag{3.5}$$

hold, then the zero solution of (1.3) is asymptotically stable.

Proof. We have shown by our last theorem that the zero solution of (1.3) is stable. For a given $\epsilon > 0$ define

$$\Gamma_0 = \{u \in \Gamma_\epsilon \text{ such that } u(m) \to 0, \text{ as } m \to \infty\}.$$

Define $F : \Gamma_0 \to \Gamma_\epsilon$ by (3.4). We must prove that for $u \in \Gamma_0$, $(Fu)(m) \to 0$ when $m \to \infty$. By definition of Γ_0, $u(m) \to 0$, as $m \to \infty$. Thus, we get

$$\begin{aligned}|(Fu)(m)| \leq\ & |Q(m, m_0)| |\omega(m_0)| + \sum_{s=m_0}^{m-1} |Q(m, s)| |B(s)| \\ & \times \left[|A(s)| (|u(s)| + |u(s - p(s))|) + \sum_{z=s-q(s)}^{s-1} |C(s, z)| |g(u(z))| \right].\end{aligned}$$

By (3.5),

$$|Q(m, m_0)| |\omega(m_0)| \to 0 \text{ when } m \to \infty.$$

Moreover, let $u \in \Gamma_0$ so that for any $\epsilon_1 \in (0, \epsilon)$, there exists $T \geq m_0$ large enough such that $s \geq T$ implies $|u(s - p(s))|, |u(s - q(s))| < \epsilon_1$. Hence, we get

$$\begin{aligned}\Lambda =\ & \sum_{s=m_0}^{m-1} |Q(m, s)| |B(s)| \\ & \times \left[|A(s)| (|u(s)| + |u(s - p(s))|) + \sum_{z=s-q(s)}^{s-1} |C(s, z)| |g(u(z))| \right] \\ \leq\ & \epsilon_1 \sum_{s=m_0}^{m-1} |Q(m, s)| |B(s)| \left(2|A(s)| + L_g \sum_{z=s-q(s)}^{s-1} |C(s, z)| \right) \\ \leq\ & \gamma\epsilon_1 < \epsilon_1.\end{aligned}$$

Thus, $\Lambda \to 0$ as $m \to \infty$.

Hence, F maps Γ_0 into itself. By the fixed point of Banach, F has a unique fixed point $u \in \Gamma_0$, which solves (1.3). □

Definition 4. *We say that the zero solution of (1.3) is exponentially stable if there exist* $\delta, \sigma > 0$ *and* $\lambda \in (0, 1)$ *such that*

$$|u(m, m_0, u_0)| < \sigma |u_0| \lambda^{m-m_0}, \ m \geq m_0, \tag{3.6}$$

whenever $|u_0| < \delta$*.*

Theorem 3. *The zero solution of (1.3) is exponentially stable if*

1) The conditions (3.1) and (3.3) hold;

2) There exists $\lambda \in (0,1)$ *such that*

$$|Q(m,m_0)| \leq \frac{1}{2}\lambda^{m-m_0}, \ \forall m \geq m_0, \tag{3.7}$$

and

$$\sum_{s=m_0}^{m-1} |B(s)| \left(\frac{\lambda^{p(s)}+1}{\lambda^{p(s)}} |A(s)| + L_g \sum_{z=s-q(s)}^{s-1} |C(s,z)| \lambda^z \right) \leq 1. \tag{3.8}$$

Proof. Since the condition (3.7) holds, we define Γ_e as the closed subspace of $\mathcal{S}$ as

$$\Gamma_e = \{u \in \mathcal{S} : \text{such that } |u(m)| \leq |u_0| \sigma \lambda^{m-m_0}, \ \forall m \geq m_0\}.$$

We will show that $F(\Gamma_e) \subset \Gamma_e$, then by (3.7), we have

$$\begin{aligned} |(Fu)(m)| &= |Q(m,m_0)| |\omega(m_0)| + \sum_{s=m_0}^{m-1} |Q(m,s)| |B(s)| \\ &\quad \times \left(|A(s)| (|u(s)| + |u(s-p(s))|) + \sum_{z=s-q(s)}^{s-1} |C(s,z)| |g(u(z))| \right) \\ &\leq \frac{1}{2} |\omega(m_0)| \sigma \lambda^{m-m_0} + \sum_{s=m_0}^{m-1} \frac{1}{2} \lambda^{m-s} |B(s)| \\ &\quad \times \left(|A(s)| \left(|u_0| \sigma \lambda^{s-m_0} + |u_0| \sigma \lambda^{s-p(s)-m_0} \right) + L_g \sum_{z=s-q(s)}^{s-1} |C(s,z)| |u_0| \sigma \lambda^{z-m_0} \right) \\ &= \frac{1}{2} |\omega(m_0)| \sigma \lambda^{m-m_0} + \frac{1}{2} |\omega(m_0)| \sigma \lambda^{m-m_0} \\ &\quad \times \sum_{s=m_0}^{m-1} |B(s)| \left(\frac{\lambda^{p(s)}+1}{\lambda^{p(s)}} |A(s)| + L_g \sum_{z=s-q(s)}^{s-1} |C(s,z)| \lambda^z \right), \end{aligned}$$

since (3.8) holds. Thus, we have

$$|(Fu)(t)| \leq \frac{1}{2}\sigma |u_0| \lambda^{m-m_0} + \frac{1}{2}\sigma |u_0| \lambda^{m-m_0} = \sigma |u_0| \lambda^{m-m_0},$$

then $F(\Gamma_e) \subset \Gamma_e$.

Hence, there exists a unique fixed point $u \in \Gamma_e$ that solves (1.3), then the zero solution of (1.3) is stable exponentially. □

4. Conclusions

In this research paper, the theoretical study of stability, asymptotic stability and exponential stability was addressed by using the fixed point theorem of Banach. Some new criteria was imposed on the fundamental matrix solution and system components to obtain the stability, asymptotic stability and exponential stability. The considered system contained two functional delays. However, the obtained results for Equation (1.3) can be extended to more than two delays.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no competing interests.

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