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Research article

A note on Boussinesq maximal estimate

Dan Li[1] **and Xiang Li**[2,*]

[1] School of Mathematics and Statistics, Beijing Technology and Business University, Beijing 100048, China

[2] School of Mathematics and Finance, Chuzhou University, Chuzhou, Anhui 239012, China

* **Correspondence:** Email: xiangli34@chzu.edu.cn; Tel: +15650771895.

Abstract: We considered the Boussinesq maximal estimate when $n \geq 1$. We obtained the Boussinesq maximal operator $\mathcal{B}_E^* f$ is bounded from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$ when $f \in L^2(\mathbb{R}^n)$ and supp $\hat{f} \subset B(0, \lambda)$.

Keywords: Boussinesq operator; maximal estimate; bounded; frequency decomposition; linearization
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1. Introduction

1.1. The pointwise convergence of Schrödinger operator

We first introduce the free Schrödinger equation

$$\begin{cases} i\partial_t u + \Delta_x u = 0, \ (x,t) \in \mathbb{R}^n \times \mathbb{R}; \\ u(x,0) = f(x), \ x \in \mathbb{R}^n. \end{cases} \tag{1.1}$$

The formal solution of (1.1) is defined by

$$e^{it\triangle} f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i(x\cdot\xi + t|\xi|^2)} \hat{f}(\xi)\, \mathrm{d}\xi,$$

where $\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\, \mathrm{d}x$.

Carleson [6] first posed the problem: Determine the optimal s such that

$$\lim_{t\to 0} e^{it\triangle} f(x) = f(x), \quad \text{a. e. } \ x \in \mathbb{R}^n \tag{1.2}$$

holds whenever $f \in H^s(\mathbb{R}^n)$, where $H^s(\mathbb{R}^n)$ is the L^2 Sobolev space, which is defined by

$$H^s(\mathbb{R}^n) := \left\{ f \in \mathcal{S}' : \|f\|_{H^s(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} \left(1 + |\xi|^2\right)^s \left|\hat{f}(\xi)\right|^2 \mathrm{d}\xi \right)^{\frac{1}{2}} < \infty \right\}.$$

We call the above problem as Carleson's problem.

Carleson [6] first showed that the almost everywhere convergence (1.2) holds for all $s \geq \frac{1}{4}$ in $\mathbb{R}$. Dahlberg-Kenig [9] proved (1.2) fails for $s < \frac{1}{4}$ when $n \geq 1$. Thus, the Carleson problem was solved in one dimension. For the situation in higher dimensions, many researchers are interested in Carleson's problem. The sufficient condition of Carleson's problem has been obtained by many references [1, 2, 5, 7, 8, 10, 11, 13, 15, 16, 18, 20–23, 27, 28] and references therein. Bourgain [3] gave counterexamples demonstrating that (1.2) fails when $s < \frac{n}{2(n+1)}$. The best sufficient condition was improved by Du-Guth-Li [12] when $n = 2$ and Du-Zhang [14] when $n \geq 3$. Hence, the Carleson problem was essentially solved, except for the endpoint.

1.2. The Boussinesq maximal estimate

As a nonlinear variant of (1.2), the Boussinesq operator acting on $f \in \mathcal{S}(\mathbb{R}^n)$ is given by

$$\mathcal{B}f(x,t) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i(x\cdot\xi + t|\xi|\sqrt{1+|\xi|^2})} \hat{f}(\xi)\, \mathrm{d}\xi,$$

which occurs in many physical situations. The name of this operator comes from the Boussinesq equation

$$u_{tt} - u_{xx} \pm u_{xxxx} = (u^2)_{xx}, \ \ \forall\ (x,t) \in \mathbb{R} \times [0, +\infty);$$

see [4] for more details.

We are motivated by subsection 1.1 to study the pointwise convergence of $\mathcal{B}f(x,t)$: Evaluate the optimal s so that

$$\lim_{t\to 0} \mathcal{B}f(x,t) = f(x), \quad \text{a. e. } \ x \in \mathbb{R}^n \tag{1.3}$$

holds for any $f \in H^s(\mathbb{R}^n)$.

Cho-Ko [7] improved the convergence on the Schrödinger operator to generalized dispersive operators excluding the Boussinesq operator. Li-Li [17] obtained the optimal $s = \frac{1}{4}$ in one dimension including the endpoint. Li-Wang [19] obtained the almost everywhere convergence (1.3) that holds for the optimal $s = \frac{1}{3}$ when $n = 2$, except for the endpoint.

In this paper, we are interested in a more general problem. Let E be a bounded set in $\mathbb{R}^{n+1}$. For $f \in \mathcal{S}(\mathbb{R}^n)$, we introduce the maximal function

$$\mathcal{B}_E^* f(x) := \sup_{(y,t)\in E} |\mathcal{B}f(x+y,t)|, \quad x \in \mathbb{R}^n.$$

Let's review the fractional Schrödinger operator, which is defined by

$$S f(x,t) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i(x\cdot\xi + t|\xi|^a)} \hat{f}(\xi)\, \mathrm{d}\xi, \quad a > 0,$$

and its maximal function, which is given by

$$S_E^* f(x) := \sup_{(y,t)\in E} |S f(x+y,t)|, \quad x \in \mathbb{R}^n.$$

Sjölin-Strömberg [24] obtained maximal function $S_E^* f$ is bounded from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$ when $n \geq 1$; see [25] for more studies. The Boussinesq maximal function is different from the fractional Schrödinger maximal function and they have different properties. Thus, we consider the Boussinesq maximal function in this paper. Our main result is as follows.

Theorem 1.1. *Assume $n \geq 1$, $\lambda \geq 1$. Let the interval $J \subset [0,1]$. Suppose B is a ball in $\mathbb{R}^n$ with radius r and set $E = B \times J = \{(y,t) : y \in B, t \in J\}$. Let $f \in L^2(\mathbb{R}^n)$ with supp $\hat{f} \subset B(0,\lambda)$, then one has*

$$\left\|\mathcal{B}_E^* f\right\|_{L^2(\mathbb{R})} \lesssim \left(|J|^{\frac{1}{4}}\lambda^{\frac{1}{2}} + r^{\frac{1}{2}}\lambda^{\frac{1}{2}} + 1\right)\|f\|_{L^2(\mathbb{R})} \text{ when } n = 1$$

and

$$\left\|\mathcal{B}_E^* f\right\|_{L^2(\mathbb{R}^n)} \lesssim \left(|J|^{\frac{1}{2}}\lambda + r\lambda + 1\right)(r\lambda + 1)^{\frac{n-2}{2}}\|f\|_{L^2(\mathbb{R}^n)} \text{ when } n \geq 2.$$

In section two we give the proof of Theorem 1.1. We use the methods of frequency decomposition, linearization of the maximal operator, TT^* and so on. In fact, in subsection 2.1 we first introduce our main lemma. In order to prove our main lemma, we shall introduce two lemmas, which are proved in section three, then we give the proof of our main lemma. In subsection 2.2 we prove Theorem 1.1.

Throughout this paper, we always use C to denote a positive constant independent of the main parameters involved, but whose value may change at each occurrence. The positive constants with subscripts, such as C_1 and C_2, do not change in different occurrences. For two real functions f and g, we always use $f \lesssim g$ or $g \gtrsim f$ to denote that f is smaller than a positive constant C times g, and we always use $f \sim g$ as shorthand for $f \lesssim g \lesssim f$. If the function f has compact support, we use supp$f$ to denote the support of f. We write $|A|$ for the Lebesgue measure of $A \subset \mathbb{R}$. We use $\mathcal{S}(\mathbb{R}^n)$ to denote the Schwartz function on $\mathbb{R}^n$. We use $B(c,r)$ to represent the ball centered at c with radius r in $\mathbb{R}^n$.

2. The Boussinesq maximal estimate

2.1. The main lemma

In order to prove Theorem 1.1, we give our main lemma as follows.

Lemma 2.1. *Assume $n \geq 1$, $\lambda \geq 1$. Let the interval $J \subset [0,1]$. Suppose B is a ball in $\mathbb{R}^n$ with radius r and $E = B \times J = \{(y,t) : y \in B, t \in J\}$. If $f \in L^2(\mathbb{R}^n)$ with supp $\hat{f} \subset B(0,\lambda)$, then*

$$\left\|\mathcal{B}_E^* f\right\|_{L^2(\mathbb{R}^n)} \lesssim \left(|J|^{\frac{n}{4}}\lambda^{\frac{n}{2}} + r^{\frac{n}{2}}\lambda^{\frac{n}{2}} + 1\right)\|f\|_{L^2(\mathbb{R}^n)}.$$

Remark 2.1. *In fact, Lemma 2.1 contains the Theorem 1.1 when $n = 1$ and $n = 2$, so it suffices to prove Theorem 1.1 when $n \geq 3$.*

Lemma 2.1 plays a key role in the proof of Theorem 1.1. In order to prove Lemma 2.1, we shall use the following Lemmas 2.2 and 2.3. We postpone the proofs of Lemmas 2.2 and 2.3 here and the details will be shown in section three.

Lemma 2.2. *Assume $n \geq 1$, $\lambda \geq 1$. Let the interval $J \subset [0,1]$. Suppose B is a ball in $\mathbb{R}^n$ with radius r and $E = B \times J = \{(y,t) : y \in B, t \in J\}$. If $f \in L^2(\mathbb{R}^n)$ with supp $\hat{f} \subset \{\xi \in \mathbb{R}^n : \frac{\lambda}{2} \leq |\xi| \leq \lambda\}$, then*

$$\left\|\mathcal{B}_E^* f\right\|_{L^2(\mathbb{R}^n)} \lesssim \left(|J|^{\frac{n}{4}}\lambda^{\frac{n}{2}} + r^{\frac{n}{2}}\lambda^{\frac{n}{2}} + 1\right)\|f\|_{L^2(\mathbb{R}^n)}.$$

The only difference between Lemma 2.1 and Lemma 2.2 is the support of $\hat{f}$ and that the condition of Lemma 2.1 is weaker than that of Lemma 2.2.

Remark 2.2. *If we take $B = B(0,\epsilon)$ with $\epsilon > 0$ small enough in Lemma 2.2, then we have*

$$\left\|\sup_{t\in J}|\mathcal{B}f(\cdot,t)|\right\|_{L^2(\mathbb{R}^n)} \lesssim \left(|J|^{\frac{n}{4}}\lambda^{\frac{n}{2}} + 1\right)\|f\|_{L^2(\mathbb{R}^n)}.$$

Lemma 2.3. *Let* $y_0 \in \mathbb{R}^n,\ t_0 \in \mathbb{R},\ 0 < r \leq 1,\ f \in L^2(\mathbb{R}^n)$ *with supp* $\hat{f} \subset B(0, \lambda)$ *and* $\lambda \geq 1$. *Set*

$$E = \left\{(y, t) \in \mathbb{R}^{n+1} : y_{y_0, j} \leq y_j \leq y_{y_0, j} + r \quad \text{for } 1 \leq j \leq n \text{ and } t_0 \leq t \leq t_0 + r^2\right\},$$

then

$$\left\|\mathcal{B}_E^* f\right\|_{L^2(\mathbb{R}^n)} \lesssim (1 + r^2\lambda^2)(1 + r\lambda)^n \|f\|_{L^2(\mathbb{R}^n)}.$$

Proof of Lemma 2.1. Let N be the smallest integer so that $|J|2^{-2N}\lambda^2 + r2^{-N}\lambda < 2$. We write $f = \sum_{j=0}^{N} f_j$ where supp $\widehat{f_j} \subset \{\xi \in \mathbb{R}^n : 2^{-j-1}\lambda \leq |\xi| \leq 2^{-j}\lambda\}$ for $0 \leq j \leq N - 1$ and supp $\widehat{f_N} \subset B(0, 2^{-N}\lambda)$. We make the following two-fold analysis:

On the one hand, we take $E = B \times J = \{(y, t) : y \in B, t \in J\}$ in Lemma 2.3, where B is the same as in Lemma 2.1, which implies that

$$\left\|\mathcal{B}_E^* f_N\right\|_{L^2(\mathbb{R}^n)} \lesssim (1 + |J|2^{-2N}\lambda^2)(1 + r2^{-N}\lambda)^n \|f_N\|_{L^2(\mathbb{R}^n)} \lesssim \|f\|_{L^2(\mathbb{R}^n)}. \tag{2.1}$$

On the other hand, according to Lemma 2.2 we have

$$\left\|\mathcal{B}_E^* f_j\right\|_{L^2(\mathbb{R}^n)} \lesssim \left(2^{-\frac{jn}{2}}|J|^{\frac{n}{4}}\lambda^{\frac{n}{2}} + r^{\frac{n}{2}}\lambda^{\frac{n}{2}}2^{-\frac{jn}{2}}\right)\|f\|_{L^2(\mathbb{R}^n)}$$

for $0 \leq j \leq N - 1$, which implies that

$$\left\|\mathcal{B}_E^*\left(\sum_{j=0}^{N-1} f_j\right)\right\|_{L^2(\mathbb{R}^n)} \lesssim \left(|J|^{\frac{n}{4}}\lambda^{\frac{n}{2}} + r^{\frac{n}{2}}\lambda^{\frac{n}{2}}\right)\|f\|_{L^2(\mathbb{R}^n)}. \tag{2.2}$$

(2.1) and (2.2) yield that

$$\left\|\mathcal{B}_E^* f\right\|_{L^2(\mathbb{R}^n)} \leq \sum_{j=0}^{N}\left\|\mathcal{B}_E^* f_j\right\|_{L^2(\mathbb{R}^n)} \lesssim \left(|J|^{\frac{n}{4}}\lambda^{\frac{n}{2}} + r^{\frac{n}{2}}\lambda^{\frac{n}{2}} + 1\right)\|f\|_{L^2(\mathbb{R}^n)}.$$

This completes the proof of Lemma 2.1.

2.2. *The proof of Theorem 1.1*

We now are ready to combine our main Lemma 2.1 and finish our proof.

Proof of Theorem 1.1. By Remark 2.1, it suffices to consider the case $n \geq 3$. By Lemma 2.1 we have

$$\left\|\mathcal{B}_E^* f\right\|^2_{L^2(\mathbb{R}^n)} \lesssim \left(|J|^{\frac{n}{2}}\lambda^n + r^n\lambda^n + 1\right)\|f\|^2_{L^2(\mathbb{R}^n)}.$$

Cover J with intervals $J_i,\ i = 1, 2, \cdots, N$, of intervals of equal length $|J_i|$ such that $|J_i|\lambda^2 = r^2\lambda^2 + 1$ with $N \leq \frac{|J|}{|J_i|} + 1$. Set $E_i := B \times J_i$, then we have

$$\begin{aligned}\left\|\mathcal{B}_{E_i}^* f\right\|^2_{L^2(\mathbb{R}^n)} &\lesssim \left(\left(|J_i|\lambda^2\right)^{\frac{n}{2}} + \left(r^2\lambda^2 + 1\right)^{\frac{n}{2}}\right)\|f\|^2_{L^2(\mathbb{R}^n)} \\ &= 2\left(r^2\lambda^2 + 1\right)^{\frac{n}{2}}\|f\|^2_{L^2(\mathbb{R}^n)},\end{aligned}$$

which implies that

$$\begin{aligned}\left\|\mathcal{B}_E^* f\right\|_{L^2(\mathbb{R}^n)}^2 &\leq \sum_{i=1}^N \left\|\mathcal{B}_{E_i}^* f\right\|_{L^2(\mathbb{R}^n)}^2 \\ &\lesssim N\left(r^2\lambda^2+1\right)^{\frac{n}{2}} \|f\|_{L^2(\mathbb{R}^n)}^2 \\ &\leq \left(\frac{|J|}{|J_i|}+1\right)\left(r^2\lambda^2+1\right)^{\frac{n}{2}} \|f\|_{L^2(\mathbb{R}^n)}^2 \\ &= \left(|J|\lambda^2\left(r^2\lambda^2+1\right)^{-1}+1\right)\left(r^2\lambda^2+1\right)^{\frac{n}{2}} \|f\|_{L^2(\mathbb{R}^n)}^2 \\ &= \left(|J|\lambda^2+r^2\lambda^2+1\right)\left(r^2\lambda^2+1\right)^{\frac{n-2}{2}} \|f\|_{L^2(\mathbb{R}^n)}^2 \\ &\leq \left(|J|^{\frac{1}{2}}\lambda+r\lambda+1\right)^2 (r\lambda+1)^{n-2} \|f\|_{L^2(\mathbb{R}^n)}^2,\end{aligned}$$

which gives the desired estimate.

3. The proofs of Lemmas 2.2 and 2.3

3.1. *The proof of Lemma 2.2*

In order to finish the proof of Lemma 2.2, we will need the following lemma, known as Van der Corput's lemma.

Lemma 3.1. *(Van der Corput's lemma [26]) For $a < b$, let $F \in C^\infty([a,b])$ be real valued and $\psi \in C^\infty([a,b])$.*

(i) *If $|F'(x)| \geq \lambda > 0,\ \forall\, x \in [a,b]$ and $F'(x)$ is monotonic on $[a,b]$, then*

$$\left|\int_a^b e^{iF(x)}\psi(x)\,dx\right| \leq \frac{C}{\lambda}\left(|\psi(b)| + \int_a^b |\psi'(x)|\ dx\right),$$

where C does not depend on F, ψ or $[a,b]$.

(ii) *If $|F''(x)| \geq \lambda > 0,\ \forall\, x \in [a,b]$, then*

$$\left|\int_a^b e^{iF(x)}\psi(x)\,dx\right| \leq \frac{C}{\lambda^{\frac{1}{2}}}\left(|\psi(b)| + \int_a^b |\psi'(x)|\ dx\right),$$

where C does not depend on F, ψ or $[a,b]$.

Proof of Lemma 2.2. Assume that χ is a smooth nonnegative function on $\mathbb{R}$, $\operatorname{supp}\chi \subset [\frac{1}{3},\frac{4}{3}]$ and $\chi \equiv 1$ on $[\frac{1}{2},1]$. We also use the same notation for the radial function on $\mathbb{R}^n$ with $\chi(\xi) = \chi(|\xi|)$, then we get the following truncated Boussinesq operator:

$$\mathcal{B}_\lambda f(x,t) := (2\pi)^{-n}\int_{\mathbb{R}^n} e^{i(x\cdot\xi + t|\xi|\sqrt{1+|\xi|^2})}\hat{f}(\xi)\chi\left(\frac{\xi}{\lambda}\right)\mathrm{d}\xi.$$

Let $t:\mathbb{R}^n \to J$ and $b:\mathbb{R}^n \to B$ be measurable functions. By linearizing the maximal operator, we have

$$\mathcal{B}_\lambda f(x+b(x),t(x)) = (2\pi)^{-n}\int_{\mathbb{R}^n} e^{i((x+b(x))\cdot\xi + t(x)|\xi|\sqrt{1+|\xi|^2})}\hat{f}(\xi)\chi\left(\frac{\xi}{\lambda}\right)\mathrm{d}\xi$$

$$= (2\pi)^{-n}\lambda^n \int_{\mathbb{R}^n} e^{i(\lambda(x+b(x))\cdot\eta+t(x)|\lambda\eta|\sqrt{1+|\lambda\eta|^2})}\hat{f}(\lambda\eta)\chi(\eta)\,\mathrm{d}\eta$$
$$=: \lambda^n T_\lambda(\hat{f}(\lambda\cdot))(x),$$

where

$$T_\lambda g(x) := \int_{\mathbb{R}^n} e^{i(\lambda(x+b(x))\cdot\xi+t(x)|\lambda\xi|\sqrt{1+|\lambda\xi|^2})}g(\xi)\chi(\xi)\,\mathrm{d}\xi.$$

We use the method of TT^* to finish the proof of Lemma 2.2. After some computation, we get that the kernel of $T_\lambda T_\lambda^*$ is

$$K_\lambda(x,y) := \int_{\mathbb{R}^n} e^{i(\lambda(x-y+b(x)-b(y))\cdot\xi+(t(x)-t(y))|\lambda\xi|\sqrt{1+|\lambda\xi|^2})}\chi^2(\xi)\,\mathrm{d}\xi.$$

We need to control $K_\lambda(x,y)$. However, it is difficult to estimate $K_\lambda(x,y)$, which leads us to majorize the kernel K_λ by a convolution kernel G_λ; that is $|K_\lambda(x,y)| \lesssim G_\lambda(x-y)$. Next, we divide the proof into two parts in order to obtain the expression of function G_λ.

On the one hand, we have that the trivial estimate

$$|K_\lambda(x,y)| \lesssim 1$$

holds for any x and y. We shall use this estimate when $\lambda|x-y| \le C_0 + 2\lambda d$, where $d = 2r$.

On the other hand, we discuss the case $\lambda|x-y| > C_0 + 2\lambda d$. Let σ be the surface measure on the unit sphere in $\mathbb{R}^n$. Clearly, polar coordinates yield that

$$K_\lambda(x,y) = \int_0^\infty e^{i\lambda r(t(x)-t(y))\sqrt{1+\lambda^2r^2}}\chi^2(r)\left(\int_{\mathbb{S}^{n-1}} e^{i\lambda r(x-y+b(x)-b(y))\cdot\xi'}\,\mathrm{d}\sigma(\xi')\right)r^{n-1}\,\mathrm{d}r$$
$$= \int_0^\infty e^{i\lambda r(t(x)-t(y))\sqrt{1+\lambda^2r^2}}\chi^2(r)\hat{\sigma}(\lambda(x-y+b(x)-b(y))r)r^{n-1}\,\mathrm{d}r.$$

Stein [26] implies that

$$\hat{\sigma}(\xi) = (2\pi)^{-n}|\xi|^{1-\frac{n}{2}}J_{\frac{n-2}{2}}(|\xi|),$$

where $J_{\frac{n-2}{2}}(|\xi|)$ is a Bessel function, which is defined by

$$J_\nu(t) = \frac{\left(\frac{t}{2}\right)^\nu}{\Gamma\left(\nu+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}\int_{-1}^1 e^{its}(1-s^2)^\nu\frac{\mathrm{d}s}{\sqrt{1-s^2}}.$$

We take C_0 large enough such that

$$J_{\frac{n-2}{2}}(r) = a_0\frac{e^{ir}}{r^{\frac{1}{2}}} + a_1\frac{e^{ir}}{r^{\frac{3}{2}}} + \cdots + a_N\frac{e^{ir}}{r^{N+\frac{1}{2}}} + b_0\frac{e^{-ir}}{r^{\frac{1}{2}}} + b_1\frac{e^{-ir}}{r^{\frac{3}{2}}} + \cdots + b_N\frac{e^{-ir}}{r^{N+\frac{1}{2}}} + R(r),$$

for $r \ge C_0$, where $|R(r)| \lesssim \frac{1}{r^{N+\frac{3}{2}}}$ (see [26]). This yields that

$$K_\lambda(x,y) = \int_0^\infty e^{i\lambda r(t(x)-t(y))\sqrt{1+\lambda^2r^2}}\chi^2(r)r^{n-1}\Bigg(a_0\frac{e^{i\lambda|x-y+b(x)-b(y)|r}}{(\lambda|x-y+b(x)-b(y)|r)^{\frac{n}{2}-\frac{1}{2}}}$$
$$+ \cdots + b_N\frac{e^{-i\lambda|x-y+b(x)-b(y)|r}}{(\lambda|x-y+b(x)-b(y)|r)^{N+\frac{n}{2}-\frac{1}{2}}} + R_1(\lambda|x-y+b(x)-b(y)|r)\Bigg)\mathrm{d}r, \tag{3.1}$$

where $R_1(r) = r^{1-\frac{n}{2}} R(r)$.

We first consider the remainder term. Since $|R(r)| \lesssim \frac{1}{r^{N+\frac{3}{2}}}$, we obtain

$$R_1(\lambda|x - y + b(x) - b(y)|r) \lesssim \frac{1}{(\lambda|x - y + b(x) - b(y)|)^{N+\frac{n}{2}+\frac{1}{2}}}.$$

Observing that $b(x), b(y) \in B$, we have $|b(x) - b(y)| \le d$, which yields

$$|x - y|\left(1 - \frac{d}{|x - y|}\right) = |x - y| - d < |x - y + b(x) - b(y)| < |x - y| + d = |x - y|\left(1 + \frac{d}{|x - y|}\right).$$

Note that $\lambda|x - y| > C_0 + 2\lambda d$. It follows that

$$\frac{1}{2}|x - y| < |x - y + b(x) - b(y)| < \frac{3}{2}|x - y|.$$

Furthermore, we conclude

$$R_1(\lambda|x - y + b(x) - b(y)|r) \lesssim \frac{1}{(\lambda|x - y|)^{N+\frac{n}{2}+\frac{1}{2}}}.$$

Henceforth, we establish the estimate of the remainder term

$$\left|K_{\lambda,\mathrm{rem}}(x, y)\right| \lesssim (\lambda|x - y|)^{-N-\frac{n}{2}-\frac{1}{2}}.$$

In order to obtain the upbound of $|K_\lambda(x, y)|$, it suffices to estimate the main term, which is defined by

$$K_{\lambda,\mathrm{main}}(x, y) := \frac{a_0}{(\lambda|x - y + b(x) - b(y)|)^{\frac{n}{2}-\frac{1}{2}}} \int_0^\infty e^{i\Phi_\lambda(r)}\chi^2(r) r^{\frac{n}{2}-\frac{1}{2}}\, \mathrm{d}r,$$

where

$$\Phi_\lambda(r) := \lambda r(t(x) - t(y))\sqrt{1 + \lambda^2 r^2} + \lambda|x - y + b(x) - b(y)|r.$$

Next, we make the following two-fold analysis:

Case 1. $|x - y| \gg \lambda|t(x) - t(y)|$. The definition of $\Phi_\lambda(r)$ implies that $\Phi'_\lambda(r) = \lambda(t(x) - t(y))\frac{1+2\lambda^2 r^2}{\sqrt{1+\lambda^2 r^2}} + \lambda|x - y + b(x) - b(y)|$, which yields

$$\left|\Phi'_\lambda(r)\right| \ge \lambda|x - y + b(x) - b(y)| - \lambda|t(x) - t(y)|\frac{1 + 2\lambda^2 r^2}{\sqrt{1 + \lambda^2 r^2}} \gtrsim \lambda|x - y|.$$

Using integration by parts, we obtain

$$\left|K_{\lambda,\mathrm{main}}(x, y)\right| \lesssim \frac{1}{(\lambda|x - y + b(x) - b(y)|)^{\frac{n}{2}-\frac{1}{2}}}(\lambda|x - y|)^{-N} \lesssim (\lambda|x - y|)^{-N} \text{ for } \forall\, N.$$

Case 2. $|x - y| \lesssim \lambda|t(x) - t(y)|$. Since $t(x), t(y) \in J$, we get $|x - y| \lesssim \lambda|J|$. It follows from the definition of $\Phi_\lambda(r)$ that $\Phi''_\lambda(r) = \lambda(t(x) - t(y))\frac{\lambda^2 r(3+2\lambda^2 r^2)}{(1+\lambda^2 r^2)^{\frac{3}{2}}}$, which implies

$$\left|\Phi''_\lambda(r)\right| \gtrsim \lambda^2|t(x) - t(y)|.$$

Using Lemma 3.1, we have

$$\left|K_{\lambda,\mathrm{main}}(x,y)\right| \lesssim \frac{1}{(\lambda|x-y|)^{\frac{n}{2}-\frac{1}{2}}}\left(\lambda^2|t(x)-t(y)|\right)^{-\frac{1}{2}} \lesssim (\lambda|x-y|)^{-\frac{n}{2}}.$$

We have established the upbound of $|K_\lambda(x,y)|$. In summary, by $|K_\lambda(x,y)| \lesssim G_\lambda(x-y)$, we may take

$$G_\lambda(x) := \chi_{\{|x|<C_0\lambda^{-1}+2d\}}(x) + \lambda^{-N}\chi_{\{|x|\geq\lambda^{-1}\}}(x)|x|^{-N} + \lambda^{-\frac{n}{2}}\chi_{\{|x|\leq C\lambda|J|\}}(x)|x|^{-\frac{n}{2}},$$

which yields

$$\begin{aligned}
\|G_\lambda\|_{L^1(\mathbb{R}^n)} &\lesssim (\lambda^{-1}+d)^n + \lambda^{-n} + \int_{|x|\leq C\lambda|J|} \lambda^{-\frac{n}{2}}|x|^{-\frac{n}{2}}\,\mathrm{d}x \\
&\lesssim (\lambda^{-1}+d)^n + \lambda^{-n} + \lambda^{-\frac{n}{2}}\int_0^{C\lambda|J|} r^{\frac{n}{2}-1}\,\mathrm{d}r \\
&\lesssim (\lambda^{-1}+d)^n + \lambda^{-n} + |J|^{\frac{n}{2}} \\
&\lesssim \lambda^{-n} + d^n + |J|^{\frac{n}{2}},
\end{aligned}$$

where in the second inequality we used polar coordinates. This implies that

$$\left\|T_\lambda T_\lambda^*\right\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} \lesssim \|G_\lambda\|_{L^1(\mathbb{R}^n)} \lesssim \lambda^{-n} + d^n + |J|^{\frac{n}{2}}.$$

It follows that

$$\|T_\lambda\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} \lesssim \lambda^{-\frac{n}{2}} + d^{\frac{n}{2}} + |J|^{\frac{n}{4}}.$$

We combine the above estimates and get

$$\begin{aligned}
\|\mathcal{B}_\lambda f(x+b(x),t(x))\|_{L^2(\mathbb{R}^n)} &\leq \lambda^n \left\|T_\lambda(\hat{f}(\lambda\cdot))\right\|_{L^2(\mathbb{R}^n)} \\
&\leq \lambda^n \|T_\lambda\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} \left\|\hat{f}(\lambda\cdot)\right\|_{L^2(\mathbb{R}^n)} \\
&\lesssim \lambda^n\left(\lambda^{-\frac{n}{2}} + d^{\frac{n}{2}} + |J|^{\frac{n}{4}}\right)\lambda^{-\frac{n}{2}}\|f\|_{L^2(\mathbb{R}^n)} \\
&= \left(1 + d^{\frac{n}{2}}\lambda^{\frac{n}{2}} + \lambda^{\frac{n}{2}}|J|^{\frac{n}{4}}\right)\|f\|_{L^2(\mathbb{R}^n)}.
\end{aligned}$$

This completes the proof of Lemma 2.2.

3.2. The proof of Lemma 2.3

Proof of Lemma 2.3. We write $y = (y_1,\cdots,y_n)$ and $y_0 = (y_{0,1},\cdots,y_{0,n})$. For $1 \leq j \leq n$, we write $\Lambda_j := e^{i\xi_j y_j} - e^{i\xi_j y_{0,j}}$ and $\Lambda_{n+1} := e^{it|\xi|\sqrt{1+|\xi|^2}} - e^{it_0|\xi|\sqrt{1+|\xi|^2}}$. It follows that

$$\begin{aligned}
\mathcal{B}f(x+y,t) &= (2\pi)^{-n}\int_{\mathbb{R}^n} e^{i\xi\cdot x}e^{i\xi_1 y_1}\cdots e^{i\xi_n y_n}e^{it|\xi|\sqrt{1+|\xi|^2}}\hat{f}(\xi)\,\mathrm{d}\xi \\
&= (2\pi)^{-n}\int_{\mathbb{R}^n} e^{i\xi\cdot x}\left(\Lambda_1 + e^{i\xi_1 y_{0,1}}\right)\cdots\left(\Lambda_n + e^{i\xi_n y_{0,n}}\right)\left(\Lambda_{n+1} + e^{it_0|\xi|\sqrt{1+|\xi|^2}}\right)\hat{f}(\xi)\,\mathrm{d}\xi.
\end{aligned}$$

Henceforth, $\mathcal{B}f(x+y,t)$ is the sum of integrals of the form

$$(2\pi)^{-n}\int_{\mathbb{R}^n} e^{i\xi\cdot x}\left(\prod_{j\in\Omega_1}\Lambda_j\right)\left(\prod_{j\in\Omega_2} e^{i\xi_j y_{0,j}}\right)\Lambda_{n+1}\hat{f}(\xi)\,\mathrm{d}\xi =: \mathcal{B}_1 f(x,y,t) \tag{3.2}$$

or

$$(2\pi)^{-n}\int_{\mathbb{R}^n} e^{i\xi\cdot x}\left(\prod_{j\in\Omega_1}\Lambda_j\right)\left(\prod_{j\in\Omega_2}e^{i\xi_j y_{0,j}}\right)e^{it_0|\xi|\sqrt{1+|\xi|^2}}\hat{f}(\xi)\,\mathrm{d}\xi =: \mathcal{B}_2 f(x,y,t). \tag{3.3}$$

Here, Ω_1 and Ω_2 are disjoint subsets of $\{1,2,3,\cdots,n\}$ and $\Omega_1\cup\Omega_2=\{1,2,3,\cdots,n\}$.

First, we give the discussion of $\mathcal{B}_1 f(x,y,t)$. For $j\in\Omega_1$, we have

$$\Lambda_j = i\xi_j\int_{y_{0,j}}^{y_j} e^{i\xi_j s_j}\,\mathrm{d}s_j,$$

and we also get

$$\Lambda_{n+1} = i|\xi|\sqrt{1+|\xi|^2}\int_{t_0}^{t} e^{i|\xi|\sqrt{1+|\xi|^2}s_{n+1}}\,\mathrm{d}s_{n+1}.$$

Assuming $\Omega_1=\{k_1,k_2,\cdots,k_m\}$, we conclude

$$\begin{aligned}\mathcal{B}_1 f(x,y,t) = &\int_{\mathbb{R}^n}\int_{y_{0,k_1}}^{y_{k_1}}\int_{y_{0,k_2}}^{y_{k_2}}\cdots\int_{y_{0,k_m}}^{y_{k_m}}\int_{t_0}^{t} e^{i\xi\cdot x}\left(\prod_{j\in\Omega_1} i\xi_j e^{i\xi_j s_j}\right)\left(\prod_{j\in\Omega_2}e^{i\xi_j y_{0,j}}\right)\\ &\times i|\xi|\sqrt{1+|\xi|^2}e^{i|\xi|\sqrt{1+|\xi|^2}s_{n+1}}\hat{f}(\xi)\,\mathrm{d}s_{k_1}\,\mathrm{d}s_{k_2}\cdots\,\mathrm{d}s_{k_m}\,\mathrm{d}s_{n+1}\,\mathrm{d}\xi.\end{aligned}$$

By changing the order of integration we get

$$|\mathcal{B}_1 f(x,y,t)| \le \int_{y_{0,k_1}}^{y_{k_1}}\cdots\int_{y_{0,k_m}}^{y_{k_m}}\int_{t_0}^{t}\left|F_{\Omega_1}(x;s_{k_1},\cdots,s_{k_m},s_{n+1})\right|\mathrm{d}s_{k_1}\cdots\,\mathrm{d}s_{k_m}\,\mathrm{d}s_{n+1},$$

where

$$\begin{aligned}&F_{\Omega_1}(x;s_{k_1},\cdots,s_{k_m},s_{n+1})\\ :=&(2\pi)^{-n}\int_{\mathbb{R}^n} e^{i\xi\cdot x}\left(\prod_{j\in\Omega_1} i\xi_j e^{i\xi_j s_j}\right)\left(\prod_{j\in\Omega_2}e^{i\xi_j y_{0,j}}\right)i|\xi|\sqrt{1+|\xi|^2}e^{i|\xi|\sqrt{1+|\xi|^2}s_{n+1}}\hat{f}(\xi)\,\mathrm{d}\xi.\end{aligned}$$

It follows that

$$\sup_{(y,t)\in E}|\mathcal{B}_1 f(x,y,t)| \le \int_{y_{0,k_1}}^{y_{0,k_1}+r}\cdots\int_{y_{0,k_m}}^{y_{0,k_m}+r}\int_{t_0}^{t_0+r^2}\left|F_{\Omega_1}(x;s_{k_1},\cdots,s_{k_m},s_{n+1})\right|\mathrm{d}s_{k_1}\cdots\,\mathrm{d}s_{k_m}\,\mathrm{d}s_{n+1}. \tag{3.4}$$

Taking L^2 norms of both sides of (3.4) and from Minkowski's inequality and Plancherel's theorem, we deduce

$$\begin{aligned}&\left\|\sup_{(y,t)\in E}|\mathcal{B}_1 f(\cdot,y,t)|\right\|_{L^2(\mathbb{R}^n)}\\ \le&\int_{y_{0,k_1}}^{y_{0,k_1}+r}\cdots\int_{y_{0,k_m}}^{y_{0,k_m}+r}\int_{t_0}^{t_0+r^2}\left\|F_{\Omega_1}(\cdot;s_{k_1},\cdots,s_{k_m},s_{n+1})\right\|_{L^2(\mathbb{R}^n)}\mathrm{d}s_{k_1}\cdots\,\mathrm{d}s_{k_m}\,\mathrm{d}s_{n+1}\end{aligned}$$

$$=(2\pi)^{-\frac{n}{2}}\int_{y_{0,k_1}}^{y_{0,k_1}+r}\cdots\int_{y_{0,k_m}}^{y_{0,k_m}+r}\int_{t_0}^{t_0+r^2}\left(\int_{\mathbb{R}^n}\left(\prod_{j\in\Omega_1}|\xi_j|^2\right)\left(|\xi|\sqrt{1+|\xi|^2}\right)^2\left|\hat{f}(\xi)\right|^2\,\mathrm{d}\xi\right)^{\frac{1}{2}}\mathrm{d}s_{k_1}\cdots\mathrm{d}s_{k_m}\,\mathrm{d}s_{n+1}$$
$$\leq r^m r^2\lambda^m\lambda^2\,\|f\|_{L^2(\mathbb{R}^n)},$$

where the last inequality follows by applying the fact that $f\in L^2(\mathbb{R}^n)$ and supp $\hat{f}\subset B(0,\lambda)$.

Next, we study $\mathcal{B}_2 f(x,y,t)$ in (3.3). The estimate of $\mathcal{B}_2 f(x,y,t)$ is similar to that of $\mathcal{B}_1 f(x,y,t)$. Since $\Omega_1=\{k_1,k_2,\cdots,k_m\}$, it follows that

$$\mathcal{B}_2 f(x,y,t)=\int_{\mathbb{R}^n}\int_{y_{0,k_1}}^{y_{k_1}}\cdots\int_{y_{0,k_m}}^{y_{k_m}}e^{i\xi\cdot x}\left(\prod_{j\in\Omega_1}i\xi_j e^{i\xi_j s_j}\right)\left(\prod_{j\in\Omega_2}e^{i\xi_j y_{0,j}}\right)e^{it_0|\xi|\sqrt{1+|\xi|^2}}\hat{f}(\xi)\,\mathrm{d}s_{k_1}\cdots\mathrm{d}s_{k_m}\,\mathrm{d}\xi.$$

Changing the order of integration again, one then obtains

$$|\mathcal{B}_2 f(x,y,t)|\leq\int_{y_{0,k_1}}^{y_{k_1}}\cdots\int_{y_{0,k_m}}^{y_{k_m}}\left|H_{\Omega_1}(x;s_{k_1},\cdots,s_{k_m})\right|\,\mathrm{d}s_{k_1}\cdots\mathrm{d}s_{k_m},$$

where

$$H_{\Omega_1}(x;s_{k_1},\cdots,s_{k_m}):=(2\pi)^{-n}\int_{\mathbb{R}^n}e^{i\xi\cdot x}\left(\prod_{j\in\Omega_1}i\xi_j e^{i\xi_j s_j}\right)\left(\prod_{j\in\Omega_2}e^{i\xi_j y_{0,j}}\right)e^{it_0|\xi|\sqrt{1+|\xi|^2}}\hat{f}(\xi)\,\mathrm{d}\xi.$$

Furthermore, we get

$$\sup_{(y,t)\in E}|\mathcal{B}_2 f(x,y,t)|\leq\int_{y_{0,k_1}}^{y_{0,k_1}+r}\cdots\int_{y_{0,k_m}}^{y_{0,k_m}+r}\left|H_{\Omega_1}(x;s_{k_1},\cdots,s_{k_m})\right|\,\mathrm{d}s_{k_1}\cdots\mathrm{d}s_{k_m}.$$

Using Minkowski's inequality and Plancherel's theorem, we then obtain

$$\left\|\sup_{(y,t)\in E}|\mathcal{B}_2 f(\cdot,y,t)|\right\|_{L^2(\mathbb{R}^n)}\leq\int_{y_{0,k_1}}^{y_{0,k_1}+r}\cdots\int_{y_{0,k_m}}^{y_{0,k_m}+r}\left\|H_{\Omega_1}(\cdot;s_{k_1},\cdots,s_{k_m})\right\|_{L^2(\mathbb{R}^n)}\mathrm{d}s_{k_1}\cdots\mathrm{d}s_{k_m}$$
$$=(2\pi)^{-\frac{n}{2}}\int_{y_{0,k_1}}^{y_{0,k_1}+r}\cdots\int_{y_{0,k_m}}^{y_{0,k_m}+r}\left(\int_{\mathbb{R}^n}\left(\prod_{j\in\Omega_1}|\xi_j|^2\right)\left|\hat{f}(\xi)\right|^2\,\mathrm{d}\xi\right)^{\frac{1}{2}}\mathrm{d}s_{k_1}\cdots\mathrm{d}s_{k_m}$$
$$\leq r^m\lambda^m\,\|f\|_{L^2(\mathbb{R}^n)}.$$

By summation of the above integrals, we conclude that

$$\left\|\mathcal{B}_E^* f\right\|_{L^2(\mathbb{R}^n)}\lesssim(1+r^2\lambda^2)(1+r\lambda)^n\,\|f\|_{L^2(\mathbb{R}^n)}.$$

Thus, Lemma 2.3 is established.

4. Conclusions

In this paper, we studied the boundedness of the Boussinesq maximal operator when $n\geq 1$. We obtained the Boussinesq maximal operator is bounded from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$ when $f\in L^2(\mathbb{R}^n)$ and supp $\hat{f}\subset B(0,\lambda)$ by using the methods of frequency decomposition, linearization of the maximal operator, TT^* and so on.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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