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Research article

The entropy weak solution to a nonlinear shallow water wave equation including the Degasperis-Procesi model

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Abstract: A nonlinear model, which characterizes motions of shallow water waves and includes the famous Degasperis-Procesi equation, is considered. The essential step is the derivation of the $L^2(\mathbb{R})$ uniform bound of solutions for the nonlinear model if its initial value belongs to space $L^2(\mathbb{R})$. Utilizing the bounded property leads to several estimates about its solutions. The viscous approximation technique is employed to establish the well-posedness of entropy weak solutions.

Keywords: Entropy solution; shallow water wave model; existence and uniqueness
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1. Introduction

This work focuses on the investigation of the equation

$$v_t - v_{txx} + mvv_x = 3\alpha v_x v_{xx} + \alpha v v_{xxx}, \tag{1.1}$$

where constants $\alpha > 0$ and $m > 0$. Equation (1.1) describes the motion of shallow water waves in certain sense [8]. In fact, the dydrodynamical equations derived in [8] includes Eq (1.1) as a special model.

If $m = 4$ and $\alpha = 1$, Eq (1.1) is turned into the Degasperis-Procesi (DP) model [12].

$$v_t - v_{txx} + 4vv_x = 3v_x v_{xx} + vv_{xxx}. \tag{1.2}$$

Degasperis et al. [13] construct a Lax pair to prove the integrability of DP model and obtain two infinite sequences of conserved quantities. The global weak solutions, global strong solutions and wave breaking conditions for (1.2) are studied within certain functional classes in [14, 22, 31].

The well-posedness and large time asymptotic features of the periodic entropy (discontinuous) solutions for Eq (1.2) is considered in [3]. Coclite and Karlsen [4] investigate entropy solutions to the DP model in the spaces $L^1(\mathbb{R}) \cap BV$ and $L^2(\mathbb{R}) \cap L^4(\mathbb{R})$, respectively. The bounded solutions in $L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and discontinuous solutions are discussed in [5]. As the DP and Camassa-Holm (CH [2]) equations possess similar dynamical properties, here we mention several works about the CH model. The wave breaking for nonlinear equations including the CH model is discovered in Constantin [7, 9]. Many dynamical results about the Camassa-Holm type models are derived and summarized in [1, 10, 11, 15, 16, 23–25, 28, 33]. Guo et al. [17] consider the dynamical properties of the CH type models with high order nonlinear terms (also see [18, 19, 30, 32]). Lai and Wu [21] study the existence of local solutions for a nonlinear model including the CH and DP model if initial data satisfy certain assumptions.

For model (1.1) endowed with initial value $v(0, x) = v_0(x) \in L^2(\mathbb{R})$, we derive that

$$c_1 \parallel v_0 \parallel_{L^2(\mathbb{R})} \leq \parallel v(t, \cdot) \parallel_{L^2(\mathbb{R})} \leq c_2 \parallel v_0 \parallel_{L^2(\mathbb{R})}, \tag{1.3}$$

in which $c_1 > 0$ and $c_2 > 0$ are constants.

The motivation of this work comes from the job in Coclite and Karlsen [5], in which the existence, uniqueness, stability of entropy solutions of DP equation are proved in $L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Under the condition $v_0(x)$ belonging to $L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, we investigate the shallow water wave (or generalized DP) equation (1.1) and prove its well-posedness of entropy (discontinuous) solutions. The novelty element in our job is that we establish inequality (1.3), namely, the $L^2(\mathbb{R})$ uniform bound of solution $v(t, x)$. The methods and ideas utilized in this work come from those presented in Coclite and Karlsen [4, 5].

The organization of our work is that section two provides several lemmas about the viscous approximations of Eq (1.1) and section three gives our main result and its proof.

2. Viscous approximations

We define the smooth function $\lambda(x)$ such that $\lambda(x) \geq 0$ for any $x \in \mathbb{R}$, $\lambda(x) = 0$ if $|x| \geq 1$ and $\int_{-\infty}^{\infty} \lambda(x)dx = 1$. For $0 < \varepsilon < \frac{1}{4}$, let $\lambda_\varepsilon(x) = \frac{1}{\varepsilon^{\frac{1}{4}}}\lambda(\frac{x}{\varepsilon^{\frac{1}{4}}})$ and $v_{0,\varepsilon} = \lambda_\varepsilon \star v_0 = \int_{\mathbb{R}} \lambda_\varepsilon(x - z, z)v_0(z)dz$. Provided that $v_0 \in H^s(\mathbb{R})$ $(s \geq 0)$, we conclude that $v_{0,\varepsilon} \in C^\infty$.

For conciseness, we employ c to represent arbitrary positive constants, which do not depend on ε and t. Let $L^p = L^p(\mathbb{R})$, $1 \leq p \leq \infty$.

Several properties of function $v_{0,\varepsilon}$ are summarized in the following conclusion.

Lemma 2.1. *[21] Assume* $1 \leq p < \infty$. *Then*

$$\begin{cases} v_{0,\varepsilon} \to v_0 \quad in \quad L^p \quad (\varepsilon \to 0), \quad \parallel v_{0,\varepsilon} \parallel_{L^\infty} \leq \parallel v_0 \parallel_{L^\infty}, \\ \parallel v_{0,\varepsilon} \parallel_{L^1} \leq \parallel v_0 \parallel_{L^1}, \quad \parallel v_{0,\varepsilon} \parallel_{L^p} \leq c \parallel v_0 \parallel_{L^p} . \end{cases}$$

Provided that $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$, the initial value problem for Eq (1.1) is written in the form

$$\begin{cases} \partial_t v - \partial^3_{txx} v + mv\partial_x v = 3\alpha\partial_x v\partial^2_{xx} v + \alpha v\partial^3_{xxx} v, \\ v(0, x) = v_0(x). \end{cases} \tag{2.1}$$

Consider the viscous approximations of system (2.1)

$$\begin{cases} \partial_t v_\varepsilon - \partial^3_{txx} v_\varepsilon + m v_\varepsilon \partial_x v_\varepsilon \\ \qquad = 3\alpha \partial_x v_\varepsilon \partial^2_{xx} v_\varepsilon + \alpha v_\varepsilon \partial^3_{xxx} v_\varepsilon + \varepsilon \partial^2_{xx} v_\varepsilon - \varepsilon \partial^4_{xxxx} v_\varepsilon, \\ v_\varepsilon(0, x) = v_{0,\varepsilon}(x). \end{cases} \tag{2.2}$$

Using the operator $\Lambda^{-2} = (1 - \frac{\partial^2}{\partial x^2})^{-1}$, we obtain that problem (2.2) becomes

$$\begin{cases} \partial_t v_\varepsilon + \frac{\alpha}{2} \partial_x (v_\varepsilon^2) + \partial_x H_\varepsilon = \varepsilon \partial^2_{xx} v_\varepsilon, \\ H_\varepsilon(t, x) = \frac{m-\alpha}{2} \Lambda^{-2} v_\varepsilon^2, \\ v_\varepsilon(0, x) = v_{0,\varepsilon}(x), \end{cases} \tag{2.3}$$

in which

$$H_\varepsilon(t, x) = \frac{m - \alpha}{4} \int_{\mathbb{R}} e^{-|x-\zeta|} v_\varepsilon^2(t, \zeta) d\zeta. \tag{2.4}$$

Lemma 2.2. *Let $v_0 \in H^s(\mathbb{R})$, $s \geq 0$ and $0 < \varepsilon < \frac{1}{4}$. Then problem (2.3) has a unique global smooth solution $v_\varepsilon(t, x)$ belonging to $C([0, \infty); H^s(\mathbb{R}))$.*

Proof. Utilizing the Theorem 2.3 in [6] completes the proof directly. □

The following lemma, which illustrates the $L^2(\mathbb{R})$ uniform bound of solutions for problem (2.3), takes a key role to discuss the dynamical features of entropy solutions in $L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ for Eq (1.1).

Lemma 2.3. *Provided that $v_0 \in L^2(\mathbb{R})$ and v_ε satisfies (2.3), $\alpha > 0$ and $m > 0$, then*

$$c_1 \parallel v_0 \parallel_{L^2(\mathbb{R})} \leq \parallel v_\varepsilon(t, \cdot) \parallel_{L^2(\mathbb{R})} \leq c_2 \parallel v_0 \parallel_{L^2(\mathbb{R})}, \tag{2.5}$$

$$\varepsilon \int_0^t \parallel \partial_x v_\varepsilon(\tau, \cdot) \parallel^2_{L^2(\mathbb{R})} d\tau \leq c_3 \parallel v_0 \parallel^2_{L^2(\mathbb{R})}, \tag{2.6}$$

in which constants $c_1 > 0$, $c_2 > 0$ and $c_3 > 0$ does not depend on ε and t.

Proof. Set $h_\varepsilon = (\frac{m}{\alpha} - \partial^2_{xx})^{-1} v_\varepsilon$. We have

$$\frac{m}{\alpha} h_\varepsilon - \partial^2_{xx} h_\varepsilon = v_\varepsilon. \tag{2.7}$$

Utilizing $(h_\varepsilon - \partial^2_{xx} h_\varepsilon)$ to multiply the first equation in (2.3) arises

$$\begin{aligned} &\int_{\mathbb{R}} \partial_t v_\varepsilon (h_\varepsilon - \partial^2_{xx} h_\varepsilon) dx - \varepsilon \int_{\mathbb{R}} \partial^2_{xx} v_\varepsilon (h_\varepsilon - \partial^2_{xx} h_\varepsilon) dx \\ &\qquad = -\alpha \int_{\mathbb{R}} v_\varepsilon \partial_x v_\varepsilon (h_\varepsilon - \partial^2_{xx} h_\varepsilon) dx - \int_{\mathbb{R}} \partial_x H_\varepsilon(t, x)(h_\varepsilon - \partial^2_{xx} h_\varepsilon) dx. \end{aligned} \tag{2.8}$$

From (2.8), we have

$$\int_{\mathbb{R}} \partial_t v_\varepsilon (h_\varepsilon - \partial^2_{xx} h_\varepsilon) dx - \varepsilon \int_{\mathbb{R}} \partial^2_{xx} v_\varepsilon (h_\varepsilon - \partial^2_{xx} h_\varepsilon) dx$$

$$
\begin{aligned}
&= \int_{\mathbb{R}} (\frac{m}{\alpha}\partial_t h_\varepsilon - \partial^3_{txx} h_\varepsilon)(h_\varepsilon - \partial^2_{xx} h_\varepsilon) dx \\
&\qquad -\varepsilon \int_{\mathbb{R}} (\frac{m}{\alpha}\partial^2_{xx} h_\varepsilon - \partial^4_{xxxx} h_\varepsilon)(h_\varepsilon - \partial^2_{xx} h_\varepsilon) dx \\
&= \int_{\mathbb{R}} (\frac{m}{\alpha} h_\varepsilon \partial_t h_\varepsilon - h_\varepsilon \partial^3_{txx} h_\varepsilon - \frac{m}{\alpha}\partial_t h_\varepsilon \partial^2_{xx} h_\varepsilon + \partial^2_{xx} h_\varepsilon \partial^3_{txx} h_\varepsilon) dx \\
&\qquad -\varepsilon \int_{\mathbb{R}} (\frac{m}{\alpha} h_\varepsilon \partial^2_{xx} h_\varepsilon - \frac{m}{\alpha}(\partial^2_{xx} h_\varepsilon)^2 - h_\varepsilon \partial^4_{xxxx} h_\varepsilon + \partial^2_{xx} h_\varepsilon \partial^4_{xxxx} h_\varepsilon) dx \\
&= \int_{\mathbb{R}} (\frac{m}{\alpha} h_\varepsilon \partial_t h_\varepsilon - (\frac{m}{\alpha} + 1) h_\varepsilon \partial^3_{txx} h_\varepsilon + \partial^2_{xx} h_\varepsilon \partial^3_{txx} h_\varepsilon) dx \\
&\qquad -\varepsilon \int_{\mathbb{R}} (\frac{m}{\alpha} h_\varepsilon \partial^2_{xx} h_\varepsilon - (\frac{m}{\alpha} + 1) h_\varepsilon \partial^4_{xxxx} h_\varepsilon + \partial^2_{xx} h_\varepsilon \partial^4_{xxxx} h_\varepsilon) dx \\
&= \int_{\mathbb{R}} (\frac{m}{\alpha} h_\varepsilon \partial_t h_\varepsilon + (\frac{m}{\alpha} + 1) \partial_x h_\varepsilon \partial^2_{tx} h_\varepsilon + \partial^2_{xx} h_\varepsilon \partial^3_{txx} h_\varepsilon) dx \\
&\qquad -\varepsilon \int_{\mathbb{R}} (-\frac{m}{\alpha} \partial_x h_\varepsilon \partial_x h_\varepsilon - (\frac{m}{\alpha} + 1) \partial^2_{xx} h_\varepsilon \partial^2_{xx} h_\varepsilon - \partial^3_{xxx} h_\varepsilon \partial^3_{xxx} h_\varepsilon) dx \\
&= \frac{1}{2}\frac{d}{dt} \int_{\mathbb{R}} \Big(\frac{m}{\alpha} h_\varepsilon^2 + (\frac{m}{\alpha} + 1)(\partial_x h_\varepsilon)^2 + (\partial^2_{xx} h_\varepsilon)^2\Big) dx \\
&\qquad +\varepsilon \int_{\mathbb{R}} \Big(\frac{m}{\alpha}(\partial_x h_\varepsilon)^2 + (\frac{m}{\alpha} + 1)(\partial^2_{xx} h_\varepsilon)^2 + (\partial^3_{xxx} h_\varepsilon)^2\Big) dx,
\end{aligned} \tag{2.9}
$$

in which we have utilized integration by parts.

For the right side in (2.8), making use of (2.7) and integration by parts gives rise to

$$
\begin{aligned}
&-\alpha \int_{\mathbb{R}} v_\varepsilon \partial_x v_\varepsilon (h_\varepsilon - \partial^2_{xx} h_\varepsilon) dx - \int_{\mathbb{R}} \partial_x H_\varepsilon(t,x)(h_\varepsilon - \partial^2_{xx} h_\varepsilon) dx \\
&= -\alpha \int_{\mathbb{R}} v_\varepsilon \partial_x v_\varepsilon (h_\varepsilon - \partial^2_{xx} h_\varepsilon) dx + \int_{\mathbb{R}} (H_\varepsilon - \partial^2_{xx} H_\varepsilon)(t,x) \partial_x h_\varepsilon dx \\
&= -\alpha \int_{\mathbb{R}} v_\varepsilon \partial_x v_\varepsilon (h_\varepsilon - \partial^2_{xx} h_\varepsilon) dx + \frac{m-\alpha}{2} \int_{\mathbb{R}} v_\varepsilon^2 \partial_x h_\varepsilon dx \\
&= \frac{\alpha}{2} \int_{\mathbb{R}} \partial_x (v_\varepsilon^2) \partial^2_{xx} h_\varepsilon dx + \frac{m}{2} \int_{\mathbb{R}} v_\varepsilon^2 \partial_x h_\varepsilon dx \\
&= \frac{\alpha}{2} \int_{\mathbb{R}} \partial_x (v_\varepsilon^2) \Big[\frac{m}{\alpha} h_\varepsilon - v_\varepsilon\Big] dx + \frac{m}{2} \int_{\mathbb{R}} v_\varepsilon^2 \partial_x h_\varepsilon dx \\
&= \frac{\alpha}{2} \int_{\mathbb{R}} v_\varepsilon^2 \partial_x v_\varepsilon dx \\
&= 0.
\end{aligned} \tag{2.10}
$$

From (2.8)–(2.10), we derive that

$$
\begin{aligned}
&\frac{m}{\alpha} \parallel h_\varepsilon \parallel^2_{L^2} + (\frac{m}{\alpha} + 1) \parallel \partial_x h_\varepsilon \parallel^2_{L^2} + \parallel \partial^2_{xx} h_\varepsilon \parallel^2_{L^2} \\
&\qquad +2\varepsilon \int_0^t \Big(\frac{m}{\alpha} \parallel \partial_x h_\varepsilon \parallel^2_{L^2} + (\frac{m}{\alpha} + 1) \parallel \partial^2_{xx} h_\varepsilon \parallel^2_{L^2} + \parallel \partial^3_{xxx} h_\varepsilon \parallel^2_{L^2} \Big) d\tau
\end{aligned}
$$

$$= \frac{m}{\alpha} \parallel h_\varepsilon(0,\cdot) \parallel_{L^2}^2 + (\frac{m}{\alpha}+1) \parallel \partial_x h_\varepsilon(0,\cdot) \parallel_{L^2}^2 + \parallel \partial_{xx}^2 h_\varepsilon(0,\cdot) \parallel_{L^2}^2 . \tag{2.11}$$

Utilizing (2.7) yields

$$\begin{aligned} \parallel v_\varepsilon(t,\cdot) \parallel_{L^2(\mathbb{R})}^2 &= \int_{\mathbb{R}} \big(-\partial_{xx}^2 h_\varepsilon + \frac{m}{\alpha} h_\varepsilon\big)^2 dx \\ &= \int_{\mathbb{R}} (\partial_{xx}^2 h_\varepsilon)^2 dx - \frac{2m}{\alpha} \int_{\mathbb{R}} h_\varepsilon \partial_{xx}^2 h_\varepsilon dx + \frac{m^2}{\alpha^2} \int_{\mathbb{R}} h_\varepsilon^2 dx \\ &= \int_{\mathbb{R}} (\partial_{xx}^2 h_\varepsilon)^2 dx + \frac{2m}{\alpha} \int_{\mathbb{R}} (\partial_x h_\varepsilon)^2 dx + \frac{m^2}{\alpha^2} \int_{\mathbb{R}} h_\varepsilon^2 dx \\ &= \frac{m^2}{\alpha^2} \parallel h_\varepsilon \parallel^2 + \frac{2m}{\alpha} \parallel \partial_x h_\varepsilon \parallel^2 + \parallel \partial_{xx}^2 h_\varepsilon \parallel^2 . \end{aligned} \tag{2.12}$$

We utilize the definition of the norm $L^2(\mathbb{R})$, the right side of (2.11) and the left side of (2.12). From (2.11), (2.12) and Lemma 2.1, we derive that there exist constants $c_1 > 0$ and $c_2 > 0$ to guarantee that

$$c_1 \parallel v_0 \parallel_{L^2(\mathbb{R})} \leq \parallel v_\varepsilon \parallel_{L^2(\mathbb{R})} \leq c_2 \parallel v_0 \parallel_{L^2(\mathbb{R})} . \tag{2.13}$$

From (2.11), we have

$$\begin{aligned} \varepsilon \int_0^t \parallel \partial_x v_\varepsilon \parallel_{L^2}^2 d\tau &\leq \varepsilon c \int_0^t \Big(\frac{m}{\alpha} \parallel \partial_x h_\varepsilon \parallel_{L^2}^2 + (\frac{m}{\alpha}+1) \parallel \partial_{xx}^2 h_\varepsilon \parallel_{L^2}^2 + \parallel \partial_{xxx}^3 h_\varepsilon \parallel_{L^2}^2 \Big) d\tau \\ &\leq \varepsilon c \Big(\parallel h_\varepsilon(0,\cdot) \parallel_{L^2}^2 + \parallel \partial_x h_\varepsilon(0,\cdot) \parallel_{L^2}^2 + \parallel \partial_{xx}^2 h_\varepsilon(0,\cdot) \parallel_{L^2}^2 \Big) \\ &\leq c \parallel v_{0,\varepsilon} \parallel_{L^2}^2 \\ &\leq c \parallel v_0 \parallel_{L^2}^2 . \end{aligned} \tag{2.14}$$

Applying (2.13) and (2.14) directly derives (2.5) and (2.6). □

We give several estimates about the nonlocal term $H_\varepsilon(t,x)$ by applying Lemma 2.3.

Lemma 2.4. *If* $v_0 \in L^2(\mathbb{R})$, *then*

$$\parallel H_\varepsilon \parallel_{L^\infty}, \quad \parallel \partial_x H_\varepsilon \parallel_{L^\infty} \leq c \parallel v_0 \parallel_{L^2}^2, \tag{2.15}$$

$$\parallel H_\varepsilon(t,\cdot) \parallel_{L^1}, \quad \parallel \partial_x H_\varepsilon(t,\cdot) \parallel_{L^1}, \quad \parallel \partial_{xx}^2 H_\varepsilon(t,\cdot) \parallel_{L^1} \leq c \parallel v_0 \parallel_{L^2}^2 . \tag{2.16}$$

Proof. Utilizing the expressions

$$\begin{aligned} H_\varepsilon(t,x) &= \frac{m-\alpha}{4} \int_{\mathbb{R}} e^{-|x-\zeta|} v_\varepsilon^2(t,\zeta) d\zeta, \\ \partial_x H_\varepsilon(t,x) &= \frac{m-\alpha}{4} \int_{\mathbb{R}} e^{-|x-\zeta|} sign(\zeta - x) v_\varepsilon^2(t,\zeta) d\zeta \end{aligned}$$

and Lemma 2.3, we complete the proof of (2.15). Noting that

$$\int_{\mathbb{R}} |H_\varepsilon(t,x)| dx, \quad \int_{\mathbb{R}} |\partial_x H_\varepsilon(t,x)| dx \leq c \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} e^{-|x-\zeta|} dx \Big) v_\varepsilon^2 d\zeta \leq c \parallel v \parallel_{L^2}^2 \leq c \parallel v_0 \parallel_{L^2}^2$$

and $\partial_{xx}^2 H_\varepsilon = H_\varepsilon - \frac{m-\alpha}{2} v_\varepsilon^2$, we finish the proof of (2.16). □

Lemma 2.5. *Provided that $v_0 \in Ł^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and v_ε satisfies (2.3), Then the inequality*

$$\| v_\varepsilon(t, \cdot) \|_{L^\infty} \leq \| v_0 \|_{L^\infty} + ct \| v_0 \|_{L^2}^2 \tag{2.17}$$

holds for any $t \geq 0$.

Proof. From problem (2.3), we have

$$\partial_t v_\varepsilon + \alpha v_\varepsilon \partial_x v_\varepsilon - \varepsilon \partial_{xx}^2 v_\varepsilon = -\partial_x H_\varepsilon. \tag{2.18}$$

Using Lemma 2.4 yields

$$\| \partial_x H_\varepsilon \|_{L^\infty(\mathbb{R})} \leq c \| v_0 \|_{L^2}^2 .$$

Considering the function $A(t) = \| v_0 \|_{L^\infty(\mathbb{R})} + ct \| v_0 \|_{L^2}^2$ arises

$$\frac{dA}{dt} = c \| v_0 \|_{L^2}^2 .$$

From Lemma 2.1, we acquire $\| v_{0,\varepsilon} \|_{L^\infty(\mathbb{R})} \leq A(0)$. Utilizing the comparison principle for parabolic equation (2.18) deduces that (2.17) holds. □

Employing Lemmas 2.3, 2.4 and the approaches utilized in [4, 5], we have the conclusion.

Lemma 2.6. *Suppose $t \in [0, T]$ and $v_0 \in Ł^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Then*

$$\| v_\varepsilon \|_{L^1(\mathbb{R})} \leq \| v_0 \|_{L^1(\mathbb{R})} + ct \| v \|_{L^2(\mathbb{R})}^2, \tag{2.19}$$

$$\partial_x v_\varepsilon(t, x) \leq \frac{1}{t} + M_T, \tag{2.20}$$

in which the constant M_T depends on T.

Since the proofs to inequalities (2.19) and (2.20) are very analogous to those of Lemma 2.5 in Coclite [4] and Lemma 6 in [5], respectively, we omit their proofs.

Let $\Omega_T = [0, T] \times \mathbb{R}$ and $\Omega_\infty = [0, \infty) \times \mathbb{R}$. According to the definitions of weak solutions in [4, 5], we state the following concepts.

Definition 2.1. *Provided that the following two assumptions hold,*
(a) $v \in L^\infty(R_+; L^2(\mathbb{R}))$,
(b) $\partial_t v + \frac{\alpha}{2}\partial_x(v^2) + \partial_x H(t, x) = 0$ in $D'(\Omega_\infty)$, namely, $\forall g(t, x) \in C_c^\infty(\Omega_\infty)$,

$$\iint_{\Omega_\infty} \Big(v\partial_t g + \frac{\alpha v^2}{2}\partial_x g - \partial_x H(t, x) g \Big) dxdt + \int_{\mathbb{R}} v_0(x) g(0, x) dx = 0, \tag{2.21}$$

then function v is called a weak solution of system (2.1).

Definition 2.2. *Assume that the following three assumptions hold.*
(a) v satisfies Definition 2.1,
(b) v belongs to $L^\infty(\Omega_T)$,
(c) Entropy $\theta(v) \in C^2(\mathbb{R})$ is a convex function with entropy flux q satisfying $q'(v) = \alpha\theta'(v)v$ and

$$\partial_t\theta(v) + \partial_x q(v) + \theta'(v)\partial_x H \leq 0 \quad in \quad D'(\Omega_\infty),$$

namely, $\forall g(t,x) \in C_c^\infty(\Omega_\infty)$, $g(t,x) \geq 0$

$$\iint\limits_{\Omega_\infty} \Big(\theta(v)\partial_t g + q(v)\partial_x g - \theta'(v)\partial_x H g\Big) dxdt + \int_{\mathbb{R}} \theta(v_0(x))g(0,x)dx \geq 0, \tag{2.22}$$

Then v is called an entropy weak solution of system (2.1).

Remark 1. Utilizing the arguments in [4, 5], for any constant $k \in \mathbb{R}$, we choose $\theta(v) = |v-k|$ and $q(v) := \frac{\alpha}{2} sign(v-k)(v^2-k^2)$, which are the Kruzkov entropies/entrop fluxes to satisfy (2.22). The assumptions (a) and (b) in Definition 2.2 guarantee that (2.22) makes sense (see Kružkov [20]). Based on the statement in [4, 20], we state that the entropy formulation (2.22) contains the weak formulation (2.21).

3. Main results

The entropy weak solutions for Eq (2.1) are usually discontinuous. However, it possesses the following $L^1(\mathbb{R})$ property, illustrating that the entropy solution for Eq (1.1) is unique.

Theorem 3.1. (L^1*-stability) For any $T > 0$, suppose that $v_1(t,x)$ and $v_2(t,x)$ are two entropy weak solutions of problem (2.5) with initial data $v_{01}, v_{02} \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, respectively. Then*

$$\| v_1(t,\cdot) - v_2(t,\cdot) \|_{L^1(\mathbb{R})} \leq e^{C_T t} \int_{-\infty}^{\infty} |v_{01}(x) - v_{02}(x)| dx, \tag{3.1}$$

in which C_T depends on v_{01}, v_{02} and T.

Utilizing the device of doubling the space variable in Kružkov [20] or some statements in [4,5], we can prove inequality (3.1). Here, we omit its proof.

We let v_{ε_n} denote any subsequence of v_ε ($\varepsilon \to 0$). The existence of v_ε is ensured by Lemma 2.2. The compensated compactness methods in [27, 29] shall be employed to handle with the problem of strong convergence for v_{ε_n}.

Lemma 3.1. *[27] Suppose that a family of functions $\{v_\varepsilon\}_{\varepsilon>0}$ satisfy*

$$\| v_\varepsilon \|_{L^\infty} \leq C_T.$$

For an arbitrary convex function $\theta \in C^2(\mathbb{R})$ and for $q(v) = \beta v\theta'(v)$ (constant $\beta > 0$), let the sequence

$$\{\partial_t\theta(v_\varepsilon) + \partial_x q(v_\varepsilon)\}_{\varepsilon>0}$$

be compact in $H^{-1}_{loc}((0,\infty)\times\mathbb{R})$. Then there exists $v \in L^\infty((0,T)\times\mathbb{R})$ to ensure that

$$v_{\varepsilon_n} \to v \quad in \quad L^p_{loc}((0,\infty)\times\mathbb{R}),$$

where $1 \leq p < \infty$.

Lemma 3.2. *[26] Suppose a bounded open subset $\Omega \in \mathbb{R}^n$, $n \geq 2$. Let distribution sequence $\{K_n\}_{n=1}^{\infty}$ be bounded in $W^{-1,\infty}(\Omega)$ and satisfy*

$$K_n = K_n^{(1)} + K_n^{(2)},$$

in which $\{K_n^{(1)}\}_{n=1}^{\infty}$ belongs to a compact subset of $H_{loc}^{-1}(\Omega)$ and $\{K_n^{(2)}\}_{n=1}^{\infty}$ belongs to a bounded subset of $L_{loc}^1(\Omega)$. Then $\{K_n\}_{n=1}^{\infty}$ belongs to a compact subset of $H_{loc}^{-1}(\Omega)$.

Lemma 3.3. *Suppose $v_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and v_ε satisfies system (2.3). For a subsequence $\{v_{\varepsilon_n}\}_{n=1}^{\infty}$ of $\{v_\varepsilon\}_{\varepsilon>0}$, an arbitrary $T > 0$ and $1 \leq p < \infty$, then there has a limit function*

$$v \in L^\infty(\mathbb{R}_+; L^2(\mathbb{R})) \cap L^\infty((0,T); L^\infty \cap L^1(\mathbb{R})) \tag{3.2}$$

to ensure that

$$v_{\varepsilon_n} \to v \quad in \quad L^p((0,T] \times \mathbb{R}). \tag{3.3}$$

Proof. Assume that $\theta : \mathbb{R} \to \mathbb{R}$ and $q'(v) = \alpha\theta'(v)v$ are defined in Definition 2.2. We set

$$\partial_t \theta(v_\varepsilon) + \partial_x q(v_\varepsilon) = K_\varepsilon^{(1)} + K_\varepsilon^{(2)},$$

where

$$\begin{cases} K_\varepsilon^{(1)} = \varepsilon\partial_{xx}^2\theta(v_\varepsilon), \\ K_\varepsilon^{(2)} = -\varepsilon\theta''(v_\varepsilon)(\partial_x v_\varepsilon)^2 + \theta'(v_\varepsilon)\partial_x H_\varepsilon(t,x). \end{cases}$$

We require that

$$\begin{cases} K_\varepsilon^{(1)} \to 0 \quad \text{in} \quad H^{-1}(\Omega_T), \\ K_\varepsilon^{(2)} \quad \text{is uniformly bounded in} \quad L^1(\Omega_T). \end{cases} \tag{3.4}$$

Utilizing Lemmas 2.3, 2.4 and 2.6 yields

$$\begin{cases} \| \varepsilon\partial_{xx}^2\theta(v_\varepsilon) \|_{H^{-1}(\mathbb{R}_+\times\mathbb{R})} \leq \sqrt{\varepsilon}c \| \theta' \|_{L^\infty} \| v_0 \|_{L^2(\mathbb{R})} \to 0, \\ \| \varepsilon\theta''(v_\varepsilon)(\partial_x v_\varepsilon)^2 \| \leq c \| \theta'' \|_{L^\infty(\mathbb{R})} \| v_0 \|_{L^2(\mathbb{R})}, \\ \| \theta'(v_\varepsilon) \|_{L^1((0,T)\times\mathbb{R})} \leq c \| \theta' \|_{L^\infty(\mathbb{R})} \| v_0 \|_{L^2(\mathbb{R})}, \end{cases}$$

which leads to (3.4). Employing Remark 1, Lemmas 3.1 and 3.2, for $1 \leq p < \infty$, we deduce that there must have a subsequence $v_{\varepsilon_n}, n = 1, 2, 3, \cdots$ and v satisfying (3.2) to guarantee that

$$v_{\varepsilon_n} \to v \quad \text{a.e in} \quad \Omega_\infty, \quad v_{\varepsilon_n} \to v \quad \text{in} \quad L_{loc}^p(\Omega_\infty). \tag{3.5}$$

From Lemmas 2.5 and 2.6, combining (3.5), we obtain (3.3). □

Lemma 3.4. *Assume $v_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and v_ε solves system (2.3). Let $\{\varepsilon_n\}_{n=1}^{\infty}$ and v be stated in Lemma 3.3. For an arbitrary $T > 0$ and $1 \leq p < \infty$, then there has a function H satisfying*

$$H_{\varepsilon_n} \to H \quad in \quad L^p([0,T); W^{1,p}(\mathbb{R})).$$

The procedures to prove Lemma 3.4 are analogous to that of Lemma 9 in [5]. Its proof is omitted here.

Theorem 3.2. *Suppose that $v_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Then system (2.1) has only one entropy weak solution.*

Proof. Provided that $g(t, x) \in C_c^\infty(\mathbb{R}_+ \times \mathbb{R})$, we deduce from (2.21) that

$$\int_{\mathbb{R}_+} \int_{\mathbb{R}} \Big(v_\varepsilon \partial_t g + \frac{\alpha}{2} v_\varepsilon^2 \partial_x g - \partial_x H_\varepsilon g + \varepsilon v_\varepsilon \partial_{xx}^2 g \Big) dx dt + \int_{\mathbb{R}} v_{0,\varepsilon} g(0, x) dx = 0.$$

We conclude that in the sense of Definition 2.1, v in Lemma 3.3 is a weak solution to system (2.1). It needs to confirm that the weak function v obeys the entropy inequalities in the sense of Definition 2.2. Let $q'(v) = \alpha v \theta'(v)$. Provided that $\theta \in C^2(\mathbb{R})$ is a convex function, we utilize the convexity of θ and system (2.3) to obtain

$$\partial_t \theta(v_\varepsilon) + \partial_x q(v_\varepsilon) + \theta'(v_\varepsilon) \partial_x H_\varepsilon = \varepsilon \partial_{xx}^2 \theta(v_\varepsilon) - \varepsilon \theta''(v_\varepsilon)(\partial_x v_\varepsilon)^2 \le \varepsilon \partial_{xx}^2 \theta(v_\varepsilon).$$

Therefore, from Lemmas 3.3 and 3.4 and the above the entropy inequality, we establish the existence of entropy solutions. Utilizing Theorem 3.1, we have the uniqueness. The proof is finished. □

4. Conclusions

In this work, we investigate a nonlinear shallow water wave equation, which includes the famous Degasperis-Procesi equation. Firstly, we derive that the viscous solutions are uniformly bounded in $L^2(\mathbb{R})$ space. Secondly, several estimates about the viscous solutions are established under the condition that the initial value belongs to space $L^2(\mathbb{R}) \cap Ł^\infty(\mathbb{R})$. Finally, we prove that the existence and uniqueness of entropy weak solutions the nonlinear equation in the space $L^2(\mathbb{R}) \cap Ł^\infty(\mathbb{R})$.

Use of AI tools declaration

The authors declare they have not used any Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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