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Research article

Generalized viscosity approximation method for solving split generalized mixed equilibrium problem with application to compressed sensing

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Abstract: In this study, we establish a new inertial generalized viscosity approximation method and prove that the resulting sequence strongly converges to a common solution of a split generalized mixed equilibrium problem, fixed point problem for a finite family of nonexpansive mappings and hierarchical fixed point problem in real Hilbert spaces. As an application, we demonstrate the use of our main finding in compressed sensing in signal processing. Additionally, we include numerical examples to evaluate the efficiency of the suggested method and then conduct a comparative analysis of its efficiency with different methods. Our findings can be used in a variety of contexts to improve results.

Keywords: projection operator; hierarchical fixed point; equilibrium problem; CQ-algorithm; approximation; iterative methods; numerical results
Mathematics Subject Classification: 47H09, 47H10

1. Introduction

Consider H a Hilbert space and Q a nonempty, closed and convex subset of H. Let $F : Q \times Q \to \mathbb{R}$ be a bifunction, $g : Q \to H$ a nonlinear mapping and $\psi : Q \to \mathbb{R}$ a function. Then, the generalized mixed equilibrium problem (GMEP) identifies $\xi \in Q$ such that

$$F(\xi, z) + \langle g\xi, z - \xi\rangle + \psi(z) - \psi(\xi) \geq 0 \text{ for all } z \in Q. \tag{1.1}$$

If $g = 0$, Problem (1.1) becomes a mixed equilibrium problem to identify $\xi \in Q$ such that

$$F(\xi, z) + \psi(z) - \psi(\xi) \geq 0 \text{ for all } z \in Q. \tag{1.2}$$

If $\psi = 0$, Problem (1.2) becomes a mixed equilibrium problem (MEP), which is to identify $\xi \in Q$ such that

$$F(\xi, z) \geq 0 \text{ for all } z \in Q. \tag{1.3}$$

If $F(t, z) = 0$ for all $t, z \in Q$, Problem (1.1) becomes a generalized vector variational inequality problem which identifying $\xi \in Q$ such that

$$\langle g\xi, z - \xi \rangle + \psi(z) - \psi(\xi) \geq 0 \text{ for all } z \in Q. \tag{1.4}$$

Censor et al. [1] proposed the split feasibility problem (SFP) for modeling inverse problems for the first time in 1994. SFPs are used in various applications, including signal processing, image restoration, computer tomography, intensity-modulated radiation therapy (IMRT) and so on; see [2]. SFP involves the use of a bounded linear operator for identifying a point in a nonempty closed and convex set in the space whose image corresponds to another nonempty closed and convex set in the image space.

Suppose that H_1 and H_2 are Hilbert spaces and Q_1 and Q_2 are nonempty, closed and convex subsets of H_1 and H_2, respectively. Suppose that $D : H_1 \to H_2$ is a bounded linear operator. Let $F_1 : Q_1 \times Q_1 \to \mathbb{R}$, $F_2 : Q_2 \times Q_2 \to \mathbb{R}$ be bifunctions, $g_1 : Q_1 \to H_1$, $g_2 : Q_2 \to H_2$ be nonlinear mappings and $\psi_1 : Q_1 \to \mathbb{R}$, $\psi_2 : Q_2 \to \mathbb{R}$ be functions. Then, the split generalized mixed equilibrium problem (SGMEP), which involves finding $\xi \in Q_1$ such that

$$F_1(\xi, z) + \langle g_1\xi, z - \xi \rangle + \psi_1(z) - \psi_1(\xi) \geq 0 \text{ for all } z \in Q_1, \tag{1.5}$$

and $w^* = D\xi \in Q_2$ solve

$$F_2(w^*, w) + \langle g_2 w^*, w - w^* \rangle + \psi_2(w) - \psi_2(w^*) \geq 0 \text{ for all } w \in Q_2. \tag{1.6}$$

Let the solution set of (1.5), (1.6) and SGMEP be denoted by $GMEP(F_1, g_1, \psi_1, Q_1)$, $GMEP(F_2, g_2, \psi_2, Q_2)$ and Θ, respectively.

Fan [3] was the first to introduce the equilibrium problem in 1972, but Blum and Oettli [4] made the most significant contributions to the issue in 1994. They studied variational principles and existence theorems for equilibrium problems, which have a significant role on the establishment of numerous domains in both pure and applied sciences; see [5, 6]. These equilibrium problems serve as generalizations of various mathematical problems, including Nash equilibrium, optimization, variational inequality, minimization, saddle point problems and so on. Equilibrium problems have several applications in image reconstruction, networks, engineering, physics, game theory, economics, transportation and elasticity. As a result, the equilibrium problem has been expanded to broader issues in various ways.

GMEP was introduced by Peng and Yao [7] in 2008 and it includes the variational inequality problem (VIP), minimization problem (MP), fixed point problem (FPP) and many more as its special cases, see [8, 9]. SGMEP includes split monotone variational inclusion problem (SMVIP), generalized mixed equilibrium problem (GMEP), mixed equilibrium problem (MEP), equilibrium problem (EP), variational inequality (VI), minimization problem, mixed variational inequality (MVI), split mixed equilibrium problem (SMEP), split generalized equilibrium problem (SGEP), split variational inequality (SVI), split minimization problem, split feasibility problem (SFP), split equilibrium problem (SEP) and many more as its special cases, see [10, 11].

Moudafi and Mainge [12] initiated the hierarchical fixed point problem (HFPP) for a nonexpansive mapping S related to another nonexpansive mapping U on Q_1, which can be defined as finding $\xi \in \text{Fix}(U)$, such that

$$\langle \xi - S\xi, \xi - w\rangle \leq 0 \text{ for all } w \in \text{ Fix}(U). \tag{1.7}$$

Let Ω represent the solution set of HFPP. Using the normal cone's definition

$$N_{Fix(S)} = \begin{cases} t \in H_1 : \langle \bar{r} - \bar{p}, t\rangle, \text{ for all } \bar{r} \in Fix(S) \text{ if } \bar{p} \in Fix(S), \\ \phi \qquad\qquad\qquad\qquad\qquad \textit{otherwise}, \end{cases} \tag{1.8}$$

one can easily see that $\xi \in \text{Fix}(U)$ satisfies a VIP by using a criterion S, namely: Identify $\xi \in \text{Fix}(U)$ and

$$0 \in (I - S)\xi + N_{Fix(S)}\xi. \tag{1.9}$$

The HFPP (1.7) is clearly identical to the problem of identifying the fixed point of a map $G = P_{Fix(U)} \circ S$, see [12], which includes monotone problems over equilibrium constraints, monotone variational inequality problems, and many more; see [13] and references therein.

The following mapping was described by Kangtunyakarn and Suantai [14] in 2009 as

$$\begin{aligned} T_{n,0} &= I \\ T_{n,1} &= \eta_{n,1}S_1T_{n,0} + (1 - \eta_{n,1})I \\ T_{n,2} &= \eta_{n,2}S_2T_{n,1} + (1 - \eta_{n,2})T_{n,1} \\ &\vdots \\ T_{n,M-1} &= \eta_{n,M-1}S_{M-1}T_{n,M-2} + (1 - \eta_{n,M-1})T_{n,M-2} \end{aligned} \tag{1.10}$$

$$K_n = T_{n,M} = \eta_{n,M}S_MT_{n,M-1} + (1 - \eta_{n,M})T_{n,M-1}, \tag{1.11}$$

where $S_j : Q_1 \to Q_1$ represents a finite collection of nonexpansive mappings, $\{\eta_{n,j}\}_{j=1}^{M} \subset (0, 1]$ with $\eta_{n,j} \to \eta_j$ and $\sum_{n=0}^{+\infty} |\eta_{n,j} - \eta_{n-1,j}| < +\infty$ for $1 \leq j \leq M$. The mapping K_n is the K-mapping generated by $S_1, S_2, ..., S_M$ and $\eta_{n,1}, \eta_{n,2}, ..., \eta_{n,M}$.

Recently, various common problems, namely the common solution of fixed point [15, 16], variational inequality [17], variational inclusion [18], equilibrium [19,20], hierarchical fixed point [14] and split feasibility [21, 22] problems with fixed point problems have been investigated by numerous authors. In 2009, Kangtunyakarn and Suantai [14] introduced an iterative technique and established a strong convergence theorem. In 2017, Kazmi et al. [23] proposed the following Krasnosel'skii-Mann iteration method to find common solutions of HFPP and SMEP.

$$\begin{cases} y_n = (1 - \tau_n)\chi_n + \tau_n(\varphi_nS\chi_n + (1 - \varphi_n)U\chi_n), \\ \chi_{n+1} = K_{Q_1}(y_n + \delta D^*(K_{Q_2} - I)Dy_n), \end{cases} \tag{1.12}$$

where $K_{Q_1} = T_{r_n}^{F_1}(I - r_ng_1)$, $K_{Q_2} = T_{r_n}^{F_2}(I - r_ng_2)$ and $\delta \in (0, \frac{1}{\|D\|^2})$. In 2017, Majee and Nahak [24] initiated the following hybrid viscosity algorithm to find a common solution of SEP and FPP with the finite family of nonexpansive mappings.

$$\begin{cases} y_n = K_{Q_1}(\chi_n + \delta D^*(K_{Q_2} - I)Dy_n), \\ t_n = \sigma_n\chi_n + (1 - \sigma_n)U_N^nU_{N-1}^n...U_2^nU_1^ny_n, \\ \chi_{n+1} = \omega_n\gamma h(\chi_n) + [I - \omega_n\mu A]t_n, \end{cases} \tag{1.13}$$

where $K_{Q_1} = T_{r_n}^{F_1}$, $K_{Q_2} = T_{r_n}^{F_2}$, $U_i^n = (1-\kappa_n^i)I + \kappa_n^i U_i$ and $\delta \in (0, \frac{1}{\|D\|^2})$. In 2018, Majee and Nahak [25] proposed the following viscosity approximation hybrid steepest-descent method to find a common solution of a SGEP and FPP for a finite collection of nonexpansive mappings.

$$\begin{cases} y_n = K_{Q_1}(\chi_n + \delta D^*(K_{Q_2} - I)Dy_n), \\ t_n = \sigma_n \chi_n + (1-\sigma_n)U_N^n U_{N-1}^n ... U_2^n U_1^n y_n, \\ \chi_{n+1} = \omega_n \gamma h(\chi_n) + \rho_n \chi_n + [(1-\rho_n)I - \omega_n \mu A]t_n, \end{cases} \tag{1.14}$$

where $K_{Q_1} = T_{r_n}^{F_1}$, $K_{Q_2} = T_{r_n}^{F_2}$, $U_i^n = (1-\kappa_n^i)I + \kappa_n^i U_i$ and $\delta \in (0, \frac{1}{\|D\|^2})$. In 2020, Kim and Majee [26] proposed the following modified Krasnosel'skii-Mann type iterative method in order to identify a common solution of SMEP and HFPP of a finite collection of k-strictly pseudocontractive operators.

$$\begin{cases} y_n = K_{Q_1}(w_n), l_n = K_{Q_2}(Dy_n), \\ u_n = y_n - \delta D^*(Dy_n - l_n), \\ \chi_{n+1} = (1-\varphi_n)u_n + \varphi_n[\sigma_n U u_n + (1-\sigma_n)U_N^n U_{N-1}^n ... U_2^n U_1^n u_n], \end{cases} \tag{1.15}$$

where $K_{Q_1} = T_{r_n}^{F_1}$, $K_{Q_2} = T_{r_n}^{F_2}$ and $U_i^n = (1-\kappa_n^i)I + \kappa_n^i P_{Q_1}(\zeta_n^i I + (1-\zeta_n^i)U_i)$. They proved its strong and weak convergence. In 2022, Yazdi and Sababe [27] proposed the following method in order to identify a common solution of a GMEP, common fixed points of a finite collection of nonexpansive mappings and a general system of variational inequalities.

$$\begin{cases} w_n = \tau_n \chi_n + (1-\tau_n)l_n, \\ F_1(t_n, z) + \langle g_1 w_n, z - \xi \rangle + \psi_1(z) - \psi_1(t_n) + \frac{1}{r_n}\langle z - t_n, t_n - w_n \rangle \geq 0 \text{ for all } z \in Q_1 \\ y_n = P_{Q_1}(I - \beta g_1)(t_n), l_n = P_{Q_1}(I - \rho g_2)y_n, \\ \chi_{n+1} = \omega_n \gamma h(\chi_n) + \rho_n \chi_n + [(1-\rho_n)I - \omega_n \mu A]U_N^n U_{N-1}^n ... U_2^n U_1^n l_n, \end{cases} \tag{1.16}$$

where $U_i^n = (1-\kappa_n^i)I + \kappa_n^i U_i$. They proved its strong convergence by taking some conditions on parameters.

The fixed point problem and its applications are very important in nonlinear analysis. In recent years, significant progress has also been made in research results; see [28–33]. We have applied our result for solving compressed sensing, and one can solve various nonlinear analysis problems using our algorithm. But, the applicability of our algorithm is not limited to the problems discussed above. It can be further used to solve many important problems, for instance, uncertain fractional-order differential equation with Caputo type [34, 35].

In recent times, numerous researchers have explored inertial-type methods, drawing inspiration from the concept of heavy ball techniques. Polyak, in their work from 1964 [36] introduced an iterative approach aimed at enhancing the convergence rate of iterative sequences through the incorporation of an inertial extrapolation factor. Inertial approaches typically involve a two-step iterative process, where the next iteration is determined based on the previous two iterations. In 2021, Rehman et al. [19] introduced an innovative approach by combining an inertial term with a subgradient extragradient algorithm. They provided a proof of weak convergence for their proposed method. In the same year, Chuasuk and Kaewcharoen [37] introduced a Krasnosel'skii-Mann-type inertial technique designed for solving SGMEP and HFPP involving k-strictly pseudocontractive operators. They demonstrated its weak convergence properties. Recently, a variety of inertial techniques have emerged to address a wide range of equilibrium problems, as documented in the literature [10, 37]. In 2023, Ugwunnadi et al. [38] introduced a Krasnosel'skii-Mann-type inertial technique for solving SMVIP and HFPP. These techniques offer valuable tools for solving mathematical problems efficiently and effectively.

In this study, influenced and inspired by aforementioned work, we give a new generalized viscosity approximation method for solving an SGMEP, fixed point problem for a finite collection of nonexpansive mappings S_j with $1 \leq j \leq M$ and an HFPP for a finite collection of μ_i-strictly pseudocontractive mappings which involve finding a point $\xi \in Q_1$ and

$$\xi \in \bigcap_{j=1}^{M} \text{Fix}(S_j) \cap \Omega \cap GMEP(F_1, g_1, \psi_1, Q_1) \text{ and } D\xi \in GMEP(F_2, g_2, \psi_2, Q_2). \tag{1.17}$$

Let the solution set of problem (1.17) be represented by Γ. We will prove strong convergence for Problem (1.17).

Remark 1.1. *In this paper, our contribution can be highlighted as*

1) For proving the convergence result, we have embedded an inertial term which accelerates the convergence speed of the algorithm. Majee and Nahak [24, 25], Kim and Majee [26], and Yazdi and Sababe [27] do not consider the inertial approach in their method.
2) We consider $K_{Q_1} = T_{r_n}^{F_1}(I - r_n g_1)$, $K_{Q_2} = T_{r_n}^{F_2}(I - r_n g_2)$ in our algorithm, and if we take $g_1 = g_2 = 0$, then various results are special cases of our result.
3) We consider τ-strictly pseudocontractive mappings for solving HFPP which include various mappings like pseudocontractive and nonexpansive mappings. Additionally, τ-strictly pseudocontractive mappings have more powerful applications than nonexpansive mappings in solving inverse problems.
4) Yazdi and Sababe [27] take the condition $\lim_{n\to+\infty} |t_{n+1} - t_n| = 0$, whereas our main proof does not require such a condition.
5) Our result improved the results of Kazmi et al. [23] from the common solution of HFPP and SMEP, Majee and Nahak [24] from common solution of SEP and finite family of FPP, Majee and Nahak [25] from the common solution of a SGEP and finite family of FPP, Kim and Majee [26] from common solution of SMEP and HFPP to common solution of SGMEP, HFPP and finite family of FPP.
6) We provide a real-life application to compressed sensing for our problem and show that our method requires less computation time to recover the signal in comparison with other methods.
7) We compare our iterative technique to other approaches and present numerical examples to show the effectiveness of our algorithm.
8) Our result generalizes the result of Kazmi et al. [23] from weak convergence to strong convergence.

In Section 1, we introduce the background and motivation for our research, highlighting the significance of GMEP and HFPP in real-world applications. Section 2 provides a comprehensive literature review, discussing previous methods and techniques proposed for solving GMEP, HFPP, and related problems. Section 3 outlines our proposed method and Algorithm 1, and we prove our main result. Section 4 discusses the practical applicability of our approach in compressed sensing. Section 5 presents numerical experiments to validate the effectiveness of the algorithm and compare it with other existing approaches.

2. Preliminaries

In this section, we consider a real Hilbert space denoted as H, equipped with an inner product denoted as $\langle .,. \rangle$ and the corresponding norm denoted as $||.||$. We assume that Q is a nonempty, closed and convex subset of this real Hilbert space H. We will use the notations $\chi_n \rightharpoonup \chi$ and $\chi_n \to \chi$ to signify weak and strong convergence, respectively, of the sequence $\{\chi_n\}$ to the limit χ. Furthermore, we denote the set of all fixed points of the mapping U as $\mathrm{Fix}(U)$.

Definition 2.1. *[39] A $\{graph(D_n)\}$ converges to $\{graph(D)\}$ in the Kuratowski-Painleve sense, if*

$$\limsup_{n\to+\infty} \mathrm{graph}(D_n) \subset \mathrm{graph}(D) \subset \liminf_{n\to+\infty} \mathrm{graph}(D_n), \tag{2.1}$$

where D_n is a sequence of maximal monotone mappings and D is a multivalued mapping.

Definition 2.2. *[40] The metric projection $P_Q : H \to Q$ is defined as*

$$||u' - P_Q u'|| = \inf\{||u' - z'||;\ z' \in Q\} \textit{ for all } u' \in H. \tag{2.2}$$

Definition 2.3. *[41] Suppose that $U : H \to H$ is an operator. Then U is called*

1) contraction on H if there is a constant $\mu \in [0, 1)$ and

$$||U\bar{u}' - U\bar{v}'|| \le \mu||\bar{u}' - \bar{v}'|| \text{ for all } \bar{u}', \bar{v}' \in H.$$

2) L-Lipschitz continuous on H if

$$||U\bar{u}' - U\bar{v}'|| \le L||\bar{u}' - \bar{v}'|| \text{ for all } \bar{u}', \bar{v}' \in H.$$

3) monotone on Q if

$$\langle U\bar{u}' - U\bar{v}', \bar{u}' - \bar{v}' \rangle \ge 0 \text{ for all } \bar{u}', \bar{v}' \in Q.$$

4) γ-inverse strongly monotone on Q if

$$\gamma||U\bar{u}' - U\bar{v}'||^2 \le \langle \bar{u}' - \bar{v}', U\bar{u}' - U\bar{v}' \rangle \text{ for all } \bar{u}', \bar{v}' \in Q.$$

5) τ-strictly pseudocontractive mapping if there exists $\tau \in [0, 1)$, such that

$$||U\bar{u}' - U\bar{v}'||^2 \le ||\bar{u}' - \bar{v}'||^2 + \tau||(I - U)\bar{u}' - (I - U)\bar{v}'||^2 \text{ for all } \bar{u}', \bar{v}' \in Q.$$

6) nonexpansive if

$$||U\bar{u}' - U\bar{v}'|| \le ||\bar{u}' - \bar{v}'|| \text{ for all } \bar{u}', \bar{v}' \in H.$$

Definition 2.4. *[42] The monotone bifunction $g : Q \times Q \to \mathbb{R}$ on Q is defined as*

$$g(\zeta, x') + g(x', \zeta) \leq 0 \, \textit{for all } \zeta, x' \in Q.$$

Definition 2.5. *[43] The normal cone of Q at $z' \in Q$ is defined as*

$$N_Q(z') = \{u' \in H : \langle u', \varrho - z' \rangle \leq 0 \, \textit{for all } \varrho \in Q\}.$$

Definition 2.6. *[44] A bounded linear operator D defined on H is called strongly positive if there is a constant $\gamma > 0$ such that*

$$\langle Dv, v \rangle \geq \gamma \|v\|^2 \, \textit{for all } v \in H. \tag{2.3}$$

Lemma 2.7. *[45] Consider a strongly positive bounded linear, self-adjoint operator denoted as D. This operator has a positive coefficient $\gamma > 0$, and $0 < \rho \leq \|D\|^{-1}$. Then, $\|I - \rho D\| \leq 1 - \rho\gamma$.*

Lemma 2.8. *[46] For $u' \in H$ and $y' \in Q$, $y' = P_Q u'$ iff $\langle u' - y', y' - z' \rangle \leq 0$ for all $z' \in Q$, where P_Q is a metric projection.*

Lemma 2.9. *[47] Assume that $\{U_i\}_{i=1}^N$ are averaged mappings with a common fixed point. Then,*

$$\bigcap_{i=1}^{N} Fix(U_i) = Fix(U_1 U_2 U_3 ... U_N). \tag{2.4}$$

Lemma 2.10. *[48] Let $u', v', z' \in H$. Then, the following conditions hold:*

1) $\|\xi u' + (1 - \xi) z'\|^2 = \xi \|u'\|^2 + (1 - \xi)\|z'\|^2 - \xi(1 - \xi)\|u' - z'\|^2$ for all $u', z' \in H$ and $\xi \in [0, 1]$.
2) $\|u' + z'\|^2 \leq \|u'\|^2 + 2 \langle z', u' + z' \rangle$ for all $u', z' \in H$.
3) (Opial's condition) Consider a sequence y_n with $y_n \rightharpoonup z'$, then the following conclusions hold:

$$\liminf_{n \to +\infty} \|y_n - z'\| < \liminf_{n \to +\infty} \|y_n - \varrho\| \text{ for all } \varrho \in H \text{ and } z' \neq \varrho.$$

Lemma 2.11. *[49] Suppose that $U : Q \to H$ is a η-strictly pseudocontractive mapping with $Fix(U) \neq \phi$. Consider a mapping S as $Sv = \tau v + (1 - \tau) Uv$ for all $v \in H$, where $\tau \in [\eta, 1)$. Then, the following conclusions hold:*

1) $Fix(P_Q U) = Fix(U)$.
2) S is nonexpansive and $Fix(U) = Fix(S)$.

Lemma 2.12. *[50] If $\{v_n\} \subset [0, +\infty)$, $\{w_n\} \subset (0, 1)$, $\{\tau_n\} \subset (0, 1)$ and $\{\eta_n\}$ are real sequences satisfying the inequality*

$$v_{n+1} \leq (1 - w_n) v_n + \eta_n + \tau_n \text{ for all } n \geq n_0. \tag{2.5}$$

Suppose $\sum_{n=0}^{+\infty} \tau_n < +\infty$, then the conclusions stated below hold:

1) If $\eta_n \leq w_n M$ for some $M \geq 0$, then $\{v_n\}$ is bounded sequence.
2) If $\sum_{n=0}^{+\infty} w_n = +\infty$ and $\lim_{n \to +\infty} \frac{\eta_n}{w_n} \leq 0$, then $\lim_{n \to +\infty} v_n = 0$.

We need the following assumptions on bifunction $g : Q \to Q$ to solve the split generalized mixed equilibrium problem:

Assumption 1.

1) g is monotone.

2) $g(u', u') \geq 0$ for all $u' \in Q$.

3) For each $u', w', y' \in Q$, $\limsup_{t \to 0^+} g(t'u' + (1-t')w', y') \leq h(w', y')$.

4) For each $u' \in Q, y' \to h(u', y')$ is lower semi-continuous and convex.

Now, we mention the following lemma which will be utilized for solving the monotone split generalized mixed equilibrium problem.

Lemma 2.13. *[51] Assume that $g : Q \times Q \to \mathbb{R}$ is a bifunction satisfying Assumption 1. Consider $g_1 : Q \to H$ a nonlinear mapping, $\psi : Q \to \mathbb{R} \cup \{+\infty\}$ a convex and proper lower semicontinuous function. Define $S_r^g(w)$ as follows:*

$$S_r^g(w) = \{x \in Q : g(x, z) + \langle g_1(x), z - x \rangle + \psi(z) - \psi(x) + \frac{1}{r} \langle z - x, x - w \rangle \geq 0 \text{ for all } z \in Q\},$$

where $w \in H$ and $r > 0$. Then, the following statements hold:

1) For every $u' \in H$, $S_r^g(u') \neq \phi$.
2) S_r^g is single-valued.
3) Fix(S_r^g)=GMEP(g, g_1, ψ).
4) Solution set GMEP(g, g_1, ψ) is closed and convex.
5) S_r^g is firmly nonexpansive, i.e., for any $u', y' \in H$

$$\|S_r^g(u') - S_r^g(y')\|^2 \leq \langle S_r^g(u') - S_r^g(y'), u' - y' \rangle .$$

Lemma 2.14. *[52] Consider C a Lipschitz maximal monotone mapping and $\{D_n\}$ a sequence of maximal monotone mappings defined on H. The statements are as follows:*

1) If D_n is graph convergent to a mapping D on H, then $C + D$ is maximal monotone and $\{C + D_n\}$ is also graph convergent to $C + D$.
2) In addition, if D is a maximal monotone mapping defined on H and $D^{-1}0 \neq \phi$, then $\{s_n^{-1}D\}$ is graph convergent to $N_{D^{-1}0}$ as $s_n \to +\infty$.

Lemma 2.15. *[53] Suppose $\{v_n\}$, $\{w_n\}$ and $\{\tau_n\}$ are bounded sequences in a Hilbert space H such that $\{\tau_n\} \subset (0, 1)$ with $0 < \liminf_{n \to +\infty} \tau_n \leq \limsup_{n \to +\infty} \tau_n < 1$. If $v_{n+1} = (1 - \tau_n)w_n + \tau_n v_n$ for all integers $n \geq 0$ and $\limsup_{n \to +\infty} \|w_{n+1} - w_n\| - \|v_{n+1} - v_n\| \leq 0$, then $\lim_{n \to +\infty} \|w_n - v_n\| = 0$.*

Lemma 2.16. *[54] Suppose that $\{U_j\} : Q \to Q$ is a finite family of nonexpansive mappings with $1 \leq j \leq M$ and $\bigcap_{j=1}^{M} Fix(U_j) \neq \phi$. Assume that the sequence $\{\eta_{n,j}\}$ converges to $\{\eta_j\}$, where $\eta_{n,j} \subset [0, 1]$ for $1 \leq j \leq M$, $\eta_j \subset (0, 1)$ for $1 \leq j \leq M - 1$ and $\eta_M \subset (0, 1]$. Consider a K-mapping generated by $U_1, U_2, ..., U_M$ and $\eta_1, \eta_2, ..., \eta_n$. Let K_n be the K-mapping generated by $U_1, U_2, ..., U_M$ and $\eta_{n,1}, \eta_{n,2}, ..., \eta_{n,M}$. Then, the conclusions stated below hold:*

1) Fix(K)=$\bigcap_{j=1}^{M}$ Fix(U_j).

2) $\lim_{n\to+\infty} \|K_n v - K v\| = 0$ for each $v \in Q_1$.

Lemma 2.17. *[55] Suppose $U : Q \to Q$ is a nonexpansive mapping. Let $\{v_n\}$ be a sequence in Q converging weakly to $v \in Q$ and $\{(I-U)v_n\}$ converging strongly to $w \in Q$, then $(I-U)v = w$ and if $w = 0$, then $v \in Fix(U)$.*

3. Main result

In this section, we propose a new inertial generalized viscosity approximation method and prove a strong convergence theorem for solving split generalized mixed equilibrium problem, common fixed point problem of a finite family of nonexpansive mappings and hierarchical fixed point problem. Let $F_1 : Q_1 \times Q_1 \to \mathbb{R}, F_2 : Q_2 \times Q_2 \to \mathbb{R}$ be bifunctions satisfying Assumption 1 and F_2 be upper semicontinuous. Suppose $D : H_1 \to H_2$ is a bounded linear operator with adjoint D^* such that $\delta \in (0, \frac{1}{L})$, where L is the spectral radius of D Let $g_1 : Q_1 \to H_1$, $g_2 : Q_2 \to H_2$ be α_1, α_2-ism mappings respectively, $h : Q_1 \to Q_1$ be a ν-contraction mapping, $U_i : Q_1 \to Q_1$ be μ_i-strictly pseudocontractive mappings for $1 \le i \le N$, $\psi_1 : Q_1 \to \mathbb{R} \cup \{+\infty\}$, $\psi_2 : Q_2 \to \mathbb{R} \cup \{+\infty\}$ be convex and proper lower semicontinuous functions, $U : Q_1 \to Q_1$ be nonexpansive mapping and $S_j : Q_1 \to Q_1$ be nonexpansive mappings for $1 \le j \le M$ and A is a strongly positive bounded linear self-adjoint operator on H_1 with constant $\bar{\gamma} > 0$ such that $0 < \gamma < \frac{\bar{\gamma}}{\nu} < \gamma + \frac{1}{\nu}$.

Algorithm 3.1. *Consider $\lambda_n \subset [0, +\infty)$ with $\sum_{n=0}^{+\infty} \lambda_n < +\infty$, $\tau \in [0, 1)$, $\varphi_n, \sigma_n, \kappa_n^i, \omega_n, \rho_n \in (0, 1)$, $\rho = \sup\{\rho_n; n \in \mathbb{N}\}$ with $\lim\limits_{n\to+\infty} |\varphi_{n+1} - \varphi_n| = 0$, $\lim\limits_{n\to+\infty} |\sigma_{n+1} - \sigma_n| = 0$ and $\sum_{n=0}^{+\infty} \sigma_n < +\infty$. Set $n = 1$. Choose $x_0, x_1 \in Q_1$ and τ_n such that $0 \le \tau_n \le \bar{\tau_n}$, where*

$$\bar{\tau_n} = \begin{cases} \min\left\{\frac{\lambda_n}{\|\chi_n - \chi_{n-1}\|}, \tau\right\} & \text{if } \chi_n \ne \chi_{n-1}, \\ \tau & \textit{otherwise}. \end{cases} \tag{3.1}$$

Step 1: Compute

$$\begin{cases} w_n = \chi_n + \tau_n(\chi_n - \chi_{n-1}), \\ y_n = K_{Q_1}(w_n), l_n = K_{Q_2}(Dy_n), \\ u_n = y_n - \delta D^*(Dy_n - l_n), \end{cases} \tag{3.2}$$

where $K_{Q_1} = T_{r_n}^{F_1}(I - r_n g_1)$, $K_{Q_2} = T_{r_n}^{F_2}(I - r_n g_2)$ with $r_n \subset \min\{\alpha_1, \alpha_2\} = 2\alpha$, $\liminf_{n\to+\infty} r_n > 0$ and $\lim\limits_{n\to+\infty} |r_{n+1} - r_n| = 0$.
Step 2: Compute

$$t_n = (1 - \varphi_n)u_n + \varphi_n[\sigma_n U u_n + (1 - \sigma_n)U_N^n U_{N-1}^n ... U_2^n U_1^n u_n], \tag{3.3}$$

where $U_i^n = (1 - \kappa_n^i)I + \kappa_n^i P_{Q_1}(\zeta_n^i I + (1 - \zeta_n^i)U_i)$ with $0 \le \mu_i \le \zeta_n^i < 1$ and $\lim\limits_{n\to+\infty} |\kappa_{n+1}^i - \kappa_n^i| = 0$ for $i \le i \le M$.
Step 3: Evaluate

$$\chi_{n+1} = \omega_n \gamma h(K_n \chi_n) + \rho_n \chi_n + [(1 - \rho_n)I - \omega_n A]K_n t_n. \tag{3.4}$$

Step 4: If $\chi_{n+1} = \chi_n$, terminate the process. Otherwise, set $n := n + 1$ and return to Step 1.

Remark 3.2. *From Eq (3.1) and* $\sum_{n=0}^{+\infty}\lambda_n < +\infty$*, we get* $\sum_{n=0}^{+\infty}\tau_n(\chi_n - \chi_{n-1}) < +\infty$.

Theorem 3.3. *Let* Γ*, the solution set defined in Eq (1.17) be nonempty. Suppose that Assumption 1 holds and the following conditions are satisfied:*

1) $\lim\limits_{n\to+\infty}\omega_n = 0$ and $\sum_{n=0}^{+\infty}\omega_n = +\infty$,
2) $0 < \liminf_{n\to+\infty}\rho_n \le \limsup_{n\to+\infty}\rho_n < 1$,
3) $\lim\limits_{n\to+\infty}\frac{\|t_n-u_n\|}{\sigma_n\varphi_n} = 0$.

Then, $\{\chi_n\}$ generated by Algorithm 3.1 converges strongly to ϱ, where $\varrho \in \Gamma$ and ϱ is the unique fixed point of contraction mapping $P_\Gamma(I + \gamma h - A)$.

Proof. We have divided the proof in various steps. We will establish the theorem for the case when $N = 2$ and, subsequently, we will illustrate how the procedure can be readily applied to the general case.

Claim 1: The sequence $\{\chi_n\}$ is bounded.

Let $\varrho \in \bigcap_{j=1}^{M}\mathrm{Fix}(S_j) \cap \Omega \cap \Theta = \Gamma$. Also, with $\omega_n \to 0$ as $n \to +\infty$, we can assume that

$$\omega_n < \frac{1-\rho}{\|A\|}, \quad \text{for all } n \tag{3.5}$$

and then

$$\omega_n < \frac{1-\rho}{\bar{\gamma}}, \quad \text{for all } n.$$

Using Lemma 2.7, we get

$$\|I - \omega_n A\| \le 1 - \omega_n\bar{\gamma}. \tag{3.6}$$

As we know that A is strongly positive bounded linear operator, then $\langle Av, v\rangle \ge \bar{\gamma}\|v\|^2$ and $\|A\| = \sup\{|\langle Av, v\rangle|;\ \|v\| = 1, v \in H_1\}$. Now consider

$$\begin{aligned}\langle((1-\rho_n)I - \omega_n A)v, v\rangle &= 1 - \rho_n - \omega_n\langle Av, v\rangle\\ &\ge 1 - \rho - \omega_n\|A\| \ge 0 \ \text{ for all } v \in H_1.\end{aligned} \tag{3.7}$$

Thus, using Eq (3.5), we get $(1-\rho_n)I - \omega_n A$ is positive operator. Also,

$$\begin{aligned}0 \le \|(1-\rho_n)I - \omega_n A\| &= \sup\{|\langle(1-\rho_n)I - \omega_n Av, v\rangle|;\ \|v\| = 1, v \in H_1\}\\ &= \sup\{|1 - \rho_n - \omega_n\langle Av, v\rangle|;\ \|v\| = 1, v \in H_1\}\\ &\le 1 - \rho_n - \omega_n\bar{\gamma}.\end{aligned} \tag{3.8}$$

Consider

$$\begin{aligned}\|w_n - \varrho\| &= \|\chi_n + \tau_n(\chi_n - \chi_{n-1}) - \varrho\|\\ &\le \|\chi_n - \varrho\| + \tau_n\|\chi_n - \chi_{n-1}\|.\end{aligned} \tag{3.9}$$

Using Lemma 2.9, $I - r_n g_1$ is a nonexpansive mapping and hence K_{Q_1} is a nonexpansive mapping. From Eq (3.2), we have

$$\begin{aligned}
\|y_n - \varrho\|^2 &= \|K_{Q_1}(w_n) - K_{Q_1}(\varrho)\|^2 \\
&= \|T_{r_n}^{F_1}(I - r_n g_1)w_n - T_{r_n}^{F_1}(I - r_n g_1)\varrho\|^2 \\
&\leq \|(w_n - \varrho) - r_n(g_1(w_n) - g_1(\varrho))\|^2 \\
&\leq \|w_n - \varrho\|^2 + r_n^2\|g_1(w_n) - g_1(\varrho)\|^2 - 2r_n\alpha_1\|g_1(w_n) - g_1(\varrho)\|^2 \\
&= \|w_n - \varrho\|^2 - r_n(2\alpha_1 - r_n)\|g_1(w_n) - g_1(\varrho)\|^2 \qquad (3.10) \\
&\leq \|w_n - \varrho\|^2. \qquad (3.11)
\end{aligned}$$

Similarly,

$$\begin{aligned}
\|l_n - D\varrho\| &= \|K_{Q_2}(Dy_n) - K_{Q_2}(D\varrho)\| \\
&\leq \|Dy_n - D\varrho\|. \qquad (3.12)
\end{aligned}$$

Using Eq (3.12), we have

$$\begin{aligned}
\langle y_n - \varrho, D^*(l_n - Dy_n)\rangle &= \langle Dy_n - D\varrho, l_n - Dy_n\rangle \\
&= \langle Dy_n - D\varrho - (l_n - Dy_n) + (l_n - Dy_n), l_n - Dy_n\rangle \\
&= \langle l_n - D\varrho, l_n - Dy_n\rangle - \|l_n - Dy_n\|^2 \\
&= \frac{1}{2}\Big[\|l_n - D\varrho\|^2 + \|l_n - Dy_n\|^2 - \|Dy_n - D\varrho\|^2\Big] \\
&\quad - \|l_n - Dy_n\|^2 \\
&\leq \frac{1}{2}\Big[\|Dy_n - D\varrho\|^2 - \|Dy_n - D\varrho\|^2\Big] - \frac{1}{2}\|l_n - Dy_n\|^2 \\
&= -\frac{1}{2}\|l_n - Dy_n\|^2. \qquad (3.13)
\end{aligned}$$

From Eqs (3.2), (3.11), (3.13) and $\delta \in (0, \frac{1}{L})$, we have

$$\begin{aligned}
\|u_n - \varrho\|^2 &= \|y_n - \delta D^*(Dy_n - l_n) - \varrho\|^2 \\
&= \|y_n - \varrho\|^2 + \delta^2\|D^*(Dy_n - l_n)\|^2 - 2\delta\,\langle y_n - \varrho, D^*(Dy_n - l_n)\rangle \\
&= \|y_n - \varrho\|^2 + \delta^2\|D^*(Dy_n - l_n)\|^2 + 2\delta\,\langle y_n - \varrho, D^*(l_n - Dy_n)\rangle \\
&\leq \|y_n - \varrho\|^2 + \delta^2 L\|l_n - Dy_n\|^2 + 2\delta[-\frac{1}{2}\|l_n - Dy_n\|^2] \\
&= \|y_n - \varrho\|^2 + (\delta^2 L - \delta)\|l_n - Dy_n\|^2 \\
&= \|y_n - \varrho\|^2 - \delta(1 - \delta L)\|l_n - Dy_n\|^2 \qquad (3.14) \\
&\leq \|w_n - \varrho\|^2. \qquad (3.15)
\end{aligned}$$

From Eqs (3.9) and (3.15), we have

$$\begin{aligned}
\|u_n - \varrho\| &\leq \|w_n - \varrho\| \\
&\leq \|\chi_n - \varrho\| + \tau_n\|\chi_n - \chi_{n-1}\|. \qquad (3.16)
\end{aligned}$$

Using Lemmas 2.9 and 2.11, we have $U_2^nU_1^n\varrho = \varrho$. From Eqs (3.3) and (3.16), we have

$$\begin{aligned}
\|t_n - \varrho\| &= \|(1-\varphi_n)u_n + \varphi_n[\sigma_n Uu_n + (1-\sigma_n)U_2^nU_1^nu_n] - \varrho\| \\
&\le (1-\varphi_n)\|u_n - \varrho\| + \varphi_n[\sigma_n\|Uu_n - \varrho\| + (1-\sigma_n)\|U_2^nU_1^nu_n - \varrho\|] \\
&\le (1-\varphi_n)\|u_n - \varrho\| + \varphi_n[\sigma_n\|Uu_n - U\varrho\| + \sigma_n\|U\varrho - \varrho\| \\
&+ (1-\sigma_n)\|u_n - \varrho\|] \\
&\le (1-\varphi_n)\|u_n - \varrho\| + \varphi_n[\|u_n - \varrho\| + \sigma_n\|U\varrho - \varrho\|] \\
&= \|u_n - \varrho\| + \varphi_n\sigma_n\|U\varrho - \varrho\| \qquad (3.17)\\
&\le \|\chi_n - \varrho\| + \tau_n\|\chi_n - \chi_{n-1}\| + \varphi_n\sigma_n\|U\varrho - \varrho\|. \qquad (3.18)
\end{aligned}$$

Using Eqs (3.4) and (3.18), we have

$$\begin{aligned}
\|\chi_{n+1} - \varrho\| &= \|\omega_n\gamma h(K_n\chi_n) + \rho_n\chi_n + [(1-\rho_n)I - \omega_nA]K_nt_n - \varrho\| \\
&\le \|\omega_n\gamma h(K_n\chi_n) - \omega_n\gamma h(\varrho)\| + \|\omega_n\gamma h(\varrho) - \omega_nA\varrho\| + \rho_n\|\chi_n - \varrho\| + [(1-\rho_n) - \omega_n\bar{\gamma}]\|K_nt_n - \varrho\| \\
&\le \omega_n\gamma\nu\|\chi_n - \varrho\| + \omega_n\|\gamma h(\varrho) - A\varrho\| + \rho_n\|\chi_n - \varrho\| + [(1-\rho_n) - \omega_n\bar{\gamma}]\|t_n - \varrho\| \\
&\le \omega_n\gamma\nu\|\chi_n - \varrho\| + \omega_n\|\gamma h(\varrho) - A\varrho\| + \rho_n\|\chi_n - \varrho\| \\
&+ [(1-\rho_n) - \omega_n\bar{\gamma}][\|\chi_n - \varrho\| + \tau_n\|\chi_n - \chi_{n-1}\| + \varphi_n\sigma_n\|U\varrho - \varrho\|] \\
&\le (1 - \omega_n(\bar{\gamma} - \gamma\nu))\|\chi_n - \varrho\| + \omega_n\|\gamma h(\varrho) - A\varrho\| \\
&+ [(1-\rho_n) - \omega_n\bar{\gamma}][\tau_n\|\chi_n - \chi_{n-1}\| + \varphi_n\sigma_n\|U\varrho - \varrho\|] \\
&\le (1 - \omega_n(\bar{\gamma} - \gamma\nu))\|\chi_n - \varrho\| + \omega_n\|\gamma h(\varrho) - A\varrho\| \\
&+ \tau_n\|\chi_n - \chi_{n-1}\| + \sigma_n\|U\varrho - \varrho\| \\
&\le (1 - \omega_n(\bar{\gamma} - \gamma\nu))\|\chi_n - \varrho\| + \omega_n\|\gamma h(\varrho) - A\varrho\| \\
&+ \tau_n\|\chi_n - \chi_{n-1}\| + \sigma_n\|U\varrho - \varrho\|. \qquad (3.19)
\end{aligned}$$

Let $v_n = \|\chi_n - \varrho\|$, $w_n = \omega_n(\bar{\gamma} - \gamma\nu)$, $\eta_n \le w_nM = \frac{\omega_n(\bar{\gamma}-\gamma\nu)\|\gamma h(\varrho) - A\varrho\|}{(\bar{\gamma}-\gamma\nu)}$ and $\tau_n^1 = \tau_n\|\chi_n - \chi_{n-1}\| + \sigma_n\|U\varrho - \varrho\|$. Thus, we have

$$v_{n+1} \le (1 - w_n)v_n + \eta_n + \tau_n^1$$

Using Lemma 2.12, Remark 3.2, condition (i) and $\sum_{n=0}^{+\infty}\sigma_n < +\infty$, we get that $\{v_n\}$ is bounded, which implies $\{\|\chi_n - \varrho\|\}$ is bounded. Hence, $\{\chi_n\}$ is bounded. Consequently, $\{t_n\}$, $\{u_n\}$, $\{w_n\}$, $\{y_n\}$, $\{h(K_n\chi_n)\}$ are also bounded.

Claim 2: $\limsup_{n\to+\infty}(\|f_{n+1} - f_n\| - \|\chi_{n+1} - \chi_n\|) \le 0$.

Consider

$$\begin{aligned}
\|w_n - \varrho\|^2 &= \|\chi_n + \tau_n(\chi_n - \chi_{n-1}) - \varrho\|^2 \\
&\le \|\chi_n - \varrho\|^2 + \tau_n^2\|\chi_n - \chi_{n-1}\|^2 + 2\tau_n\|\chi_n - \chi_{n-1}\|\|\chi_n - \varrho\| \\
&= \|\chi_n - \varrho\|^2 + \tau_n\|\chi_n - \chi_{n-1}\|[\tau_n\|\chi_n - \chi_{n-1}\| + 2\|\chi_n - \varrho\|] \\
&\le \|\chi_n - \varrho\|^2 + \tau_n\|\chi_n - \chi_{n-1}\|M, \qquad (3.20)
\end{aligned}$$

where $M = \sup\{\tau_n\|\chi_n - \chi_{n-1}\| + 2\|\chi_n - \varrho\|;\ n \in \mathbb{N}\}$.

Also,

$$\|w_{n+1} - w_n\| = \|\chi_{n+1} + \tau_{n+1}(\chi_{n+1} - \chi_n) - (\chi_n + \tau_n(\chi_n - \chi_{n-1}))\|$$

$$\leq \|\chi_{n+1} - \chi_n\| + \tau_{n+1}\|\chi_{n+1} - \chi_n\| + \tau_n\|\chi_n - \chi_{n-1}\|. \tag{3.21}$$

As $y_n = T^{F_1}_{r_n}(I - r_n g_1)(w_n)$ and $y_{n+1} = T^{F_1}_{r_{n+1}}(I - r_{n+1} g_1)(w_{n+1})$, we get

$$F_1(y_n, z) + \langle g_1(w_n), z - y_n\rangle + \psi_1(z) - \psi_1(y_n) + \frac{1}{r_n}\langle z - y_n, y_n - w_n\rangle \geq 0 \text{ for all } z \in Q_1, \tag{3.22}$$

and

$$\begin{aligned} &F_1(y_{n+1}, z) + \langle g_1(w_{n+1}), z - y_{n+1}\rangle + \psi_1(z) - \psi_1(y_{n+1}) \\ &\qquad + \frac{1}{r_{n+1}}\langle z - y_{n+1}, y_{n+1} - w_{n+1}\rangle \geq 0 \text{ for all } z \in Q_1. \end{aligned} \tag{3.23}$$

Putting $z = y_{n+1}$ and $z = y_n$ in Eqs (3.22) and (3.23), respectively, we get

$$\begin{aligned} &F_1(y_n, y_{n+1}) + \langle g_1(w_n), y_{n+1} - y_n\rangle \\ &\qquad + \psi_1(y_{n+1}) - \psi_1(y_n) + \frac{1}{r_n}\langle y_{n+1} - y_n, y_n - w_n\rangle \geq 0\,, \end{aligned} \tag{3.24}$$

and

$$\begin{aligned} &F_1(y_{n+1}, y_n) + \langle g_1(w_{n+1}), y_n - y_{n+1}\rangle + \psi_1(y_n) - \psi_1(y_{n+1}) \\ &\qquad + \frac{1}{r_{n+1}}\langle y_n - y_{n+1}, y_{n+1} - w_{n+1}\rangle \geq 0. \end{aligned} \tag{3.25}$$

Adding Eqs (3.24) and (3.25) and using the monotonicity of F_1, we get

$$\langle g_1(w_{n+1}) - g_1(w_n), y_n - y_{n+1}\rangle + \left\langle y_n - y_{n+1}, \frac{y_{n+1} - w_{n+1}}{r_{n+1}} - \frac{y_n - w_n}{r_n}\right\rangle \geq 0. \tag{3.26}$$

Upon rearranging the terms in Eq (3.26), we get

$$\begin{aligned} 0 &\leq \left\langle y_n - y_{n+1}, r_n(g_1(w_{n+1}) - g_1(w_n)) + \frac{r_n}{r_{n+1}}(y_{n+1} - w_{n+1}) - (y_n - w_n)\right\rangle \\ &\leq \left\langle y_{n+1} - y_n, y_n - y_{n+1} + (1 - \frac{r_n}{r_{n+1}})y_{n+1}\right\rangle \\ &+ \left\langle y_{n+1} - y_n, (w_{n+1} - r_n g_1(w_{n+1})) - (w_n - r_n g_1(w_n)) - w_{n+1} + \frac{r_n}{r_{n+1}}w_{n+1}\right\rangle \\ &\leq \left\langle y_{n+1} - y_n, y_n - y_{n+1} + (1 - \frac{r_n}{r_{n+1}})(y_{n+1} - w_{n+1})\right\rangle \\ &+ \langle y_{n+1} - y_n, (w_{n+1} - r_n g_1(w_{n+1})) - (w_n - r_n g_1(w_n))\rangle\,. \end{aligned} \tag{3.27}$$

Hence, we get

$$\|y_{n+1} - y_n\|^2 \leq \|y_{n+1} - y_n\|\left[\|w_{n+1} - w_n\| + \left|1 - \frac{r_n}{r_{n+1}}\right|\|y_{n+1} - w_{n+1}\|\right]. \tag{3.28}$$

Subsequently, we have

$$\begin{aligned}\|y_{n+1}-y_n\| &\le \|w_{n+1}-w_n\| + \left|1-\frac{r_n}{r_{n+1}}\right|\|y_{n+1}-w_{n+1}\| \\ &\le \|w_{n+1}-w_n\| + \frac{1}{r_{n+1}}\Big|r_{n+1}-r_n\Big|\|y_{n+1}-w_{n+1}\|. \qquad (3.29)\end{aligned}$$

Assume that for any $n > 0$, there is a real number c_1 such that $r_n > c_1 > 0$. From Eq (3.29), we get

$$\|y_{n+1}-y_n\| \le \|w_{n+1}-w_n\| + \frac{1}{c_1}\Big|r_{n+1}-r_n\Big|M_1, \qquad (3.30)$$

where $M_1 = \sup\{\|w_{n+1}-y_{n+1}\|;\ n \in \mathbb{N}\}$. In a similar way, we can deduce that

$$\|l_{n+1}-l_n\| \le \|Dy_{n+1}-Dy_n\| + \frac{1}{c_2}\Big|r_{n+1}-r_n\Big|M_2, \qquad (3.31)$$

where $M_2 = \sup\{\|l_{n+1}-Dy_{n+1}\|;\ n \in \mathbb{N}\}$, $l_n = T^{F_2}_{r_n}(I-r_ng_2)Dy_n$ and $l_{n+1} = T^{F_2}_{r_{n+1}}(I-r_{n+1}g_2)Dy_{n+1}$. Also,

$$\begin{aligned}\|u_{n+1}-u_n\|^2 &= \|y_{n+1}-\delta D^*(Dy_{n+1}-l_{n+1})-(y_n-\delta D^*(Dy_n-l_n))\|^2 \\ &\le \|y_{n+1}-y_n\|^2 + \delta^2\|D^*\|^2\|D(y_{n+1})-D(y_n)-(l_{n+1}-l_n)\|^2 \\ &+ 2\delta\left\langle Dy_{n+1}-Dy_n+(l_{n+1}-Dy_{n+1}),(l_{n+1}-Dy_{n+1})-(l_n-Dy_n)\right\rangle \\ &- \left\langle (l_n-Dy_n),(l_{n+1}-Dy_{n+1})-(l_n-Dy_n)\right\rangle \\ &- 2\delta\left\langle (l_{n+1}-Dy_{n+1})-(l_n-Dy_n),(l_{n+1}-Dy_{n+1})-(l_n-Dy_n)\right\rangle \\ &\le \|y_{n+1}-y_n\|^2 + \delta^2\|D^*\|^2\|D(y_{n+1})-D(y_n)-(l_{n+1}-l_n)\|^2 \\ &+ 2\delta\Big[\frac{1}{2}\|l_{n+1}-l_n\|^2 + \frac{1}{2}\|D(y_{n+1})-D(y_n)-(l_{n+1}-l_n)\|^2 \\ &- \frac{1}{2}\|Dy_{n+1}-Dy_n\|^2\Big] - 2\delta\|(l_{n+1}-Dy_{n+1})-(l_n-Dy_n)\|^2 \\ &= \|y_{n+1}-y_n\|^2 - \delta(1-\delta\|D\|^2)\|D(y_{n+1})-D(y_n)-(l_{n+1}-l_n)\|^2 \\ &+ \delta(\|l_{n+1}-l_n\|^2 - \|D(y_{n+1})-D(y_n)\|^2) \\ &\le \|y_{n+1}-y_n\|^2 - \delta(1-\delta L)\|D(y_{n+1})-D(y_n)-(l_{n+1}-l_n)\|^2 \\ &+ \frac{M_2}{c_2}\delta|r_{n+1}-r_n|(\|l_{n+1}-l_n\| + \|D(y_{n+1})-D(y_n)\|) \\ &\le \|y_{n+1}-y_n\|^2 + \frac{M_2}{c_2}\delta|r_{n+1}-r_n|(\|l_{n+1}-l_n\| + \|D(y_{n+1})-D(y_n)\|). \qquad (3.32)\end{aligned}$$

Using the inequality $\sqrt{|c'|+|d'|} \le \sqrt{|c'|} + \sqrt{|d'|}$, we get

$$\|u_{n+1}-u_n\| \le \|y_{n+1}-y_n\| + \sqrt{\frac{M_2}{c_2}\delta|r_{n+1}-r_n|(\|l_{n+1}-l_n\| + \|D(y_{n+1})-D(y_n)\|)}. \qquad (3.33)$$

Using Eq (3.30), we get

$$\|u_{n+1}-u_n\| \le \|w_{n+1}-w_n\| + \sqrt{\frac{M_2}{c_2}\delta|r_{n+1}-r_n|(\|l_{n+1}-l_n\| + \|D(y_{n+1})-D(y_n)\|)}$$

$$+\frac{1}{c_1}\Big|r_{n+1}-r_n\Big|M_1. \tag{3.34}$$

Choose constant M_3 such that

$$\sqrt{\frac{M_2}{c_2}\delta(\|l_{n+1}-l_n\|+\|D(y_{n+1})-D(y_n)\|)} \leq M_3. \tag{3.35}$$

From Eqs (3.34) and (3.35), we have

$$\begin{aligned}
\|u_{n+1}-u_n\| &\leq \|w_{n+1}-w_n\|+M_3\sqrt{|r_{n+1}-r_n|}+\frac{1}{c_1}\Big|r_{n+1}-r_n\Big|M_1\\
&\leq \|\chi_{n+1}-\chi_n\|+\tau_{n+1}\|\chi_{n+1}-\chi_n\|+\tau_n\|\chi_n-\chi_{n-1}\|\\
&+M_3\sqrt{|r_{n+1}-r_n|}+\frac{1}{c_1}\Big|r_{n+1}-r_n\Big|M_1.
\end{aligned} \tag{3.36}$$

Let $s_n=\sigma_n Uu_n+(1-\sigma_n)U_2^nU_1^nu_n$, then $t_n=(1-\varphi_n)u_n+\varphi_n s_n$, and we estimate

$$\begin{aligned}
\|s_{n+1}-s_n\| &= \|\sigma_{n+1}Uu_{n+1}+(1-\sigma_{n+1})U_2^{n+1}U_1^{n+1}u_{n+1}-(\sigma_nUu_n\\
&+(1-\sigma_n)U_2^nU_1^nu_n)\|\\
&= \|\sigma_{n+1}Uu_{n+1}+\sigma_{n+1}Uu_n-\sigma_{n+1}Uu_n+(1-\sigma_{n+1})U_2^{n+1}U_1^{n+1}u_{n+1}\\
&+(1-\sigma_{n+1})U_2^nU_1^nu_n-(1-\sigma_{n+1})U_2^nU_1^nu_n-(\sigma_nUu_n\\
&+(1-\sigma_n)U_2^nU_1^nu_n)\|\\
&\leq \sigma_{n+1}\|Uu_{n+1}-Uu_n\|+|\sigma_{n+1}-\sigma_n|\|U_2^nU_1^nu_n-Uu_n\|\\
&+(1-\sigma_{n+1})\|U_2^{n+1}U_1^{n+1}u_{n+1}-U_2^nU_1^nu_n\|.
\end{aligned} \tag{3.37}$$

In a similar way,

$$\begin{aligned}
\|t_{n+1}-t_n\| &= \|(1-\varphi_{n+1})u_{n+1}+\varphi_{n+1}s_{n+1}-((1-\varphi_n)u_n+\varphi_ns_n)\|\\
&\leq (1-\varphi_{n+1})\|u_{n+1}-u_n\|+(\varphi_{n+1}-\varphi_n)\|u_n-s_n\|+\varphi_{n+1}\|s_{n+1}-s_n\|.
\end{aligned} \tag{3.38}$$

Using Eqs (3.37) and (3.38), we get

$$\begin{aligned}
\|t_{n+1}-t_n\| &\leq (1-\varphi_{n+1})\|u_{n+1}-u_n\|+(\varphi_{n+1}-\varphi_n)\|u_n-s_n\|\\
&+\varphi_{n+1}[\sigma_{n+1}\|Uu_{n+1}-Uu_n\|+|\sigma_{n+1}-\sigma_n|\|U_2^nU_1^nu_n-Uu_n\|\\
&+(1-\sigma_{n+1})\|U_2^{n+1}U_1^{n+1}u_{n+1}-U_2^nU_1^nu_n\|].
\end{aligned} \tag{3.39}$$

Now, consider

$$\begin{aligned}
\|U_2^{n+1}U_1^{n+1}u_{n+1}-U_2^nU_1^nu_n\| &\leq \|U_2^{n+1}U_1^{n+1}u_{n+1}-U_2^nU_1^nu_{n+1}\|\\
&+\|U_2^nU_1^nu_{n+1}-U_2^nU_1^nu_n\|\\
&\leq \|U_2^{n+1}U_1^{n+1}u_{n+1}-U_2^{n+1}U_1^nu_{n+1}\|\\
&+\|U_2^{n+1}U_1^nu_{n+1}-U_2^nU_1^nu_{n+1}\|+\|u_{n+1}-u_n\|\\
&\leq \|U_1^{n+1}u_{n+1}-U_1^nu_{n+1}\|
\end{aligned}$$

$$+\|U_2^{n+1}U_1^n u_{n+1} - U_2^n U_1^n u_{n+1}\| + \|u_{n+1} - u_n\|. \tag{3.40}$$

Using the definition of U_i^n, we estimate that

$$\begin{aligned}\|U_1^{n+1}u_{n+1} - U_1^n u_{n+1}\| &= \|(1-\kappa_{n+1}^1)I + \kappa_{n+1}^1 P_{Q_1}(\zeta_{n+1}^1 I + (1-\zeta_{n+1}^1)U_1)u_{n+1} \\ &\quad - ((1-\kappa_n^1)I + \kappa_n^1 P_{Q_1}(\zeta_n^1 I + (1-\zeta_n^1)U_1)u_{n+1})\| \\ &\leq \|(1-\kappa_{n+1}^1)I + \kappa_{n+1}^1 P_{Q_1}(\zeta_n^1 I + (1-\zeta_n^1)U_1)u_{n+1} \\ &\quad + \kappa_{n+1}^1 P_{Q_1}(\zeta_n^1 I + (1-\zeta_n^1)U_1)u_{n+1} - \kappa_{n+1}^1 P_{Q_1}(\zeta_{n+1}^1 I + (1 \\ &\quad - \zeta_{n+1}^1)U_1)u_{n+1} - ((1-\kappa_n^1)I + \kappa_n^1 P_{Q_1}(\zeta_n^1 I + (1-\zeta_n^1)U_1)u_{n+1})\| \\ &\leq |\kappa_{n+1}^1 - \kappa_n^1|[\|u_{n+1}\| + \|P_{Q_1}(\zeta_n^1 I + (1-\zeta_n^1)U_1)u_{n+1}\|] \\ &\leq |\kappa_{n+1}^1 - \kappa_n^1|J_1,\end{aligned} \tag{3.41}$$

where $J_1 = \|P_{Q_1}(\zeta_n^1 I + (1-\zeta_n^1)U_1)u_{n+1}\| + \|u_{n+1}\|$. As $\lim_{n\to+\infty}|\kappa_{n+1}^1 - \kappa_n^1| = 0$, $\{u_n\}$ and $\{P_{Q_1}(\zeta_n^1 I + (1-\zeta_n^1)U_1)u_{n+1}\}$ are bounded, we have

$$\lim_{n\to+\infty}\|U_1^{n+1}u_{n+1} - U_1^n u_{n+1}\| = 0. \tag{3.42}$$

Similarly,

$$\begin{aligned}\|U_2^{n+1}U_1^n u_{n+1} - U_2^n U_1^n u_{n+1}\| &\leq |\kappa_{n+1}^2 - \kappa_n^2|[\|U_1^n u_{n+1}\| \\ &\quad + \|P_{Q_1}(\zeta_n^2 I + (1-\zeta_n^2)U_2)U_1^n u_{n+1}\|] \\ &\leq |\kappa_{n+1}^2 - \kappa_n^2|J_2,\end{aligned} \tag{3.43}$$

where $J_2 = \|U_1^n u_{n+1}\| + \|P_{Q_1}(\zeta_n^2 I + (1-\zeta_n^2)U_1)u_{n+1}\|$. Using Eqs (3.36), (3.39), (3.40), (3.41) and (3.43), we get

$$\begin{aligned}\|t_{n+1} - t_n\| &\leq (1-\varphi_{n+1})\|u_{n+1} - u_n\| + (\varphi_{n+1} - \varphi_n)\|u_n - s_n\| + \varphi_{n+1}[\|u_{n+1} - u_n\| \\ &\quad + (1-\sigma_{n+1})(|\kappa_{n+1}^1 - \kappa_n^1|J_1 + |\kappa_{n+1}^2 - \kappa_n^2|J_2) \\ &\quad + |\sigma_{n+1} - \sigma_n|\|U_2^n U_1^n u_n - Uu_n\|] \\ &\leq (1-\varphi_{n+1})\|u_{n+1} - u_n\| + (\varphi_{n+1} - \varphi_n)\|u_n - s_n\| + \varphi_{n+1}[\|u_{n+1} - u_n\| \\ &\quad + |\kappa_{n+1}^1 - \kappa_n^1|J_1 + |\kappa_{n+1}^2 - \kappa_n^2|J_2 + |\sigma_{n+1} - \sigma_n|\|U_2^n U_1^n u_n - Uu_n\|] \\ &\leq \|u_{n+1} - u_n\| + (\varphi_{n+1} - \varphi_n)\|u_n - s_n\| + |\sigma_{n+1} - \sigma_n|\|U_2^n U_1^n u_n - Uu_n\| \\ &\quad + |\kappa_{n+1}^1 - \kappa_n^1|J_1 + |\kappa_{n+1}^2 - \kappa_n^2|J_2 \\ &\leq \|u_{n+1} - u_n\| + (\varphi_{n+1} - \varphi_n)\|u_n - s_n\| + |\sigma_{n+1} - \sigma_n|\|U_2^n U_1^n u_n - Uu_n\| \\ &\quad + |\kappa_{n+1}^1 - \kappa_n^1|J_1 + |\kappa_{n+1}^2 - \kappa_n^2|J_2 \\ &\leq \|\chi_{n+1} - \chi_n\| + \tau_{n+1}\|\chi_{n+1} - \chi_n\| + \tau_n\|\chi_n - \chi_{n-1}\| \\ &\quad + M_3\sqrt{|r_{n+1} - r_n|} + \frac{1}{c_1}\left|r_{n+1} - r_n\right|M_1 \\ &\quad + (\varphi_{n+1} - \varphi_n)\|u_n - s_n\| + |\sigma_{n+1} - \sigma_n|\|U_2^n U_1^n u_n - Uu_n\| \\ &\quad + |\kappa_{n+1}^1 - \kappa_n^1|J_1 + |\kappa_{n+1}^2 - \kappa_n^2|J_2.\end{aligned} \tag{3.44}$$

Let $\chi_{n+1} = (1-\rho_n) f_n + \rho_n \chi_n$. Then, $f_n = \frac{\chi_{n+1}-\rho_n\chi_n}{1-\rho_n}$ and

$$\begin{aligned}
\|f_{n+1} - f_n\| \leq & \left\| \frac{\omega_{n+1}\gamma h(K_{n+1}\chi_{n+1}) + [(1-\rho_{n+1})I - \omega_{n+1}A]K_{n+1}t_{n+1}}{1-\rho_{n+1}} \right. \\
& \left. - \frac{\omega_n \gamma h(K_n\chi_n) + [(1-\rho_n)I - \omega_n A]K_n t_n}{1-\rho_n} \right\| \\
& \leq \frac{\omega_{n+1}}{1-\rho_{n+1}}[\|\gamma h(K_{n+1}\chi_{n+1})\| + \|AK_{n+1}t_{n+1}\|] \\
& + \frac{\omega_n}{1-\rho_n}[\|\gamma h(K_n\chi_n)\| + \|AK_n t_n\|] \\
& + \|K_{n+1}t_{n+1} - K_{n+1}t_n\| + \|K_{n+1}t_n - K_n t_n\| \\
& \leq \frac{\omega_{n+1}}{1-\rho_{n+1}}[\|\gamma h(K_{n+1}\chi_{n+1})\| + \|AK_{n+1}t_{n+1}\|] \\
& + \frac{\omega_n}{1-\rho_n}[\|\gamma h(K_n\chi_n)\| + \|AK_n t_n\|] \\
& + \|t_{n+1} - t_n\| + \|K_{n+1}t_n - K_n t_n\|. \qquad (3.45)
\end{aligned}$$

Now, calculating $\|K_{n+1}t_n - K_n t_n\|$ for each $j \in \{2, 3, ..., M-2\}$, we get

$$\begin{aligned}
\|T_{n+1,M-j}t_n - T_{n,M-j}t_n\| = & \|\eta_{n+1,M-j}S_{M-j}T_{n+1,M-j-1}t_n + (1-\eta_{n+1,M-j})T_{n+1,M-j-1}t_n \\
& - \eta_{n,M-j}S_{M-j}T_{n,M-j-1}t_n - (1-\eta_{n,M-j})T_{n,M-j-1}t_n\| \\
& \leq \|\eta_{n+1,M-j}S_{M-j}T_{n+1,M-j-1}t_n + \eta_{n+1,M-j}S_{M-j}T_{n,M-j-1}t_n \\
& - \eta_{n+1,M-j}S_{M-j}T_{n,M-j-1}t_n + (1-\eta_{n+1,M-j})T_{n+1,M-j-1}t_n \\
& - \eta_{n,M-j}S_{M-j}T_{n,M-j-1}t_n + (1-\eta_{n+1,M-j})T_{n,M-j-1}t_n \\
& - (1-\eta_{n+1,M-j})T_{n,M-j-1}t_n - (1-\eta_{n,M-j})T_{n,M-j-1}t_n\| \\
& \leq \eta_{n+1,M-j}\|S_{M-j}T_{n+1,M-j-1}t_n - S_{M-j}T_{n,M-j-1}t_n\| \\
& + (1-\eta_{n+1,M-j})\|T_{n+1,M-j-1}t_n - T_{n,M-j-1}t_n\| \\
& + |\eta_{n+1,M-j} - \eta_{n,M-j}|[\|T_{n,M-j-1}t_n\| + \|S_{M-j}T_{n,M-j-1}t_n\|] \\
& \leq \|T_{n+1,M-j-1}t_n - T_{n,M-j-1}t_n\| + |\eta_{n+1,M-j} - \eta_{n,M-j}|M_4, \qquad (3.46)
\end{aligned}$$

where $M_4 = \sup\{\sum_{j=2}^{M} \|S_j T_{n,j-1}\| + \|T_{n,j-1}t_n\| + \|S_1 t_n\| + \|t_n\|\}$.

Consider

$$\begin{aligned}
\|T_{n+1,1}t_n - T_{n,1}t_n\| = & \|\eta_{n+1,1}S_1 t_n + (1-\eta_{n+1,1})t_n - \eta_{n,1}S_1 t_n - (1-\eta_{n,1})t_n\| \\
& \leq |\eta_{n+1,1} - \eta_{n,1}|[\|S_1 t_n\| + \|t_n\|] \\
& \leq |\eta_{n+1,1} - \eta_{n,1}|M_4. \qquad (3.47)
\end{aligned}$$

Also,

$$\begin{aligned}
\|K_{n+1}t_n - K_n t_n\| = & \|T_{n+1,M}t_n - T_{n,M}t_n\| \\
& \leq \|T_{n+1,M-1}t_n - T_{n,M-1}t_n\| + M_4|\eta_{n+1,M} - \eta_{n,M}| \\
& \leq \|T_{n+1,M-2}t_n - T_{n,M-2}t_n\|
\end{aligned}$$

$$
\begin{aligned}
&+ M_4|\eta_{n+1,M} - \eta_{n,M}| + M_4|\eta_{n+1,M-1} - \eta_{n,M-1}| \\
&\ \vdots \\
&\le \|T_{n+1,1}t_n - T_{n,1}t_n\| + M_4 \sum_{j=2}^{M} |\eta_{n+1,j} - \eta_{n,j}| \\
&\le M_4 \sum_{j=1}^{M} |\eta_{n+1,j} - \eta_{n,j}|. \qquad (3.48)
\end{aligned}
$$

From Eqs (3.44), (3.45) and (3.48), we get

$$
\begin{aligned}
\|f_{n+1} - f_n\| \le\ & \frac{\omega_{n+1}}{1-\rho_{n+1}}[\|\gamma h(K_{n+1}\chi_{n+1})\| + \|AK_{n+1}t_{n+1}\|] \\
&+ \frac{\omega_n}{1-\rho_n}[\|\gamma h(K_n\chi_n)\| + \|AK_nt_n\|] + \|\chi_{n+1} - \chi_n\| + \tau_{n+1}\|\chi_{n+1} - \chi_n\| \\
&+ \tau_n\|\chi_n - \chi_{n-1}\| + M_3\sqrt{|r_{n+1} - r_n|} + \frac{1}{c_1}\big|r_{n+1} - r_n\big|M_1 \\
&+ (\varphi_{n+1} - \varphi_n)\|u_n - s_n\| + |\sigma_{n+1} - \sigma_n|\|U_2^nU_1^nu_n - Uu_n\| \\
&+ |\kappa_{n+1}^1 - \kappa_n^1|J_1 + |\kappa_{n+1}^2 - \kappa_n^2|J_2 + M_4\sum_{j=1}^{M}|\eta_{n+1,j} - \eta_{n,j}|. \qquad (3.49)
\end{aligned}
$$

Using Remark 3.2, $\lim_{n\to+\infty}|\kappa_{n+1}^i - \kappa_n^i| = 0$, for $i = 1, 2$, $\lim_{n\to+\infty}\omega_n = 0$, $\lim_{n\to+\infty}|r_{n+1} - r_n| = 0$, $\lim_{n\to+\infty}|\sigma_{n+1} - \sigma_n| = 0$, $\lim_{n\to+\infty}|\varphi_{n+1} - \varphi_n| = 0$ and taking lim sup in Eq (3.49), we have

$$
\limsup_{n\to+\infty}(\|f_{n+1} - f_n\| - \|\chi_{n+1} - \chi_n\|) \le 0.
$$

Claim 3: $\lim_{n\to+\infty}\|\chi_n - \chi_{n-1}\| = \lim_{n\to+\infty}\|f_n - \chi_n\| = 0$ and $\lim_{n\to+\infty}\|t_n - u_n\| = \lim_{n\to+\infty}\|w_n - \chi_n\| = \lim_{n\to+\infty}\|\chi_n - K_nt_n\| = 0$.
Using Lemma 2.15, we have

$$
\lim_{n\to+\infty}\|f_n - \chi_n\| = 0. \qquad (3.50)
$$

Also, $\chi_{n+1} = (1-\rho_n)f_n + \rho_n\chi_n$, which implies $\|\chi_{n+1} - \chi_n\| = \|(1-\rho_n)(f_n - \chi_n)\|$. Now using Eq (3.50), we have

$$
\lim_{n\to+\infty}\|\chi_{n+1} - \chi_n\| = 0. \qquad (3.51)
$$

From Eq (3.2), we have $\|w_n - \chi_n\| = \|\tau_n(\chi_n - \chi_{n-1})\|$. Taking the limit $n \to +\infty$, we get

$$
\lim_{n\to+\infty}\|w_n - \chi_n\| = 0. \qquad (3.52)
$$

Also,

$$
\|\chi_n - K_nt_n\| = \|\chi_n - \chi_{n+1} + \chi_{n+1} - K_nt_n\|
$$

$$
\begin{aligned}
&\leq \|\chi_n - \chi_{n+1}\| + \|\omega_n \gamma h(K_n\chi_n) + \rho_n\chi_n \\
&+ [(1-\rho_n)I - \omega_n A]K_n t_n - K_n t_n\| \\
&\leq \|\chi_n - \chi_{n+1}\| + \omega_n\|\gamma h(K_n\chi_n) - AK_n t_n\| + \rho_n\|\chi_n - K_n t_n\|,
\end{aligned}
$$

which implies

$$
(1-\rho_n)\|\chi_n - K_n t_n\| \leq \|\chi_n - \chi_{n+1}\| + \omega_n\|\gamma h(K_n\chi_n) - AK_n t_n\|. \tag{3.53}
$$

Taking the limit $n \to +\infty$ and using $\lim_{n\to+\infty} \omega_n = 0$, we have

$$
\lim_{n\to+\infty} \|\chi_n - K_n t_n\| = 0. \tag{3.54}
$$

Consider

$$
\begin{aligned}
\|\chi_{n+1} - \varrho\|^2 &= \|\omega_n\gamma h(K_n\chi_n) + \rho_n\chi_n + [(1-\rho_n)I - \omega_n A]K_n t_n - \varrho\|^2 \\
&\leq \|(1-\omega_n A)(K_n t_n - \varrho) + \rho_n(\chi_n - K_n t_n)\|^2 \\
&+ 2\omega_n \langle \gamma h(K_n\chi_n) - A\varrho, \chi_{n+1} - \varrho\rangle \\
&\leq (1-\omega_n\bar{\gamma})^2\|t_n - \varrho\|^2 + \rho_n^2\|\chi_n - K_n t_n\|^2 \\
&+ 2\rho_n(1-\omega_n\bar{\gamma})\|t_n - \varrho\|\|\chi_n - K_n t_n\| + 2\omega_n \langle \gamma h(K_n\chi_n) - A\varrho, \chi_{n+1} - \varrho\rangle .
\end{aligned} \tag{3.55}
$$

Also,

$$
\begin{aligned}
\|y_n - \varrho\|^2 &= \|T_{r_n}^{F_1}(I - r_n g_1)w_n - T_{r_n}^{F_1}(I - r_n g_1)\varrho\|^2 \\
&\leq \langle y_n - \varrho, (I - r_n g_1)w_n - (I - r_n g_1)\varrho\rangle \\
&= \frac{1}{2}\Big[\|y_n - \varrho\|^2 + \|(I - r_n g_1)w_n - (I - r_n g_1)\varrho\|^2 \\
&- \|y_n - \varrho - ((I - r_n g_1)w_n - (I - r_n g_1)\varrho)\|^2\Big] \\
&\leq \|(I - r_n g_1)w_n - (I - r_n g_1)\varrho\|^2 - \|(y_n - w_n) - r_n(g_1(w_n) - g_1(\varrho))\|^2,
\end{aligned} \tag{3.56}
$$

which implies

$$
\|y_n - \varrho\|^2 \leq \|w_n - \varrho\|^2 - \{\|w_n - y_n\|^2 - 2r_n\|w_n - y_n\|\|g_1 w_n - g_1\varrho\|\}. \tag{3.57}
$$

Using Eq (3.14), we have

$$
\begin{aligned}
\|u_n - \varrho\|^2 &= \|y_n + \delta D^*(l_n - Dy_n) - \varrho\|^2 \\
&\leq \langle u_n - \varrho, y_n - \delta D^*(Dy_n - l_n) - \varrho\rangle \\
&= \frac{1}{2}\Big[\|u_n - \varrho\|^2 + \|y_n - \delta D^*(Dy_n - l_n) - \varrho\|^2 - \|u_n - \varrho - (y_n - \delta D^*(Dy_n - l_n) - \varrho)\|^2\Big] \\
&\leq \frac{1}{2}\Big[\|u_n - \varrho\|^2 + \|y_n - \varrho\|^2 - \|u_n - \varrho - (y_n - \delta D^*(Dy_n - l_n) - \varrho)\|^2\Big] \\
&= \frac{1}{2}\Big[\|u_n - \varrho\|^2 + \|y_n - \varrho\|^2 - \{\|u_n - y_n\|^2 + \delta^2\|D^*(l_n - Dy_n)\|^2 - 2\delta\langle u_n - y_n, D^*(Dy_n - l_n)\rangle\}\Big],
\end{aligned} \tag{3.58}
$$

which implies

$$\|u_n - \varrho\|^2 \leq \|y_n - \varrho\|^2 - \|u_n - y_n\|^2 + 2\delta\|Dy_n - l_n\|\|u_n - y_n\|\|D^*\|. \tag{3.59}$$

Using Eqs (3.11), (3.14), (3.17) and (3.55), we get

$$\begin{aligned}
\|\chi_{n+1} - \varrho\|^2 &= (1-\omega_n\bar{\gamma})^2(\|u_n - \varrho\| + \varphi_n\sigma_n\|U\varrho - \varrho\|)^2 + \rho_n^2\|\chi_n - K_n t_n\|^2 \\
&+ 2\rho_n(1-\omega_n\bar{\gamma})\|t_n - \varrho\|\|\chi_n - K_n t_n\| \\
&+ 2\omega_n \langle h(K_n\chi_n) - A\varrho, \chi_{n+1} - \varrho\rangle \\
&\leq (1-\omega_n\bar{\gamma})^2(\|u_n - \varrho\|^2 + \varphi_n^2\sigma_n^2\|U\varrho - \varrho\|^2 \\
&+ 2\varphi_n\sigma_n\|U\varrho - \varrho\|\|u_n - \varrho\|) + \rho_n^2\|\chi_n - K_n t_n\|^2 \\
&+ 2\rho_n(1-\omega_n\bar{\gamma})\|t_n - \varrho\|\|\chi_n - K_n t_n\| + 2\omega_n \langle h(K_n\chi_n) - A\varrho, \chi_{n+1} - \varrho\rangle \qquad (3.60) \\
&\leq (1-\omega_n\bar{\gamma})^2(\|w_n - \varrho\|^2 - \delta(1-\delta L)\|l_n - Dy_n\|^2) \\
&+ \varphi_n^2\sigma_n^2\|U\varrho - \varrho\|^2 + 2\varphi_n\sigma_n\|U\varrho - \varrho\|\|u_n - \varrho\| \\
&+ \rho_n^2\|\chi_n - K_n t_n\|^2 + 2\rho_n(1-\omega_n\bar{\gamma})\|t_n - \varrho\|\|\chi_n - K_n t_n\| \\
&+ 2\omega_n \langle h(K_n\chi_n) - A\varrho, \chi_{n+1} - \varrho\rangle. \qquad (3.61)
\end{aligned}$$

Using Eqs (3.20) and (3.60), we get

$$\begin{aligned}
(1-\omega_n\bar{\gamma})^2\delta(1-\delta L)\|l_n - Dy_n\|^2 &\leq \|\chi_n - \varrho\|^2 - \|\chi_{n+1} - \varrho\|^2 + (\omega_n\bar{\gamma})^2\|\chi_n - \varrho\|^2 \\
&+ (1-\omega_n\bar{\gamma})^2(\tau_n\|\chi_n - \chi_{n-1}\|M \\
&+ \varphi_n^2\sigma_n^2\|U\varrho - \varrho\|^2 + 2\varphi_n\sigma_n\|U\varrho - \varrho\|\|u_n - \varrho\|) \\
&+ \rho_n^2\|\chi_n - K_n t_n\|^2 \\
&+ 2\rho_n(1-\omega_n\bar{\gamma})\|t_n - \varrho\|\|\chi_n - K_n t_n\| \\
&+ 2\omega_n \langle h(K_n\chi_n) - A\varrho, \chi_{n+1} - \varrho\rangle \\
&\leq (\|\chi_n - \varrho\| + \|\chi_{n+1} - \varrho\|)\|\chi_n - \chi_{n+1}\| \\
&+ (\omega_n\bar{\gamma})^2\|\chi_n - \varrho\|^2 + (1-\omega_n\bar{\gamma})^2(\tau_n\|\chi_n - \chi_{n-1}\|M \\
&+ \varphi_n^2\sigma_n^2\|U\varrho - \varrho\|^2 + 2\varphi_n\sigma_n\|U\varrho - \varrho\|\|u_n - \varrho\|) \\
&+ \rho_n^2\|\chi_n - K_n t_n\|^2 \\
&+ 2\rho_n(1-\omega_n\bar{\gamma})\|t_n - \varrho\|\|\chi_n - K_n t_n\| \\
&+ 2\omega_n \langle h(K_n\chi_n) - A\varrho, \chi_{n+1} - \varrho\rangle \qquad (3.62)
\end{aligned}$$

and using Eqs (3.20), (3.59) and (3.60), we estimate

$$\begin{aligned}
(1-\omega_n\bar{\gamma})^2\|u_n - y_n\|^2 &\leq \|\chi_n - \varrho\|^2 - \|\chi_{n+1} - \varrho\|^2 + \rho_n^2\|\chi_n - K_n t_n\|^2 \\
&+ (1-\omega_n\bar{\gamma})^2(2\delta\|Dy_n - l_n\|\|u_n - y_n\|\|D^*\| + \varphi_n^2\sigma_n^2\|U\varrho - \varrho\|^2 \\
&+ 2\varphi_n\sigma_n\|U\varrho - \varrho\|\|u_n - \varrho\|) + 2\rho_n(1-\omega_n\bar{\gamma})\|t_n - \varrho\|\|\chi_n - K_n t_n\| \\
&+ 2\omega_n \langle h(K_n\chi_n) - A\varrho, \chi_{n+1} - \varrho\rangle \\
&\leq (\|\chi_n - \varrho\| + \|\chi_{n+1} - \varrho\|)\|\chi_n - \chi_{n+1}\| + \rho_n^2\|\chi_n - K_n t_n\|^2 \\
&+ (1-\omega_n\bar{\gamma})^2(2\delta\|Dy_n - l_n\|\|u_n - y_n\|\|D^*\| + \varphi_n^2\sigma_n^2\|U\varrho - \varrho\|^2
\end{aligned}$$

$$+ 2\varphi_n\sigma_n\|U\varrho - \varrho\|\|u_n - \varrho\|) + 2\rho_n(1 - \omega_n\bar{\gamma})\|t_n - \varrho\|\|\chi_n - K_n t_n\|$$
$$+ 2\omega_n \langle h(K_n\chi_n) - A\varrho, \chi_{n+1} - \varrho\rangle . \tag{3.63}$$

Using Eqs (3.14), (3.20) and (3.57), we have

$$\begin{aligned}
(1 - \omega_n\bar{\gamma})^2 r_n(2\alpha - r_n)\|g_1(w_n) - g_1(\varrho)\|^2 \le\ & \|\chi_n - \varrho\|^2 - \|\chi_{n+1} - \varrho\|^2 + (\omega_n\bar{\gamma})^2\|\chi_n - \varrho\|^2 \\
& + (1 - \omega_n\bar{\gamma})^2(\tau_n\|\chi_n - \chi_{n-1}\|M + \varphi_n^2\sigma_n^2\|U\varrho - \varrho\|^2 \\
& + 2\varphi_n\sigma_n\|U\varrho - \varrho\|\|u_n - \varrho\|) + \rho_n^2\|\chi_n - K_n t_n\|^2 \\
& + 2\rho_n(1 - \omega_n\bar{\gamma})\|t_n - \varrho\|\|\chi_n - K_n t_n\| \\
& + 2\omega_n \langle h(K_n\chi_n) - A\varrho, \chi_{n+1} - \varrho\rangle \\
\le\ & (\|\chi_n - \varrho\| + \|\chi_{n+1} - \varrho\|)\|\chi_n - \chi_{n+1}\| \\
& + (\omega_n\bar{\gamma})^2\|\chi_n - \varrho\|^2 \\
& + (1 - \omega_n\bar{\gamma})^2(\tau_n\|\chi_n - \chi_{n-1}\|M \\
& + \varphi_n^2\sigma_n^2\|U\varrho - \varrho\|^2 \\
& + 2\varphi_n\sigma_n\|U\varrho - \varrho\|\|u_n - \varrho\|) \\
& + \rho_n^2\|\chi_n - K_n t_n\|^2 \\
& + 2\rho_n(1 - \omega_n\bar{\gamma})\|t_n - \varrho\|\|\chi_n - K_n t_n\| \\
& + 2\omega_n \langle h(K_n\chi_n) - A\varrho, \chi_{n+1} - \varrho\rangle .
\end{aligned} \tag{3.64}$$

Using Eqs (3.20), (3.60) and (3.57), we have

$$\begin{aligned}
(1 - \omega_n\bar{\gamma})^2\|w_n - y_n\|^2 \le\ & \|\chi_n - \varrho\|^2 - \|\chi_{n+1} - \varrho\|^2 \\
& + \rho_n^2\|\chi_n - K_n t_n\|^2 + (\omega_n\bar{\gamma})^2\|\chi_n - \varrho\|^2 \\
& + (1 - \omega_n\bar{\gamma})^2[\tau_n\|\chi_n - \chi_{n-1}\|M + 2r_n\|w_n - y_n\|\|g_1(w_n) - g_1(\varrho)\| \\
& + \varphi_n^2\sigma_n^2\|U\varrho - \varrho\|^2 + 2\varphi_n\sigma_n\|U\varrho - \varrho\|\|u_n - \varrho\|] \\
& + \rho_n^2\|\chi_n - K_n t_n\|^2 + 2\rho_n(1 - \omega_n\bar{\gamma})\|t_n - \varrho\|\|\chi_n - K_n t_n\| \\
& + 2\omega_n \langle h(K_n\chi_n) - A\varrho, \chi_{n+1} - \varrho\rangle \\
\le\ & (\|\chi_n - \varrho\| + \|\chi_{n+1} - \varrho\|)\|\chi_n - \chi_{n+1}\| + \rho_n^2\|\chi_n - K_n t_n\|^2 \\
& + \varphi_n^2\sigma_n\|U\varrho - \varrho\|^2 \\
& + 2\varphi_n^2\sigma_n\|U\varrho - \varrho\| + \rho_n^2\|\chi_n - K_n t_n\|^2 + (\omega_n\bar{\gamma})^2\|\chi_n - \varrho\|^2 \\
& + 2(1 - \omega_n\bar{\gamma})^2 r_n\|w_n - y_n\|\|g_1(w_n) - g_1(\varrho)\| \\
& + (1 - \omega_n\bar{\gamma})^2(\tau_n\|\chi_n - \chi_{n-1}\|M + \varphi_n^2\sigma_n^2\|U\varrho - \varrho\|^2 \\
& + 2\varphi_n\sigma_n\|U\varrho - \varrho\|\|u_n - \varrho\|) \\
& + \rho_n^2\|\chi_n - K_n t_n\|^2 + 2\rho_n(1 - \omega_n\bar{\gamma})\|t_n - \varrho\|\|\chi_n - K_n t_n\| \\
& + 2\omega_n \langle h(K_n\chi_n) - A\varrho, \chi_{n+1} - \varrho\rangle .
\end{aligned} \tag{3.65}$$

From Eqs (3.54), (3.62) and (3.64) and using $\lim_{n\to+\infty}\sigma_n < +\infty$, $\lim_{n\to+\infty}\|\chi_n - \chi_{n-1}\| = 0$, $\lim_{n\to+\infty}\omega_n = 0$ and Remark 3.2, we get

$$\lim_{n\to+\infty}\|l_n - Dy_n\| = 0 \text{ and } \lim_{n\to+\infty}\|g_1(w_n) - g_1(\varrho)\| = 0. \tag{3.66}$$

Using Eqs (3.54), (3.62) and (3.66) and using $\lim_{n\to+\infty}\sigma_n < +\infty$, $\lim_{n\to+\infty}\|\chi_n - \chi_{n-1}\| = 0$, $\lim_{n\to+\infty}\omega_n = 0$ and Remark 3.2, we get

$$\lim_{n\to+\infty}\|u_n - y_n\| = 0 \text{ and } \lim_{n\to+\infty}\|w_n - y_n\| = 0. \tag{3.67}$$

Using the triangle inequality and Eqs (3.52) and (3.67), we get

$$\lim_{n\to+\infty}\|u_n - \chi_n\| = 0. \tag{3.68}$$

As we know that U and $U_2^nU_1^n$ are nonexpansive mappings and $\{u_n\}$ is bounded, one may suppose that there is a nonnegative real number k such that $\|Uu_n - U_2^nU_1^nu_n\| \le k$ for all $n \ge 0$. Now, consider

$$\begin{aligned}\|t_n - U_2^nU_1^nu_n\| &= \|(1-\varphi_n)u_n + \varphi_n[\sigma_nUu_n + (1-\sigma_n)U_2^nU_1^nu_n] - U_2^nU_1^nu_n\|\\ &\le (1-\varphi_n)\|u_n - U_2^nU_1^nu_n\| + \varphi_n\sigma_n\|Uu_n - U_2^nU_1^nu_n\|\\ &\le (1-\varphi_n)\|u_n - t_n\| + (1-\varphi_n)\|t_n - U_2^nU_1^nu_n\| + \varphi_n\sigma_n\|Uu_n - U_2^nU_1^nu_n\|.\end{aligned} \tag{3.69}$$

Subsequently, we have

$$\|t_n - U_2^nU_1^nu_n\| \le \frac{(1-\varphi_n)}{\varphi_n}\|u_n - t_n\| + \sigma_nk. \tag{3.70}$$

Using condition (*iii*) and $\lim_{n\to+\infty}\sigma_n < +\infty$, we get

$$\lim_{n\to+\infty}\|t_n - U_2^nU_1^nu_n\| = 0. \tag{3.71}$$

Also,

$$\begin{aligned}\|t_n - u_n\| &= \|(1-\varphi_n)u_n + \varphi_n[\sigma_nUu_n + (1-\sigma_n)U_2^nU_1^nu_n] - u_n\|\\ &\le \varphi_n\sigma_n\|Uu_n - u_n\| + (1-\sigma_n)\varphi_n\|U_2^nU_1^nu_n - u_n\|\\ &\le \varphi_n\sigma_n[\|Uu_n - t_n\| + \|t_n - u_n\|] + (1-\sigma_n)\varphi_n[\|U_2^nU_1^nu_n - t_n\| + \|t_n - u_n\|].\end{aligned} \tag{3.72}$$

Hence, we have

$$(1-\varphi_n)\|t_n - u_n\| \le \sigma_n\|Uu_n - t_n\| + (1-\sigma_n)\|U_2^nU_1^nu_n - t_n\|. \tag{3.73}$$

From Eq (3.71) and $\lim_{n\to+\infty}\sigma_n = 0$, we have

$$\lim_{n\to+\infty}\|t_n - u_n\| = 0. \tag{3.74}$$

Claim 4: $u' \in GMEP(F_1, g_1, \psi_1, Q_1)$.
As we know that $\{u_n\}$ is bounded, there exists a subsequence $\{u_{n_i}\}$ of $\{u_n\}$ that converges weakly to some $u' \in Q_1$. Also, from Eq (3.54), we have $K_nt_{n_i} \rightharpoonup u'$. Now, we show $u' \in GMEP(F_1, g_1, \psi_1, Q_1)$. Using Lemma 2.13, we have

$$F_1(u_n, z) + \langle g_1(y_n), z - u_n\rangle + \psi_1(z) - \psi_1(u_n) + \frac{1}{r_n}\langle z - u_n, u_n - y_n\rangle \ge 0 \text{ for all } z \in Q_1.$$

Using the monotonicity of F_1, we have

$$\langle g_1(y_n), z - u_n\rangle + \psi_1(z) - \psi_1(u_n) + \frac{1}{r_n}\langle z - u_n, u_n - y_n\rangle \geq F_1(z, u_n) \text{ for all } z \in Q_1.$$

Replacing n by n_k, we have

$$\left\langle g_1(z_{n_k}), z - u_{n_k}\right\rangle + \psi_1(z) - \psi_1(u_{n_k}) + \frac{1}{r_{n_k}}\left\langle z - u_{n_k}, u_{n_k} - z_{n_k}\right\rangle \geq F_1(z, u_{n_k}) \text{ for all } z \in Q_1.$$

Let m with $0 < m \leq 1$ and $u \in Q_1$ satisfying $u_m = mu + (1-m)u'$, then $u_m \in Q_1$ and, from the above inequality, we have

$$\begin{aligned}\langle u_m - u_{n_k}, g_1(u_m)\rangle &\geq \langle u_m - u_{n_k}, g_1(u_m)\rangle + \psi_1(u_{n_k}) - \psi_1(u_m) - \left\langle g_1(z_{n_k}), z - u_{n_k}\right\rangle \\ &\quad + \left\langle u_m - u_{n_k}, \frac{1}{r_{n_k}}(u_{n_k} - z_{n_k})\right\rangle + F_1(u_m, u_{n_k}) \\ &= \langle u_m - u_{n_k}, g_1(u_m) - g_1(u_{n_k})\rangle + \langle u_m - u_{n_k}, g_1(u_{n_k}) - g_1(z_{n_k})\rangle \\ &\quad + \psi_1(u_{n_k}) - \psi_1(u_m) + \left\langle u_m - u_{n_k}, \frac{1}{r_{n_k}}(u_{n_k} - z_{n_k})\right\rangle + F_1(u_m, u_{n_k}). \end{aligned} \tag{3.75}$$

Using the Lipschitz continuity of g_1 and Eq (3.67), we have $\|g_1 u_{n_k} - g_1 z_{n_k}\| = 0$ as $k \to +\infty$. Further, as F_1 is monotone and ϕ_1 is convex and lower semicontinuous, the above equation implies

$$\langle u_m - u', g_1(u_m)\rangle \geq F_1(u_m, u') + \psi_1(u') - \psi_1(u_m). \tag{3.76}$$

Consider for $m > 0$

$$\begin{aligned} 0 = F_1(u_m, u_m) &\leq mF_1(u_m, u) + (1-m)F_1(u_m, u') \\ &\leq mF_1(u_m, u) + (1-m)(\langle u_m - u', g_1(u_m)\rangle - \psi_1(u') + \psi_1(u_m)) \\ &\leq mF_1(u_m, u) + (1-m)m(\langle u - u', g_1(u_m)\rangle - \psi_1(u') + \psi_1(u)). \end{aligned} \tag{3.77}$$

Taking $m \to 0_+$, we get

$$F_1(u', u) + \langle u - u', g_1(u')\rangle - \psi_1(u') + \psi_1(u) \geq 0 \text{ for all } u \in Q_1. \tag{3.78}$$

Hence, $u' \in GMEP(F_1, g_1, \psi_1, Q_1)$.

Claim 5: Now we will prove $Du' \in GMEP(F_2, g_2, \psi_2, Q_2)$.

As D is a bounded linear operator, and using Eqs (3.66) and (3.67), this implies $Dz_{n_k} \rightharpoonup Du'$. Taking $l'_{n_k} = Dz_k - T^{F_2}_{r_{n_k}}(I - r_{n_k} g_2)Dz_k$ and using Eq (3.66) we have $\lim_{n\to+\infty} l'_{n_k} = 0$ and $T^{F_2}_{r_{n_k}}(I - r_{n_k} g_2)Dz_k = Dz_k - l'_{n_k}$. Now, using Lemma 2.13, we get

$$\begin{aligned} &F_2(Dz_{n_k} - l'_{n_k}, s) + \left\langle s - Dz_{n_k} + l'_{n_k}, g_2(z_{n_k})\right\rangle - \psi_2(Dz_{n_k} - l'_{n_k}) + \psi_2(s) \\ &\quad + \frac{1}{r_{n_k}}\left\langle s - (Dz_{n_k} - l'_{n_k}), Dz_{n_k} - l'_{n_k} - Dz_{n_k}\right\rangle \geq 0, \text{ for all } s \in Q_2. \end{aligned} \tag{3.79}$$

As F_2 is upper semicontinuous, we use $\limsup$ in the above equation as $k \to +\infty$. Also, with $\liminf_{n\to+\infty} r_n > 0$, we have

$$F_2(Du', s) + \langle s - Du', g_2(u')\rangle - \psi_2(Du') + \psi_2(s) \geq 0 \text{ for all } s \in Q_2. \tag{3.80}$$

Hence, $Du' \in GMEP(F_2, g_2, \psi_2, Q_2)$.

Claim 6: Now we will prove $u' \in Fix(K)$.

Assume that K is the K-mappings generated by $S_1, S_2, ..., S_M$ and $\eta_1, \eta_2, ..., \eta_M$. Now, using Lemma 2.16, we have

$$K_{n_j} x \to Kx \text{ and } Fix(K) = \bigcap_{j=1}^{M} \text{Fix}(S_j). \tag{3.81}$$

We have to show $u' \in \text{Fix}(K)$. We will do it by contradiction. Assume that $u' \notin \text{Fix}(K)$, which implies $Ku' \neq u'$. Now using opial conditions, we get

$$\begin{aligned}
\liminf_{j\to+\infty} \|t_{n_j} - u'\| &< \liminf_{j\to+\infty} \|t_{n_j} - Ku'\| \\
&\leq \liminf_{j\to+\infty} \|t_{n_j} - K_{n_j} t_{n_j}\| + \|K_{n_j} t_{n_j} - K_{n_j} u'\| + \|K_{n_j} u' - Ku'\| \\
&\leq \liminf_{j\to+\infty} \|K_{n_j} t_{n_j} - K_{n_j} u'\| \\
&\leq \liminf_{j\to+\infty} \|t_{n_j} - u'\|,
\end{aligned} \tag{3.82}$$

which is a contradiction. Thus, $u' \in \text{Fix}(K) = \bigcap_{j=1}^{M} \text{Fix}(S_j)$.

Claim 7: We claim that $u' \in \text{Fix}(U_1) \cap \text{Fix}(U_2)$. As the sequence $\{\chi_n\}$ is bounded, then there is a subsequence $\{x_{n_k}\}$ of $\{\chi_n\}$ such that $\{x_{n_k}\} \rightharpoonup u'$ as $k \to +\infty$. Also, κ_n^i is bounded, which implies $\kappa_{n_k}^i \to \kappa_{+\infty}^i$ for $i = 1, 2$ and $k \to +\infty$, where $0 < \kappa_{+\infty}^i < 1$. Consider $U_i^{+\infty} = (1 - \kappa_{+\infty}^i)I + \kappa_{+\infty}^i P_{Q_1}(\zeta_{+\infty}^i I + (1 - \zeta_{+\infty}^i)U_i)$ for $i = 1, 2$. Using Lemma 2.11, we conclude that $\text{Fix}(P_{Q_1}(\zeta_{+\infty}^i I + (1 - \zeta_{+\infty}^i)U_i)) = \text{Fix}(U_i)$. As $P_{Q_1}(\zeta_{+\infty}^i I + (1 - \zeta_{+\infty}^i)U_i)$ is a nonexpansive mapping, $\text{Fix}(U_i^{+\infty}) = \text{Fix}(U_i)$ and $U_i^{+\infty}$ is averaged. Further,

$$\text{Fix}(U_1^{+\infty}) \cap \text{Fix}(U_2^{+\infty}) = \text{Fix}(U_1) \cap \text{Fix}(U_2) = \text{Fix}(U) \neq \phi. \tag{3.83}$$

Using Lemma 2.9, we get

$$\text{Fix}(U_1^{+\infty} U_2^{+\infty}) = \text{Fix}(U_1^{+\infty}) \cap \text{Fix}(U_2^{+\infty}) = \text{Fix}(U) \neq \phi. \tag{3.84}$$

Additionally,

$$\|U_i^{n_k} s - U_i^{+\infty} s\| \leq |\kappa_{n_k}^i - \kappa_{+\infty}^i|(\|s\| + \|P_{Q_1}(\zeta_n^i s + (1 - \zeta_n^i U_i(s))\|). \tag{3.85}$$

Subsequently, we get

$$\lim_{j\to+\infty} \sup_{s\in K} \|U_i^{n_k} s - U_i^{+\infty} s\| = 0, \tag{3.86}$$

where K is any bounded subset of H_1. Note that

$$\begin{aligned}
\|x_{n_k} - U_2^{+\infty} U_1^{+\infty} x_{n_k}\| &\leq \|x_{n_k} - U_2^{n_k} U_1^{n_k} x_{n_k}\| + \|U_2^{n_k} U_1^{n_k} x_{n_k} - U_2^{+\infty} U_1^{n_k} x_{n_k}\| \\
&\quad + \|U_2^{+\infty} U_1^{n_k} x_{n_k} - U_2^{+\infty} U_1^{+\infty} x_{n_k}\| \\
&\leq \|x_{n_k} - U_2^{n_k} U_1^{n_k} x_{n_k}\| + \|U_2^{n_k} U_1^{n_k} x_{n_k} - U_2^{+\infty} U_1^{n_k} x_{n_k}\| \\
&\quad + \|U_2^{+\infty} U_1^{n_k} x_{n_k} - U_2^{+\infty} U_1^{+\infty} x_{n_k}\|
\end{aligned}$$

$$\leq \|x_{n_k} - U_2^{n_k} U_1^{n_k} x_{n_k}\| + \sup_{s\in K_1} \|U_2^{n_k} s - U_2^{+\infty} s\| + \sup_{s\in K_2} \|U_1^{n_k} s - U_1^{+\infty} s\|, \tag{3.87}$$

where K_1 and K_2 are bounded subsets including $\{U_1^{n_k} x_{n_k}\}$ and $\{x_{n_k}\}$ respectively. From Eqs (3.71), (3.86) and (3.87), we conclude that

$$\lim_{k\to+\infty} \|x_{n_k} - U_2^{+\infty} U_1^{+\infty} x_{n_k}\| = 0. \tag{3.88}$$

Subsequently, using Lemma 2.17, we get $u' \in \mathrm{Fix}(U_1^{+\infty} U_2^{+\infty}) = \mathrm{Fix}(U_1) \cap \mathrm{Fix}(U_2)$.
Claim 8: Next, we will show $u' \in \Omega$. From Eq (3.3), we get

$$t_n - u_n = \varphi_n[\sigma_n(U - I)u_n + (1 - \sigma_n)(U_2^n U_1^n u_n - u_n)] \tag{3.89}$$

and hence

$$\frac{1}{\varphi_n \sigma_n}(u_n - t_n) = (I - U)u_n + (1 - \sigma_n)(I - U_2^n U_1^n)u_n. \tag{3.90}$$

Using Lemma 2.14 (i), the sequence $\left\{\frac{(1-\sigma_n)}{\sigma_n}(I - U_2^n U_1^n)\right\}$ is graph convergent to $N_{\mathrm{Fix}(U_1)\cap\mathrm{Fix}(U_2)}$, and using Lemma 2.14 (ii), one can conclude that the sequence $(I - U) + \left\{\frac{(1-\sigma_n)}{\sigma_n}(I - U_2^n U_1^n)\right\}$ is graph convergent to $(I - U) + N_{\mathrm{Fix}(U_1)\cap\mathrm{Fix}(U_2)}$. Replacing n by n_j and taking the limit $j \to +\infty$ in Eq (3.90) and using condition (iii), we have

By substituting n_j for n and taking the limit as j tends to infinity in Eq (3.90) while utilizing condition (iii), we obtain:

$$0 \in (I - U)u' + N_{\mathrm{Fix}(U_1)\cap\mathrm{Fix}(U_2)} u', \tag{3.91}$$

which implies $u' \in \Omega$. From Claims 5–8, we have $u' \in \Gamma$.
Claim 9: Now we show $\limsup_{n\to+\infty} \langle(\gamma h - A)u', \chi_n - v\rangle \leq 0$, where $u' = P_\Gamma(I + \gamma h - A)u'$. As the sequence $\{t_n\}$ weakly converges to u' and using Lemma 2.8, we have

$$\begin{aligned}\limsup_{n\to+\infty} \langle(\gamma h - A)\varrho, \chi_n - \varrho\rangle &= \limsup_{n\to+\infty} \langle(\gamma h - A)\varrho, K_n t_n - \varrho\rangle \\ &\leq \limsup_{n\to+\infty} \langle(\gamma h - A)\varrho, t_n - \varrho\rangle \\ &= 0.\end{aligned} \tag{3.92}$$

As h is a contraction mapping, one can easily prove $P_\Gamma(I+\gamma h-A)$ is also a contraction mapping from H_1 to itself. Using the Banach contraction principle, there exists a $u' \in H_1$ such that $u' = P_\Gamma(I + \gamma h - A)u'$.
Claim 10: Next we show $\chi_n \to \varrho$.
Consider

$$\begin{aligned}\|\chi_{n+1} - \varrho\|^2 = &\langle\omega_n(\gamma h(K_n\chi_n) - A\varrho) + \rho_n(\chi_n - \varrho), \chi_{n+1} - \varrho\rangle \\ &+ \langle[(1 - \rho_n)I - \omega_n A](K_n t_n - \varrho), \chi_{n+1} - \varrho\rangle\end{aligned}$$

$$
\begin{aligned}
&\leq \omega_n \langle \gamma h(K_n\chi_n) - A\varrho, \chi_{n+1} - \varrho\rangle + \rho_n \langle \chi_n - \varrho, \chi_{n+1} - \varrho\rangle \\
&+ [(1-\rho_n)I - \omega_n A] \langle K_n t_n - \varrho, \chi_{n+1} - \varrho\rangle \\
&\leq \omega_n \langle \gamma h(K_n\chi_n) - A\varrho, \chi_{n+1} - \varrho\rangle \\
&+ \rho_n\|\chi_n - \varrho\|\|\chi_{n+1} - \varrho\| + [(1-\rho_n)I - \omega_n\bar{\gamma}]\|K_n t_n - \varrho\|\|\chi_{n+1} - \varrho\| \\
&\leq \omega_n \langle \gamma h(K_n\chi_n) - \gamma h\varrho, \chi_{n+1} - \varrho\rangle + \omega_n \langle \gamma h\varrho - A\varrho, \chi_{n+1} - \varrho\rangle \\
&+ \rho_n\|\chi_n - \varrho\|\|\chi_{n+1} - \varrho\| + [(1-\rho_n)I - \omega_n\bar{\gamma}]\|t_n - \varrho\|\|\chi_{n+1} - \varrho\| \\
&\leq \omega_n\gamma\nu\|\chi_n - \varrho\|\|\chi_{n+1} - \varrho\| + \omega_n \langle \gamma h\varrho - A\varrho, \chi_{n+1} - \varrho\rangle \\
&+ \rho_n\|\chi_n - \varrho\|\|\chi_{n+1} - \varrho\| + [(1-\rho_n)I \\
&- \omega_n\bar{\gamma}]\|\chi_{n+1} - \varrho\|(\|\chi_n - \varrho\| + \tau_n\|\chi_n - \chi_{n-1}\| + \varphi_n\sigma_n\|U\varrho - \varrho\|) \\
&\leq (1 - \omega_n(\bar{\gamma} - \gamma\nu))\|\chi_n - \varrho\|\|\chi_{n+1} - \varrho\| + \omega_n \langle \gamma h\varrho - A\varrho, \chi_{n+1} - \varrho\rangle \\
&+ [(1-\rho_n)I - \omega_n\bar{\gamma}]\|\chi_{n+1} - \varrho\|(\tau_n\|\chi_n - \chi_{n-1}\| + \varphi_n\sigma_n\|U\varrho - \varrho\|) \\
&\leq (1 - \omega_n(\bar{\gamma} - \gamma\nu))\frac{1}{2}[\|\chi_n - \varrho\|^2 + \|\chi_{n+1} - \varrho\|^2] \\
&+ \omega_n \langle \gamma h\varrho - A\varrho, \chi_{n+1} - \varrho\rangle \\
&+ [1 - \rho_n - \omega_n\bar{\gamma}]\|\chi_{n+1} - \varrho\|(\tau_n\|\chi_n - \chi_{n-1}\| + \varphi_n\sigma_n\|U\varrho - \varrho\|), \qquad (3.93)
\end{aligned}
$$

which implies

$$
\begin{aligned}
\|\chi_{n+1} - \varrho\|^2 \leq\ & (1 - \omega_n(\bar{\gamma} - \gamma\nu))\|\chi_n - \varrho\|^2 \\
&+ \frac{1}{(\bar{\gamma} - \gamma\nu)}(\bar{\gamma} - \gamma\nu)\omega_n \langle \gamma h\varrho - A\varrho, \chi_{n+1} - \varrho\rangle \\
&+ M_5(\tau_n\|\chi_n - \chi_{n-1}\| + \sigma_n\|U\varrho - \varrho\|), \qquad (3.94)
\end{aligned}
$$

where $M_5 = \sup\{\|\chi_n - \varrho\| : n \in \mathbb{N}\}$. Hence, we get

$$
a_{n+1} \leq (1 - b_n)a_n + d_n + c_n, \qquad (3.95)
$$

where $a_n = \|\chi_n - \varrho\|^2$, $b_n = \omega_n(\bar{\gamma} - \gamma\nu)$, $d_n = \frac{1}{(\bar{\gamma}-\gamma\nu)}(\bar{\gamma} - \gamma\nu)\omega_n \langle \gamma h\varrho - A\varrho, \chi_{n+1} - \varrho\rangle$ and $c_n = M_5(\tau_n\|\chi_n - \chi_{n-1}\| + \sigma_n\|U\varrho - \varrho\|)$. From Remark (3.2) and $\sum_{n=0}^{+\infty} \sigma_n < +\infty$, we have $\sum_{n=0}^{+\infty} c_n < +\infty$. From Eq (3.92), we get $\limsup_{n\to+\infty} \frac{d_n}{b_n} \leq 0$. Also, $\sum_{n=0}^{+} \infty b_n = +\infty$ and from Lemma 2.12 (*ii*), we obtain

$$
\lim_{n\to+\infty} a_n = \lim_{n\to+\infty} \|\chi_n - \varrho\|^2 = 0. \qquad (3.96)
$$

Therefore, $\chi_n \to \varrho$. □

Corollary 3.4. *Let $x_0, x_1 \in Q_1$ and τ_n such that $0 \leq \tau_n \leq \bar{\tau}_n$. Define a sequence $\{\chi_n\}$ as:*

$$
\begin{cases}
w_n = \chi_n + \tau_n(\chi_n - \chi_{n-1}), \\
u_n = K_{Q_1}(w_n), \\
t_n = (1 - \varphi_n)u_n + \varphi_n[\sigma_n U u_n + (1 - \sigma_n)U_N^n U_{N-1}^n \ldots U_2^n U_1^n u_n], \\
\chi_{n+1} = \omega_n\gamma h(K_n\chi_n) + \rho_n\chi_n + [(1-\rho_n)I - \omega_n A]K_n t_n.
\end{cases} \qquad (3.97)
$$

$$\bar{\tau_n} = \begin{cases} \min\left\{\frac{\lambda_n}{\|\chi_n-\chi_{n-1}\|}, \tau\right\} & \text{if } \chi_n \neq \chi_{n-1}, \\ \tau & \text{if } otherwise, \end{cases} \tag{3.98}$$

where $K_{Q_1} = T_{r_n}^{F_1}(I - r_n g_1)$, $\liminf_{n\to+\infty} r_n > 0$ and $\lim\limits_{n\to+\infty} |r_{n+1} - r_n| = 0$, $U_i^n = (1-\kappa_n^i)I + \kappa_n^i P_{Q_1}(\zeta_n^i I + (1-\zeta_n^i)U_i)$ with $0 \le \mu_i \le \zeta_n^i < 1$ and $\lim\limits_{n\to+\infty} |\kappa_{n+1}^i - \kappa_n^i| = 0$ for $i \le i \le M$. Also $\lambda_n \subset [0,+\infty)$ with $\sum_{n=0}^{+\infty} \lambda_n < +\infty$, $\tau \in [0,1)$, $\varphi_n, \sigma_n, \kappa_n^i, \omega_n, \rho_n \in (0,1)$, $\rho = \sup\{\rho_n; n \in \mathbb{N}\}$ with $\lim\limits_{n\to+\infty} |\varphi_{n+1} - \varphi_n| = 0$, $\eta_{n,j} \to \eta_j$, $\sum_{n=0}^{+\infty} |\eta_{n,j} - \eta_{n-1,j}| < +\infty$, $\lim\limits_{n\to+\infty} |\sigma_{n+1} - \sigma_n| = 0$ and $\sum_{n=0}^{+\infty} \sigma_n < +\infty$. Under the assumptions that conditions (i)–(iii) of Theorem 3.3 are satisfied, we can conclude that the sequence χ_n generated by Eq (3.97) strongly converges to the element $\xi \in \Delta$. This element ξ represents the unique solution to the fixed-point problem associated with the contraction mapping $P_\Delta(I + \gamma h - A)$. In other words, ξ is the solution to the variational inequality stated below:

$$\langle (A - \gamma h)\xi, y - \xi \rangle \ge 0, \text{ for all } y \in \Delta.$$

Proof. By taking $D = O$, $H_1 = H_2$, $Q_1 = Q_2$, $F_1 = F_2$, $g_1 = g_2$ and $\psi_1 = \psi_2$ in Theorem 3.3, we get the required conclusion. □

Corollary 3.5. *Let* $x_0, x_1 \in Q_1$ *and* τ_n *such that* $0 \le \tau_n \le \bar{\tau_n}$. *Define a sequence* $\{\chi_n\}$ *as:*

$$\begin{cases} w_n = \chi_n + \tau_n(\chi_n - \chi_{n-1}), \\ y_n = K_{Q_1}(w_n), l_n = K_{Q_1}(y_n), \\ u_n = y_n - \delta(y_n - l_n), \\ t_n = (1-\varphi_n)u_n + \varphi_n[\sigma_n U u_n + (1-\sigma_n)U_N^n U_{N-1}^n ... U_2^n U_1^n u_n], \\ \chi_{n+1} = \omega_n \gamma h(K_n \chi_n) + \rho_n \chi_n + [(1-\rho_n)I - \omega_n A]K_n t_n. \end{cases} \tag{3.99}$$

$$\bar{\tau_n} = \begin{cases} \min\left\{\frac{\lambda_n}{\|\chi_n-\chi_{n-1}\|}, \tau\right\} & \text{if } \chi_n \neq \chi_{n-1}, \\ \tau & \text{if } otherwise, \end{cases} \tag{3.100}$$

where $K_{Q_1} = T_{r_n}^{F_1}(I - r_n g_1)$, $\liminf_{n\to+\infty} r_n > 0$ and $\lim\limits_{n\to+\infty} |r_{n+1} - r_n| = 0$, $U_i^n = (1-\kappa_n^i)I + \kappa_n^i P_{Q_1}(\zeta_n^i I + (1-\zeta_n^i)U_i)$ with $0 \le \mu_i \le \zeta_n^i < 1$ and $\lim\limits_{n\to+\infty} |\kappa_{n+1}^i - \kappa_n^i| = 0$ for $i \le i \le M$. Also $\lambda_n \subset [0,+\infty)$ with $\sum_{n=0}^{+\infty} \lambda_n < +\infty$, $\tau \in [0,1)$, $\varphi_n, \sigma_n, \kappa_n^i, \omega_n, \rho_n \in (0,1)$, $\rho = \sup\{\rho_n; n \in \mathbb{N}\}$ with $\lim\limits_{n\to+\infty} |\varphi_{n+1} - \varphi_n| = 0$, $\eta_{n,j} \to \eta_j$, $\sum_{n=0}^{+\infty} |\eta_{n,j} - \eta_{n-1,j}| < +\infty$, $\lim\limits_{n\to+\infty} |\sigma_{n+1} - \sigma_n| = 0$ and $\sum_{n=0}^{+\infty} \sigma_n < +\infty$. Under the assumptions that conditions (i)–(iii) of Theorem 3.3 are satisfied, we can conclude that the sequence χ_n generated by Eq (3.97) strongly converges to the element $\xi \in \Delta$. This element ξ represents the unique solution to the fixed-point problem associated with the contraction mapping $P_\Delta(I + \gamma h - A)$. In other words, ξ is the solution to the variational inequality stated below:

$$\langle (A - \gamma h)\xi, y - \xi \rangle \ge 0, \text{ for all } y \in \Delta.$$

Proof. By taking $D = I$, $H_1 = H_2$, $Q_1 = Q_2$, $F_1 = F_2$, $g_1 = g_2$ and $\psi_1 = \psi_2$ in Theorem 3.3, we get the required conclusion. □

Remark 3.6.

1) Theorem 3.3 generalizes and enhances the findings of Rizvi [56] from a nonexpansive mapping to a finite family of nonexpansive mappings. Furthermore, our findings extend the outcomes of Rizvi [56] from a common solution of SMEP and HFPP to a common solution of HFPP, SGMEP and FPP for a finite collection of nonexpansive mappings.

2) Theorem 3.3 generalizes the Husain and Singh [57] result from a common solution of SMEP and HFPP to a common solution of HFPP, SGMEP and FPP for a finite family of nonexpansive operators. In addition, we consider HFPP for a finite collection of strictly pseudocontractive operators, which is more general than the nonexpansive mappings taken in Husain and Singh [57] result.

3) Theorem 3.3 generalizes and enhances the findings of Kim and Majee [26] (Theorem 3.6) from a common solution of SEP and HFPP to a common solution of HFPP, SGMEP and FPP for a finite collection of nonexpansive operators.

4) Theorem 3.3 generalizes and enhances the result of Majee and Nahak [24] from a common solution of SEP and HFPP to a common solution of HFPP, SGMEP and FPP for a finite collection of nonexpansive operators.

4. Application in compressed sensing in signal processing

Compressed sensing in signal processing [58] can be represented by the following linear equation:

$$y = Dx + \epsilon. \tag{4.1}$$

Here, ϵ is the noise, D is an $M \times N$ matrix with $M < N$, $x \in \mathbb{R}^N$ is a recovered vector with m non-zero components and $y \in \mathbb{R}^M$ is the observed data. The problem described in Eq (4.1) can be considered as a LASSO problem:

$$\min_{x\in\mathbb{R}^N} \frac{1}{2}\|y - Dx\|_2^2 \text{ subject to } \|x\|_1 \leq u. \tag{4.2}$$

Here, $u > 0$ is constant.

In this case, a uniform distribution in the interval [-1, 1] is used to construct the sparse vector $x \in \mathbb{R}^N$, which has m non-zero members. A normal distribution with a zero mean and a unit variance is used to produce the matrix D. As $\delta \in (0, 1/L)$, it is randomly generated in MATLAB. By applying white Gaussian noise with a signal-to-noise ratio (SNR) of 40, the observation y is produced. The process starts with an initial point $x_1 = ones_{N\times 1}$ and $u = m$. Specifically, the LASSO problem can be seen as an SFP (Split Feasibility Problem) if $Q_1 = \{x \in \mathbb{R}^N : \|x\|_1 \leq u\}$ and $Q_2 = \{y\}$. In this connection, we can solve Eq (4.2) using the CQ technique. The stopping criterion is given by the mean squared error (MSE):

$$E_n = \frac{1}{N}\|\chi_n - \varrho\|_2^2 < \Lambda,$$

where Λ is a tolerance and χ_n is the estimated signal of x. Note that if in Problem (1.5)–(1.6) we set $g_1 = g_2 = \psi_1 = \psi_2 = 0$, we obtain the split equilibrium problems (SEQ) and if, in addition, $F_1(v, w) = I_{Q_1}(v) - I_{Q_1}(w)$ and $F_2(v', w') = I_{Q_2}(v') - I_{Q_2}(w')$, where I_{Q_1} and I_{Q_2} are identity operators on Q_1 and Q_2 respectively, then SEQ becomes SFP. Hence, we can apply our algorithm to the SFP with the resolvent operator $T_r^{F_1}$ and $T_r^{F_2}$ being the projection onto Q_1 and Q_2, respectively. In order to implement our algorithm, we choose the following parameters: $\lambda_n = \frac{1}{n^2}$, $\tau = 0.5$, $U = U_i = S_j = I$ for all i, j so that $K_n = I$ (Identity mapping), $A = I$ (Identity operator), $\rho_n = \omega_n = \frac{1}{n+1}$ and $h(x) = x/2$ so that $\gamma = 1$ and $\Lambda = 10^{-10}$.

Figure 1 represents the original signal, observed value and recovered signals by Algorithm 3.1, the Chuasuk Algorithm [37], the Kim Algorithm [26] and the Majee Algorithm [24]. Table 1 and Figure 2 give the mean square error of Algorithm 3.1, the Chuasuk Algorithm [37], the Kim Algorithm [26] and the Majee Algorithm [24]. The experiment shows that all three methods are effective in recovering the signal, however, the time taken by the Chuasuk Algorithm [37] (Average time = 7.8654s), the Majee Algorithm [24] (Average time = 10.9854s) and the Kim Algorithm [26] (Average time = 13.5864s) is more than the time taken by the proposed algorithm (Average time = 5.8754s).

Table 1. Numerical results for MSE versus the number of iterations (n) when $N = 1024$, $M = 512$ and $m = 60$.

	Number of Iterations	CPU Time (Seconds)
Algorithm 3.1	11	0.2965
Chuasuk Algorithm	27	0.7008
Majee Algorithm	17	0.8665
Kim Algorithm	41	0.9545

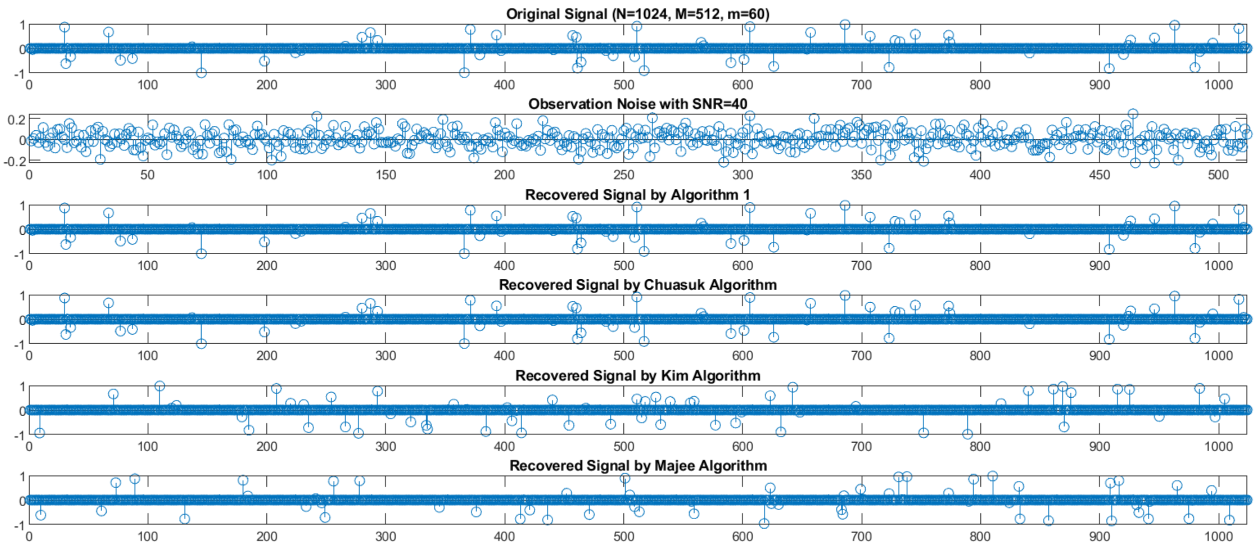

Figure 1. From top to bottom: original signal, observed value and recovered signals by Algorithm 3.1, the Chuasuk Algorithm [37], the Kim Algorithm [26] and the Majee Algorithm [24].

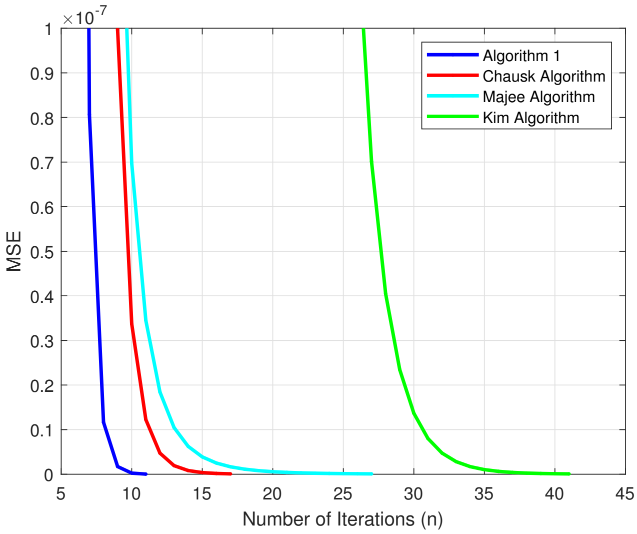

Figure 2. Numerical results for MSE versus the number of iterations (n) when $N = 1024$, $M = 512$, and $m = 60$.

5. Numercial examples

In this section, we first conduct a comparison of the convergence rates between our algorithm and those presented in the Chuasuk Algorithm [37], the Kim Algorithm [26] (Theorem 3.6), and the Majee Algorithm [24]. We implemented the proposed algorithm using MATLAB 9.10.0 ($R2021a$) on a laptop equipped with an Intel Core i5 CPU running at 1.60GHz, 256 GB SSD and 1 TB hard-disk capacity. The operating system used is Microsoft Windows 11, version 21H2. Secondly, we present numerical experiments related to compressed sensing.

Example 5.1. *Assume that* $H_1 = H_2 = \mathbb{R}^5$, *and*

$$Q_1 = Q_2 = \{x \in \mathbb{R}^5 : \sum_{i=1}^{5} x_i \geq -1, -6 \leq x_i \leq 6, 1 \leq i \leq 5\}.$$

Let $g_1 : Q_1 \to \mathbb{R}$, $g_2 : Q_2 \to \mathbb{R}$ be inverse strongly monotone mappings defined by $g_1(x) = 3x$ and $g_2(x) = 3x$. Suppose $F_1 : Q_1 \times Q_1 \to \mathbb{R}, F_2 : Q_2 \times Q_2 \to \mathbb{R}$ are the bifunctions defined by $F_1(x, y) = F_2(x, y) = \langle Px + Qy + q, y - x\rangle$, arising from Nash Cournot Oligopolistic market equilibrium model [17] where $q \in \mathbb{R}^5$ and $P, Q \in \mathbb{R}^{5\times5}$ are two matrices of order 5 with Q being symmetric, positive semidefinite and $Q - P$ being negative semidefinite. Obviously, bifunction g satisfies Assumption 1 and $A : \mathbb{R} \to \mathbb{R}$ is defined by $A(x) = x$ with constant $\bar{\gamma} = 1$. Let $\psi_1 : Q_1 \to \mathbb{R} \cup \{+\infty\}$, $\psi_2 : Q_2 \to \mathbb{R} \cup \{+\infty\}$ be defined by $\psi_1(x) = \psi_2(x) = 0$, $D : \mathbb{R} \to \mathbb{R}$ be defined by $D(x) = x$, $D^*(x) = x$, then $T_r^{F_1}(x) = T_r^{F_2}(x) = ((P + Q + 3I)r + I)^{-1}x$. Let $h : Q_1 \to Q_1$ be $\frac{1}{2}$-contraction defined by $h(x) = \frac{x}{2}$ and $S_j : Q_1 \to Q_1$ be pseudocontractive mappings defined by $S_j(x) = \frac{x}{6(j+1)}$, for $j = 1, 2$. Assume that $U : Q_1 \to Q_1$ and $U_i : Q_1 \to Q_1$ are nonexpansive mappings defined by $U(x) = \frac{x}{4}$ and $U_i(x) = \frac{x}{10i}$, for $i, = 1, 2$, $x = (x_1, x_2, x_3, x_4, x_5)^T$. Choose $\delta = \frac{1}{16}$, $r_n = 1$, $\lambda_n = \frac{1}{n^2}$, $\tau = 0.5$, $\kappa_n^i = \frac{n+i}{n+5+i}$, $\zeta_n^i = \frac{1}{20}$, $\eta_n^j = \frac{1}{20n+5j}$ for $i, j = 1, 2$, $\sigma_n = \frac{1}{(n+1)^2}$, $\varphi_n = \frac{5}{6}$, $\rho_n = \frac{n+1}{2(n+50)}$ and $\omega_n = \frac{1}{n+200}$. One can easily see that $\text{Fix}(\Gamma) = \{0\} \neq \phi$. We can obtain $K_{Q_1}(x) = K_{Q_2}(x) = -2(P + Q + 4I)^{-1}x$. Take $P = I_5$, $Q = 0_{5\times5}$, $x_0 = (0.5, 0.5, 0.5, 0.5, 0.5)^T$, $x_1 = (0.8, 0.8, 0.8, 0.8, 0.8)^T$ and $q = [0, 0, 0, 0, 0]^{\mathrm{T}}$. We take a stopping criterion of $E_n = \|\chi_n - \chi_{n+1}\| < 10^{-4}$ and plot the graphs between number of iterations n and errors E_n. We do comparative analysis of the numerical result of Algorithm 3.1 with the Chuasuk Algorithm [37], the Kim Algorithm [26] (Theorem (3.6)) and the Majee Algorithm [24]. Table 2 and Figure 3 represent the comparative analysis.

Table 2. Example 5.1: Comparison of Algorithm 3.1 with the Chuasuk Algorithm [37], the Kim Algorithm [26] and the Majee Algorithm [24].

	Number of Iterations	CPU Time (Seconds)
Algorithm 3.1	38	0.2755
Chuasuk Algorithm	172	0.9870
Majee Algorithm	93	0.8106
Kim Algorithm	307	1.0956

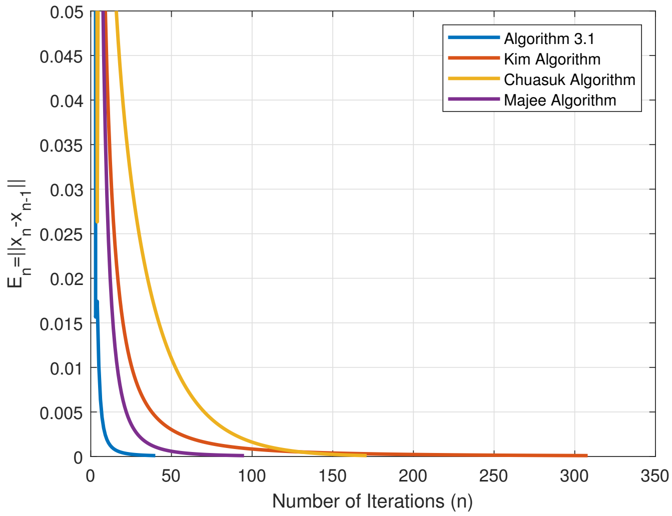

Figure 3. Example 5.1: Comparison of Algorithm 3.1 with the Chuasuk Algorithm [37], the Kim Algorithm [26] and the Majee Algorithm [24].

Example 5.2. *Assume that $H_1 = H_2 = l_2$ are real Hilbert spaces with square-summable infinite sequences of real numbers as its elements and $Q_1 = Q_2 = \{v \in l_2 : \|v\| \leq 3\}$. Let $g_1 : [-5, 5] \to \mathbb{R}$, $g_2 : [-5, 5] \to \mathbb{R}$ be ism mappings defined by $g_1(x) = 10x$ and $g_2(x) = 2x$. Suppose $F_1 : Q_1 \times Q_1 \to \mathbb{R}$, $F_2 : Q_2 \times Q_2 \to \mathbb{R}$ are the bifunctions defined by $F_1(x, y) = -5x^2 + xy + 4y^2$, $F_2(x, y) = -3x^2 + xy + 2y^2$ for all $x = (x_1, x_2, x_3, ..., x_n, ...)$ and $y = (y_1, y_2, y_3, ..., y_n, ...)$ with $\|.\| : l_2 \to \mathbb{R}$ and $\langle ., . \rangle : l_2 \times l_2 \to \mathbb{R}$ given by $\|x\| = (\sum_{k=1}^{+\infty} |x_k|^2)^{\frac{1}{2}}$ and $\langle x, y \rangle = \sum_{k=1}^{+\infty} x_k y_k$, where $x = \{x_k\}_{k=1}^{+\infty}$, $y = \{y_k\}_{k=1}^{+\infty}$. Suppose that $A : \mathbb{R} \to \mathbb{R}$ is defined by $A(x) = x$ for all $x = (x_1, x_2, x_3, ..., x_n, ...)$ with constant $\bar{\gamma} = 1$. Let $\psi_1 : Q_1 \to \mathbb{R} \cup \{+\infty\}$, $\psi_2 : Q_2 \to \mathbb{R} \cup \{+\infty\}$ be given by $\psi_1(x) = x^2$, $\psi_2(x) = 2x^2$, $D : \mathbb{R} \to \mathbb{R}$ be defined by $D(x) = -5x$, $D^* = -5x$, then $T_r^{F_1}(x) = \frac{x}{21r+1}$ and $T_r^{F_2}(x) = \frac{x}{11r+1}$. Let $h : Q_1 \to Q_1$ be $\frac{1}{2}$-contraction defined by $h(x) = \frac{x}{2}$ and $S_j : Q_1 \to Q_1$ be pseudocontractive mappings defined by $S_j(x) = \frac{x}{2(j+1)}$, for $j = 1, 2$. Assume that $U : Q_1 \to Q_1$ and $U_i : Q_1 \to Q_1$ are nonexpansive mappings defined by $U(x) = x$*

and $U_i(x) = \frac{x}{100i}$, *for* $i, = 1, 2$. *Choose* $\delta = \frac{1}{16}$, $r_n = 1$, $\lambda_n = \frac{1}{n^2}$, $\tau = 0.9$, $\kappa_n^i = \frac{n+i}{n+5+i}$, $\zeta_n^i = \frac{7}{8i}$, $\eta_n^j = \frac{1}{20n+5j}$ *for* $i, j = 1, 2$, $\sigma_n = \frac{1}{n^2}$, $\varphi_n = \frac{5}{6}$, $\rho_n = \frac{n+1}{2000(n+5)}$ *and* $\omega_n = \frac{1}{700n+4}$. *One can easily see that* $Fix(\Gamma) = \{0\} \neq \phi$. *We can obtain* $K_{Q_1}(x) = \frac{-9x}{21}$ *and* $K_{Q_2}(x) = \frac{-x}{12}$. *We take a stopping criterion of* $E_n = \|\chi_n - \chi_{n+1}\| < 10^{-4}$ *and plot the graphs between errors* E_n *and the number of iterations* n. *Take initial values* $x_0 = (0.5, 0.5, 0.5, 0.5, ..., 0.5, ...)$ *and* $x_1 = (0.8, 0.8, 0.8, 0.8, ..., 0.8, ...)$. *We do comparative analysis of the numerical result obtain from Algorithm 3.1 with the Chuasuk [37], the Kim [26] (Theorem (3.6)) and the Majee [24] algorithms. Table 3 and Figure 4 show the numerical results*

Table 3. Example 5.2: Comparative analysis of Algorithm 3.1 with the Chuasuk Algorithm [37], the Kim Algorithm [26] and the Majee Algorithm [24].

	Number of Iterations	CPU Time (Seconds)
Algorithm 3.2	12	0.03876
Chuasuk Algorithm	38	1.756
Kim Algorithm	25	0.1638
Majee Algorithm	21	0.1548

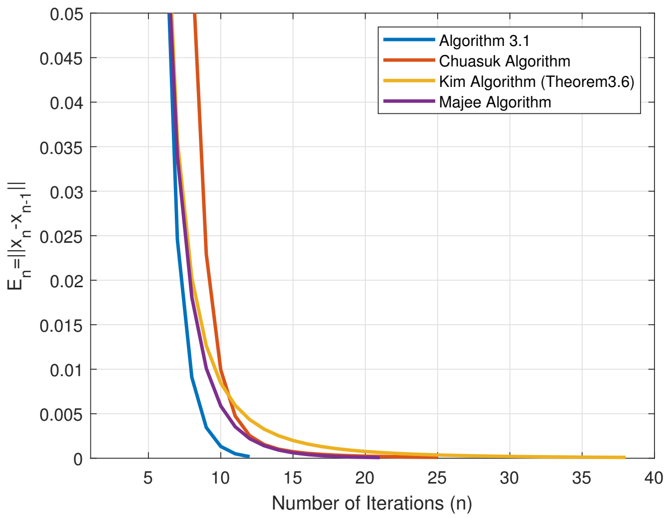

Figure 4. Example 5.2: Algorithm comparison 3.1 with the Chuasuk Algorithm [37], the Kim Algorithm [26] and the Majee Algorithm [24].

6. Conclusions

This paper discussed a new inertial generalized viscosity approximation method for solving split generalized mixed equilibrium problem, fixed point problem for a finite family of nonexpansive mappings and hierarchical fixed point problem in real Hilbert spaces. Under certain appropriate conditions, we have established the result of strong convergence. We have demonstrated the use of our main finding with compressed sensing in signal processing. We have explained the numerical effectiveness of our approach in comparison to another method. The results discussed in this paper enhance and summarize previously published findings in the literature.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interests.

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