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Research article

An eigenvalue problem related to the variable exponent double-phase operator

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Abstract: In this paper, we studied a double-phase eigenvalue problem with large variable exponents. Let $\lambda^1_{(p_n(\cdot),\, q_n(\cdot))}$ be the first eigenvalues and u_n be the first eigenfunctions, normalized by $\|u_n\|_{\mathcal{H}_n} = 1$. Under some assumptions on the variable exponents $p_n(\cdot)$ and $q_n(\cdot)$, we showed that $\lambda^1_{(p_n(\cdot),\, q_n(\cdot))}$ converges to Λ_∞, u_n converges to u_∞ uniformly in the space $C^\alpha(\Omega)\,(0 < \alpha < 1)$ and u_∞ is a nontrivial viscosity solution to a Dirichlet ∞-Laplacian problem. Even in the case where the variable exponents reduce to the constant exponents, our work is the first one dealing with a double-phase eigenvalue problem with large exponents.

Keywords: double-phase problem; eigenvalues; ∞-Laplacian; viscosity solution; variable exponents
Mathematics Subject Classification: 35D40, 35P30, 46E30

1. Introduction

An expedient feature of the p-Laplacian eigenvalue problem is that the eigenfunctions may be multiplied by constant factors (in other words, the fact that if u is an eigenfunction, so is ku). Unfortunately, the $p(x)$-Laplacian eigenvalue problem does not possess this expedient property. It is important to stress that the loss of the property under consideration is not only a consequence of the dependence on x, but it can also occur in presence of unbalanced growth. For example, the double phase operator (that does not depend on x)

$$\operatorname{div}\left(|\nabla u|^{p-2}\,\nabla u + \mu(x)\,|\nabla u|^{q-2}\,\nabla u\right), \tag{1.1}$$

loses this property. In this paper we are interested in considering that the operator has both peculiarities: It depends on x and it is unbalanced.

Let $\Omega \subset \mathbb{R}^N (N \geq 2)$ be a bounded domain with Lipschitz boundary $\partial\Omega$. This article studies an eigenvalue problem coming from the minimization of the Rayleigh quotient:

$$\frac{||\nabla u||_{\mathcal{H}_n}}{||u||_{\mathcal{H}_n}}, \tag{1.2}$$

among all $u \in W_0^{1,\mathcal{H}_n}(\Omega), u \not\equiv 0$. These functions belong to an appropriate Musielak-Orlicz Sobolev space with variable exponents; see its definition in section two. The function $a : \bar{\Omega} \to [0, +\infty)$ is a C^1 differentiable function.

Put

$$\begin{aligned} &K_n(u) := ||\nabla u||_{\mathcal{H}_n},\ k_n(u) := ||u||_{\mathcal{H}_n}, \\ &S_n(u) := \frac{\int_\Omega \left[p_n(x)\left|\frac{\nabla u}{K_n(u)}\right|^{p_n(x)-2} + q_n(x)a(x)\left|\frac{\nabla u}{K_n(u)}\right|^{q_n(x)-2}\right]dx}{\int_\Omega \left[p_n(x)\left|\frac{u}{k_n(u)}\right|^{p_n(x)-2} + q_n(x)a(x)\left|\frac{u}{k_n(u)}\right|^{q_n(x)-2}\right]dx}, \\ &\mathcal{H}_n := t^{p_n(x)} + a(x)t^{q_n(x)} \end{aligned} \tag{1.3}$$

and define the first eigenvalue as

$$\lambda^1_{(p_n(\cdot),q_n(\cdot))} = \inf_{u\in W_0^{1,\mathcal{H}_n}(\Omega)\setminus\{0\}} \frac{||\nabla u||_{\mathcal{H}_n}}{||u||_{\mathcal{H}_n}}. \tag{1.4}$$

By a similar proof of Proposition 3.1 in [1], we can show that the following equation

$$\begin{aligned} &-\operatorname{div}\left[\left(p_n(x)\left|\frac{\nabla u}{K_n(u)}\right|^{p_n(x)-2} + q_n(x)a(x)\left|\frac{\nabla u}{K_n(u)}\right|^{q_n(x)-2}\right)\frac{\nabla u}{K_n(u)}\right] \\ =&\lambda_{(p_n(\cdot),\,q_n(\cdot))}S_n(u)\frac{u}{k_n(u)}\left(p_n(x)\left|\frac{u}{k_n(u)}\right|^{p_n(x)-2} + q_n(x)a(x)\left|\frac{u}{k_n(u)}\right|^{q_n(x)-2}\right), u \in W_0^{1,\mathcal{H}_n}(\Omega) \end{aligned} \tag{1.5}$$

is the Euler-Lagrange equation corresponding to the minimization of the Rayleigh quotient (1.2), where $\lambda_{(p_n(\cdot),\,q_n(\cdot))} = \lambda^1_{(p_n(\cdot),\,q_n(\cdot))}$.

Here, we impose the following hypotheses on the variable exponents $p_n(x)$ and $q_n(x)$.

(H1): Assume that $p_n(x)$ and $q_n(x)$ are two sequences of C^1 functions in $\overline{\Omega}$, $q_n(\cdot) > p_n(\cdot)$ for every $n \geq 1$ and

$$p_n(x), q_n(x) \to +\infty, \text{uniformly for all } x \in \Omega, \tag{1.6}$$

$$\frac{\nabla p_n(x)}{p_n(x)} \to \xi_1(x),\ \text{uniformly for all } x \in \Omega, \tag{1.7}$$

$$\frac{\nabla q_n(x)}{q_n(x)} \to \xi_2(x),\ \text{uniformly for all } x \in \Omega. \tag{1.8}$$

(H2): The following two quotients are bounded, namely,

$$\limsup_{n\to+\infty} \frac{p_n^+}{p_n^-} \leq k_1,\ \limsup_{n\to+\infty} \frac{q_n^+}{q_n^-} \leq k_2, \tag{1.9}$$

where for a function g we denote

$$g^- = \min_{x\in\overline{\Omega}} g(x),\ g^+ = \max_{x\in\overline{\Omega}} g(x).$$

(H3): We also assume that

$$p_n^- > 1, q_n^- > 1, \frac{q_n^+}{p_n^-} < 1 + \frac{1}{N}, \tag{1.10}$$

then we can find a positive and continuous function $\theta\,(0 < \theta < +\infty)$, such that

$$\lim_{n\to\infty} \frac{q_n(x)}{p_n(x)} = \theta(x) \tag{1.11}$$

uniformly for all $x \in \Omega$.

The differential operator in (1.5) is the double-phase operator with variable exponents, which can be given by

$$\operatorname{div}\left(|\nabla u|^{p_n(x)-2}\,\nabla u + \mu(x)\,|\nabla u|^{q_n(x)-2}\,\nabla u\right). \tag{1.12}$$

This operator is the classical double phase operator (1.1) when $p_n(x)$ and $q_n(x)$ are constants. Moreover, special cases of (1.12), studied extensively in the literature, occur when $\inf_{\overline{\Omega}}\mu > 0$ (the weighted $(q(x), p(x))$-Laplacian) or when $\mu \equiv 0$ (the $p(x)$-Laplacian).

The energy functional related to the double-phase operator (1.12) is given by

$$\int_\Omega |\nabla u|^{p_n(x)} + \mu(x)|\nabla u|^{q_n(x)}dx, \tag{1.13}$$

whose integrand switches two different elliptic behaviors. The integral functional (1.13) was first introduced by Zhikov [2–5], who obtained that the energy density changed its ellipticity and growth properties according to the point in order to provide models for strongly anisotropic materials. Moreover, double phase differential operators (1.12) and corresponding energy functionals (1.13) have several physical applications. We refer to the works of [6] on transonic flows, [7] on quantum physics and [8] on reaction diffusion systems. Finally, we mention a recent paper that is very close to our topic. For related works dealing with the double phase eigenvalue problems, we refer to the works of Colasuonno-Squassina [9], who proved the existence and properties of related variational eigenvalues. By using the Rayleigh quotient of two norms of Musielak-Orlicz space, the author of this paper has defined the eigenvalue, which has the same properties as the p-Laplace operator. Recently, Liu-Papageorgiou has considered an eigenvalue problem for the Dirichlet $(p, q(\cdot))-$Laplacian by using the Nehari method (see [10]), a nonlinear eigenvalue problem for the Dirichlet $(p, q)-$Laplacian with a sign-changing Carathéodory reaction (see [11]) and a nonlinear eigenvalue problem driven by the anisotropic $(p(\cdot), q(\cdot))-$Laplacian (see [12]). Motivated by [9], Yu [13] discuss the asymptotic behavior of an eigenvalue for the double phase operator. However, to the author's knowledge, the eigenvalue problem for variable exponents double phase operator has remained open. Our article fits into this general field of investigation.

Assume that $\delta : \Omega \to [0, \infty)$ is the distance function, which is given by

$$\delta(x) := \operatorname{dist}(x, \partial\Omega) = \inf_{y\in\partial\Omega} |x - y|.$$

This function is a Lipschitz continuous function. For all $x \in \Omega$, we get $|\nabla\delta| = 1$. Define

$$\Lambda_\infty := \inf_{\varphi\in W_0^{1,\infty}(\Omega)\backslash\{0\}} \frac{\|\nabla\varphi\|_{L^\infty(\Omega)}}{\|\varphi\|_{L^\infty(\Omega)}}. \tag{1.14}$$

It is known from the paper [1] that

$$\Lambda_\infty = \frac{\|\nabla\delta\|_{L^\infty(\Omega)}}{\|\delta\|_{L^\infty(\Omega)}} = \frac{1}{\max_{x\in\Omega}\{\mathrm{dist}(x,\partial\Omega)\}}. \tag{1.15}$$

Define

$$\triangle_\infty u_\infty := \sum_{i,j=1}^{N}(u_\infty)_{x_i}(u_\infty)_{x_j}(u_\infty)_{x_ix_j},$$

$$k_\infty(u) := \|u\|_{L^\infty(\Omega)} = \operatorname*{ess\,sup}_{x\in\Omega}|u|, \tag{1.16}$$

$$K_\infty(u) := \|\nabla u\|_{L^\infty(\Omega)} = \operatorname*{ess\,sup}_{x\in\Omega}|\nabla u|, \tag{1.17}$$

and

$$k_\infty(u_\infty) := \|u_\infty\|_{L^\infty(\Omega)} = \operatorname*{ess\,sup}_{x\in\Omega}|u_\infty|.$$

The following are the main results of this paper.

Theorem 1.1. *Let $u \in C(\Omega)$ be a weak solution of problem (1.5), then it is also a viscosity solution of the problem (3.2).*

Theorem 1.2. *Let hypotheses (H1)–(H3) be satisfied, $\lambda^1_{(p_n(\cdot),\,q_n(\cdot))}$ and Λ_∞ be defined by (1.4) and (1.14), respectively. In addition, assume that u_n normalized by $\|u_n\|_{\mathcal{H}_n} = 1$ is the positive first eigenfunction, then,*

(1)

$$\lim_{n\to\infty}\lambda^1_{(p_n(\cdot),q_n(\cdot))} = \Lambda_\infty; \tag{1.18}$$

(2) there exists a nonnegative function u_∞ such that $u_\infty \in C^\alpha(\Omega)\setminus\{0\}$ and $\|u_\infty\|_{L^\infty(\Omega)} = 1$;

(3) we can extract a subsequence, which is still denoted by u_n, such that

$$u_n \to u_\infty, \tag{1.19}$$

in the space $C^\alpha(\Omega)$, where $\alpha\,(0<\alpha<1)$ is a constant;

(4) we can obtain that the function $u_\infty(x)$ is a nontrivial viscosity solution of the problem

$$\begin{cases} \min\Big\{-\Lambda_\infty u_\infty + |\nabla u_\infty|, -\Lambda_\infty(u_\infty)^{\theta(x_0)} + |\nabla u_\infty|, \\ -\triangle_\infty u_\infty - [\ln(|\nabla\psi(x_0)|) - \ln(K_\infty(u_\infty))]|\nabla\psi(x_0)|^2\nabla\psi(x_0)\cdot\xi_2(x_0)\Big\} = 0, & x\in\Omega, \\ \qquad u_\infty = 0, & x\in\partial\Omega. \end{cases} \tag{1.20}$$

To the best of our knowledge, this is the first work dealing with the double phase eigenvalue problem (1.5). The rest of this paper is organized as follows. In section two, we collect some notations and facts about the Musielak-Orlicz space $L^{\mathcal{H}}(\Omega)$ and $W^{1,\mathcal{H}}_0(\Omega)$, which will be used in this paper. Section three and section four are devoted to prove Theorems 1.1 and 1.2, respectively.

2. Preliminaries

In this section, we recall some known results about the Musielak-Orlicz spaces $L^{\mathcal{H}}(\Omega)$ and $W_0^{1,\mathcal{H}}(\Omega)$. For more detail, please see references [9, 14–17].

We follow the notation of [9]. Let $N(\Omega)$ denote the set of all generalized N-functions. Let us introduce the nonlinear function $\mathcal{H} : \Omega \times [0, +\infty) \to [0, +\infty)$ defined as

$$\mathcal{H}(x,t) := t^{p(x)} + a(x)t^{q(x)}, \quad \text{for all } (x,t) \in \Omega \times [0, +\infty),$$

with $1 < p(x) < q(x)$ and $0 \le a(\cdot) \in L^1(\Omega)$. It is clear that $\mathcal{H} \in N(\Omega)$ is a locally integrable and generalized N-function. In addition, it fulfills the Δ_2 condition, namely,

$$\mathcal{H}(x, 2t) \le 2^{q^+}\mathcal{H}(x,t).$$

Therefore, in correspondence to $\mathcal{H}$, we define the Musielak-Orlicz space $L^{\mathcal{H}}(\Omega)$ as

$$L^{\mathcal{H}}(\Omega) := \{u : \Omega \to \mathbb{R} \text{ measurable} : \rho_{\mathcal{H}}(u) < +\infty\},$$

which can be equipped with the norm

$$\|u\|_{\mathcal{H}} := \inf\{\gamma > 0 : \rho_{\mathcal{H}}(u/\gamma) \le 1\},$$

where

$$\rho_{\mathcal{H}}(u) := \int_\Omega \mathcal{H}(x, |u|)dx,$$

which is called $\mathcal{H}$-modular.

Similarly, we can define the Musielak-Orlicz Sobolev spaces. The space $W^{1,\mathcal{H}}(\Omega)$ is given by

$$W^{1,\mathcal{H}}(\Omega) = \left\{u \in L^{\mathcal{H}}(\Omega) \, \text{such that} \, |\nabla u| \in L^{\mathcal{H}}(\Omega)\right\},$$

with the norm

$$\|u\|_{1,\mathcal{H}} := \|u\|_{\mathcal{H}} + \|\nabla u\|_{\mathcal{H}}.$$

We denote by $W_0^{1,\mathcal{H}}(\Omega)$ the completion of $C_0^\infty(\Omega)$ in $W^{1,\mathcal{H}}(\Omega)$. With these norms, the spaces $L^{\mathcal{H}}(\Omega)$,$W^{1,\mathcal{H}}(\Omega)$ and $W_0^{1,\mathcal{H}}(\Omega)$ are separable, reflexive and uniformly convex Banach spaces.

From Proposition 2.16 (ii) in [18], if

$$\frac{q^+}{p^-} < 1 + \frac{1}{N},$$

then the following Poincar$\acute{e}$-type inequality

$$\|u\|_{\mathcal{H}} \le C\|\nabla u\|_{\mathcal{H}}$$

holds for all $u \in W_0^{1,\mathcal{H}}(\Omega)$, where C is a positive constant independent of u. Therefore, in this paper, we equip $W_0^{1,\mathcal{H}}(\Omega)$ with the equivalent norm $\|\nabla u\|_{\mathcal{H}}$ for all $u \in W_0^{1,\mathcal{H}}(\Omega)$.

Proposition 2.1. *[18] If $u \in L^{\mathcal{H}}(\Omega)$ and $\rho_{\mathcal{H}}(u)$ is the $\mathcal{H}$-modular, then the following properties hold.*
(1) If $u \ne 0$, then $\|u\|_{\mathcal{H}} = \lambda$ if, and only if, $\varrho_{\mathcal{H}}(\frac{u}{\lambda}) = 1$;
(2) $\|u\|_{\mathcal{H}} < 1\ (= 1; > 1)$ if, and only if, $\varrho_{\mathcal{H}}(u) < 1\ (= 1; > 1)$;
(3) If $\|u\|_{\mathcal{H}} \le 1$, then $\|u\|_{\mathcal{H}}^{q^+} \le \rho_{\mathcal{H}}(u) \le \|u\|_{\mathcal{H}}^{p^-}$;
(4) If $\|u\|_{\mathcal{H}} \ge 1$, then $\|u\|_{\mathcal{H}}^{p^-} \le \rho_{\mathcal{H}}(u) \le \|u\|_{\mathcal{H}}^{q^+}$;
(5) $\|u\|_{\mathcal{H}} \to 0$ if, and only if, $\rho_{\mathcal{H}}(u) \to 0$;
(6) $\|u\|_{\mathcal{H}} \to 0$ if, and only if, $\rho_{\mathcal{H}}(u) \to 0$.

3. The proof of Theorem 1.1

Given $u \in C(\Omega) \bigcap W_0^{1,\mathcal{H}_n}(\Omega)$ and $\phi \in C^2(\Omega)$. Define

$$\begin{aligned}\triangle_{p_n(x)}\phi &:= \operatorname{div}(|\nabla\phi|^{p_n(x)-2}\nabla\phi)\\ &= |\nabla\phi|^{p_n(x)-4}\{|\nabla\phi|^2\triangle\phi + (p_n(x)-2)\triangle_\infty\phi + |\nabla\phi|^2\ln(|\nabla\phi|)\nabla\phi\cdot\nabla p_n\},\end{aligned}$$

$$\begin{aligned}\triangle_{q_n(x)}\phi &:= \operatorname{div}(|\nabla\phi|^{q_n(x)-2}\nabla\phi)\\ &= |\nabla\phi|^{q_n(x)-4}\{|\nabla\phi|^2\triangle\phi + (q_n(x)-2)\triangle_\infty\phi + |\nabla\phi|^2\ln(|\nabla\phi|)\nabla\phi\cdot\nabla q_n\},\end{aligned}$$

and

$$\triangle_\infty\phi := \sum_{i,j=1}^{N}\frac{\partial\phi}{\partial x_i}\frac{\partial\phi}{\partial x_j}\frac{\partial^2\phi}{\partial x_i\partial x_j},$$

where $\triangle_\infty\phi$ is the ∞-Laplacian.

Here, we are now in a position to give the following definition of weak solutions to problem (1.5).

Definition 3.1. *We call $u \in W_0^{1,\mathcal{H}_n}(\Omega)\backslash\{0\}$ a weak solution of problem (1.5) if*

$$\begin{aligned}&\int_\Omega\left(p_n(x)\left|\frac{\nabla u}{K(u)}\right|^{p_n(x)-2} + q_n(x)a(x)\left|\frac{\nabla u}{K(u)}\right|^{q_n(x)-2}\right)\frac{\nabla u\cdot\nabla v}{K(u)}dx\\ =&\lambda_{(p_n(\cdot),\,q_n(\cdot))}S(u)\int_\Omega\left(p_n(x)\left|\frac{u}{k(u)}\right|^{p_n(x)-2} + q_n(x)a(x)\left|\frac{u}{k(u)}\right|^{q_n(x)-2}\right)\frac{uv}{k(u)}dx\end{aligned}\tag{3.1}$$

is satisfied for all test functions $v \in W_0^{1,\mathcal{H}_n}(\Omega)$. If $u \neq 0$, we say that $\lambda_{(p_n(\cdot),q_n(\cdot))}$ is an eigenvalue of (1.5) and that u is an eigenfunction corresponding to $\lambda_{(p_n(\cdot),q_n(\cdot))}$.

In (1.5), we replace u by ϕ and keep S_n, K_n and k_n unchanged, then

$$\begin{cases}-p_n(x)(K(u))^{1-p_n(x)}\triangle_{p_n(x)}\phi - q_n(x)a(x)(K(u))^{1-q_n(x)}\triangle_{q_n(x)}\phi\\ -q_n(x)(K(u))^{1-q_n(x)}|\nabla\phi(x)|^{q_n(x)-2}\nabla\phi(x)\cdot\nabla a(x)\\ -(K(u))^{1-p_n(x)}|\nabla\phi|^{p_n(x)-2}\nabla\phi(x)\cdot\nabla p_n(x)\\ -a(x)(K(u))^{1-q_n(x)}|\nabla\phi(x)|^{q_n(x)-2}\nabla\phi(x)\cdot\nabla q_n(x)\\ +p_n(x)(K(u))^{1-p_n(x)}\ln(K(u))|\nabla\phi(x)|^{p_n(x)-2}\nabla\phi(x)\cdot\nabla p_n(x)\\ +q_n(x)a(x)(K(u))^{1-q_n(x)}\ln(K(u))|\nabla\phi(x)|^{q_n(x)-2}\nabla\phi(x)\cdot\nabla q_n(x)\\ -\lambda_{(p_n(\cdot),q_n(\cdot))}S(u)(p_n(x)(k(u))^{1-p_n(x)}|\phi|^{p_n(x)-2}\phi\\ +q_n(x)a(x)(k(u))^{1-q_n(x)}|\phi(x)|^{q_n(x)-2}\phi(x)) = 0, & x\in\Omega,\\ \qquad\qquad\phi = 0, & x\in\partial\Omega.\end{cases}$$

We first recall the definition of viscosity solutions. Assume we are given a continuous function

$$F : \mathbb{R}^N\times\mathbb{R}\times\mathbb{R}^N\times\mathcal{S}(N)\to\mathbb{R},$$

where $\mathcal{S}(N)$ denotes the set of $N\times N$ symmetric matrices.

Consider the problem

$$F(x,u,\nabla u,D^2u) = 0,\tag{3.2}$$

where

$$
\begin{aligned}
F(x,u,\nabla u,D^2u) = &- p_n(x)(K(u))^{1-p_n(x)}\{|\nabla u|^{p_n(x)-4}[|\nabla u|^2\triangle u\\
&+ (p_n(x)-2)\triangle_\infty u + |\nabla u|^2\ln(|\nabla u|)\nabla u\cdot\nabla p_n(x)]\}\\
&- q_n(x)a(x)(K(u))^{1-q_n(x)}\{|\nabla u|^{q_n(x)-4}[|\nabla u|^2\triangle u\\
&+ (q_n(x)-2)\triangle_\infty u + |\nabla u|^2\ln(|\nabla u|)\nabla u\cdot\nabla q_n(x)]\}\\
&- q_n(x)(K(u))^{1-q_n(x)}|\nabla u|^{q_n(x)-2}\nabla u\cdot\nabla a(x) - (K(u))^{1-p_n(x)}|\nabla u|^{p_n(x)-2}\nabla u\cdot\nabla p_n(x)\\
&- a(x)(K(u))^{1-q_n(x)}|\nabla u|^{q_n(x)-2}\nabla u\cdot\nabla q_n(x)\\
&+ p_n(x)(K(u))^{1-p_n(x)}\ln(K(u))|\nabla u|^{p_n(x)-2}\nabla u\cdot\nabla p_n(x)\\
&+ q_n(x)a(x)(K(u))^{1-q_n(x)}\ln(K(u))|\nabla u|^{q_n(x)-2}\nabla u\cdot\nabla q_n(x)\\
&- \lambda_{(p_n(\cdot),q_n(\cdot))}S(u)(p_n(x)(k(u))^{1-p_n(x)}|u|^{p_n(x)-2}u\\
&+ q_n(x)a(x)(k(u))^{1-q_n(x)}|u|^{q_n(x)-2}u).
\end{aligned}
\tag{3.3}
$$

Definition 3.2. *Assume that $x_0\in\Omega$, $u\in C(\Omega)$, $\psi\in C^2(\Omega)$ and $\varphi\in C^2(\Omega)$.*

(1) Let $u(x_0)=\psi(x_0)$ and suppose that $u-\psi$ attains its strict maximum value at x_0. If

$$F(x_0,\psi(x_0),\nabla\psi(x_0),D^2\psi(x_0))\le 0$$

for all of such x_0, then the function u is said to be a viscosity subsolution of Eq (3.2).

(2) Let $u(x_0)=\varphi(x_0)$ and suppose that $u-\varphi$ attains its strict minimum value at x_0. If

$$F(x_0,\varphi(x_0),\nabla\varphi(x_0),D^2\varphi(x_0))\ge 0$$

for all of such x_0, then the function u is said to be a viscosity supersolution of Eq (3.2).

(3) If u is both a subsolution and a supersolution of the problem (3.2), then u is a viscosity solution of the problem (3.2).

Proof of Theorem 1.1. **Claim:** u is a viscosity supersolution of (3.2).

Let $x_0\in\Omega$ and $\varphi\in C^2(\Omega)$. Assume that $u(x_0)=\varphi(x_0)$ and the function $u-\varphi$ obtains its strict minimum value at the point x_0. Our goal is to show that

$$F(x_0,u(x_0),\nabla\varphi(x_0),D^2\varphi(x_0))\ge 0. \tag{3.4}$$

If

$$F(x_0,u(x_0),\nabla\varphi(x_0),D^2\varphi(x_0))<0,$$

then by continuity there exists a positive constant r such that $B(x_0,2r)\subset\Omega$, $u>\varphi$ in this ball, except for the point x_0 and

$$F(x,u(x),\nabla\varphi(x),D^2\varphi(x))<0,$$

for all $x\in B(x_0,2r)$. Thus, if $x\in B(x_0,r)$, we have

$$
\begin{aligned}
&-\mathrm{div}\left[\left(p_n(x)\left|\frac{\nabla\varphi(x)}{K_n(u)}\right|^{p_n(x)-2}+q_n(x)a(x)\left|\frac{\nabla\varphi(x)}{K_n(u)}\right|^{q_n(x)-2}\right)\frac{\nabla\varphi(x)}{K_n(u)}\right]\\
&-\lambda_{(p_n(\cdot),\,q_n(\cdot))}S_n(u)\left(p_n(x)\left|\frac{u(x)}{k_n(u)}\right|^{p_n(x)-2}+q_n(x)a(x)\left|\frac{u(x)}{k_n(u)}\right|^{q_n(x)-2}\right)\frac{u(x)}{k_n(u)}<0.
\end{aligned}
$$

If $x \in \partial B(x_0, r)$, the minimum value of the function $u - \varphi$ is defined as m. Let $\Phi(x) := \varphi(x) + \frac{m}{2}$. Note that $m > 0$ and the above inequality still holds if the function $\varphi(x)$ is replaced by $\Phi(x)$, namely,

$$\begin{aligned} &-\mathrm{div}\left[\left(p_n(x)\left|\frac{\nabla\Phi(x)}{K_n(u)}\right|^{p_n(x)-2} + q_n(x)a(x)\left|\frac{\nabla\Phi(x)}{K_n(u)}\right|^{q_n(x)-2}\right)\frac{\nabla\Phi(x)}{K_n(u)}\right] \\ &-\lambda_{(p_n(\cdot),q_n(\cdot))}S_n(u)\left(p_n(x)\left|\frac{u(x)}{k_n(u)}\right|^{p_n(x)-2} + q_n(x)a(x)\left|\frac{u(x)}{k_n(u)}\right|^{q_n(x)-2}\right)\frac{u(x)}{k_n(u)} < 0. \end{aligned} \tag{3.5}$$

Define $\eta(x) := (\Phi - u)^+ \geq 0$, then if $x \in \partial B(x_0, r)$, we have $\eta(x) \equiv 0$.

Let

$$\Omega_1 = \{x | x \in B(x_0, r) \text{ and } \Phi(x) > u(x)\}.$$

We multiply (3.5) by the function $\eta(x)$ and integrate over $B(x_0, r)$, then the inequality

$$\begin{aligned} &\int_{\Omega_1}\left(p_n(x)\left|\frac{\nabla\Phi}{K_n(u)}\right|^{p_n(x)-2} + q_n(x)a(x)\left|\frac{\nabla\Phi}{K_n(u)}\right|^{q_n(x)-2}\right)\frac{\nabla\Phi}{K_n(u)}\cdot\nabla(\Phi - u)dx \\ &-\int_{\Omega_1}\lambda_{(p_n(\cdot),q_n(\cdot))}S_n(u)\left(p_n(x)\left|\frac{u}{k_n(u)}\right|^{p_n(x)-2} + q_n(x)a(x)\left|\frac{u}{k_n(u)}\right|^{q_n(x)-2}\right)\frac{u}{k_n(u)}(\Phi - u)dx < 0 \end{aligned} \tag{3.6}$$

is true.

If we define

$$\eta_1(x) = \begin{cases} (\Phi - u)^+, & x \in B(x_0, r), \\ 0, & x \in \Omega \setminus B(x_0, r), \end{cases}$$

and use $\eta_1(x)$ as a test function in (3.1), then we get

$$\begin{aligned} &\int_{\Omega_1}\left(p_n(x)\left|\frac{\nabla u}{K_n(u)}\right|^{p_n(x)-2} + q_n(x)a(x)\left|\frac{\nabla u}{K_n(u)}\right|^{q_n(x)-2}\right)\frac{\nabla u}{K_n(u)}\cdot\nabla(\Phi - u)dx \\ &-\int_{\Omega_1}\lambda_{(p_n(\cdot),q_n(\cdot))}S_n(u)\left(p_n(x)\left|\frac{u}{k_n(u)}\right|^{p_n(x)-2} + q_n(x)a(x)\left|\frac{u}{k_n(u)}\right|^{q_n(x)-2}\right)\frac{u}{k_n(u)}(\Phi - u)dx = 0. \end{aligned} \tag{3.7}$$

Subtracting (3.7) from (3.6), we arrive at

$$\begin{aligned} &\int_{\Omega_1} p_n(x)\left(\left|\frac{\nabla\Phi}{K_n(u)}\right|^{p_n(x)-2}\frac{\nabla\Phi}{K_n(u)} - \left|\frac{\nabla u}{K_n(u)}\right|^{p_n(x)-2}\frac{\nabla u}{K_n(u)}\right)\cdot\nabla(\Phi - u)dx \\ &+\int_{\Omega_1} q_n(x)a(x)\left(\left|\frac{\nabla\Phi}{K_n(u)}\right|^{q_n(x)-2}\frac{\nabla\Phi}{K_n(u)} - \left|\frac{\nabla u}{K_n(u)}\right|^{q_n(x)-2}\frac{\nabla u}{K_n(u)}\right)\nabla(\Phi - u)dx < 0. \end{aligned} \tag{3.8}$$

The first integral is nonnegative due to the elementary inequality

$$\langle |a|^{p-2}a - |b|^{p-2}b, a - b\rangle \geq 0, \tag{3.9}$$

which holds for all $p > 1$. Here, we take $p = p_n(x)$. We get a contradiction. Hence, (3.4) holds. Similarly, we conclude that u is a viscosity subsolution of (3.2) and we omit the details. □

4. The proof of Theorem 1.2

Let $n \in N$ be large enough such that $p_n \geq r > N$, which results in $W_0^{1,\mathcal{H}_n}(\Omega) \hookrightarrow W_0^{1,r}(\Omega)$ (see Proposition 2.16 (1) of Blanco, Gasiński, Harjulehto and Winkert [18]). It follows that u_n are continuous functions. The reason is that the space $W_0^{1,r}(\Omega) \hookrightarrow\hookrightarrow C^{\alpha}(\Omega)$, $0 < \alpha < 1$. Moreover, it is known (see [9]) that for each $n \in \mathbb{N}$ fixed, we have $u_n > 0$.

In order to prove Theorem 1.2, we only need to prove the following conclusions.

Lemma 4.1. *Let $h : \overline{\Omega} \to (1, \infty)$ be a given continuous function, then*

$$\||\nabla v|^s|_{\frac{p(x)}{s}}^{\frac{1}{s}} \leq \|\nabla v\|_{\mathcal{H}}, \tag{4.1}$$

for all $v \in W_0^{1,\mathcal{H}}(\Omega)$ and $s \in (1, p^-)$.

Proof. Since $\left\|\frac{\nabla v}{\|\nabla v\|_{\mathcal{H}}}\right\|_{\mathcal{H}} = 1$, it follows from Proposition 2.1 that

$$\int_\Omega \left[\left(\frac{|\nabla v|}{\|\nabla v\|_{\mathcal{H}}}\right)^{p(x)} + a(x)\left(\frac{|\nabla v|}{\|\nabla v\|_{\mathcal{H}}}\right)^{q(x)}\right] dx = 1. \tag{4.2}$$

Thus,

$$\int_\Omega \left[\left(\frac{|\nabla v|}{\|\nabla v\|_{\mathcal{H}}}\right)^{s}\right]^{\frac{p(x)}{s}} \frac{dx}{\frac{p(x)}{s}} \leq 1. \tag{4.3}$$

Invoking Proposition 2.1 again, we conclude that

$$\left|\left(\frac{|\nabla v|}{\|\nabla v\|_{\mathcal{H}}}\right)^{s}\right|_{\frac{p(x)}{s}} \leq 1,$$

which implies (4.1). □

Lemma 4.2. *If $u \in L^\infty(\Omega)$, then we have*

$$\lim_{n\to\infty} k_n(u) = k_\infty(u). \tag{4.4}$$

Proof. **Step1:** To show that the following inequality holds,

$$\limsup_{n\to\infty} k_n(u) \leq k_\infty(u). \tag{4.5}$$

If $k_n(u) \leq k_\infty(u)$, the above inequality is true. Thus, we can assume that $k_n(u) > k_\infty(u)$, and since $q_n(x) > p_n(x) > 1$, we have

$$\begin{aligned} 1 &= \left(\int_\Omega \left|\frac{u}{k_n(u)}\right|^{p_n(x)} + a(x)\left|\frac{u}{k_n(u)}\right|^{q_n(x)} dx\right)^{\frac{1}{p_n^-}} \\ &\leq \left[\int_\Omega \left(\frac{k_\infty(u)}{k_n(u)}\right)^{p_n(x)} + a(x)\left(\frac{k_\infty(u)}{k_n(u)}\right)^{q_n(x)} dx\right]^{\frac{1}{p_n^-}} \end{aligned}$$

$$\le \left[\int_\Omega \left(\frac{k_\infty(u)}{k_n(u)}\right)^{p_n^-} + a(x)\left(\frac{k_\infty(u)}{k_n(u)}\right)^{p_n^-} dx \right]^{\frac{1}{p_n^-}}$$
$$= \frac{k_\infty(u)}{k_n(u)}\left(|\Omega| + \int_\Omega a(x)dx\right)^{\frac{1}{p_n^-}},$$

which implies (4.5) holds.
Step2: To show that the following inequality holds,

$$\liminf_{n\to\infty} k_n(u) \ge k_\infty(u). \tag{4.6}$$

Case1: $k_\infty(u) = 0$. It is easy to find that (4.6) holds.
Case2: $k_\infty(u) > 0$. Given $\varepsilon > 0$, there exists a nonempty set $\Omega_\varepsilon \subset \Omega$ such that, for all $x \in \Omega_\varepsilon$, $|u| > k_\infty(u) - \varepsilon$. Ignoring those indices n that $k_n(u) \ge k_\infty(u) - \varepsilon$, we have

$$1 = \left(\int_\Omega \left|\frac{u}{k_n(u)}\right|^{p_n(x)} + a(x)\left|\frac{u}{k_n(u)}\right|^{q_n(x)} dx\right)^{\frac{1}{p_n^-}}$$
$$\ge \left(\int_{\Omega_\varepsilon} \left|\frac{u}{k_n(u)}\right|^{p_n(x)} + a(x)\left|\frac{u}{k_n(u)}\right|^{q_n(x)} dx\right)^{\frac{1}{p_n^-}}$$
$$\ge \left(\int_{\Omega_\varepsilon} \left|\frac{k_\infty(u)-\varepsilon}{k_n(u)}\right|^{p_n(x)} + a(x)\left|\frac{k_\infty(u)-\varepsilon}{k_n(u)}\right|^{q_n(x)} dx\right)^{\frac{1}{p_n^-}}$$
$$\ge \left(\int_{\Omega_\varepsilon} \left|\frac{k_\infty(u)-\varepsilon}{k_n(u)}\right|^{p_n^-} + a(x)\left|\frac{k_\infty(u)-\varepsilon}{k_n(u)}\right|^{p_n^-} dx\right)^{\frac{1}{p_n^-}}$$
$$= \frac{k_\infty(u)-\varepsilon}{k_n(u)}\left(|\Omega_\varepsilon| + \int_{\Omega_\varepsilon} a(x)dx\right)^{\frac{1}{p_n^-}},$$

which gives

$$\liminf_{n\to\infty} k_n(u) \ge k_\infty(u) - \varepsilon.$$

The arbitrariness of ε implies that (4.6) is true. Consequently, (4.4) holds. □

Remark 4.1. *If $|\nabla u| \in L^\infty(\Omega)$, we can argue as Lemma 4.2 to obtain that*

$$\lim_{n\to\infty} K_n(u) = K_\infty(u). \tag{4.7}$$

Lemma 4.3. *If the assumptions of Theorem 1.2 hold, then*
(1) (1.18) holds;
(2) there exists a nonnegative function u_∞ such that $u_\infty \in C^\alpha(\Omega) \setminus \{0\}$ and $\|u_\infty\|_{L^\infty(\Omega)} = 1$;
(3) we can extract a subsequence, which is still denoted by u_n, such that

$$u_n \to u_\infty$$

in the space $C^\alpha(\Omega)$, where $\alpha\,(0 < \alpha < 1)$ is a constant.

Proof. Assume for simplicity that the following inequality holds

$$\int_\Omega dx = 1.$$

Step 1: To show that,

$$\limsup_{n\to\infty} \lambda^1_{(p_n(\cdot),\, q_n(\cdot))} \le \Lambda_\infty. \tag{4.8}$$

Inserting $u(x) = \delta(x)$ into (1.4) gives

$$\lambda^1_{(p_n(\cdot),\, q_n(\cdot))} \le \frac{\|\nabla\delta\|_{\mathcal{H}_n}}{\|\delta\|_{\mathcal{H}_n}}.$$

Note that by Lemma 4.2 and Remark 4.1, we have

$$\limsup_{n\to\infty} \lambda^1_{(p_n(\cdot),\, q_n(\cdot))} \le \frac{\|\nabla\delta\|_{L^\infty(\Omega)}}{\|\delta\|_{L^\infty(\Omega)}} = \Lambda_\infty.$$

Step 2: We now claim that $u_\infty \in W_0^{1,\infty}(\Omega)$.

Since (4.8) holds, for all $n \in N$ sufficiently large, we can assume that $\lambda^1_{(p_n(\cdot),\, q_n(\cdot))} \le \Lambda_\infty + 1$. Thus, we have

$$\Lambda_\infty + 1 \ge \lambda^1_{(p_n(\cdot),\, q_n(\cdot))} = \frac{\|\nabla u_n\|_{\mathcal{H}_n}}{\|u_n\|_{\mathcal{H}_n}} = \|\nabla u_n\|_{\mathcal{H}_n}.$$

Note that the sequence $\{\|\nabla u_n\|_{\mathcal{H}_n}\}$ is bounded.

Let $r \in [1, \infty)$ be arbitrary. We can find an integer n_r, for all $n \ge n_r$, such that $p_n(\cdot) \ge r$ and

$$W_0^{1,\mathcal{H}_n}(\Omega) \hookrightarrow W_0^{1,r}(\Omega) \hookrightarrow\hookrightarrow L^r(\Omega).$$

Hence, the sequence $\{u_n\}$ is bounded in the reflexive Banach space $W_0^{1,r}(\Omega)$. We can find a subsequence, still defined by $\{u_n\}$, and a function $u_\infty \in W_0^{1,r}(\Omega)$, such that $\nabla u_n \rightharpoonup \nabla u_\infty$ in $W_0^{1,r}(\Omega)$ and $u_n \to u_\infty$ in $L^r(\Omega)$.

Define

$$s_n(x) := \frac{p_n(x)}{p_n(x) - r},\ x \in \Omega,$$

and it follows that

$$s_n^+ = \frac{p_n^-}{p_n^- - r},\ s_n^- = \frac{p_n^+}{p_n^+ - r}$$

and

$$|1|_{s_n(x)} \le \max\{|\Omega|^{\frac{1}{s_n^+}}, |\Omega|^{\frac{1}{s_n^-}}\}. \tag{4.9}$$

Using Hölder's inequality and the above inequality, we have

$$\begin{aligned}\left(\int_\Omega |\nabla u_n|^r dx\right)^{\frac{1}{r}} &\le \left(\frac{1}{s_n^-} + \frac{r}{p_n^-}\right)|1|^{\frac{1}{r}}_{s_n(x)} \||\nabla u_n|^r|^{\frac{1}{r}}_{\frac{p_n(x)}{r}} \\ &\le \left(\frac{1}{s_n^-} + \frac{r}{p_n^-}\right)\max\{|\Omega|^{\frac{1}{s_n^+}}, |\Omega|^{\frac{1}{s_n^-}}\}^{\frac{1}{r}} \||\nabla u_n|^r|^{\frac{1}{r}}_{\frac{p_n(x)}{r}}. \end{aligned} \tag{4.10}$$

Thus, (4.1) and (4.10) ensure that

$$\|\nabla u_n\|_{L^r(\Omega)} \le 2(1+|\Omega|)\|\nabla u_n\|_{\mathcal{H}_n} \le 2(1+|\Omega|)\Lambda_\infty + 1. \tag{4.11}$$

We choose an arbitrary positive real number r_1 such that $B(x,r_1) \subset \Omega$, where the point $x \in \Omega$ is a Lebesgue point such that $|\nabla u_\infty| \in L^1(\Omega)$, then we find that

$$\begin{aligned}\frac{1}{|B(x,r_1)|}\int_{B(x,r_1)} |\nabla u_\infty(y)|dy &\le \liminf_{n\to\infty} \frac{1}{|B(x,r_1)|}\int_{B(x,r_1)} |\nabla u_n(y)|dy \\ &\le \liminf_{n\to\infty} |B(x,r_1)|^{-\frac{1}{r}}\|\nabla u_n\|_{L^r(\Omega)} \\ &\le |B(x,r)|^{-\frac{1}{r}} 2(1+|\Omega|)(\Lambda_\infty+1). \end{aligned} \tag{4.12}$$

Passing to the limit as $r \to +\infty$ in the above inequality, gives

$$\frac{1}{|B(x,r_1)|}\int_{B(x,r_1)} |\nabla u_\infty(y)|dy \le 2(1+|\Omega|)(\Lambda_\infty+1).$$

Letting $r_1 \to 0^+$ in the above inequality, gives

$$|\nabla u_\infty(x)| \le 2(1+|\Omega|)(\Lambda_\infty+1),$$

for a.e. $x \in \Omega$, which implies that $\nabla u_\infty \in L^\infty(\Omega)$, as claimed.

Step 3: We want to prove that $u_n \to u_\infty$ in $C^\alpha(\Omega)$ $(0<\alpha<1)$ and $\|u_\infty\|_{L^\infty(\Omega)} = 1$.

Keeping in mind that $r \in [1,\infty)$ is an arbitrary constant, we can assume that $r > N$. Therefore, this combined with the fact that $W_0^{1,r}(\Omega) \hookrightarrow\hookrightarrow C^\alpha(\Omega)\,(0<\alpha<1)$ implies that there exists a nonnegative function $u_\infty \in C^\alpha(\Omega)\setminus\{0\}$, such that $u_n \to u_\infty$ in $C^\alpha(\Omega)$ and u_n converges uniformly to u_∞ in Ω. Given $\varepsilon \in (0,1)$, we can find a constant $N_\varepsilon \in \mathbb{N}$ such that

$$|u_n(x) - u_\infty(x)| < \varepsilon, \tag{4.13}$$

for all $x \in \Omega, n \ge N_\varepsilon$. It follows that

$$\begin{aligned}[\rho_{\mathcal{H}_n}(u_n - u_\infty)]^{\frac{1}{p_n^-}} &= \left[\int_\Omega |u_n-u_\infty|^{p_n(x)} + a(x)|u_n-u_\infty|^{q_n(x)}dx\right]^{\frac{1}{p_n^-}} \\ &\le \left[\int_\Omega \varepsilon^{p_n(x)} + a(x)\varepsilon^{q_n(x)}dx\right]^{\frac{1}{p_n^-}} \\ &\le \varepsilon\left[\int_\Omega (1+a(x))dx\right]^{\frac{1}{p_n^-}} \\ &\le \left[\int_\Omega (1+a(x))dx\right]^{\frac{1}{p_n^-}} \end{aligned} \tag{4.14}$$

and

$$\begin{aligned}[\rho_{\mathcal{H}_n}(u_n - u_\infty)]^{\frac{1}{q_n^+}} &\le \varepsilon^{\frac{p_n^-}{q_n^+}}\left[\int_\Omega (1+a(x))dx\right]^{\frac{1}{q_n^+}} \\ &\le \left[\int_\Omega (1+a(x))dx\right]^{\frac{1}{q_n^+}}, \end{aligned} \tag{4.15}$$

for all $n \geq N_\varepsilon$. Letting $n \to \infty$ in (4.14) and (4.15) yields

$$\lim_{n\to\infty}[\rho_{\mathcal{H}_n}(u_n - u_\infty)]^{\frac{1}{p_n^-}} = \lim_{n\to\infty}[\rho_{\mathcal{H}_n}(u_n - u_\infty)]^{\frac{1}{q_n^+}} = 0. \tag{4.16}$$

Thus, the inequality

$$\begin{aligned}
|\|u_n\|_{\mathcal{H}_n} - \|u_\infty\|_{L^\infty(\Omega)}| &\leq |\|u_n\|_{\mathcal{H}_n} - \|u_\infty\|_{\mathcal{H}_n}| + |\|u_\infty\|_{\mathcal{H}_n} - \|u_\infty\|_{L^\infty(\Omega)}| \\
&\leq \|u_n - u_\infty\|_{\mathcal{H}_n} + |\|u_\infty\|_{\mathcal{H}_n} - \|u_\infty\|_{L^\infty(\Omega)}| \\
&\leq \left\{[\rho_{\mathcal{H}_n}(u_n - u_\infty)]^{\frac{1}{p_n^-}} + [\rho_{\mathcal{H}_n}(u_n - u_\infty)]^{\frac{1}{q_n^+}}\right\} + |\|u_\infty\|_{\mathcal{H}_n} - \|u_\infty\|_{L^\infty(\Omega)}|. \\
&\leq \left\{[\rho_{\mathcal{H}_n}(u_n - u_\infty)]^{\frac{1}{p_n^-}} + [\rho_{\mathcal{H}_n}(u_n - u_\infty)]^{\frac{1}{q_n^+}}\right\} + |k_n(u_\infty) - \|u_\infty\|_{L^\infty(\Omega)}|
\end{aligned}$$

holds. In view of Lemma 4.2 and (4.16), we can get

$$\|u_\infty\|_{L^\infty(\Omega)} = \lim_{n\to\infty} \|u_n\|_{\mathcal{H}_n} = 1. \tag{4.17}$$

Step 4: To show that $\liminf_{n\to\infty} \lambda^1_{(p_n(\cdot),\, q_n(\cdot))} \geq \Lambda_\infty$.

Since $\nabla u_n \rightharpoonup \nabla u_\infty$ in $W_0^{1,r}(\Omega)$, $\|u_n\|_{\mathcal{H}_n} = 1$ and the inequality (4.11) holds, we have

$$\|\nabla u_\infty\|_{L^r(\Omega)} \leq \liminf_{n\to\infty} \|\nabla u_n\|_{L^r(\Omega)} \leq \liminf_{n\to\infty} \|\nabla u_n\|_{\mathcal{H}_n} = \liminf_{n\to\infty} \lambda^1_{(p_n(\cdot),q_n(\cdot))}.$$

Letting $r \to \infty$ and using Proposition 7 in [19] and equality (4.17), we get

$$\Lambda_\infty \leq \frac{\|\nabla u_\infty\|_{L^\infty(\Omega)}}{\|u_\infty\|_{L^\infty(\Omega)}} \leq \liminf_{n\to\infty} \lambda^1_{(p_n(\cdot),q_n(\cdot))}. \tag{4.18}$$

Thus, (4.8) and (4.18) imply that (1.18) holds. The proof is complete. □

Remark 4.2. *We can again argue with Step 3 to obtain*

$$\|\nabla u_\infty\|_{L^\infty(\Omega)} = \lim_{n\to\infty} \|\nabla u_n\|_{\mathcal{H}_n}. \tag{4.19}$$

The function $u_\infty(x)$ also has the following property.

Lemma 4.4. *If the assumptions of Theorem 1.2 hold, we can deduce that $u_\infty(x)$ is a nontrivial viscosity solution of the problem (1.20).*

Proof. For the first part we only need to show that u_∞ is a viscosity subsolution of (1.20). Let $x_0 \in \Omega$ and $\psi \in C^2(\Omega)$. Assume that $u_\infty - \psi$ attains its strict maximum value of zero at x_0, namely, $u_\infty(x_0) - \psi(x_0) = 0$.

Claim: We want to show that

$$\begin{aligned}
&\max\Big\{\Lambda_\infty \psi(x_0) - |\nabla\psi(x_0)|, (\psi(x_0))^{\theta(x_0)} K_\infty(u_\infty) - |\nabla\psi(x_0)|, \\
&\triangle_\infty \psi(x_0) + [\ln(|\nabla\psi(x_0)|) - \ln(K_\infty(u_\infty))]|\nabla\psi(x_0)|^2 \nabla\psi(x_0) \cdot \xi_2(x_0)\Big\} \leq 0.
\end{aligned} \tag{4.20}$$

By Lemma 4.3, we know that the convergence of u_n to u_∞ in Ω is uniform. Therefore, there exists a sequence $\{x_n\} \subset \Omega$ such that $x_n \to x_0$ (as $n \to \infty$), $u_n(x_n) = \psi(x_n)$ and $u_n - \psi$ attains its strict maximum value at x_n.

Employing Theorem 1.1, it turns out that for any $n \in \mathbb{N}$ large enough, u_n are continuous viscosity solutions of (1.5) with $\lambda_{(p_n(\cdot),\, q_n(\cdot))} = \lambda^1_{(p_n(\cdot),\, q_n(\cdot))}$. Thus, we have

$$
\begin{aligned}
&- p_n(x_n)(K_n(u_n))^{1-p_n(x_n)}|\nabla\psi(x_n)|^{p_n(x_n)-4}\{|\nabla\psi(x_n)|^2\triangle\psi(x_n) + (p_n(x_n)-2)\triangle_\infty\psi(x_n)\\
&+ [\ln(|\nabla\psi(x_n)|) - \ln(K_n(u_n))]|\nabla\psi(x_n)|^2\nabla\psi(x_n)\cdot\nabla p_n(x_n)\}\\
&- q_n(x_n)a(x_n)(K_n(u_n))^{1-q_n(x_n)}|\nabla\psi(x_n)|^{q_n(x_n)-4}\{|\nabla\psi(x_n)|^2\triangle\psi(x_n) + (q_n(x_n)-2)\triangle_\infty\psi(x_n)\\
&+ [\ln(|\nabla\psi(x_n)|) - \ln(K_n(u_n))]|\nabla\psi(x_n)|^2\nabla\psi(x_n)\cdot\nabla q_n(x_n)\}\\
&- q_n(x_n)(K_n(u_n))^{1-q_n(x_n)}|\nabla\psi(x_n)|^{q_n(x_n)-2}\nabla\psi(x_n)\cdot\nabla a(x_n)\\
&- (K_n(u_n))^{1-p_n(x_n)}|\nabla\psi(x_n)|^{p_n(x_n)-2}\nabla\psi(x_n)\cdot\nabla p_n(x_n)\\
&- a(x_n)(K_n(u_n))^{1-q_n(x_n)}|\nabla\psi(x_n)|^{q_n(x_n)-2}\nabla\psi(x_n)\cdot\nabla q_n(x_n)\\
&- \lambda^1_{(p_n(\cdot),\, q_n(\cdot))}S_n(u_n)p_n(x_n)(k_n(u_n))^{1-p_n(x_n)}|\psi(x_n)|^{p_n(x_n)-2}\psi(x_n)\\
&- \lambda^1_{(p_n(\cdot),\, q_n(\cdot))}S_n(u_n)q_n(x_n)a(x_n)(k_n(u_n))^{1-q_n(x_n)}|\psi(x_n)|^{q_n(x_n)-2}\psi(x_n) \geq 0.
\end{aligned}
\tag{4.21}
$$

Case 1: $\psi(x_0) = u_\infty(x_0) > 0$.

Continuing (4.21), for $n \in \mathbb{N}$ sufficiently large, we have $|\nabla\psi(x_n)| > 0$. Let us assume the assertion is not true, then by (4.21) and continuity, we have $\psi(x_0) \leq 0$. This leads to a contradiction.

Dividing both sides of (4.21) by

$$
p_n(x_n)(p_n(x_n)-2)(K_n(u_n))^{1-p_n(x_n)}|\nabla\psi(x_n)|^{p_n(x_n)-4},
$$

we see that the following inequality holds

$$
\begin{aligned}
&- \frac{|\nabla\psi(x_n)|^2\triangle\psi(x_n)}{p_n(x_n)-2} - \triangle_\infty\psi(x_n) - [\ln(|\nabla\psi(x_n)|) - \ln(K_n(u_n))]|\nabla\psi(x_n)|^2\nabla\psi(x_n)\cdot\frac{\nabla p_n(x_n)}{p_n(x_n)-2}\\
&- \frac{q_n(x_n)}{p_n(x_n)}\left|\frac{\nabla\psi(x_n)}{K_n(u_n)}\right|^{q_n(x_n)-p_n(x_n)}\Big\{a(x_n)\frac{|\nabla\psi(x_n)|^2\triangle\psi(x_n)}{p_n(x_n)-2} + a(x_n)\Big(\frac{q_n(x_n)-2}{p_n(x_n)-2}\Big)\triangle_\infty\psi(x_n)\\
&+ \frac{|\nabla\psi(x_n)|^2\nabla\psi(x_n)\cdot\nabla a(x_n)}{p_n(x_n)-2} + a(x_n)\frac{|\nabla\psi(x_n)|^2}{p_n(x_n)-2}\nabla\psi(x_n)\cdot\frac{\nabla q_n(x_n)}{q_n(x_n)}\\
&+ a(x_n)[\ln(|\nabla\psi(x_n)|) - \ln(K_n(u_n))]|\nabla\psi(x_n)|^2\nabla\psi(x_n)\cdot\frac{\nabla q_n(x_n)}{p_n(x_n)-2}\Big\} - \frac{|\nabla\psi(x_n)|^2}{p_n(x_n)}\frac{\nabla\psi(x_n)\cdot\nabla p_n(x_n)}{p_n(x_n)-2}\\
\geq\;& \Big(\lambda^1_{(p_n(\cdot),q_n(\cdot))}\Big)^3 S_n(u_n)\left|\frac{\lambda^1_{(p_n(\cdot),q_n(\cdot))}\psi(x_n)}{\nabla\psi(x_n)}\right|^{p_n(x_n)-4}\frac{|\psi(x_n)|^2\psi(x_n)}{p_n(x_n)-2}\\
&+ \Big(\lambda^1_{(p_n(\cdot),q_n(\cdot))}\Big)^3 S_n(u_n)\frac{q_n(x_n)}{p_n(x_n)}a(x_n)\frac{|\psi(x_n)|^2\psi(x_n)}{p_n(x_n)-2}\Big[\Big(\frac{|\psi(x_n)|}{k_n(u_n)}\Big)^{(q_n(x_n)-4)/(p_n(x_n)-4)}\frac{K_n(u_n)}{|\nabla\psi(x_n)|}\Big]^{p_n(x_n)-4}\\
\geq\;& 0.
\end{aligned}
\tag{4.22}
$$

Now, letting $n \to \infty$, we deduce that

$$
\begin{aligned}
&- \frac{|\nabla\psi(x_n)|^2 \triangle\psi(x_n)}{p_n(x_n)-2} - \triangle_\infty\psi(x_n) \to -\triangle_\infty\psi(x_0),\\
&- [\ln(|\nabla\psi(x_n)|) - \ln(K_n(u_n))]|\nabla\psi(x_n)|^2\nabla\psi(x_n)\cdot\frac{\nabla p_n(x_n)}{p_n(x_n)-2}\\
\to &- [\ln(|\nabla\psi(x_0)|) - \ln(K_\infty(u_\infty))]|\nabla\psi(x_0)|^2\nabla\psi(x_0)\cdot\xi_1(x_0),\\
&- \frac{q_n(x_n)}{p_n(x_n)}\Big\{a(x_n)\frac{|\nabla\psi(x_n)|^2\triangle\psi(x_n)}{p_n(x_n)-2} + a(x_n)\Big(\frac{q_n(x_n)-2}{p_n(x_n)-2}\Big)\triangle_\infty\psi(x_n) + \frac{|\psi(x_n)|^2\nabla\psi(x_n)\cdot\nabla a(x_n)}{p_n(x_n)-2}\\
&+ a(x_n)[\ln(|\nabla\psi(x_n)|) - \ln(K_n(u_n))]|\nabla\psi(x_n)|^2\nabla\psi(x_n)\cdot\frac{\nabla q_n(x_n)}{p_n(x_n)-2}\\
&+ a(x_n)\frac{|\nabla\psi(x_n)|^2}{p_n(x_n)-2}\nabla\psi(x_n)\cdot\frac{\nabla q_n(x_n)}{q_n(x_n)}\Big\}\\
\to &- \theta^2(x_0)a(x_0)\left\{\triangle_\infty\psi(x_0) + [\ln(|\nabla\psi(x_0)|) - \ln(K_\infty(u_\infty))]|\nabla\psi(x_0)|^2\nabla\psi(x_0)\cdot\xi_2(x_0)\right\},\\
&- \frac{|\nabla\psi(x_n)|^2}{p_n(x_n)}\frac{\nabla\psi(x_n)\cdot\nabla p_n(x_n)}{p_n(x_n)-2} \to 0.
\end{aligned}
$$

Taking the lower limit in inequality (4.22) and employing the limits above, we have

$$
\begin{aligned}
&- \left|\frac{\nabla\psi(x_0)}{K_\infty(u_\infty)}\right|^{\liminf_{n\to\infty}(q_n(x_n)-p_n(x_n))}\theta^2(x_0)a(x_0)\\
&\cdot\left\{\triangle_\infty\psi(x_0) + [\ln(|\nabla\psi(x_0)|) - \ln(K_\infty(u_\infty))]|\nabla\psi(x_0)|^2\nabla\psi(x_0)\cdot\xi_2(x_0)\right\}\\
&- \left\{\triangle_\infty\psi(x_0) + [\ln(|\nabla\psi(x_0)|) - \ln(K_\infty(u_\infty))]|\nabla\psi(x_0)|^2\nabla\psi(x_0)\cdot\xi_1(x_0)\right\}\\
=& - \left(\left|\frac{\nabla\psi(x_0)}{K_\infty(u_\infty)}\right|^{\liminf_{n\to\infty}(q_n(x_n)-p_n(x_n))}\theta^2(x_0)a(x_0) + 1\right)\triangle_\infty\psi(x_0)\\
&- [\ln(|\nabla\psi(x_0)|) - \ln(K_\infty(u_\infty))]|\nabla\psi(x_0)|^2\nabla\psi(x_0)\left(\xi_1(x_0) + \theta^2(x_0)a(x_0)\left|\frac{\nabla\psi(x_0)}{K_\infty(u_\infty)}\right|^{\liminf_{n\to\infty}(q_n(x_n)-p_n(x_n))}\xi_2(x_0)\right)\\
\geq&(\Lambda_\infty)^3\liminf_{n\to\infty}S_n(u_n)\left|\frac{\lambda^1_{(p_n(\cdot),q_n(\cdot))}\psi(x_n)}{\nabla\psi(x_n)}\right|^{p_n(x_n)-4}\frac{|\psi(x_n)|^2\psi(x_n)}{p_n(x_n)-2}\\
&+ (\Lambda_\infty)^3\theta(x_0)a(x_0)\liminf_{n\to\infty}S_n(u_n)\frac{|\psi(x_n)|^2\psi(x_n)}{p_n(x_n)-2}\left[\left(\frac{|\psi(x_n)|}{k_n(u_n)}\right)^{(q_n(x_n)-4)/(p_n(x_n)-4)}\frac{K_n(u_n)}{|\nabla\psi(x_n)|}\right]^{p_n(x_n)-4}\\
\geq&0. \qquad (4.23)
\end{aligned}
$$

Note that by (4.17), (4.19) and $u_\infty(x_0) = \psi(x_0) > 0$, we have

$$
\begin{aligned}
\left|\frac{\nabla\psi(x_0)}{K_\infty(u_\infty)}\right|^{\liminf_{n\to\infty}(q_n(x_n)-p_n(x_n))} &= \left|\frac{\nabla\psi(x_0)}{\Lambda_\infty k_\infty(u_\infty)}\right|^{\liminf_{n\to\infty}(q_n(x_n)-p_n(x_n))}\\
&\leq \left|\frac{\nabla\psi(x_0)}{\Lambda_\infty u_\infty(x_0)}\right|^{\liminf_{n\to\infty}(q_n(x_n)-p_n(x_n))}\\
&= \left|\frac{\nabla\psi(x_0)}{\Lambda_\infty\psi(x_0)}\right|^{\lim_{n\to\infty}(q_n(x_n)-p_n(x_n))} \qquad (4.24)
\end{aligned}
$$

and

$$\left|\frac{\nabla\psi(x_0)}{K_\infty(u_\infty)}\right|^{\liminf_{n\to\infty}(q_n(x_n)-p_n(x_n))} = \left|\frac{\nabla\psi(x_0)}{\Lambda_\infty(k_\infty(u_\infty))^{\theta(x_0)}}\right|^{\liminf_{n\to\infty}(q_n(x_n)-p_n(x_n))}$$
$$\leq \left|\frac{\nabla\psi(x_0)}{\Lambda_\infty(u_\infty(x_0))^{\theta(x_0)}}\right|^{\liminf_{n\to\infty}(q_n(x_n)-p_n(x_n))}$$
$$= \left|\frac{\nabla\psi(x_0)}{\Lambda_\infty(\psi(x_0))^{\theta(x_0)}}\right|^{\liminf_{n\to\infty}(q_n(x_n)-p_n(x_n))}. \tag{4.25}$$

Claim:

$$\Lambda_\infty\psi(x_0) - |\nabla\psi(x_0)| \leq 0. \tag{4.26}$$

Assume that $\Lambda_\infty\psi(x_0) > |\nabla\psi(x_0)|$, then (4.24) and (1.11) imply

$$\left|\frac{\nabla\psi(x_0)}{K_\infty(u_\infty)}\right|^{\liminf_{n\to\infty}(q_n(x_n)-p_n(x_n))} = 0 \tag{4.27}$$

and

$$\lim_{n\to\infty}\left|\frac{\lambda^1_{(p_n(\cdot),q_n(\cdot))}\psi(x_n)}{\nabla\psi(x_n)}\right|^{(p_n(x_n)-4)\backslash(q_n(x_n)-4)} = \left(\frac{\Lambda_\infty\psi(x_0)}{|\nabla\psi(x_0)|}\right)^{\frac{1}{\theta(x_0)}} > 1. \tag{4.28}$$

Thus, choosing $\varepsilon > 0$ small enough, we have

$$\left|\frac{\lambda^1_{(p_n(\cdot),q_n(\cdot))}\psi(x_n)}{\nabla\psi(x_n)}\right|^{(p_n(x_n)-4)\backslash(q_n(x_n)-4)} \geq 1+\varepsilon, \tag{4.29}$$

for all $n \in \mathbb{N}$ sufficiently large. By (4.29), we get

$$\liminf_{n\to\infty}\left|\frac{\lambda^1_{(p_n,q_n)}\psi(x_n)}{\nabla\psi(x_n)}\right|^{p_n(x_n)-4}\frac{|\psi(x_n)|^2\psi(x_n)}{p_n(x_n)-2}$$
$$=\liminf_{n\to\infty}\frac{\left(\left|\frac{\lambda^1_{(p_n,q_n)}\psi(x_n)}{\nabla\psi(x_n)}\right|^{(p_n(x_n)-4)\backslash(q_n(x_n)-4)}\right)^{q_n(x_n)-4}}{q_n(x_n)-4}\frac{|\psi(x_n)|^2\psi(x_n)}{\frac{p_n(x_n)-2}{q_n(x_n)-4}}$$
$$\geq R\psi(x_0)^3\lim_{n\to\infty}\frac{(1+\varepsilon)^{q_n(x_n)-4}}{q_n(x_n)-4}$$
$$=+\infty. \tag{4.30}$$

From (4.23), (4.27) and (4.30), we see that

$$-\left\{\triangle_\infty\psi(x_0) + [\ln(|\nabla\psi(x_0)|) - \ln(K_\infty(u_\infty))]|\nabla\psi(x_0)|^2\nabla\psi(x_0)\cdot\xi_1(x_0)\right\} \geq +\infty, \tag{4.31}$$

which is a contradiction. Hence, (4.26) holds.

Claim:

$$(\psi(x_0))^{\theta(x_0)}K_\infty(u_\infty) - |\nabla\psi(x_0)| \leq 0. \tag{4.32}$$

Suppose that the above inequality is not true, then we have

$$\begin{aligned}
&\lim_{n\to\infty}\left[\left(\frac{\psi(x_n)}{k_n(u_n)}\right)^{(q_n(x_n)-4)/(p_n(x_n)-4)}\frac{K_n(u_n)}{|\nabla\psi(x_n)|}\right]^{(p_n(x_n)-4)/(q_n(x_n)-4)}\\
&=\lim_{n\to\infty}\left[(\psi(x_n))^{(q_n(x_n)-4)/(p_n(x_n)-4)}\frac{K_n(u_n)}{|\nabla\psi(x_n)|}\right]^{(p_n(x_n)-4)/(q_n(x_n)-4)}\\
&=\left[(\psi(x_0))^{\theta(x_0)}\frac{K_\infty(u_\infty)}{|\nabla\psi(x_0)|}\right]^{\frac{1}{\theta(x_0)}}>1.
\end{aligned}$$

Thus, choosing $\varepsilon_1 > 0$ small enough, we have

$$\left[\left(\frac{\psi(x_n)}{k_n(u_n)}\right)^{(q_n(x_n)-4)/(p_n(x_n)-4)}\frac{K_n(u_n)}{|\nabla\psi(x_n)|}\right]^{(p_n(x_n)-4)/(q_n(x_n)-4)}\geq 1+\varepsilon_1, \tag{4.33}$$

for all $n \in \mathbb{N}$ sufficiently large. We are led to

$$\begin{aligned}
&\liminf_{n\to\infty}\left[\left(\frac{|\psi(x_n)|}{k_n(u_n)}\right)^{(q_n(x_n)-4)/(p_n(x_n)-4)}\frac{K_n(u_n)}{|\nabla\psi(x_n)|}\right]^{p_n(x_n)-4}\frac{|\psi(x_n)|^2\psi(x_n)}{p_n(x_n)-2}\\
&\geq\liminf_{n\to\infty}\frac{(1+\varepsilon_1)^{q_n(x_n)-4}}{q_n(x_n)-4}\frac{|\psi(x_n)|^2\psi(x_n)}{\frac{p_n(x_n)-2}{q_n(x_n)-4}}\\
&=\theta(x_0)\psi(x_0)^3\lim_{n\to\infty}\frac{(1+\varepsilon_1)^{q_n(x_n)-4}}{q_n(x_n)-4}\\
&=+\infty.
\end{aligned}\tag{4.34}$$

In view of $(\psi(x_0))^{\theta(x_0)}K_\infty(u_\infty)-|\nabla\psi(x_0)|>0$ and (4.25),

$$\left|\frac{\nabla\psi(x_0)}{K_\infty(u_\infty)}\right|^{\liminf_{n\to\infty}(q_n(x_n)-p_n(x_n))}=0.$$

Therefore, this fact along with (4.23) shows that (4.31) holds. This is a contradiction. Thus we deduce that (4.32) holds.

Claim:

$$\triangle_\infty\psi(x_0)+[\ln(|\nabla\psi(x_0)|)-\ln(K_\infty(u_\infty))]|\nabla\psi(x_0)|^2\nabla\psi(x_0)\cdot\xi_2(x_0)\leq 0. \tag{4.35}$$

Taking (4.24) and (4.26) into account, we have

$$\left|\frac{\nabla\psi(x_0)}{K_\infty(u_\infty)}\right|^{\liminf_{n\to\infty}(q_n(x_n)-p_n(x_n))}=+\infty. \tag{4.36}$$

At the same time, by (4.25) and (4.32), we also deduce that (4.36) holds. If we assume that inequality (4.35) does not hold, then by (4.23) and (4.36), there is a contradiction. Thus, we deduce that (4.35) holds.

Case 2: $\psi(x_0) = u_\infty(x_0) = 0$.

Note that if $|\nabla\psi(x_0)| = 0$ (in this case, we have $\triangle_\infty\psi(x_0)=0$), the inequality (4.20) trivially holds. Hence, let us assume that $|\nabla\psi(x_0)| > 0$, then $|\nabla\psi(x_n)| > 0$ for $n \in \mathbb{N}$ large enough. We can use very similar arguments as Case 1 to conclude that (4.20) holds. The same argument can be used in order to show that u_∞ is a viscosity supersolution of (1.20). □

By Lemmas 4.3 and 4.4, it follows that Theorem 1.2 holds.

Remark 4.3. *In the particular case where $p_n(x) = np(x)$ and $q_n(x) = nq(x)$, Theorems 1.1 and 1.2 are also true.*

5. Conclusions

In this paper, we studied a double-phase eigenvalue problem with large variable exponents. As we know, for p-Laplace operator eigenvalue problems, there is an important feature that if u is an eigenfunction, so is ku, where k is an arbitrary constant. However, the double-phase operator with variable exponents looses this property. To overcome the above mentioned shortcoming, we defined the eigenvalue by using the Rayleigh quotient of two norms of Musielak-Orlicz space. Moreover, in the particular case where $p_n(\cdot) = p_n$ and $q_n(\cdot) = q_n$, Theorems 1.1 and 1.2 are also true (see [13]).

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interests.

References

1. G. Franzina, P. Lindqvist, An eigenvalue problem with variable exponents, *Nonlinear Anal.*, **85** (2013), 1–16. http://doi.org/10.1016/j.na.2013.02.011
2. V. V. Zhikov, Averaging of functionals of the calculus of variations and elasticity theory, *Math. USSR Izv.*, **29** (1986), 675–710. http://doi.org/10.1070/IM1987v029n01ABEH000958
3. V. V. Zhikov, On Lavrentiev's phenomenon, *Russ. J. Math. Phys.*, **3** (1995), 249–269.
4. V. V. Zhikov, On some variational problems, *Russ. J. Math. Phys.*, **5** (1997).
5. V. V. Zhikov, S. M. Kozlov, O. A. Oleinik, *Homogenization of differential operators and integral functionals*, Heidelberg: Springer Berlin, 1994. https://doi.org/10.1007/978-3-642-84659-5
6. A. Bahrouni, V. D. Rădulescu, D. D. Repovš, Double phase transonic flow problems with variable growth: Nonlinear patterns and stationary waves, *Nonlinearity*, **32** (2019), 2481. http://doi.org/10.1088/1361-6544/ab0b03
7. V. Benci, P. D'Avenia, D. Fortunato, L. Pisani, Solitons in several space dimensions: Derrick's problem and infinitely many solutions, *Arch. Ration. Mech. Anal.*, **154** (2000), 297–324. https://doi.org/10.1007/s002050000101
8. L. Cherfils, Y. Il'yasov, On the stationary solutions of generalized reaction diffusion equations with p, q- Laplacian, *Commun. Pure Appl. Anal.*, **4** (2005), 9–22. http://doi.org/10.3934/cpaa.2005.4.9
9. F. Colasuonno, M. Squassina, Eigenvalues for double phase variational integrals, *Ann. Mat. Pura Appl.*, **195** (2016), 1917–1959. https://doi.org/10.1007/s10231-015-0542-7

10. Z. H. Liu, N. S. Papageorgiou, Unbounded spectrum for a nonlinear eigenvalue problem with a variable exponent, *Appl. Math. Lett.*, **145** (2023), 108768. https://doi.org/10.1016/j.aml.2023.108768
11. Z. H. Liu, N. S. Papageorgiou, On a nonhomogeneous, nonlinear Dirichlet eigenvalue problem, *Math. Nachr.*, **296** (2023), 3986–4001.https://doi.org/10.1002/mana.202200040
12. Z. H. Liu, N. S. Papageorgiou, On an anisotropic eigenvalue problem, *Results Math.*, **78** (2023). https://doi.org/10.1007/s00025-023-01954-y
13. L. J. Yu, A double phase eigenvalue problem with large exponents, *Open Math.*, **21** (2023), 20230138. http://doi.org/10.1515/math-2023-0138
14. L. Diening, P. Harjulehto, P. Hasto, M. Ruzicka, Lebesgue and Sobolev spaces with variable exponents, In: *Lecture notes in mathematics*, Heidelberg: Springer Berlin, **2017** (2011). http://doi.org/10.1007/978-3-642-18363-8
15. X. L. Fan, An imbedding theorem for Musielak-Sobolev spaces, *Nonlinear Anal.*, **75** (2012), 1959–1971. http://doi.org/10.1016/j.na.2011.09.045
16. J. Musielak, Orlicz spaces and modular spaces, In: *Lecture notes in mathematics*, Heidelberg: Springer-Verlag, **1034** (1983). https://doi.org/10.1007/BFb0072210
17. B. B. Wang, D. C. Liu, P. Zhao, Hölder continuity for nonlinear elliptic problem in Musielak-Orlicz-Sobolev space, *J. Differ. Equ.*, **266** (2019), 4835–4863. https://doi.org/10.1016/j.jde.2018.10.013
18. Á. Crespo-Blanco, L. Gasiński, P. Harjulehto, P. Winkert, A new class of double phase variable exponent problems: Existence and uniqueness, *J. Differ. Equ.*, **323** (2022), 182–228. http://doi.org/10.1016/j.jde.2022.03.029
19. L. J. Yu, The asymptotic behaviour of the $p(x)$-Laplacian Steklov eigenvalue problem, *Discrete Contin. Dyn. Syst. Ser. B*, **25** (2020), 2621–2637. http://doi.org/10.3934/dcdsb.2020025

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