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Research article

Bifurcations of traveling wave solutions for the mixed Korteweg-de Vries equation

Hui Wang* **and Xue Wang**

College of Science, Henan University of Engineering, Zhengzhou, 451191, China

* **Correspondence:** Email: supwanghui@163.com.

Abstract: In this paper, the bifurcation theory of planar dynamical systems is employed to investigate the mixed Korteweg-de Vries (KdV) equation. Under different parameter conditions, the bifurcation curves and phase portraits of corresponding Hamiltonian system are given. Furthermore, many different types of exact traveling waves are obtained, which include hyperbolic function solution, triangular function solution, rational solution and doubly periodic solutions in terms of the Jacobian elliptic functions. Furthermore, as all parameters in the representations of exact solutions are free variables, the solutions obtained show more complex dynamical behaviors, and could be applicable to explain diversity in qualitative features of wave phenomena.

Keywords: planar dynamical systems; bifurcation; the mixed KdV equation; solitary wave solutions; periodic wave solutions
Mathematics Subject Classification: 35C07, 35C08, 74J35

1. Introduction

The investigation of the exact solutions for nonlinear evolution equations (NEEs) plays an important role in the study of nonlinear physical phenomena. Exact solutions, such as multidimensional transonic shock wave solutions [1], positive ground state solution [2] and single peaked traveling wave solutions [3], can provide us with a deeper understanding of complex physical phenomena. Due to its high degree of nonlinearity, searching for exact solutions of NEEs, and conducting specific research on certain characteristics of the solutions has always been a fundamental and challenging task. In recent years, important process has been made in understanding nonlinear partial differential equations. Various powerful methods have been presented in finding the explicit exact solutions, such as the inverse scattering transformation [4], Bäcklund and Darboux transformations [5], direct integral method [6], algebraic geometric method [7], the Fan sub-equation method [8], the Hirota bilinear method [9], Painlevé analysis [10] and so on. Among them, bifurcation theory of planar dynamical system is a very useful method in seeking for explicit traveling wave solutions, from which different types of exact solutions can be obtained, including the solitary solution, rational function solutions, hyperbolic function solutions, triangle function solutions and Jacobian elliptic function solutions with double periods [11, 12].

The mixed Korteweg-de Vries (KdV) equation is an extension of the nonlinear crystal propagation equation [13]

$$u_t + a_1 u u_x + a_2 u^2 u_x + \beta u_{xxx} = 0,$$

which has the following form

$$u_t + a_0 u_x + a_1 u u_x + a_2 u^2 u_x + \beta u_{xxx} = 0, \tag{1.1}$$

where $u = u(x,t), a_0, a_1, a_2, \beta \in \mathbb{R}$ and $a_0 a_1 a_2 \neq 0, \beta > 0$. It has a broad background in hydrodynamics, plasma physics, ocean dynamics. It models a variety of nonlinear phenomena, including interfacial solitary waves, dust-acoustic solitary waves, ion-acoustic waves in plasmas with a negative ion, and so on. All this time, Scientists have taken a deep interest in the study on KdV and KdV-like equations. By using the theory of planar dynamical systems, Zhang and Bi [14] investigated a compound KdV-type nonlinear wave equation, and obtained the bifurcation boundaries of the system. Khan, Saifullah, Ahamd, et al. [15] studied multiple bifurcation solitons, lumps and rogue wave solutions of the generalized perturbed KdV equation with the Hirota bilinear technique. Notably, Chen and Li [16] considered the generalized KdV-mKdV-like equation

$$u_t + \tilde{\alpha} u_x + \tilde{\beta} u^p u_x + \tilde{\gamma} u^{2p} u_x + u_{xxx} = 0 \tag{1.2}$$

When $p = 1$, Eq (1.2) turns into Eq (1.1) ($\beta = 1$). Chen and Li concentrated on obtaining solitary wave solutions and rational solutions (1-blow-up wave solutions, 2-blow-up wave solutions), but they did not investigate the periodic solutions. On the other hand, Wick-type stochastic KdV equation is also an interesting research subject. Ghany used many tools, such as white noise analysis, Hermite transforms and the modified tanh-coth methd, to obtain some white noise functional solutions for generalized stochastic Hirota-Satsuma coupled KdV equations [17], stochastic space-time fractional KdV equation [18] and stochastic fractional 2D KdV equations [19].

The outline of this paper is organized as follows. In Section 2, phase portraits and bifurcations of the mixed KdV equation are given according to the bifurcation theory of planar dynamical system. In Section 3, the exact representations of bounded traveling wave solutions for (1.1) under different parametric regions are investigated, and many different types of exact solutions are obtained, such as hyperbolic function solution, triangular function solution, rational solution and doubly periodic solutions in terms of the Jacobian elliptic function. At last, a short conclusion is given.

2. Phase portraits and Bifurcations of the mixed KdV equation

Using of the traveling wave transformation $u(x,t) = \phi(\xi)$, $\xi = x - ct$, where c is the wave velocity, (1.1) is reduced to

$$(a_0 - c)\phi' + a_1 \phi \phi' + a_2 \phi^2 \phi' + \beta \phi''' = 0. \tag{2.1}$$

Integrating (2.1) with respect to ξ once and letting the integral constant be zero yields

$$(a_0 - c)\phi + \frac{1}{2}a_1\phi^2 + \frac{1}{3}a_2\phi^3 + \beta\phi'' = 0, \tag{2.2}$$

which is equivalent to

$$\phi'' = b_1\phi + b_2\phi^2 + b_3\phi^3, \tag{2.3}$$

where $b_1 = -\frac{(a_0-c)}{\beta}$, $b_2 = -\frac{a_1}{2\beta}$, $b_3 = -\frac{a_2}{3\beta}$ and easily find that $b_2b_3 \neq 0$.

Furthermore, letting $\phi' = \frac{d\phi}{d\xi} = y$, then (2.3) is equivalent to the following Hamiltonian system

$$\frac{d\phi}{d\xi} = y, \frac{dy}{d\xi} = b_1\phi + b_2\phi^2 + b_3\phi^3 = \phi(b_1 + b_2\phi + b_3\phi^2), \tag{2.4}$$

which has the Hamiltonian function

$$H(\phi, y) = y^2 - (b_1\phi^2 + \frac{2}{3}b_2\phi^3 + \frac{1}{2}b_3\phi^4) = h, \tag{2.5}$$

where h is the Hamiltonian constant. Hamiltonian function represents a family of orbits with different phase diagrams, which are determined by parameters $h, b_i, i = 1, 2, 3$.

Notice that the invariance of (2.4) under the transformation $\phi \to -\phi$, $y \to -y$, $b_2 \to -b_2$ enable us just consider the case $b_2 > 0$.

According to the bifurcation theory of planar dynamical systems [20–22], we have the following propositions on the distribution of the equilibrium points of (2.4).

Proposition 2.1. Suppose that $b_3 \neq 0$, then

(2.1a) For $\Delta = b_2^2 - 4b_1b_3 > 0$, $b_1 \neq 0$, (2.4) has three equilibria at $E_1(\phi_1^*, 0)$, $E_2(\phi_2^*, 0)$ and $E_3(\phi_3^*, 0)$, where $\phi_1^* = 0$, $\phi_2^* = \frac{-b_2+\sqrt{\Delta}}{2b_3}$, $\phi_3^* = \frac{-b_2-\sqrt{\Delta}}{2b_3}$.

(2.1b) For $\Delta > 0$, $b_1 = 0$, (2.4) has two equilibria at $E_1(\phi_1^*, 0)$, $E_4(\phi_4^*, 0)$, where $\phi_4^* = -\frac{b_2}{b_3}$.

(2.1c) For $\Delta = 0$, (2.4) has two equilibria at $E_1(\phi_1^*, 0)$ and $E_5(\phi_5^*, 0)$, where $\phi_5^* = -\frac{b_2}{2b_3}$.

(2.1d) For $\Delta < 0$, (2.4) has a unique equilibrium point at $E_1(\phi_1^*, 0)$.

Proof. Obviously, all the equilibrium points of (2.4) lie in the ϕ-axis and their abscissas are the real zeros of $f(\phi) = \phi(b_1 + b_2\phi + b_3\phi^2)$.

Proposition 2.2. Suppose that $b_3 > 0$, then

(2.2a) For $\Delta > 0$, E_1, E_3 are both saddles, and E_2 is a center for $b_1 > 0$, while E_2, E_3 are both saddles, and E_1 is a center for $b_1 < 0$.

(2.2b) For $\Delta > 0$, $b_1 = 0$, E_1 is a cusp, and E_4 is a saddle.

(2.2c) For $\Delta = 0$, E_1 is a saddle for $b_1 > 0$ and a center for $b_1 < 0$, while E_5 is a cusp.

(2.2d) For $\Delta < 0$, E_1 is a saddle for $b_1 > 0$ and a center for $b_1 < 0$.

Proof. According to the Hamiltonian system (2.4), let $E(\phi_e, 0)$ be an equilibrium point of (2.4) and $M(\phi_e, 0)$ be the coefficient matrix of the linearized system of (2.4) at the equilibrium point $E(\phi_e, 0)$. We have

$$M(\phi_e, 0) = \begin{pmatrix} 0 & 1 \\ b_1 + b_2\phi_e + b_3\phi_e^2 & 0 \end{pmatrix},$$

and

$$J(\phi_e, 0) = \det M(\phi_e, 0) = \begin{cases} -b_1, & \phi_e = \phi_1^* \\ \frac{\sqrt{\Delta}(b_2 - \sqrt{\Delta})}{2b_3}, & \phi_e = \phi_2^* \\ \frac{-\sqrt{\Delta}(b_2 + \sqrt{\Delta})}{2b_3}, & \phi_e = \phi_3^* \\ \frac{-b_2^2}{b_3}, & \phi_e = \phi_4^* \\ 0, & \phi_e = \phi_5^* \end{cases}.$$

By the bifurcation theory of planar dynamical systems, the equilibrium $E(\phi_e, 0)$ of the Hamiltonian system is a center (saddle) if $J(\phi_e, 0) > 0 (< 0)$, and a cusp if $J(\phi_e, 0) = 0$, then we have the proposition above.

Proposition 2.3. Suppose that $b_3 < 0$, then

(2.3a) For $\Delta > 0$, E_1 is a saddle, and E_2, E_3 are centers for $b_1 > 0$, while E_1, E_3 are both centers, and E_2 is a saddle for $b_1 < 0$.

(2.3b) For $\Delta > 0$, $b_1 = 0$, E_1 is a cusp, and E_4 is a center.

(2.3c) For $\Delta = 0$, E_1 is a saddle for $b_1 > 0$ and a center for $b_1 < 0$, while E_5 is a cusp.

(2.3d) For $\Delta < 0$, E_1 is a saddle for $b_1 > 0$ and a center for $b_1 < 0$.

Proof. The proof is similar to Propisition 2.2, we omit it here.

Using the qualitative analysis above, we can obtain the bifurcation curves and phase portraits under various parameter conditions shown as follows.

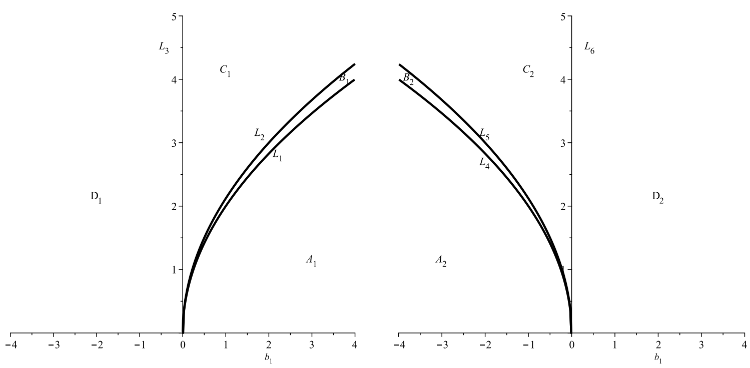
Case (I): For $b_3 > 0$, there are three bifurcation curves (Figure 1a).

$$L_1 : b_1 = \frac{b_2^2}{4b_3},$$
$$L_2 : b_1 = \frac{2b_2^2}{9b_3},$$
$$L_3 : b_1 = 0, b_2 > 0,$$

which separate the upper half (b_1, b_2)-plane into four subregions

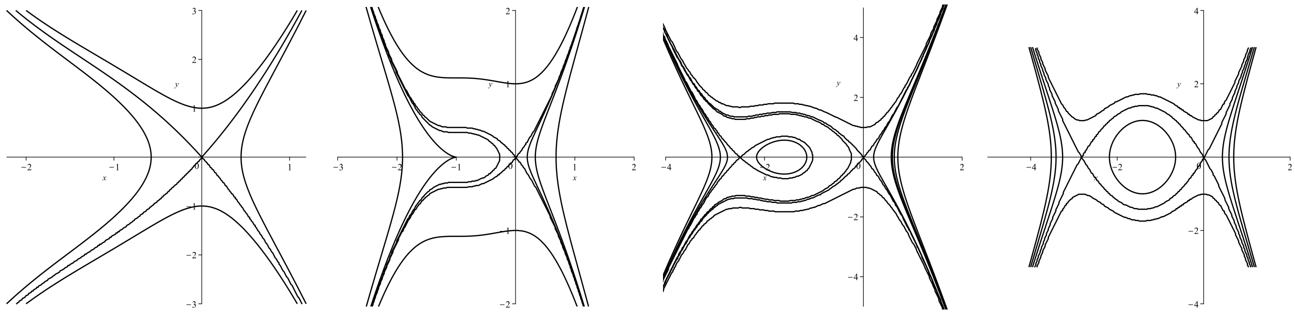
$$A_1 : 0 < b_2 < 2\sqrt{b_1 b_3}, b_1 > 0,$$
$$B_1 : 2\sqrt{b_1 b_3} < b_2 < \frac{3\sqrt{2b_1 b_3}}{2}, b_1 > 0,$$
$$C_1 : b_2 > \frac{3\sqrt{2b_1 b_3}}{2}, b_1 > 0,$$
$$D_1 : b_2 > 0, b_1 < 0.$$

The phase portraits of (2.4) are shown in Figure 2.

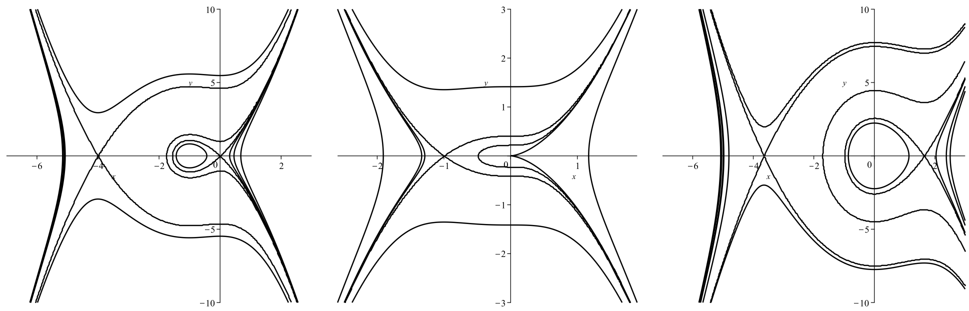


(a) $b_3 > 0$ (b) $b_3 < 0$

Figure 1. The bifurcation set of (2.4) in (b_1, b_2)-parameter plane.



(a) $(b_1, b_2) \in A_1$ **(b)** $(b_1, b_2) \in L_1$ **(c)** $(b_1, b_2) \in B_1$ **(d)** $(b_1, b_2) \in L_2$



(e) $(b_1, b_2) \in C_1$ **(f)** $(b_1, b_2) \in L_3$ **(g)** $(b_1, b_2) \in D_1$

Figure 2. The phase portraits of (2.4) under the condition that $b_3 > 0$.

Case (II): For $b_3 < 0$, there are another three bifurcation curves (Figure 1b).

$$\begin{aligned} L_4 &: b_1 = \frac{b_2^2}{4b_3}, \\ L_5 &: b_1 = \frac{2b_2^2}{9b_3}, \\ L_6 &: b_1 = 0, b_2 > 0, \end{aligned}$$

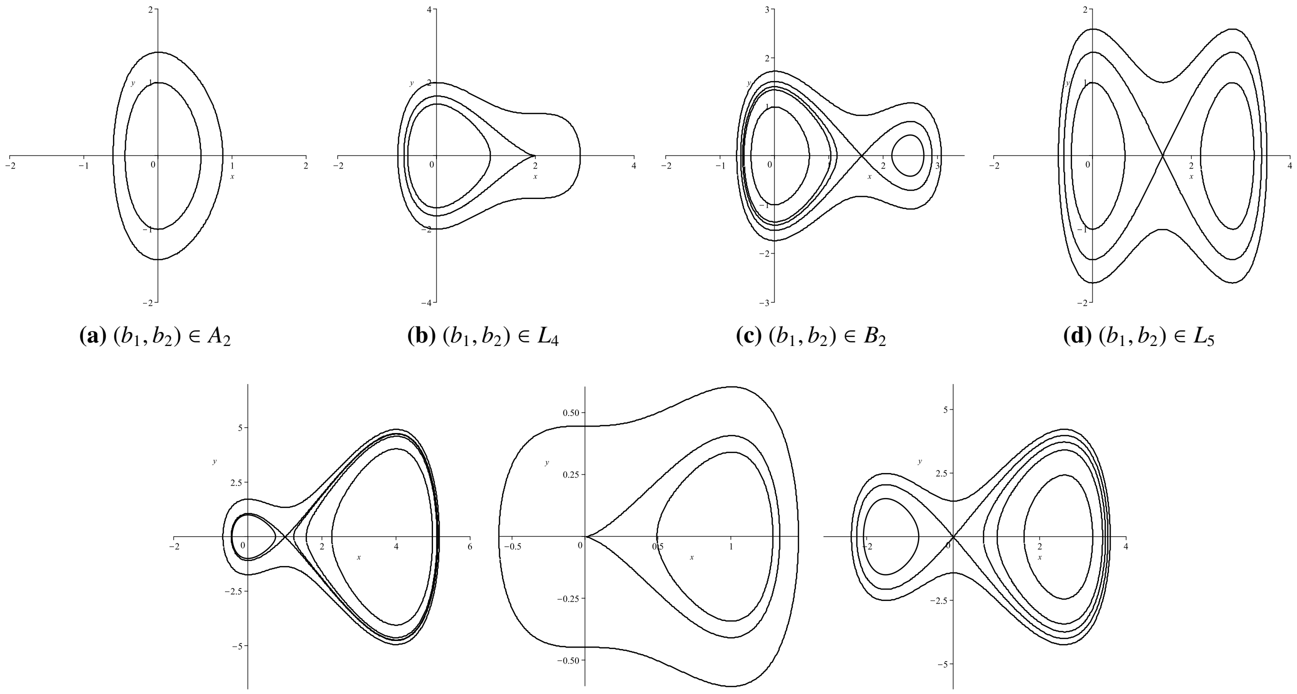
which separate the upper half (b_1, b_2)-plane into four subregions

$$\begin{aligned} A_2 &: 0 < b_2 < 2\sqrt{b_1 b_3}, b_1 < 0, \\ B_2 &: 2\sqrt{b_1 b_3} < b_2 < \frac{3\sqrt{2b_1 b_3}}{2}, b_1 < 0, \\ C_2 &: b_2 > \frac{3\sqrt{2b_1 b_3}}{2}, b_1 < 0, \\ D_2 &: b_2 > 0, b_1 > 0. \end{aligned}$$

The phase portraits of (2.4) are shown in Figure 3. According to the bifurcation theory of planar dynamical systems, a homoclinic (heteroclinic) orbit corresponds to a solitary (kink) wave solution, while a periodic orbit corresponds to a periodic wave solution. Obviously, there are three homoclinic orbits in Figure 2(c), (e) and (g), respectively. Two heteroclinic orbits intersect in Figure 2(d), and there are infinite periodic orbits in Figure 3, which means that there are infinitely many periodic wave solutions in the system (2.4).

3. Exact solutions of the mixed KdV equation determined by the phase portraits

In this section, we search for various types of traveling wave solutions based on the phase diagrams of corresponding Hamiltonian systems.



(a) $(b_1, b_2) \in A_2$ **(b)** $(b_1, b_2) \in L_4$ **(c)** $(b_1, b_2) \in B_2$ **(d)** $(b_1, b_2) \in L_5$

(e) $(b_1, b_2) \in C_2$ **(f)** $(b_1, b_2) \in L_6$ **(g)** $(b_1, b_2) \in D_2$

Figure 3. The phase portraits of (2.4) under the condition that $b_3 < 0$.

Denote that $h_i = H(\phi_i^*, 0), i = 1, 2, 3, 4, 5$, easily find that

$$h_1 = 0, h_2 = -\frac{\left(-b_2+\sqrt{\Delta}\right)^2\left(b_2\sqrt{\Delta}+6b_1b_3-b_2^2\right)}{48b_3^3}, h_3 = \frac{\left(b_2+\sqrt{\Delta}\right)^2\left(b_2\sqrt{\Delta}-6b_1b_3+b_2^2\right)}{48b_3^3}, h_4 = \frac{b_2^4}{6b_3^3}, h_5 = \frac{-b_1^2}{6b_3}.$$

Case (I): $b_3 > 0$.

(1) When $(b_1, b_2) \in A_1 \bigcup L_1 \bigcup B_1$ (Figure 2(a)–(c)), for $h = h_1 = 0$, from the first equation in (2.4), we have

$$\int_{-\infty}^{\phi} \frac{d\phi}{\phi\sqrt{(\phi + \frac{2b_2}{3b_3})^2 + \frac{18b_1b_3-4b_2^2}{9b_3^2}}} = \pm\int_0^{\xi} \sqrt{\frac{b_3}{2}}d\xi. \tag{3.1}$$

Thus we obtain two unbounded solutions to (2.4)

$$\phi_1^{\pm} = \frac{6b_1b_2 \pm 3b_1\sqrt{18b_1b_3 - 4b_2^2}\sinh(\sqrt{b_1}\xi)}{(9b_1b_3 - 2b_2^2)\cosh(\sqrt{b_1}\xi)^2 - 9b_1b_3}. \tag{3.2}$$

(2) When $(b_1, b_2) \in L_1$ (Figure 2(b)), for $h = h_5$, we obtain a unbounded solution to (2.4)

$$\phi_2 = -\sqrt{\frac{b_1}{b_3}}\frac{2b_1\xi^2 + 3}{2b_1\xi^2 - 9}. \tag{3.3}$$

(3) When $(b_1, b_2) \in B_1$ (Figure 2(c)), for $h = h_3$, it can be observed that there are two independent orbits (a homoclinic orbit to saddle point E_3 and a special orbit) with three intersections with the ϕ-axis, so (2.5) can be written in the following form

$$y^2 = \frac{b_3}{2}(\phi - \phi_3^*)^2(\varphi_2 - \phi)(\varphi_3 - \phi), \tag{3.4}$$

where $\phi_3^* < \varphi_2 < \varphi_3$. Then

$$\int_{\phi_3^*}^{\phi} \frac{d\phi}{(\phi - \phi_3^*)\sqrt{(\varphi_2 - \phi)(\varphi_3 - \phi)}} = \int_0^{\xi} \sqrt{\frac{b_3}{2}} d\xi. \tag{3.5}$$

We obtain a smooth soliton solution with peak form

$$\phi_3 = \phi_3^* + \frac{4\nu_1 \exp(\sqrt{\frac{b_3\nu_1}{2}}\xi)}{2\mu_1 \exp(\sqrt{\frac{b_3\nu_3}{2}}\xi) + \exp(\sqrt{2b_3\nu_1}\xi) + (\varphi_2 - \varphi_3)^2}, \tag{3.6}$$

where $\mu_1 = \varphi_2 + \varphi_3 - 2\phi_3^*$, $\nu_1 = (\varphi_2 - \phi_3^*)(\varphi_3 - \phi_3^*)$.

(4) When $(b_1, b_2) \in L_2$ (Figure 2(d)), we have

$$b_2^2 = \frac{9}{2}b_1 b_3, h_2 = -\frac{b_1^2}{8b_3}, h_1 = h_3 = 0. \tag{3.7}$$

(4.1) For $h = h_1 = 0$, it's clearly that there are two heteroclinic orbits connecting saddles E_1 with E_3, then (2.5) can be written as following form

$$y^2 = \frac{b_3}{2}\phi^2(\phi + \sqrt{\frac{2b_1}{b_3}})^2. \tag{3.8}$$

Then

$$\int_{\phi}^{0} \frac{d\phi}{\sqrt{\phi^2(\phi + \sqrt{\frac{2b_1}{b_3}})^2}} = \pm \int_{\xi}^{0} \sqrt{\frac{b_3}{2}} d\xi. \tag{3.9}$$

We obtain two smooth kink wave soliton solutions

$$\phi_4^{\pm} = -\sqrt{\frac{b_1}{2b_3}}(1 \pm \tanh(\frac{\sqrt{b_1}}{2}\xi)). \tag{3.10}$$

(4.2) For $h \in (h_2, 0)$, we have

$$y^2 = \frac{b_3}{2}[(\phi + \sqrt{\frac{b_1}{2b_3}})^2 - \frac{b_1 - \sqrt{-8hb_3}}{2b_3}][(\phi + \sqrt{\frac{b_1}{2b_3}})^2 - \frac{b_1 + \sqrt{-8hb_3}}{2b_3}]. \tag{3.11}$$

Then

$$\int_0^{\xi} \sqrt{\frac{b_3}{2}} d\xi = \int_{-B}^{\phi} \frac{d(\frac{\tilde{\phi}}{B})}{A\sqrt{(1 - (\frac{\tilde{\phi}}{B})^2)(1 - (\frac{B}{A})^2(\frac{\tilde{\phi}}{B})^2)}}, \tag{3.12}$$

where $A = \sqrt{\frac{\nu_2}{2b_3}}$, $B = \sqrt{\frac{\mu_2}{2b_3}}$, $\mu_2 = b_1 - \sqrt{-8hb_3}$, $\nu_2 = b_1 + \sqrt{-8hb_3}$ and $\tilde{\phi} = \phi + \sqrt{\frac{b_1}{2b_3}}$. Then we obtain a family of doubly periodic solutions

$$\phi_6 = -\sqrt{\frac{b_1}{2b_3}} + \sqrt{\frac{\mu_2}{2b_3}}\mathrm{sn}(\frac{\sqrt{\nu_2}}{2}\xi, \sqrt{\frac{\mu_2}{\nu_2}}), \tag{3.13}$$

where $\mathrm{sn}(x,k)$ and below $\mathrm{cn}(x,k)$, $\mathrm{dn}(x,k)$ are Jacobian elliptic functions with modulus $k \in (0,1)$.

(5) For $(b_1, b_2) \in C_1$ (Figure 2(e)) and $h = h_1 = 0$, we have

$$y^2 = \frac{b_3}{2}\phi^2(\phi + \frac{2b_2 + \sqrt{4b_2^2 - 18b_1b_3}}{3b_3})(\phi + \frac{2b_2 - \sqrt{4b_2^2 - 18b_1b_3}}{3b_3}). \tag{3.14}$$

Then we obtain two smooth soliton solutions

$$\phi_7^{\pm} = \frac{3b_1(-2b_2 \pm \sqrt{4b_2^2 - 18b_1b_3}\sinh(\sqrt{b_1}\xi))}{(2b_2^2 - 9b_1b_3)\cosh(\sqrt{b_1}\xi)^2 + 9b_1b_3}. \tag{3.15}$$

(6) For $(b_1, b_2) \in L_3$ (Figure 2(f)) and $h = h_1 = 0$, we obtain an unbounded solution

$$\phi_8 = \frac{12b_2}{2b_2^2\xi^2 - 9b_3}. \tag{3.16}$$

(7) For $(b_1, b_2) \in D_1$ (Figure 2(g)) and $h = h_2$, it can be observed that there are two independent orbits (a homoclinic orbit to saddle point E_2 and a special orbit) with three intersections with the ϕ-axis, so (2.5) can be written in the following form

$$y^2 = \frac{b_3}{2}(\phi_2^* - \phi)^2(\phi - \psi_2)(\phi - \psi_3), \tag{3.17}$$

where $\psi_3 < \psi_2 < \phi_2^*$. Then

$$\int_{\psi_2}^{\phi} \frac{d\phi}{(\phi_2^* - \phi)\sqrt{(\phi - \psi_2)(\phi - \psi_3)}} = \int_0^{\xi} \sqrt{\frac{b_3}{2}}d\xi. \tag{3.18}$$

We obtain a smooth soliton solution with peak form

$$\phi_9 = \phi_2^* - \frac{4\nu_3\exp(\sqrt{\frac{b_3\nu_3}{2}}\xi)}{2\mu_3\exp(\sqrt{\frac{b_3\nu_3}{2}}\xi) + \exp(\sqrt{2b_3\nu_3}\xi) + (\psi_2 - \psi_3)^2}, \tag{3.19}$$

where $\mu_3 = \psi_2 + \psi_3 - 2\phi_2^*$, $\nu_3 = (\phi_2^* - \psi_2)(\phi_2^* - \psi_3)$.

Case (II): $b_3 < 0$.

(1) When $(b_1, b_2) \in L_4$ (Figure 3(b)), for $h = h_5 = -\frac{b_1^2}{6b_3}$, (2.4) has a soliton solution with valley form

$$\phi_{10} = -\sqrt{\frac{b_1}{b_3}}\frac{2b_1\xi^2 + 3}{2b_1\xi^2 - 9}. \tag{3.20}$$

(2) When $(b_1, b_2) \in L_5$ (Figure 3(d)), where $b_2 = \frac{3\sqrt{2b_1b_3}}{2}$, $b_1 < 0$ and

$$\phi_2^* = -\sqrt{\frac{b_1}{2b_3}}, \phi_3^* = -\sqrt{\frac{2b_1}{b_3}}, h_2 = -\frac{b_1^2}{8b_3}, h_3 = 0. \tag{3.21}$$

(2.1) For $h = h_2$, we have

$$y^2 = -\frac{b_3}{2}(\phi - \phi_2^*)^2(-\phi^2 - \sqrt{\frac{2b_1}{b_3}}\phi + \frac{b_1}{2b_3}), \tag{3.22}$$

then there are two soliton solutions with peak form and valley form, respectively

$$\phi_{11}^{\pm} = \sqrt{\frac{b_1}{2b_3}}(1 \pm \sqrt{2}\mathrm{sech}(\sqrt{\frac{-b_1}{2}}\xi)). \tag{3.23}$$

(2.2) For $h \in (0, h_2)$, we have

$$y^2 = -\frac{b_3}{2}(\frac{b_1 - \sqrt{-8hb_3}}{2b_3} - (\phi + \sqrt{\frac{b_1}{2b_3}})^2)((\phi + \sqrt{\frac{b_1}{2b_3}})^2 - \frac{b_1 + \sqrt{-8hb_3}}{2b_3}). \tag{3.24}$$

Then

$$\int_0^{\xi} \sqrt{-\frac{b_3}{2}}d\xi = \int_{-\tilde{B}}^{\phi} \frac{d(\frac{\tilde{\phi}}{\tilde{B}})}{\tilde{B}\sqrt{(1 - (\frac{\tilde{\phi}}{\tilde{B}})^2)((\frac{\tilde{\phi}}{\tilde{B}})^2 - (\frac{\tilde{A}}{\tilde{B}})^2)}}, \tag{3.25}$$

where $\tilde{A} = \sqrt{\frac{\nu_4}{2b_3}}$, $\tilde{B} = \sqrt{\frac{-\mu_4}{2b_3}}$, $\mu_4 = -b_1 + \sqrt{-8hb_3}$, $\nu_4 = b_1 + \sqrt{-8hb_3}$ and $\tilde{\phi} = \phi + \sqrt{\frac{b_1}{2b_3}}$.

Then with the help of Maple, we get a family of doubly periodic solutions

$$\phi_{12} = -\sqrt{\frac{b_1}{2b_3}} + \sqrt{\frac{\mu_4}{-2b_3}}\mathrm{dn}(\frac{\sqrt{\mu_4}}{2}\xi, \sqrt{\frac{\mu_4 + \nu_4}{\mu_4}}). \tag{3.26}$$

(2.3) For $h > h_2$, we have

$$y^2 = -\frac{b_3}{2}(\tilde{B}^2 - \tilde{\phi}^2)(\tilde{\phi}^2 + \tilde{C}^2), \tag{3.27}$$

where $\tilde{C} = \sqrt{\frac{-\nu_4}{2b_3}}$.

Then

$$\int_0^{\xi} \sqrt{-\frac{b_3}{2}}\sqrt{\tilde{B}^2 + \tilde{C}^2}d\xi = \int_{-\tilde{B}}^{\phi} \frac{d(\frac{\tilde{\phi}}{\tilde{B}})}{\frac{\tilde{B}}{\sqrt{\tilde{B}^2+\tilde{C}^2}}\sqrt{(1 - (\frac{\tilde{\phi}}{\tilde{B}})^2)((\frac{\tilde{\phi}}{\tilde{B}})^2 + (\frac{\tilde{C}}{\tilde{B}})^2)}}. \tag{3.28}$$

Then with the help of Maple, we get a family of doubly periodic solutions

$$\phi_{13} = -\sqrt{\frac{b_1}{2b_3}} + \sqrt{\frac{\mu_4}{-2b_3}}\mathrm{cn}(\frac{\sqrt{\mu_4 + \nu_4}}{2}\xi, \sqrt{\frac{\mu_4}{\mu_4 + \nu_4}}). \tag{3.29}$$

(3) When $(b_1, b_2) \in B_2 \bigcup C_2$ (Figure 3(c)), for $h = h_2$, it can be observed that there is a closed homoclinic orbits to saddle point E_2 with three intersections with the ϕ-axis, so (2.5) can be written in the following form

$$y^2 = -\frac{b_3}{2}(\phi - \chi_1)(\phi - \phi_2^*)^2(\chi_3 - \phi), \tag{3.30}$$

where $\chi_1 < \phi_2^* < \chi_3$. Then we obtain two smooth soliton solutions with peak form

$$\phi_{14}^{\pm} = \phi_2^* - \frac{2\nu_5(\mu_5 \pm (\chi_1 - \chi_3)\sinh(\sqrt{\frac{-b_3\nu_5}{2}}\xi))}{(\chi_1 - \chi_3)^2\cosh(\sqrt{\frac{-b_3\nu_5}{2}}\xi)^2 - 4\nu_5}, \tag{3.31}$$

where $\mu_5 = \chi_1 + \chi_3 - 2\phi_2^*$, $\nu_5 = (\chi_2 - \phi_2^*)(\phi_2^* - \chi_1)$.

(4) When $(b_1, b_2) \in L_6$ (Figure 3(f)), for $h = h_1 = 0$, we have a soliton solution

$$\phi_{15} = \frac{12b_2}{2b_2^2\xi^2 - 9b_3}. \tag{3.32}$$

(5) For $(b_1, b_2) \in D_2$ (Figure 3(g)), for $h = h_1 = 0$, we have

$$y^2 = -\frac{b_3}{2}\phi^2\left(\phi - \frac{2b_2 - \sqrt{4b_2^2 - 18b_1b_3}}{-3b_3}\right)\left(\frac{2b_2 + \sqrt{4b_2^2 - 18b_1b_3}}{-3b_3} - \phi\right). \tag{3.33}$$

Then we obtain two soliton solutions

$$\phi_{16}^{\pm} = \frac{3b_1(-2b_2 \pm \sqrt{4b_2^2 - 18b_1b_3}\sinh(\sqrt{b_1}\xi))}{(2b_2^2 - 9b_1b_3)\cosh(\sqrt{b_1}\xi)^2 + 9b_1b_3}. \tag{3.34}$$

4. Conclusions

In this work, all bifurcations of phase portraits in different subregions for the mixed KdV equation are studied using the approach of dynamical systems. Many different types of traveling wave solutions are obtained with the aid of Maple, such as hyperbolic function solution, triangular function solution, rational solution and Jacobian elliptic function solution with double periods. Moreover, these dynamical behaviors can provide us with a deeper understanding of complex physical phenomena.

Use of AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

This work does not have any conflicts of interest.

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