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Research article

Hopf bifurcation exploration and control technique in a predator-prey system incorporating delay

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Abstract: Recently, delayed dynamical model has witnessed a great interest from many scholars in biological and mathematical areas due to its potential application in describing the interaction of different biological populations. In this article, relying the previous studies, we set up two new predator-prey systems incorporating delay. By virtue of fixed point theory, inequality tactics and an appropriate function, we explore well-posedness (includes existence and uniqueness, boundedness and non-negativeness) of the solution of the two formulated delayed predator-prey systems. With the aid of bifurcation theorem and stability theory of delayed differential equations, we gain the parameter conditions on the emergence of stability and bifurcation phenomenon of the two formulated delayed predator-prey systems. By applying two controllers (hybrid controller and extended delayed feedback controller) we can efficaciously regulate the region of stability and the time of occurrence of bifurcation phenomenon for the two delayed predator-prey systems. The effect of delay on stabilizing the system and adjusting bifurcation is investigated. Computer simulation plots are provided to sustain the acquired prime outcomes. The conclusions of this article are completely new and can provide some momentous instructions in dominating and balancing the densities of predator and prey.

Keywords: predator-prey system; well-posedness; Hopf bifurcation; stability; hybrid controller; delay
Mathematics Subject Classification: 34C23, 34K18, 37GK15, 39A11, 92B20

1. Introduction

As is known to us, predator-prey models play a vital role in describing the interaction between predator population and prey population in real natural world. In order to expose the internal change process and development law of predator population and prey population, a great deal of predator-prey models have been established. Through the discussion on predator-prey models, we can find the impact of parameters on the biological population densities in some specific environment. During the past decades, a lot of work on predator-prey models has been carried out and abundant fruits have been resulted. For example, Balc [1] explored the stability, well-posedness, and bifurcation issue of a fractional prey-predator model. Pandey et al. [2] explored the rich dynamics (e.g., transcritical, saddle-node, Hopf-bifurcation, etc.) of a delayed predator-prey model. Rao and Kang [3] established the conditions for the existence of a unique ergodic stationary distribution and the extinction conditions of predator species and prey species for a stochastic predator-prey model. Sarkar and Khajanchi [4] dealt with the spatiotemporal dynamical trait of a prey-predator model involving the fear effect. For more concrete examples, one can see [5–8].

In 2020, Sen et al. [9] formulated the following predator-prey system:

$$\begin{cases} \dfrac{du_1(t)}{dt} = u_1(t)(h_1 - a_1u_1(t)) - \dfrac{d_1u_1(t)u_2(t)}{1 + bu_1(t)}, \\ \dfrac{du_2(t)}{dt} = u_2(t)[h_2 - a_2u_2(t)] + \dfrac{dd_1u_1(t)u_2(t)}{1 + bu_1(t)}, \end{cases} \tag{1.1}$$

where $u_1(t)$ stands for the density of prey at time t and $u_2(t)$ stands for the density of predator at time t, h_1 is the intrinsic growth rate of prey and h_2 is the intrinsic growth rate of predator, a_1 denotes the intra-species competition of prey and a_2 denotes the intra-species competition of predator, b denotes the handling parameter, which is the product of the handling time and the searching efficiency, $d > 0$ is the conversion efficiency and d_1 represents the searching efficiency by an individual predator per unit time. All other parameters are positive real numbers. In details, one can see [9–12].

In many cases, the densities of prey and predator are affected due to the time delay of population development, then it is necessary to introduce the delay into the predator-prey models. Based on this viewpoint, we can establish more suitable delayed predator-prey models. Assume that the density of prey is affected by the self feedback time from u_1 to u_1, then we can modify model (1.1) as follows:

$$\begin{cases} \dfrac{du_1(t)}{dt} = u_1(t)(h_1 - a_1u_1(t)) - \dfrac{d_1u_1(t)u_2(t)}{1 + bu_1(t)}, \\ \dfrac{du_2(t)}{dt} = u_2(t)[h_2 - a_2u_2(t - \delta)] + \dfrac{dd_1u_1(t)u_2(t)}{1 + bu_1(t)}, \end{cases} \tag{1.2}$$

where $\delta > 0$ is a time delay. All other parameters are positive real numbers. Assume that the density of prey is affected by the self feedback time from u_1 to u_1 and the density of predator is affected by the self feedback time from u_2 to u_2, then we can modify model (1.1) as follows:

$$\begin{cases} \dfrac{du_1(t)}{dt} = u_1(t)(h_1 - a_1u_1(t-\delta)) - \dfrac{d_1u_1(t)u_2(t)}{1+bu_1(t)}, \\ \dfrac{du_2(t)}{dt} = u_2(t)[h_2 - a_2u_2(t-\delta)] + \dfrac{dd_1u_1(t)u_2(t)}{1+bu_1(t)}, \end{cases} \tag{1.3}$$

where $\delta > 0$ is a time delay. All other parameters are positive real numbers.

Many studies show that delay is often a vital factor that affects the dynamical behavior of the delayed dynamical model. In many instances, delay will make the system lose its stability, produce periodic vibration, generate chaotic behavior and so on [13–22]. In particular, delay-induced Hopf bifurcation is an important dynamical peculiarity. Biologically, delay-induced Hopf bifurcation plays a vital role in describing the balanced relationship among the concentrations of numerous biological populations. In the light of this viewpoint, we argue that exploring the delay-induced Hopf bifurcation in abundant predator-prey models has very important theoretical significance. Inspired by the above idea, we are going to investigate the delay-induced Hopf bifurcation and control of bifurcation for models (1.2) and (1.3). Specifically, we are to deal with the following three core points: (1) Study the well-posedness (e.g., non-negativeness, boundedness, existence and uniqueness) of solution to models (1.2) and (1.3). (2) Explore the emergence of Hopf bifurcation and stability of models (1.2) and (1.3). (3) Construct two different controllers to control the region of stability and the time of generation of bifurcation behavior of models (1.2) and (1.3).

The key highlights of this study are stated as follows: (I) Depending on the previous studies, a new bifurcation and stability criterion without relying on time delay for model (1.2) is built. (II) By virtue of two different controllers, the domain of stability and the time of generation of Hopf bifurcation of models (1.2) and (1.3) are effectively under control. (III) The impact of time delay on dominating Hopf bifurcation phenomenon and stabilizing the densities of predators and preys of models (1.2) and (1.3) is presented.

This structure of this article is presented as follows: The well-posedness involving existence and uniqueness, non-negativeness and boundedness of the solution of system (1.2) is discussed in Section 2. Section 3 explores the bifurcation phenomenon and stability nature of system (1.2). Section 4 focuses on the control problem of bifurcation phenomenon for system (1.2) by virtue of a reasonable hybrid controller incorporating state feedback and parameter perturbation involving delay. Section 5 handles the control problem of bifurcation phenomenon and stability for system (1.3). Section 6 handles the control problem of bifurcation phenomenon for system (1.3) by virtue of a reasonable hybrid controller incorporating state feedback and parameter perturbation involving delay. Section 7 carries out numerical experiments to verify the rationality of the acquired key outcomes. A brief conclusion is included to finish this work in Section 8.

2. Well-posedness

In this part, we are going to explore the well-posedness of solutions to model (1.2) and model (1.3) (include boundedness, existence and uniqueness, non-negativeness) via making use of fixed point theory, inequality technique and construction of a reasonable function.

Theorem 2.1. *Denote* $\Psi = \{u_1, u_2 \in R^2 : \max\{|u_1|, |u_2|\} \le \mathcal{U}\}$*, where* $\mathcal{U} > 0$ *denotes a constant. For each* $(u_{10}, u_{20}) \in \Psi$*, system (1.2) under the initial state* (u_{10}, u_{20}) *owns a unique solution* $U = (u_1, u_2) \in \Psi$.

Proof. Define the following mapping:

$$f(U) = (f_1(U), f_2(U)), \tag{2.1}$$

where

$$\begin{cases} f_1(U) = u_1(t)(h_1 - a_1u_1(t)) - \dfrac{d_1u_1(t)u_2(t)}{1 + bu_1(t)}, \\ f_2(U) = u_2(t)[h_2 - a_2u_2(t-\delta)] + \dfrac{dd_1u_1(t)u_2(t)}{1 + bu_1(t)}. \end{cases} \tag{2.2}$$

For every $U, \bar{U} \in \Psi$, we can get

$$\begin{aligned}
&\|f(U) - f(\bar{U})\| \\
&= \left|u_1(h_1 - a_1u_1) - \frac{d_1u_1u_2}{1 + bu_1} - \left[\bar{u}_1(h_1 - a_1\bar{u}_1) - \frac{d_1\bar{u}_1\bar{u}_2}{1 + b\bar{u}_1}\right]\right| \\
&\quad + \left|u_2[h_2 - a_2u_2(t-\delta)] + \frac{dd_1u_1u_2}{1 + bu_1} - \left[\bar{u}_2[h_2 - a_2\bar{u}_2(t-\delta)] + \frac{dd_1\bar{u}_1\bar{u}_2}{1 + b\bar{u}_1}\right]\right| \\
&= \left|u_1h_1 - a_1u_1^2 - \frac{d_1u_1u_2}{1 + bu_1} - \bar{u}_1h_1 + a_1\bar{u}_1^2 + \frac{d_1\bar{u}_1\bar{u}_2}{1 + b\bar{u}_1}\right| \\
&\quad + \left|u_2h_2 - a_2u_2(t)u_2(t-\delta) + \frac{dd_1u_1u_2}{1 + bu_1} - \bar{u}_2h_2 + a_2\bar{u}_2(t)\bar{u}_2(t-\delta) - \frac{dd_1\bar{u}_1\bar{u}_2}{1 + b\bar{u}_1}\right| \\
&= \left|h_1(u_1 - \bar{u}_1) - a_1\left(u_1^2 - \bar{u}_1^2\right) - d_1\left(\frac{u_1u_2}{1 + bu_1} - \frac{\bar{u}_1\bar{u}_2}{1 + b\bar{u}_1}\right)\right| \\
&\quad + \left|h_2(u_2 - \bar{u}_2) - a_2[u_2(t)u_2(t-\delta) - \bar{u}_2(t)\bar{u}_2(t-\delta)] + dd_1\left(\frac{u_1u_2}{1 + bu_1} - \frac{\bar{u}_1\bar{u}_2}{1 + b\bar{u}_1}\right)\right| \\
&\le h_1|u_1 - \bar{u}_1| + 2\mathcal{U}a_1|u_1 - \bar{u}_1| \\
&\quad + d_1\left|\frac{u_1u_2(1 + b\bar{u}_1) - \bar{u}_1\bar{u}_2(1 + bu_1)}{(1 + bu_1)(1 + b\bar{u}_1)}\right| \\
&\quad + h_2|u_2 - \bar{u}_2| + 2\mathcal{U}a_2|u_2 - \bar{u}_2| \\
&\quad + dd_1\left|\frac{u_1u_2(1 + b\bar{u}_1) - \bar{u}_1\bar{u}_2(1 + bu_1)}{(1 + bu_1)(1 + b\bar{u}_1)}\right| \\
&\le (h_1 + 2\mathcal{U}a_1)|u_1 - \bar{u}_1| + d_1|u_1u_2(1 + b\bar{u}_1) - \bar{u}_1\bar{u}_2(1 + bu_1)| \\
&\quad + (h_2 + 2\mathcal{U}a_2)|u_2 - \bar{u}_2| + dd_1|u_1u_2(1 + b\bar{u}_1) - \bar{u}_1\bar{u}_2(1 + bu_1)| \\
&= (h_1 + 2\mathcal{U}a_1)|u_1 - \bar{u}_1| + (h_2 + 2\mathcal{U}a_2)|u_2 - \bar{u}_2| \\
&\quad + d_1(1 + d)|u_1u_2(1 + b\bar{u}_1) - \bar{u}_1u_2 + \bar{u}_1u_2 - \bar{u}_1\bar{u}_2 + bu_1\bar{u}_1(u_2 - \bar{u}_2)| \\
&\le (h_1 + 2\mathcal{U}a_1)|u_1 - \bar{u}_1| + (h_2 + 2\mathcal{U}a_2)|u_2 - \bar{u}_2| \\
&\quad + d_1(1 + d)|\mathcal{U}(u_1 - \bar{u}_1)| \\
&\quad + d_1(1 + d)|\mathcal{U}(u_2 - \bar{u}_2)| \\
&\quad + d_1(1 + d)\left|b\mathcal{U}^2(u_2 - \bar{u}_2)\right|
\end{aligned}$$

$$
\begin{aligned}
&= [h_1 + 2\mathcal{U}a_1 + d_1(1+d)\mathcal{U}]\,|u_1 - \bar{u}_1| \\
&\quad + \left[h_2 + 2\mathcal{U}a_2 + d_1(1+d)\left(\mathcal{U} + b\mathcal{U}^2\right)\right]|u_2 - \bar{u}_2| \\
&\le \rho\|U - \bar{U}\|,
\end{aligned} \tag{2.3}
$$

where

$$
\rho = \max\left\{h_1 + 2\mathcal{U}a_1 + d_1(1+d)\mathcal{U}, h_2 + 2\mathcal{U}a_2 + d_1(1+d)\left(\mathcal{U} + b\mathcal{U}^2\right)\right\}. \tag{2.4}
$$

Thus $f(U)$ obeys the Lipschitz condition for U. Using fixed point theorem, one can conclude that Theorem 2.1 is right.

Theorem 2.2. *Every solution of system (1.2) starting with R_+^2 is non-negative.*

Proof. In view of the first equation of system (1.2), we can get

$$
\frac{du_1}{dt} = u_1(h_1 - a_1u_1) - \frac{d_1u_1u_2}{1 + bu_1}, \tag{2.5}
$$

then

$$
\frac{du_1}{u_1} = \left(h_1 - a_1u_1 - \frac{d_1u_2}{1 + bu_1}\right)dt, \tag{2.6}
$$

which leads to

$$
\int_0^t \frac{du_1}{u_1} = \int_0^t \left[h_1 - a_1u_1(s) - \frac{d_1u_2(s)}{1 + bu_1(s)}\right]ds, \tag{2.7}
$$

and then one gets

$$
\frac{u_1(t)}{u_1(0)} = \exp\left\{\int_0^t \left[h_1 - a_1u_1(s) - \frac{d_1u_2(s)}{1 + bu_1(s)}\right]ds\right\}. \tag{2.8}
$$

Thus,

$$
u_1(t) = u_1(0)\exp\left\{\int_0^t \left[h_1 - a_1u_1(s) - \frac{d_1u_2(s)}{1 + bu_1(s)}\right]ds\right\} > 0. \tag{2.9}
$$

In a same way, we know

$$
u_2(t) = u_2(0)\exp\left\{\int_0^t \left[h_2 - a_2u_2(s-\delta) + \frac{dd_1u_1(s)}{1 + bu_1(s)}\right]ds\right\} > 0. \tag{2.10}
$$

Thus, Theorem 2.2 is correct.

Theorem 2.3. *The solutions of system (1.2) are uniformly bounded.*

Proof. We consider two cases: $d > 1$ and $0 < d < 1$.

Case 1. If $d > 1$, let $W(t) = u_1(t) + u_2(t)$. Then

$$\begin{aligned}
\frac{dW}{dt} &= \frac{du_1}{dt} + \frac{du_2}{dt} \\
&= u_1(h_1 - a_1u_1) - \frac{d_1u_1u_2}{1+bu_1} + u_2[h_2 - a_2u_2(t-\delta)] + \frac{dd_1u_1u_2}{1+bu_1} \\
&= u_1(h_1 - a_1u_1) + u_2[h_2 - a_2u_2(t-\delta)] - \frac{d_1u_1u_2}{1+bu_1}(1-d) \\
&\leq u_1(h_1 - a_1u_1) + u_2(h_2 - a_2u_2) - \frac{d_1u_1u_2}{bu_1}(1-d) \\
&= u_1h_1 - a_1u_1^2 + u_2h_2 - a_2u_2^2 - \frac{d_1u_2}{b}(1-d) \\
&= -h_1(u_1+u_2) + 2u_1h_1 - a_1u_1^2 + u_2\left[h_1 + h_2 - \frac{d_1}{b}(1-d)\right] - a_2u_2^2 \\
&\leq -h_1(u_1+u_2) + \frac{h_1^2}{a_1} + \frac{\left[h_1 + h_2 - \frac{d_1}{b}(1-d)\right]^2}{4a_2}, \qquad (2.11)
\end{aligned}$$

where

$$\frac{h_1^2}{a_1} = \max_{t\in R^+}\left\{2u_1h_1 - a_1u_1^2\right\},$$

$$\frac{\left[h_1 + h_2 - \frac{d_1}{b}(1-d)\right]^2}{4a_2} = \max_{t\in R^+}\left\{u_2\left[h_1 + h_2 - \frac{d_1}{b}(1-d)\right] - a_2u_2^2\right\}.$$

Let

$$L = \frac{h_1^2}{a_1} + \frac{\left[h_1 + h_2 - \frac{d_1}{b}(1-d)\right]^2}{4a_2}. \qquad (2.12)$$

Then,

$$\frac{dW}{dt} \leq -h_1W + L. \qquad (2.13)$$

According to the differential inequality theorem, we get

$$0 \leq W(t) \leq \frac{L}{h_1}\left(1 - e^{-h_1t}\right) + W(0)e^{-h_1t}, \qquad (2.14)$$

then

$$0 \leq W(t) \leq \frac{L}{h_1}, \quad t \to \infty. \qquad (2.15)$$

Case 2. If $0 < d < 1$, let $W(t) = u_1(t) + u_2(t)$. Then,

$$\begin{aligned}
\frac{dW}{dt} &= \frac{du_1}{dt} + \frac{du_2}{dt} \\
&= u_1(h_1 - a_1u_1) - \frac{d_1u_1u_2}{1+bu_1} + u_2[h_2 - a_2u_2(t-\delta)] + \frac{dd_1u_1u_2}{1+bu_1} \\
&= u_1(h_1 - a_1u_1) + u_2[h_2 - a_2u_2(t-\delta)] + \frac{d_1u_1u_2}{1+bu_1}(d-1)
\end{aligned}$$

$$
\begin{aligned}
&\le u_1(h_1 - a_1u_1) + u_2(h_2 - a_2u_2)\\
&= u_1h_1 - a_1u_1^2 + u_2h_2 - a_2u_2^2\\
&= -h_1(u_1+u_2) + 2u_1h_1 + u_2(h_1+h_2) - a_1u_1^2 - a_2u_2^2\\
&\le -h_1(u_1+u_2) + \frac{h_1^2}{a_1} + \frac{(h_1+h_2)^2}{4a_2}, \qquad (2.16)
\end{aligned}
$$

where

$$
\begin{aligned}
\frac{h_1^2}{a_1} &= \max_{t\in R^+}\left\{2u_1h_1 - a_1u_1^2\right\},\\
\frac{(h_1+h_2)^2}{4a_2} &= \max_{t\in R^+}\left\{u_2(h_1+h_2) - a_2u_2^2\right\}.
\end{aligned}
$$

Let

$$L = \frac{h_1^2}{a_1} + \frac{(h_1+h_2)^2}{4a_2}, \tag{2.17}$$

so

$$\frac{dW}{dt} \le -h_1W + L. \tag{2.18}$$

According to the differential inequality theorem, one gets

$$0 \le W(t) \le \frac{L}{h_1}\left(1 - e^{-h_1t}\right) + W(0)e^{-h_1t}, \tag{2.19}$$

which results in

$$0 \le W(t) \le \frac{L}{h_1}, \quad t \to \infty. \tag{2.20}$$

Based on these two cases, we can conclude that Theorem 2.3 is correct.

3. Exploration of bifurcation of model (1.2)

In this section, we are going to explore the bifurcation and stability issue of model (1.2). Firstly, we assume that $E(u_{1\star}, u_{2\star})$ is the equilibrium point of model (1.2), then $u_{1\star}, u_{2\star}$ obey the following condition:

$$
\begin{cases}
u_{1\star}(h_1 - a_1u_{1\star}) - \dfrac{d_1u_{1\star}u_{2\star}}{1+bu_{1\star}} = 0,\\
u_{2\star}(h_2 - a_2u_{2\star}) + \dfrac{dd_1u_{1\star}u_{2\star}}{1+bu_{1\star}} = 0.
\end{cases} \tag{3.1}
$$

Let

$$
\begin{cases}
\bar{u}_1(t) = u_1(t) - u_{1\star},\\
\bar{u}_2(t) = u_2(t) - u_{2\star}.
\end{cases} \tag{3.2}
$$

Substitute system (3.2) into system (1.2), we gain the linear system of model (1.2) at $E(u_{1\star}, u_{2\star})$:

$$
\begin{cases}
\dfrac{d\bar{u}_1}{dt} = b_1\bar{u}_1 - b_2\bar{u}_2,\\
\dfrac{d\bar{u}_2}{dt} = b_3\bar{u}_1 + b_4\bar{u}_2 - b_5\bar{u}_2(t-\delta),
\end{cases} \tag{3.3}
$$

where

$$\begin{cases} b_1 = h_1 - 2a_1u_{1\star} - \dfrac{d_1u_{2\star}}{1+bu_{1\star}} + \dfrac{bd_1u_{1\star}u_{2\star}}{(1+bu_{1\star})^2}, \\ b_2 = \dfrac{d_1u_{1\star}}{1+bu_{1\star}}, \\ b_3 = \dfrac{d_1u_{2\star}}{1+bu_{1\star}} + \dfrac{bdd_1u_{1\star}u_{2\star}}{(1+bu_{1\star})^2}, \\ b_4 = h_2 - a_2u_{2\star} + \dfrac{dd_1u_{1\star}}{1+bu_{1\star}}, \\ b_5 = a_2u_{2\star}. \end{cases} \tag{3.4}$$

The characteristic equation of system (3.3) owns the following expression:

$$\det\begin{bmatrix} \lambda - b_1 & b_2 \\ -b_3 & \lambda - b_4 + b_5e^{-\lambda\delta} \end{bmatrix} = 0, \tag{3.5}$$

which leads to

$$\lambda^2 + (-b_1 - b_4)\lambda + (b_5\lambda - b_1b_5)e^{-\lambda\delta} + b_1b_4 + b_2b_3 = 0. \tag{3.6}$$

If $\delta = 0$, then Eq (3.6) becomes

$$\lambda^2 + (b_5 - b_1 - b_4)\lambda + b_1b_4 + b_2b_3 - b_1b_5 = 0. \tag{3.7}$$

If

$$(\mathcal{A}_1)\begin{cases} b_5 - b_1 - b_4 > 0, \\ b_1b_4 + b_2b_3 - b_1b_5 > 0, \end{cases} \tag{3.8}$$

is fulfilled, then the two roots λ_1, λ_2 of Eq (3.7) have negative real parts. Thus the equilibrium point $E(u_{1\star}, u_{2\star})$ of system (1.2) with $\delta = 0$ is locally asymptotically stable.

Assume that $\lambda = i\varepsilon$ is the root of Eq (3.6), then Eq (3.6) becomes

$$-\varepsilon^2 + (-b_1 - b_4)i\varepsilon + (b_5i\varepsilon - b_1b_5)e^{-i\varepsilon\delta} + b_1b_4 + b_2b_3 = 0. \tag{3.9}$$

It follows from (3.9) that

$$\begin{cases} b_5\varepsilon\sin\varepsilon\delta - b_1b_5\cos\varepsilon\delta = \varepsilon^2 - b_1b_4 - b_2b_3, \\ b_5\varepsilon\cos\varepsilon\delta + b_1b_5\sin\varepsilon\delta = (b_1 + b_4)\varepsilon. \end{cases} \tag{3.10}$$

Then

$$\varepsilon^4 + (b_1^2 + b_4^2 - b_5^2 - 2b_2b_3)\varepsilon^2 + (b_1b_4 + b_2b_3)^2 - (b_1b_5)^2 = 0. \tag{3.11}$$

Let

$$\Pi_1(\varepsilon) = \varepsilon^4 + (b_1^2 + b_4^2 - b_5^2 - 2b_2b_3)\varepsilon^2 + (b_1b_4 + b_2b_3)^2 - (b_1b_5)^2. \tag{3.12}$$

Assume that

$$(\mathcal{A}_2)\ \ |b_1b_4 + b_2b_3| < |b_1b_5|.$$

By virtue of $(\mathcal{A}_2)$, we know $\Pi_1(0) = (b_1b_4 + b_2b_3)^2 - (b_1b_5)^2 < 0$, since $\lim_{\varepsilon\to\infty}\Pi_1(\varepsilon) > 0$, then we will know Eq (3.11) has at least one positive real root. Therefore Eq (3.6) has at least one pair of purely imaginary roots. Without loss of generality, we can assume that Eq (3.11) has four positive real roots (say ε_j, $j = 1, 2, 3, 4$). Relying on (3.10), we know

$$\delta_j^{(n)} = \frac{1}{\varepsilon_j}\left[\arcsin\left(\frac{\varepsilon_j^3 + (b_1^2 - b_2 b_3)\varepsilon_j}{b_5\varepsilon_j^2 + b_1^2 b_5}\right) + 2n\pi\right], \tag{3.13}$$

where $j = 1,2,3,4; n = 0,1,2,\cdots$. Assume $\delta_0 = \min_{\{j=1,2,3,4;n=0,1,2,\cdots\}}\{\delta_j^{(n)}\}$ and suppose that when $\delta = \delta_0$, Eq (3.6) has a pair of imaginary roots $\pm i\varepsilon_0$.

Next we present the following assumption:

$$(\mathcal{A}_3)\ \ G_{1R}G_{2R} + G_{1I}G_{2I} > 0,$$

where

$$\begin{cases} G_{1R} = b_5\cos\varepsilon_0\delta_0 - b_1 - b_4, \\ G_{1I} = 2\varepsilon_0 - b_5\sin\varepsilon_0\delta_0, \\ G_{2R} = -\varepsilon_0^2 b_5\cos\varepsilon_0\delta_0 - b_1 b_5\varepsilon_0\sin\varepsilon_0\delta_0, \\ G_{2I} = \varepsilon_0^2 b_5\cos\varepsilon_0\delta_0 - b_1 b_5\varepsilon_0\cos\varepsilon_0\delta_0. \end{cases} \tag{3.14}$$

Lemma 3.1. *Suppose that $\lambda(\theta) = \epsilon_1(\delta) + i\epsilon_2(\delta)$ is the root of Eq (3.6) at $\delta = \delta_0$ such that $\epsilon_1(\delta_0) = 0$, $\epsilon_2(\delta_0) = \varepsilon_0$, then $Re\left(\frac{d\lambda}{d\delta}\right)\Big|_{\delta=\delta_0,\varepsilon=\varepsilon_0} > 0$.*

Proof. By Eq (3.6), we can get

$$(2\lambda - b_1 - b_4)\frac{\mathrm{d}\lambda}{\mathrm{d}\delta} + b_5 e^{-\lambda\delta}\frac{\mathrm{d}\lambda}{\mathrm{d}\delta} - (\delta\frac{\mathrm{d}\lambda}{\mathrm{d}\delta} + \lambda)(b_5\lambda - b_1 b_5)e^{-\lambda\delta} = 0. \tag{3.15}$$

It means that

$$\left(\frac{d\lambda}{d\delta}\right)^{-1} = \frac{G_1(\lambda)}{G_2(\lambda)} - \frac{\delta}{\lambda}, \tag{3.16}$$

where

$$\begin{cases} G_1(\lambda) = 2\lambda - b_1 - b_4 + b_5 e^{\lambda\delta}, \\ G_2(\lambda) = \lambda(b_5\lambda - b_1 b_5)e^{\lambda\delta}. \end{cases} \tag{3.17}$$

Hence

$$\mathrm{Re}\left[\left(\frac{d\lambda}{d\delta}\right)^{-1}\right]_{\delta=\delta_0,\varepsilon=\varepsilon_0} = \mathrm{Re}\left[\frac{G_1(\lambda)}{G_2(\lambda)}\right]_{\delta=\delta_0,\varepsilon=\varepsilon_0} = \frac{G_{1R}G_{2R} + G_{1I}G_{2I}}{G_{2R}^2 + G_{2I}^2}. \tag{3.18}$$

By the assumption $(\mathcal{A}_3)$, we get

$$\mathrm{Re}\left[\left(\frac{d\lambda}{d\delta}\right)^{-1}\right]_{\delta=\delta_0,\varepsilon=\varepsilon_0} > 0, \tag{3.19}$$

which ends the proof. According to the above discussion, the following outcome is easily derived.

Theorem 3.1. *Suppose that $(\mathcal{A}_1)$–$(\mathcal{A}_3)$ hold, then the equilibrium point $E(u_{1\star}, u_{2\star})$ of model (1.2) holds a locally asymptotically stable state if $\delta \in [0, \delta_0)$ and model (1.2) generates a cluster of Hopf bifurcations around the equilibrium point $E(u_{1\star}, u_{2\star})$ when $\delta = \delta_0$.*

4. Control of bifurcation for model (1.2) using hybrid controller

In this section, we are to study the Hopf bifurcation issue of system (1.2) by using a reasonable hybrid controller consisting of parameter perturbation with delay and state feedback. By virtue of the idea in [19,20], we formulate the following controlled predator-prey model:

$$\begin{cases} \dfrac{du_1}{dt} = \alpha_1 u_1(h_1 - a_1 u_1) - \dfrac{d_1 u_1 u_2}{1 + bu_1} + k[u_1(t-\delta) - u_1(t)], \\ \dfrac{du_2}{dt} = u_2[h_2 - a_2 u_2(t-\delta)] + \dfrac{dd_1 u_1 u_2}{1 + bu_1}. \end{cases} \tag{4.1}$$

System (4.1) owns the same equilibrium point $E(u_{1\star}, u_{2\star})$ as that in system (1.2). Let

$$\begin{cases} u_{1\star} = u_1(t) - \bar{u}_1(t), \\ u_{2\star} = u_2(t) - \bar{u}_2(t). \end{cases} \tag{4.2}$$

The linear system of system (4.1) near $E(u_{1\star}, u_{2\star})$ can be expressed as follows:

$$\begin{cases} \dfrac{d\bar{u}_1}{dt} = c_1\bar{u}_1 - c_2\bar{u}_2 + k\bar{u}_1(t-\delta), \\ \dfrac{d\bar{u}_2}{dt} = c_3\bar{u}_1 + c_4\bar{u}_2 - c_5\bar{u}_2(t-\delta), \end{cases} \tag{4.3}$$

where

$$\begin{cases} c_1 = \alpha_1 h_1 - k - 2a_1 u_{1*}\alpha_1 - \dfrac{d_1 u_{2\star}}{1 + bu_{1\star}} + \dfrac{bd_1 u_{1\star} u_{2\star}}{(1 + bu_{1\star})^2}, \\ c_2 = \dfrac{d_1 u_{1\star}}{1 + bu_{1\star}}, \\ c_3 = \dfrac{d_1 u_{2\star}}{1 + bu_{1\star}} + \dfrac{bdd_1 u_{1\star} u_{2\star}}{(1 + bu_{1\star})^2}, \\ c_4 = h_2 - a_2 u_{2\star} + \dfrac{dd_1 u_{1\star}}{1 + bu_{1\star}}, \\ c_5 = a_2 u_{2\star}. \end{cases} \tag{4.4}$$

The characteristic equation of system (4.3) owns the following expression:

$$\det\begin{bmatrix} \lambda - c_1 - ke^{-\lambda\delta} & c_2 \\ -c_3 & \lambda - c_4 + c_5 e^{-\lambda\delta} \end{bmatrix} = 0, \tag{4.5}$$

which leads to

$$(c_5 - k)\lambda + kc_4 - c_1c_5 + [\lambda^2 + (-c_1 - c_4)\lambda + c_2c_3 + c_1c_4]e^{\lambda\delta} - kc_5 e^{-\lambda\delta} = 0. \tag{4.6}$$

If $\delta = 0$, then Eq (4.6) reads as:

$$\lambda^2 + (c_5 - k - c_1 - c_4)\lambda + kc_4 + c_2c_3 - c_1c_5 + c_1c_4 - kc_5 = 0. \tag{4.7}$$

If

$$(\mathcal{A}_4)\begin{cases} c_5 - k - c_1 - c_4 > 0, \\ kc_4 + c_2c_3 - c_1c_5 + c_1c_4 - kc_5 > 0, \end{cases} \tag{4.8}$$

is fulfilled, there are two roots λ_1 and λ_2 of Eq (4.6) that have negative real parts. Thus the equilibrium point $E(u_{1\star}, u_{2\star})$ of system (4.1) with $\delta = 0$ holds a locally asymptotically stable state. Suppose that $\lambda = i\varepsilon^*$ is the root of Eq (4.6), then Eq (4.6) becomes:

$$(c_5 - k)i\varepsilon^* + kc_4 - c_1c_5 + [-\varepsilon^{*2} + (-c_1 - c_4)i\varepsilon^* + c_2c_3 + c_1c_4]e^{i\varepsilon^*\delta} - kc_5 e^{-i\varepsilon^*\delta} = 0. \tag{4.9}$$

It follows from (4.9) that

$$\begin{cases} (-\varepsilon^{*2} + c_2c_3 + c_1c_4 - kc_5)\cos\varepsilon^*\delta - \varepsilon^*(-c_1 - c_4)\sin\varepsilon^*\delta = c_1c_5 - kc_4, \\ \varepsilon^*(-c_1 - c_4)\cos\varepsilon^*\delta + (-\varepsilon^{*2} + c_2c_3 + c_1c_4 + kc_5)\sin\varepsilon^*\delta = (k - c_5)\varepsilon^*. \end{cases} \tag{4.10}$$

By (4.10), we can get

$$\begin{cases} E_1\cos(\varepsilon^*\delta)-E_2\sin(\varepsilon^*\delta)=E_3, \\ E_2\cos(\varepsilon^*\delta)+E_4\sin(\varepsilon^*\delta)=E_5, \end{cases} \tag{4.11}$$

where

$$\begin{cases} E_1=-\varepsilon^{*2}+c_2c_3+c_1c_4-kc_5, \\ E_2=\varepsilon^*(-c_1-c_4), \\ E_3=c_1c_5-kc_4, \\ E_4=-\varepsilon^{*2}+c_2c_3+c_1c_4+kc_5, \\ E_5=(k-c_5)\varepsilon^*. \end{cases} \tag{4.12}$$

So there is

$$\begin{cases} \cos\varepsilon^*\delta=\dfrac{E_1E_2E_5+E_1E_3E_4}{E_1(E_2^2+E_1E_4)}, \\ \sin\varepsilon^*\delta=\dfrac{E_1^2E_5-E_1E_2E_3}{E_1(E_2^2+E_1E_4)}. \end{cases} \tag{4.13}$$

Because of $\cos^2\varepsilon^*\delta+\sin^2\varepsilon^*\delta=1$, we can get

$$\left[\frac{E_1E_2E_5+E_1E_3E_4}{E_1(E_2^2+E_1E_4)}\right]^2+\left[\frac{E_1^2E_5-E_1E_2E_3}{E_1(E_2^2+E_1E_4)}\right]^2=1. \tag{4.14}$$

So

$$\begin{aligned} &E_1^2E_2^2E_5^2+2E_1^2E_2E_3E_4E_5+E_1^2E_3^2E_4^2+E_1^2E_2^2E_3^2 \\ &-2E_1^3E_2E_3E_5+E_1^4E_5^2-E_1^2E_2^4-2E_1^3E_2^2E_4-E_1^4E_4^2=0. \end{aligned} \tag{4.15}$$

According to Eq (4.12), one gets

$$\begin{cases} E_1=-\varepsilon^{*2}+g_1, \\ E_2=g_2\varepsilon^*, \\ E_3=c_1c_5-kc_4, \\ E_4=-\varepsilon^{*2}+g_3, \\ E_5=g_4\varepsilon^*, \end{cases} \tag{4.16}$$

where

$$\begin{cases} g_1=c_2c_3+c_1c_4-kc_5, \\ g_2=-c_1-c_4, \\ g_3=c_2c_3+c_1c_4+kc_5, \\ g_4=k-c_5. \end{cases} \tag{4.17}$$

Using (4.15) and (4.16), we know

$$-\varepsilon^{*12}+D_1\varepsilon^{*10}+D_2\varepsilon^{*8}+D_3\varepsilon^{*6}+D_4\varepsilon^{*4}+D_5\varepsilon^{*2}+D_6=0, \tag{4.18}$$

therefore, the results can be obtained as follows:

$$\varepsilon^{*12}-D_1\varepsilon^{*10}-D_2\varepsilon^{*8}-D_3\varepsilon^{*6}-D_4\varepsilon^{*4}-D_5\varepsilon^{*2}-D_6=0, \tag{4.19}$$

where

$$\begin{cases} D_1 = g_1^2 - 2g_2^2 + 2g_3 + 4g_1, \\ D_2 = E_3^2 - 2g_2E_3g_4 + g_2^2g_4^2 - 4g_1g_4^2 + 2g_2E_3g_4 - g_2^4 + 6g_1g_2^2 + 2g_2^2E_3 \\ \quad - 6g_1^2 - 8g_1g_3 - g_3^2, \\ D_3 = -2g_1E_3^2 - 2g_3E_3^2 + 4g_1g_2E_3g_4 + 2g_2g_3E_3g_4 - 2g_1g_2^2g_4^2 + 6g_1^2g_4^2 \\ \quad - 6g_1g_2E_3g_4 + g_2^2E_3^2 + 2g_1g_2^4 - 6g_1^2g_2^2 - 6g_1g_2^2g_3 + 4g_1^3 + 12g_1^2g_3 + 4g_1g_3^2, \\ D_4 = g_1^2E_3^2 + 4g_1g_3E_3^2 + g_3^2E_3^2 - 2g_1^2g_2E_3g_4 - 4g_1g_2g_3E_3g_4 \\ \quad + g_1^2g_2^2g_4^2 - 4g_1^3g_4^2 + 6g_1^2g_2E_3g_4 - 2g_1g_2^2E_3^2 - g_1^2g_2^4 \\ \quad + 2g_1^3g_2^2 + 6g_1^2g_2^2g_3 - g_1^4 - 8g_1^3g_3 - 6g_1^2g_3^2, \\ D_5 = -2g_1^2g_3E_3^2 - 2g_1g_3^2E_3^2 + 2g_1^2g_2g_3E_3g_4 + g_1^4g_4^2 \\ \quad - 2g_1^3g_2E_3g_4 + g_1^2g_2^2E_3^2 - 2g_1^3g_2^2g_3 + 2g_1^4g_3 + 4g_1^3g_3^2, \\ D_6 = g_1^2g_3^2E_3^2 - g_1^4g_3^2. \end{cases} \tag{4.20}$$

Let

$$\Pi_2(\varepsilon^*) = \varepsilon^{*12} - D_1\varepsilon^{*10} - D_2\varepsilon^{*8} - D_3\varepsilon^{*6} - D_4\varepsilon^{*4} - D_5\varepsilon^{*2} - D_6. \tag{4.21}$$

We can make the following assumption:

$$(\mathcal{A}_5)\ \ |g_1g_3E_3| > \left|g_1^2g_3\right|.$$

If $(\mathcal{A}_5)$ holds, then $\Pi_2(0) = -D_6 < 0$. Since $\lim_{\varepsilon^*\to\infty}\Pi_2(\varepsilon^*) = +\infty > 0$, then Eq (4.19) has at least one pair of positive real roots, and Eq (4.6) has at least one pair of pure roots. So we can assume that Eq (4.19) has 12 positive solid roots (say ε_j^*, $j = 1, 2, 3, ..., 12$). It is available according to Eq (4.11),

$$\delta_j^{(k)} = \frac{1}{\varepsilon_j^*}\left[\arccos\left(\frac{E_1(\varepsilon_j^*)E_2(\varepsilon_j^*)E_5(\varepsilon_j^*) + E_1(\varepsilon_j^*)E_3E_4(\varepsilon_j^*)}{E_1(\varepsilon_j^*)\left(E_2^2(\varepsilon_j^*) + E_1(\varepsilon_j^*)E_4(\varepsilon_j^*)\right)} + 2k\pi\right)\right], \tag{4.22}$$

where $j = 1, 2, 3, \cdots, 12$; $k = 0, 1, 2, \cdots$. Let $\delta_* = \min_{\{j=1,2,3,\cdots,12;k=0,1,2,\cdots\}}\{\delta_j^{(k)}\}$, and assume that when $\delta = \delta_*$, Eq (4.6) has at least one pair of pure real roots $\pm i\varepsilon_0^*$.

Next the following assumption is given:

$$(\mathcal{A}_6)\ \ H_{1R}H_{2R} + H_{1I}H_{2I} > 0,$$

where

$$\begin{cases} \mathcal{H}_{1R} = c_5 - k - 2\varepsilon_0^*\sin\varepsilon_0^*\delta_* - (c_1 + c_4)\cos\varepsilon_0^*\delta_*, \\ \mathcal{H}_{1I} = 2\varepsilon_0^*\cos\varepsilon_0^*\delta_* - (c_1 + c_4)\sin\varepsilon_0^*\delta_*, \\ \mathcal{H}_{2R} = \left[\varepsilon^{*3}_0 + (c_2c_3 + c_1c_4)\varepsilon_0^* - kc_5\varepsilon_0^*\right]\sin\varepsilon_0^*\delta_* - (c_1 + c_4)\varepsilon^{*2}_0\cos\varepsilon_0^*\delta_*, \\ \mathcal{H}_{2I} = \left[-\varepsilon^{*3}_0 - (c_2c_3 + c_1c_4)\varepsilon_0^* - kc_5\varepsilon_0^*\right]\cos\varepsilon_0^*\delta_* - (c_1 + c_4)\varepsilon^{*2}_0\sin\varepsilon_0^*\delta_*. \end{cases} \tag{4.23}$$

Lemma 4.1. *Suppose that $\lambda(\theta) = \bar{\epsilon}_1(\delta) + i\bar{\epsilon}_2(\delta)$ is the root of Eq (4.6) at $\delta = \delta_*$ such that $\bar{\epsilon}_1(\delta_*) = 0$, $\bar{\epsilon}_2(\delta_*) = \varepsilon_0^*$, then $Re\left(\frac{d\lambda}{d\delta}\right)\Big|_{\delta=\delta_*,\varepsilon=\varepsilon_0^*} > 0$.*

Proof. By Eq (4.6), one gets

$$(c_5 - k)\frac{\mathrm{d}\lambda}{\mathrm{d}\delta} + (2\lambda - c_1 - c_4)e^{\lambda\delta}\frac{\mathrm{d}\lambda}{\mathrm{d}\delta} + \left[\lambda^2 + (-c_1 - c_4)\lambda + c_2c_3 + c_1c_4\right]$$

$$\times e^{\lambda\delta}\left(\lambda+\delta\frac{\mathrm{d}\lambda}{\mathrm{d}\delta}\right)+kc_5e^{-\lambda\delta}\left(\lambda+\delta\frac{\mathrm{d}\lambda}{\mathrm{d}\delta}\right)=0, \tag{4.24}$$

which implies

$$\left(\frac{d\lambda}{d\delta}\right)^{-1}=\frac{\mathcal{H}_1(\lambda)}{\mathcal{H}_2(\lambda)}-\frac{\delta}{\lambda}, \tag{4.25}$$

where

$$\begin{cases}\mathcal{H}_1(\lambda)=c_5-k+(2\lambda-c_1-c_4)e^{\lambda\delta},\\ \mathcal{H}_2(\lambda)=\left[-\lambda^2+(-c_1-c_4)\lambda+c_2c_3+c_1c_4\right]\lambda e^{\lambda\delta}-kc_5\lambda e^{-\lambda\delta}.\end{cases} \tag{4.26}$$

Hence

$$\operatorname{Re}\left[\left(\frac{d\lambda}{d\delta}\right)^{-1}\right]_{\delta=\delta_*,\varepsilon=\varepsilon_0^*}=\operatorname{Re}\left[\frac{\mathcal{H}_1(\lambda)}{\mathcal{H}_2(\lambda)}\right]_{\delta=\delta_*,\varepsilon=\varepsilon_0^*}=\frac{\mathcal{H}_{1R}\mathcal{H}_{2R}+\mathcal{H}_{1I}\mathcal{H}_{2I}}{\mathcal{H}_{2R}^2+\mathcal{H}_{2I}^2}. \tag{4.27}$$

According to ($\mathcal{A}_6$), one gets

$$\operatorname{Re}\left[\left(\frac{d\lambda}{d\delta}\right)^{-1}\right]_{\delta=\delta_*,\varepsilon=\varepsilon_0^*}>0, \tag{4.28}$$

which completes the proof.

Depending on the analysis above, the following conclusion is acquired:

Theorem 4.1. *Suppose that ($\mathcal{A}_4$)–($\mathcal{A}_6$) hold, then the equilibrium point $E(u_{1\star},u_{2\star})$ of model (4.1) is locally asymptotically stable if $\delta\in[0,\delta_*)$ and model (4.1) generates a cluster of Hopf bifurcations near the equilibrium point $E(u_{1\star},u_{2\star})$ when $\delta=\delta_*$.*

5. Exploration of bifurcation for model (1.3)

In this section, we are going to explore the Hopf bifurcation phenomenon of system (1.3). System (1.3) owns the same equilibrium point $E(u_{1\star},u_{2\star})$ as that in system (1.2). Let

$$\begin{cases}u_{1\star}=u_1(t)-\bar{u}_1(t),\\ u_{2\star}=u_2(t)-\bar{u}_2(t).\end{cases} \tag{5.1}$$

The linear system of system (1.3) near $E(u_{1\star},u_{2\star})$ can be expressed as follows:

$$\begin{cases}\dfrac{du_1}{dt}=e_1\bar{u}_1-e_2\bar{u}_2-e_3\bar{u}_1(t-\delta),\\ \dfrac{du_2}{dt}=e_4\bar{u}_1+e_5\bar{u}_2-e_6\bar{u}_2(t-\delta),\end{cases} \tag{5.2}$$

where

$$\begin{cases}e_1=h_1-a_1u_{1\star}-\dfrac{d_1u_{2\star}}{1+bu_{1\star}}+\dfrac{bd_1u_{1\star}u_{2\star}}{(1+bu_{1\star})^2},\\ e_2=\dfrac{d_1u_{1\star}}{1+bu_{1\star}},\\ e_3=a_1u_{1\star},\\ e_4=\dfrac{dd_1u_{2\star}}{1+bu_{1\star}}-\dfrac{bdd_1u_{1\star}u_{2\star}}{(1+bu_{1\star})^2},\\ e_5=h_2-a_2u_{2\star}+\dfrac{dd_1u_{1\star}}{1+bu_{1\star}},\\ e_6=a_2u_{2\star}.\end{cases} \tag{5.3}$$

The characteristic equation of system (5.2) owns the following expression:

$$\det\begin{bmatrix} \lambda - e_1 + e_3 e^{-\lambda\delta} & e_2 \\ -e_4 & \lambda - e_5 + e_6 e^{-\lambda\delta} \end{bmatrix} = 0, \tag{5.4}$$

which leads to:

$$\lambda^2 + (-e_1 - e_5)\lambda + (e_3\lambda + e_6\lambda + e_1e_6 - e_3e_5)e^{-\lambda\delta} + e_3e_6e^{-2\lambda\delta} + e_1e_5 + e_2e_4 = 0, \tag{5.5}$$

that is,

$$(e_3 + e_6)\lambda + e_1e_6 - e_3e_5 + \left[\lambda^2 + (-e_1 - e_5)\lambda + e_1e_5 + e_2e_4\right]e^{\lambda\delta} + e_3e_6e^{-\lambda\delta} = 0. \tag{5.6}$$

If $\delta = 0$, then Eq (5.6) reads as:

$$\lambda^2 + (e_3 + e_6 - e_1 - e_5)\lambda + e_1e_6 - e_3e_5 + e_1e_5 + e_2e_4 + e_3e_6 = 0. \tag{5.7}$$

If

$$(\mathcal{A}_7)\begin{cases} e_3 + e_6 - e_1 - e_5 > 0, \\ e_1e_6 - e_3e_5 + e_1e_5 + e_2e_4 + e_3e_6 > 0, \end{cases} \tag{5.8}$$

is fulfilled, there are two roots λ_1, λ_2 of Eq (5.7) that have negative real parts. Thus the equilibrium point $E(u_{1\star}, u_{2\star})$ of system (1.3) with $\delta = 0$ is locally asymptotically stable.

Suppose that $\lambda = i\varepsilon^\tau$ is the root of Eq (5.6), then Eq (5.6) becomes:

$$(e_3 + e_6)i\varepsilon^\tau + e_1e_6 - e_3e_5 + \left[-\varepsilon^{\tau 2} + (-e_1 - e_5)i\varepsilon^\tau + e_1e_5 + e_2e_4\right]e^{i\varepsilon^\tau\delta} + e_3e_6e^{-i\varepsilon^\tau\delta} = 0. \tag{5.9}$$

By (5.9), we have

$$\begin{cases} (-\varepsilon^{\tau 2} + e_1e_5 + e_2e_4 + e_3e_6)\cos\varepsilon^\tau\delta + \varepsilon^\tau(e_1 + e_5)\sin\varepsilon^\tau\delta = e_3e_5 - e_1e_6, \\ -\varepsilon^\tau(e_1 + e_5)\cos\varepsilon^\tau\delta + (-\varepsilon^{\tau 2} + e_1e_5 + e_2e_4 - e_3e_6)\sin\varepsilon^\tau\delta = -(e_3 + e_6)\varepsilon^\tau, \end{cases} \tag{5.10}$$

which means

$$\begin{cases} Y_1\cos\varepsilon^\tau\delta + Y_2\sin\varepsilon^\tau\delta = Y_3, \\ -Y_2\cos\varepsilon^\tau\delta + Y_4\sin\varepsilon^\tau\delta = Y_5, \end{cases} \tag{5.11}$$

where

$$\begin{cases} Y_1 = -\varepsilon^{\tau 2} + e_1e_5 + e_2e_4 + e_3e_6, \\ Y_2 = \varepsilon^\tau(e_1 + e_5), \\ Y_3 = e_3e_5 - e_1e_6, \\ Y_4 = -\varepsilon^{\tau 2} + e_1e_5 + e_2e_4 - e_3e_6, \\ Y_5 = -(e_3 + e_6)\varepsilon^\tau. \end{cases} \tag{5.12}$$

So, we can get

$$\begin{cases} \cos\varepsilon^\tau\delta = \dfrac{Y_1Y_3Y_4 - Y_1Y_2Y_5}{Y_1(Y_2^2 + Y_1Y_4)}, \\ \sin\varepsilon^\tau\delta = \dfrac{Y_1^2Y_5 + Y_1Y_2Y_3}{Y_1(Y_2^2 + Y_1Y_4)}. \end{cases} \tag{5.13}$$

Because of $\cos^2\varepsilon^\tau\delta + \sin^2\varepsilon^\tau\delta = 1$, then

$$\left[\frac{Y_1Y_3Y_4 - Y_1Y_2Y_5}{Y_1(Y_2^2 + Y_1Y_4)}\right]^2 + \left[\frac{Y_1^2Y_5 + Y_1Y_2Y_3}{Y_1(Y_2^2 + Y_1Y_4)}\right]^2 = 1. \tag{5.14}$$

It follows from (5.14) that

$$Y_1^2Y_2^2Y_5^2 - 2Y_1^2Y_2Y_3Y_4Y_5 + Y_1^2Y_3^2Y_4^2 + Y_1^2Y_2^2Y_3^2 + 2Y_1^3Y_2Y_3Y_5 \\ +Y_1^4Y_5^2 - Y_1^2Y_2^4 - 2Y_1^3Y_2^2Y_4 - Y_1^4Y_4^2 = 0. \tag{5.15}$$

By (5.12), one gets

$$\begin{cases} Y_1 = -\varepsilon^{\tau 2} + y_1, \\ Y_2 = y_2\varepsilon^{\tau}, \\ Y_4 = -\varepsilon^{\tau 2} + y_3, \\ Y_5 = y_4\varepsilon^{\tau}, \end{cases} \tag{5.16}$$

where

$$\begin{cases} y_1 = e_1e_5 + e_2e_4 + e_3e_6, \\ y_2 = e_1 + e_5, \\ y_3 = e_1e_5 + e_2e_4 - e_3e_6, \\ y_4 = -(e_3 + e_6). \end{cases} \tag{5.17}$$

Using (5.15) and (5.16), we know

$$-\varepsilon^{\tau 12} + N_1\varepsilon^{\tau 10} + N_2\varepsilon^{\tau 8} + N_3\varepsilon^{\tau 6} + N_4\varepsilon^{\tau 4} + N_5\varepsilon^{\tau 2} + N_6 = 0. \tag{5.18}$$

Therefore, the results can be obtained as follows

$$\varepsilon^{\tau 12} - N_1\varepsilon^{\tau 10} - N_2\varepsilon^{\tau 8} - N_3\varepsilon^{\tau 6} - N_4\varepsilon^{\tau 4} - N_5\varepsilon^{\tau 2} - N_6 = 0, \tag{5.19}$$

where

$$\begin{cases} N_1 = 2y_3 + 4y_1 - 2y_2^2 + 4y_4^2, \\ N_2 = Y_3^2 + 2y_2Y_3y_4 + y_2^2y_4^2 - 4y_1y_4^2 - 2y_2Y_3y_4 \\ \qquad - y_2^4 + 6y_1y_2^2 + 2y_2^2Y_3 - 6y_1^2 - 8y_1y_3 - y_3^2, \\ N_3 = -2y_1Y_3^2 - 2y_3Y_3^2 - 4y_1y_2Y_3y_4 - 2y_2y_3Y_3y_4 \\ \qquad + 2y_1y_2^2y_4^2 + 6y_1^2y_4^2 + 6y_1y_2Y_3y_4 \\ \qquad + y_2^2Y_3^2 + 2y_1y_2^4 - 6y_1^2y_2^2 \\ \qquad - 6y_1y_2^2y_3 + 4y_1^3 + 12y_1^2y_3 + 4y_1y_3^2, \\ N_4 = y_1^2Y_3^2 + 4y_1y_3Y_3^2 + y_3^2Y_3^2 \\ \qquad + 2y_1^2y_2Y_3y_4 + 4y_1y_2y_3Y_3y_4 + y_1^2y_2^2y_4^2 \\ \qquad - 4y_1^3y_4^2 - 6y_1^2y_2Y_3y_4 - 2y_1y_2^2Y_3^2 \\ \qquad - y_1^2y_2^4 + 2y_1^3y_2^2 + 6y_1^2y_2^2y_3 \\ \qquad - y_1^4 - 8y_1^3y_3 - 6y_1^2y_3^2, \\ N_5 = -2y_1^2y_3Y_3^2 - 2y_1y_3^2Y_3^2 \\ \qquad - 2y_1^2y_2y_3Y_3y_4 + y_1^4y_4^2 + 2y_1^3y_2Y_3y_4 \\ \qquad + y_1^2y_2^2Y_3^2 - 2y_1^3y_2^2y_3 + 2y_1^4y_3 + 4y_1^3y_3^2, \\ N_6 = y_1^2y_3^2Y_3^2 - y_1^4y_3^2. \end{cases} \tag{5.20}$$

Let

$$\Pi_3(\varepsilon^{\tau}) = \varepsilon^{\tau 12} - N_1\varepsilon^{\tau 10} - N_2\varepsilon^{\tau 8} - N_3\varepsilon^{\tau 6} - N_4\varepsilon^{\tau 4} - N_5\varepsilon^{\tau 2} - N_6. \tag{5.21}$$

We can make the following assumptions:

$$(\mathcal{A}_8)\ \ |y_1y_3Y_3| > \left|y_1^2y_3\right|.$$

If $(\mathcal{A}_8)$ holds, then $\Pi_3(0) = -N_6 < 0$, since $\lim_{\varepsilon^\tau\to\infty}\Pi_3(\varepsilon^\tau) = +\infty > 0$, then Eq (5.19) has at least one pair of positive real roots, and Eq (5.6) has at least one pair of pure roots. So we can assume that Eq (5.19) has 12 positive real roots (say ε_j^τ, $j = 1, 2, 3, ..., 12$).

By Eq (5.11), one gets

$$\delta_j^{(m)} = \frac{1}{\varepsilon_j^\tau}\left[\arccos\left(\frac{Y_1(\varepsilon_j)Y_3Y_4(\varepsilon_j^\tau) - Y_1(\varepsilon_j^\tau)Y_2(\varepsilon_j^\tau)Y_5(\varepsilon_j^\tau)}{Y_1(\varepsilon_j^\tau)\left(Y_2^2(\varepsilon_j^\tau) + Y_1(\varepsilon_j^\tau)Y_4(\varepsilon_j^\tau)\right)} + 2m\pi\right)\right], \tag{5.22}$$

where $j = 1, 2, 3, \cdots, 12; m = 0, 1, 2, \cdots$.

Let $\delta_\star = \min_{\{j=1,2,3,\cdots,12;m=0,1,2,\cdots\}}\{\delta_j^{(m)}\}$, and assume that when $\delta = \delta_\star$, Eq (5.6) has at least one pair of pure of real roots $\pm i\varepsilon_0^\tau$.

Next the following assumption is needed:

$$(\mathcal{A}_9)\ \ F_{1R}F_{2R} + F_{1I}F_{2I} > 0,$$

where

$$\begin{cases} \mathcal{F}_{1R} = e_3 + e_6 - 2\varepsilon_0^\tau \sin\varepsilon_0^\tau\delta_\star - (e_1 + e_5)\cos\varepsilon_0^\tau\delta_\star, \\ \mathcal{F}_{1I} = 2\varepsilon_0^\tau \cos\varepsilon_0^\tau\delta_\star - (e_1 + e_5)\sin\varepsilon_0^\tau\delta_\star, \\ \mathcal{F}_{2R} = \left[\varepsilon_0^{\tau 3} + (e_1e_5 + e_2e_4 + e_3e_6)\varepsilon_0^\tau\right]\sin\varepsilon_0^\tau\delta_\star - (e_1 + e_5)\varepsilon_0^\tau\cos\varepsilon_0^\tau\delta_\star, \\ \mathcal{F}_{2I} = \left[-\varepsilon_0^{\tau 3} + (e_3e_6 - e_1e_5 - e_2e_4)\varepsilon_0^\tau\right]\cos\varepsilon_0^\tau\delta_\star - (e_1 + e_5)\varepsilon_0^\tau\sin\varepsilon_0^\tau\delta_\star. \end{cases} \tag{5.23}$$

Lemma 5.1. *Suppose that $\lambda(\theta) = \xi_1(\theta) + i\xi_2(\theta)$ is the root of Eq (5.6) at $\delta = \delta_\star$, such that $\xi_1(\delta_\star) = 0$, $\xi_2(\delta_\star) = \varepsilon_0^\tau$, then $Re\left(\frac{d\lambda}{d\delta}\right)\Big|_{\delta=\delta_\star,\varepsilon=\varepsilon_0^\tau} > 0$.*

Proof. By Eq (5.6), one gets

$$\begin{aligned} &(e_3 + e_6)\frac{\mathrm{d}\lambda}{\mathrm{d}\delta} + (2\lambda - e_1 - e_5)e^{\lambda\delta}\frac{\mathrm{d}\lambda}{\mathrm{d}\delta} + \left[\lambda^2 + (-e_1 - e_5)\lambda + e_2e_4 + e_1e_5\right] \\ &\times e^{\lambda\delta}\left(\lambda + \delta\frac{\mathrm{d}\lambda}{\mathrm{d}\delta}\right) - e_3e_6e^{-\lambda\delta}\left(\lambda + \delta\frac{\mathrm{d}\lambda}{\mathrm{d}\delta}\right) = 0, \end{aligned} \tag{5.24}$$

which implies

$$\left(\frac{d\lambda}{d\delta}\right)^{-1} = \frac{\mathcal{F}_1(\lambda)}{\mathcal{F}_2(\lambda)} - \frac{\delta}{\lambda}, \tag{5.25}$$

where

$$\begin{cases} \mathcal{F}_1(\lambda) = e_3 + e_6 + (2\lambda - e_1 - e_5)e^{\lambda\delta}, \\ \mathcal{F}_2(\lambda) = -\left[\lambda^2 + (-e_1 - e_5)\lambda + e_2e_4 + e_1e_5\right]\lambda e^{\lambda\delta} + e_3e_6\lambda e^{-\lambda\delta}. \end{cases} \tag{5.26}$$

Hence

$$\mathrm{Re}\left[\left(\frac{d\lambda}{d\delta}\right)^{-1}\right]_{\delta=\delta_\star,\varepsilon=\varepsilon_0^\tau} = \mathrm{Re}\left[\frac{\mathcal{F}_1(\lambda)}{\mathcal{F}_2(\lambda)}\right]_{\delta=\delta_\star,\varepsilon=\varepsilon_0^\tau} = \frac{\mathcal{F}_{1R}\mathcal{F}_{2R} + \mathcal{F}_{1I}\mathcal{F}_{2I}}{\mathcal{F}_{2R}^2 + \mathcal{F}_{2I}^2}. \tag{5.27}$$

By ($\mathcal{A}_9$), we have

$$\mathrm{Re}\left[\left(\frac{d\lambda}{d\delta}\right)^{-1}\right]_{\delta=\delta_\star,\varepsilon=\varepsilon_0^\tau} > 0, \tag{5.28}$$

which ends the proof.

Depending on the discussion above, the following results is obtained:

Theorem 5.1. *Suppose that ($\mathcal{A}_7$)–($\mathcal{A}_9$) are fulfilled, then the equilibrium point $E(u_{1\star}, u_{2\star})$ of model (5.1) is locally asymptotically stable if $\delta \in [0, \delta_\star)$ and model (5.1) generates a cluster of Hopf bifurcations near the equilibrium point $E(u_{1\star}, u_{2\star})$ when $\delta = \delta_\star$.*

Remark 5.1. *In model (1.3), there is only one delay. If there are two different delays in model (1.3), we can deal with the effect of two delays on the stability and bifurcation. We leave it for future work.*

6. Control of bifurcation for model (1.3) using extended delayed feedback controller

In this section, we are to study the Hopf bifurcation issue of system (1.3) by using a reasonable extended delayed feedback controller consisting of parameter perturbation with delay. By virtue of the idea in [23–25], we formulate the following controlled predator-prey model:

$$\begin{cases} \dfrac{du_1}{dt} = u_1[h_1 - a_1u_1(t-\delta)] - \dfrac{d_1u_1u_2}{1+bu_1} + k_1[u_1(t-\delta) - u_1(t)], \\ \dfrac{du_2}{dt} = u_2[h_2 - a_2u_2(t-\delta)] + \dfrac{dd_1u_1u_2}{1+bu_1} + k_2[u_2(t-\delta) - u_2(t)]. \end{cases} \tag{6.1}$$

System (6.1) owns the same equilibrium point $E(u_{1\star}, u_{2\star})$ as that of system (1.3). Let

$$\begin{cases} u_{1\star} = u_1(t) - \bar{u}_1(t), \\ u_{2\star} = u_2(t) - \bar{u}_2(t). \end{cases} \tag{6.2}$$

The linear system of system (6.1) around $E(u_{1\star}, u_{2\star})$ takes the following expression:

$$\begin{cases} \dfrac{d\bar{u}_1}{dt} = \tau_1\bar{u}_1 - \tau_2\bar{u}_2 + \tau_3\bar{u}_1(t-\delta), \\ \dfrac{d\bar{u}_2}{dt} = \tau_4\bar{u}_1 + \tau_5\bar{u}_2 + \tau_6\bar{u}_2(t-\delta), \end{cases} \tag{6.3}$$

where

$$\begin{cases} \tau_1 = h_1 - k_1 - a_1u_{1\star} - \dfrac{d_1u_{2\star}}{1+bu_{1\star}} + \dfrac{bd_1u_{1\star}u_{2\star}}{(1+bu_{1\star})^2}, \\ \tau_2 = \dfrac{d_1u_{1\star}}{1+bu_{1\star}}, \\ \tau_3 = k_1 - a_1u_{1\star}, \\ \tau_4 = \dfrac{dd_1u_{2\star}}{1+bu_{1\star}} - \dfrac{bdd_1u_{1\star}u_{2\star}}{(1+bu_{1\star})^2}, \\ \tau_5 = h_2 - a_2u_{2\star} + \dfrac{dd_1u_{1\star}}{1+bu_{1\star}} - k_2, \\ \tau_6 = k_2 - a_2u_{2\star}. \end{cases} \tag{6.4}$$

The characteristic equation of system (6.3) owns the following expression:

$$\det\begin{bmatrix} \lambda-\tau_1+\tau_3 e^{-\lambda\delta} & \tau_2 \\ -\tau_4 & \lambda-\tau_5-\tau_6 e^{-\lambda\delta} \end{bmatrix}=0, \tag{6.5}$$

which leads to:

$$\lambda^2+(-\tau_1-\tau_5)\lambda+(-\tau_3\lambda-\tau_6\lambda+\tau_1\tau_6+\tau_3\tau_5)e^{-\lambda\delta}+\tau_3\tau_6 e^{-2\lambda\delta}+\tau_1\tau_5+\tau_2\tau_4=0, \tag{6.6}$$

that is

$$(\tau_3+\tau_6)\lambda-(\tau_1\tau_6+\tau_3\tau_5)+\left[\lambda^2+(-\tau_1-\tau_5)\lambda+\tau_1\tau_5+\tau_2\tau_4\right]e^{\lambda\delta}-\tau_3\tau_6 e^{-\lambda\delta}=0. \tag{6.7}$$

If $\delta=0$, then Eq (6.7) reads as:

$$\lambda^2-(\tau_1+\tau_3+\tau_5+\tau_6)\lambda+\tau_1\tau_6+\tau_3\tau_5+\tau_1\tau_5+\tau_2\tau_4+\tau_3\tau_6=0. \tag{6.8}$$

If

$$(\mathcal{A}_{10})\ \begin{cases} -(\tau_1+\tau_3+\tau_5+\tau_6)>0, \\ \tau_1\tau_6+\tau_3\tau_5+\tau_1\tau_5+\tau_2\tau_4+\tau_3\tau_6>0, \end{cases} \tag{6.9}$$

is fulfilled, then the two roots λ_1, λ_2 of Eq (6.7) have negative real parts. Thus the equilibrium point $E(u_{1\star}, u_{2\star})$ of system (6.1) with $\delta=0$ is locally asymptotically stable.

Suppose that $\lambda=i\varepsilon^{\mu}$ is the root of Eq (6.7), then Eq (6.7) becomes:

$$(\tau_3+\tau_6)i\varepsilon^{\mu}-(\tau_1\tau_6+\tau_3\tau_5)-\left[\varepsilon^{\mu 2}+(-\tau_1-\tau_5)i\varepsilon^{\mu}+\tau_1\tau_5+\tau_2\tau_4\right]e^{i\varepsilon^{\mu}\delta}-\tau_3\tau_6 e^{-i\varepsilon^{\mu}\delta}=0. \tag{6.10}$$

By (6.10), we have

$$\begin{cases} \left(\varepsilon^{\mu 2}-\tau_1\tau_5-\tau_2\tau_4-\tau_3\tau_6\right)\cos\varepsilon^{\mu}\delta-\varepsilon^{\mu}(\tau_1+\tau_5)\sin\varepsilon^{\mu}\delta=\tau_1\tau_6+\tau_3\tau_5, \\ \varepsilon^{\mu}(\tau_1+\tau_5)\cos\varepsilon^{\mu}\delta+\left(\varepsilon^{\mu 2}-\tau_1\tau_5-\tau_2\tau_4+\tau_3\tau_6\right)\sin\varepsilon^{\mu}\delta=-(\tau_3+\tau_6)\varepsilon^{\beta}, \end{cases} \tag{6.11}$$

which means

$$\begin{cases} T_1\cos\varepsilon^{\mu}\delta-T_2\sin\varepsilon^{\mu}\delta=T_3, \\ T_2\cos\varepsilon^{\mu}\delta+T_4\sin\varepsilon^{\mu}\delta=T_5, \end{cases} \tag{6.12}$$

where

$$\begin{cases} T_1=-\varepsilon^{\mu 2}-\tau_1\tau_5-\tau_2\tau_4-\tau_3\tau_6, \\ T_2=(\tau_1+\tau_5)\varepsilon^{\mu}, \\ T_3=\tau_3\tau_5+\tau_1\tau_6, \\ T_4=-\varepsilon^{\mu 2}-\tau_1\tau_5-\tau_2\tau_4+\tau_3\tau_6, \\ T_5=-(\tau_3+\tau_6)\varepsilon^{\mu}. \end{cases} \tag{6.13}$$

So, we can get

$$\begin{cases} \cos\varepsilon^{\mu}\delta=\dfrac{T_1T_3T_4+T_1T_2T_5}{T_1(T_2^2+T_1T_4)}, \\ \sin\varepsilon^{\mu}\delta=\dfrac{T_1^2T_5-T_1T_2T_3}{T_1(T_2^2+T_1T_4)}. \end{cases} \tag{6.14}$$

Because of $\cos^2 \varepsilon^{\mu}\delta + \sin^2 \varepsilon^{\mu}\delta = 1$,

$$\left[\frac{T_1T_3T_4 + T_1T_2T_5}{T_1(T_2^2 + T_1T_4)}\right]^2 + \left[\frac{T_1^2T_5 - T_1T_2T_3}{T_1(T_2^2 + T_1T_4)}\right]^2 = 1. \tag{6.15}$$

It follows from (6.15) that

$$\begin{aligned} &T_1^2T_2^2T_5^2 + 2T_1^2T_2T_3T_4T_5 + T_1^2T_3^2T_4^2 + T_1^2T_2^2T_3^2 - 2T_1^3T_2T_3T_5 \\ &+T_1^4T_5^2 - T_1^2T_2^4 - 2T_1^3T_2^2T_4 - T_1^4T_4^2 = 0. \end{aligned} \tag{6.16}$$

By (6.13), one gets

$$\begin{cases} T_1 = -\varepsilon^{\mu 2} + x_1, \\ T_2 = x_2\varepsilon^{\mu}, \\ T_4 = -\varepsilon^{\mu 2} + x_3, \\ T_5 = x_4\varepsilon^{\mu}, \end{cases} \tag{6.17}$$

where

$$\begin{cases} x_1 = -\tau_1\tau_5 - \tau_2\tau_4 - \tau_3\tau_6, \\ x_2 = \tau_1 + \tau_5, \\ x_3 = -\tau_1\tau_5 - \tau_2\tau_4 + \tau_3\tau_6, \\ x_4 = -(\tau_3 + \tau_6). \end{cases} \tag{6.18}$$

Using (6.16) and (6.17), we know

$$-\varepsilon^{\mu 12} + X_1\varepsilon^{\mu 10} + X_2\varepsilon^{\mu 8} + X_3\varepsilon^{\mu 6} + X_4\varepsilon^{\mu 4} + X_5\varepsilon^{\mu 2} + X_6 = 0, \tag{6.19}$$

therefore, the results can be obtained as follows:

$$\varepsilon^{\mu 12} - X_1\varepsilon^{\mu 10} - X_2\varepsilon^{\mu 8} - X_3\varepsilon^{\mu 6} - X_4\varepsilon^{\mu 4} - X_5\varepsilon^{\mu 2} - X_6 = 0, \tag{6.20}$$

where

$$\begin{cases} X_1 = x_4^2 - 2x_2^2 - 2x_3 + 4x_1, \\ X_2 = T_3^2 + 2x_2T_3x_4 + x_2^2x_4^2 + 4x_1x_4^2 - 2x_2T_3x_4 \\ \qquad - x_2^4 - 6x_1x_2^2 - 2x_2^2T_3 - 6x_1^2 - 8x_1x_3 - x_3^2, \\ X_3 = 2x_1T_3^2 + 2x_3T_3^2 + 4x_1x_2T_3x_4 + 2x_2x_3T_3x_4 \\ \qquad + 2x_1x_2^2x_4^2 + 6x_1^2x_4^2 - 6x_1x_2T_3x_4 + x_2^2T_3^2 \\ \qquad - 2x_1x_2^4 - 6x_1^2x_2^2 - 6x_1x_2^2x_3 - 4x_1^3 - 12x_1^2x_3 - 4x_1x_3^2, \\ X_4 = x_1^2T_3^2 + 4x_1x_3T_3^2 + x_3^2T_3^2 + 2x_1^2x_2T_3x_4 \\ \qquad + 4x_1x_2x_3T_3x_4 + x_1^2x_2^2x_4^2 + 4x_1^3x_4^2 - 6x_1^2x_2T_3x_4 \\ \qquad + 2x_1x_2^2T_3^2 - x_1^2x_2^4 - 2x_1^3x_2^2 - 6x_1^2x_2^2x_3 \\ \qquad - x_1^4 - 8x_1^3x_3 - 6x_1^2x_3^2, \\ X_5 = x_1^2x_3T_3^2 + 2x_1x_3^2T_3^2 + 2x_1^2x_2x_3T_3x_4 \\ \qquad + x_1^4x_4^2 - 2x_1^3x_2T_3x_4 + x_1^2x_2^2T_3^2 \\ \qquad - 2x_1^3x_2^2x_3 - 2x_1^4x_3 - 4x_1^3x_3^2, \\ X_6 = x_1^2x_3^2T_3^2 - x_1^4x_3^2. \end{cases} \tag{6.21}$$

Let

$$\Pi_4(\varepsilon^\mu) = \varepsilon^{\mu 12} - X_1\varepsilon^{\mu 10} - X_2\varepsilon^{\mu 8} - X_3\varepsilon^{\mu 6} - X_4\varepsilon^{\mu 4} - X_5\varepsilon^{\mu 2} - X_6. \tag{6.22}$$

We can make the following assumption:

$$(\mathcal{A}_{11})\ \ |x_1x_3T_3| > \left|x_1^2x_3\right|.$$

If $(\mathcal{A}_{11})$ holds, then $\Pi_4(0) = -X_6 < 0$. Notice that $\lim_{\varepsilon^\mu\to\infty}\Pi_4(\varepsilon^\mu) = +\infty > 0$, then Eq (6.20) has at least one pair of positive real roots, and Eq (6.7) has at least one pair of purely real roots. So we can assume that Eq (6.20) has 12 positive real roots (say ε_j^μ, $j = 1, 2, 3, \cdots, 12$).

By Eq (6.14), one gets

$$\delta_j^{(\theta)} = \frac{1}{\varepsilon_j^\mu}\left[\arccos\left(\frac{X_1(\varepsilon_j^\mu)X_3X_4(\varepsilon_j^\mu) - X_1(\varepsilon_j^\mu)X_2(\varepsilon_j^\mu)X_5(\varepsilon_j^\mu)}{X_1(\varepsilon_j^\mu)\left(X_2^2(\varepsilon_j^\mu) + X_1(\varepsilon_j^\mu)X_4(\varepsilon_j^\mu)\right)} + 2\theta\pi\right)\right], \tag{6.23}$$

where $j = 1, 2, 3, \cdots, 12; \theta = 0, 1, 2, \cdots$.

Let $\delta_{0*} = \min_{\{j=1,2,3,\cdots,12;\theta=0,1,2,\cdots\}}\{\delta_j^{(\theta)}\}$, and assume that when $\delta = \delta_{0*}$, Eq (6.7) has at least one pair of pure real roots $\pm\varepsilon_0^\mu$.

Next the following assumption is needed:

$$(\mathcal{A}_{12})\ \ M_{1R}M_{2R} + M_{1I}M_{2I} > 0,$$

where

$$\begin{cases} \mathcal{M}_{1R} = \tau_3 + \tau_6 + 2\varepsilon_0^\mu \sin\varepsilon_0^\mu\delta_{0*} + (\tau_1 + \tau_5)\cos\varepsilon_0^\mu\delta_{0*}, \\ \mathcal{M}_{1I} = -2\varepsilon_0^\mu\cos\varepsilon_0^\mu\delta_{0*} + (\tau_1 + \tau_5)\sin\varepsilon_0^\mu\delta_{0*}, \\ \mathcal{M}_{2R} = \left[\varepsilon_0^{\mu 3} - (\tau_1\tau_5 + \tau_2\tau_4 + \tau_3\tau_6)\varepsilon_0^\mu\right]\sin\varepsilon_0^\mu\delta_{0*} + (\tau_1 + \tau_5)\varepsilon_0^\mu\cos\varepsilon_0^\mu\delta_{0*}, \\ \mathcal{M}_{2I} = \left[-\varepsilon_0^{\mu 3} + (\tau_1\tau_5 + \tau_2\tau_4 - \tau_3\tau_6)\varepsilon_0^\mu\right]\cos\varepsilon_0^\mu\delta_{0*} + (\tau_1 + \tau_5)\varepsilon_0^\mu\sin\varepsilon_0^\mu\delta_{0*}. \end{cases} \tag{6.24}$$

Lemma 6.1. *Suppose that $\lambda(\theta) = \bar{\xi}_1(\theta) + i\bar{\xi}_2(\theta)$ is the root of Eq (6.7) at $\delta = \delta_{0*}$ such that $\bar{\xi}_1(\delta_{0*}) = 0$, $\bar{\xi}_2(\delta_{0*}) = \varepsilon_0^\mu$, then $Re\left(\frac{d\lambda}{d\delta}\right)\Big|_{\delta=\delta_{0*},\varepsilon=\varepsilon_0^\mu} > 0$.*

Proof. By Eq (6.7), one gets

$$\begin{aligned} &(\tau_3 + \tau_6)\frac{\mathrm{d}\lambda}{\mathrm{d}\delta} - (2\lambda - \tau_1 - \tau_5)e^{\lambda\delta}\frac{\mathrm{d}\lambda}{\mathrm{d}\delta} - \left[\lambda^2 + (-\tau_1 - \tau_5)\lambda + \tau_2\tau_4 + \tau_1\tau_5\right] \\ &\times e^{\lambda\delta}\left(\lambda + \delta\frac{\mathrm{d}\lambda}{\mathrm{d}\delta}\right) - e\tau_3\tau_6e^{-\lambda\delta}\left(\lambda + \delta\frac{\mathrm{d}\lambda}{\mathrm{d}\delta}\right) = 0, \end{aligned} \tag{6.25}$$

which implies

$$\left(\frac{d\lambda}{d\delta}\right)^{-1} = \frac{\mathcal{M}_1(\lambda)}{\mathcal{M}_2(\lambda)} - \frac{\delta}{\lambda}, \tag{6.26}$$

where

$$\begin{cases} \mathcal{M}_1(\lambda) = \tau_3 + \tau_6 - (2\lambda - \tau_1 - \tau_5)e^{\lambda\delta}, \\ \mathcal{M}_2(\lambda) = \left[\lambda^2 + (-\tau_1 - \tau_5)\lambda + \tau_2\tau_4 + \tau_1\tau_5\right]\lambda e^{\lambda\delta} - \tau_3\tau_6\lambda e^{-\lambda\delta}. \end{cases} \tag{6.27}$$

Hence

$$\mathrm{Re}\left[\left(\frac{d\lambda}{d\delta}\right)^{-1}\right]_{\delta=\delta_{0*},\varepsilon=\varepsilon_0^\mu} = \mathrm{Re}\left[\frac{\mathcal{F}_1(\lambda)}{\mathcal{F}_2(\lambda)}\right]_{\delta=\delta_{0*},\varepsilon=\varepsilon_0^\mu} = \frac{\mathcal{M}_{1R}\mathcal{M}_{2R} + \mathcal{M}_{1I}\mathcal{M}_{2I}}{\mathcal{M}_{2R}^2 + \mathcal{M}_{2I}^2}. \tag{6.28}$$

By $(\mathcal{A}_{12})$, we have

$$\mathrm{Re}\left[\left(\frac{d\lambda}{d\delta}\right)^{-1}\right]_{\delta=\delta_{0*},\varepsilon=\varepsilon_0^{\mu}} > 0, \tag{6.29}$$

which completes the proof.

Depending on the study above, the following conclusion is acquired:

Theorem 6.1. *Suppose that $(\mathcal{A}_{10})$–$(\mathcal{A}_{12})$ hold, then the equilibrium point $E(u_{1\star}, u_{2\star})$ of model (6.1) is locally asymptotically stable if $\delta \in [0, \delta_{0*})$ and model (6.1) generates a cluster of Hopf bifurcations at the equilibrium point $E(u_{1\star}, u_{2\star})$ when $\delta = \delta_{0*}$.*

Remark 6.1. *In this paper, some mathematical formulas and assumptions are very complicated (for example, $(\mathcal{A}_6)$, $(\mathcal{A}_{12})$, etc.), but we can check their correctness using computerized calculations.*

Remark 6.2. *The control methods in this paper can be applied to control the bifurcation or chaos of fractional-order dynamical system.*

7. Simulation outcomes

In this section, to verify the obtained key outcomes of this paper, we give some computer simulations.

Example 7.1. Consider the following predator-prey system incorporating delay:

$$\begin{cases} \dfrac{du_1(t)}{dt} = u_1(t)(h_1 - a_1u_1(t)) - \dfrac{d_1u_1(t)u_2(t)}{1 + bu_1(t)}, \\ \dfrac{du_2(t)}{dt} = u_2(t)[h_2 - a_2u_2(t-\delta)] + \dfrac{dd_1u_1(t)u_2(t)}{1 + bu_1(t)}, \end{cases} \tag{7.1}$$

where $h_1 = 0.5, h_2 = 0.5, a_1 = 2, a_2 = 2, d_1 = 0.4, b = 0.1, d = 0.45$. Clearly, model (7.1) admits a unique positive equilibrium point $E(0.1975, 0.2674)$. One can easily derive that the conditions $(\mathcal{A}_1)$–$(\mathcal{A}_3)$ of Theorem 3.1 hold. Making use of computer software, one can obtain that $\delta_0 \approx 2.9$. To verify the correctness of the gained outcomes of Theorem 3.1, we choose two nonidentical values of delay. One is $\delta = 2.8$ and the other is $\delta = 2.97$. If $\delta = 2.8 < \delta_0 \approx 2.9$, we gain computer simulation diagrams that are given in Figure 1. From Figure 1, we can easily understand that $u_1 \to 0.1975, u_2 \to 0.2674$ when $t \to +\infty$. Namely, unique positive equilibrium point $E(0.1975, 0.2674)$ of model (7.1) maintains locally asymptotically stable status. If $\delta = 2.97 > \delta_0 \approx 2.9$, we gain computer simulation diagrams that are given in Figure 2. From Figure 2, we are able to see that u_1 is to keep a periodic quavering level around the value 0.1975, u_2 is to keep a periodic quavering level around the value 0.2674. In other words, a cluster of periodic solutions (namely, Hopf bifurcations) arise near the positive equilibrium point $E(0.1975, 0.2674)$.

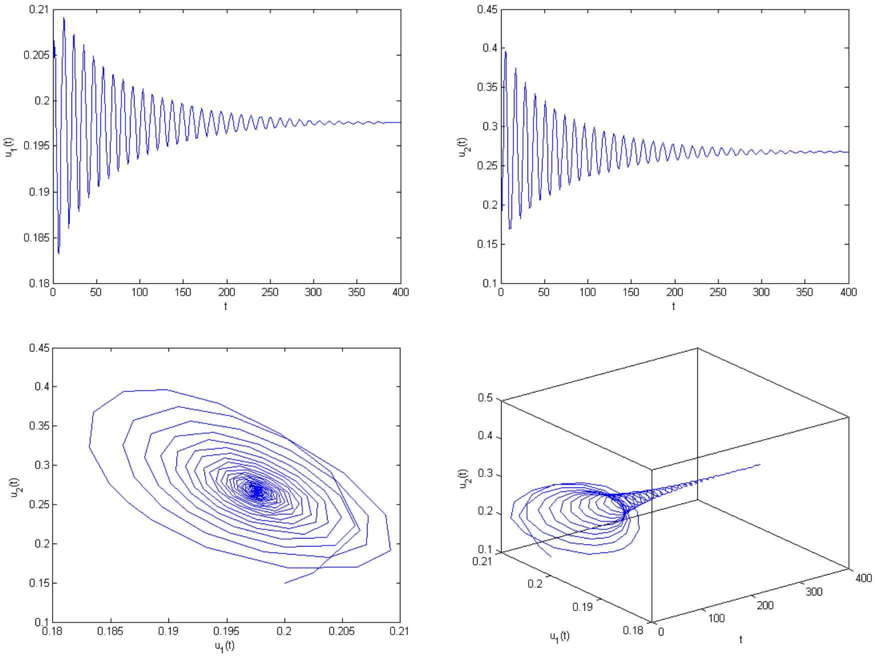

Figure 1. Computer experiment results of model (7.1) including the delay $\delta = 2.8 < \delta_0 = 2.9$. The positive equilibrium point $E(0.1975, 0.2674)$ keeps locally asymptotically stable status.

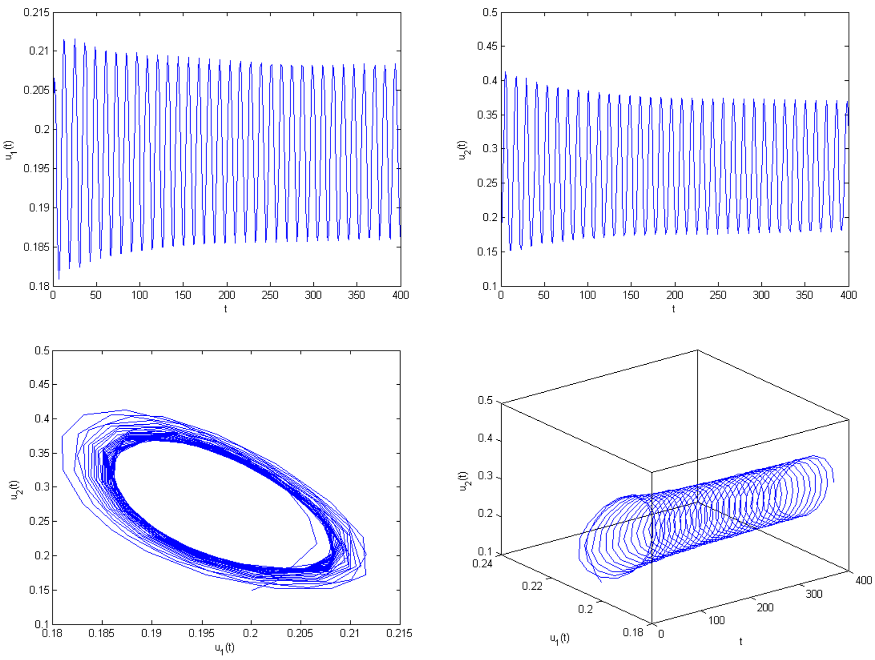

Figure 2. Computer experiment results of model (7.1) including the delay $\delta = 2.97 > \delta_0 = 2.9$. A cluster of periodic solutions (i.e., Hopf bifurcations) arise near the positive equilibrium point $E(0.1975, 0.2674)$.

Example 7.2. Consider the following controlled predator-prey system incorporating delay:

$$\begin{cases} \dfrac{du_1(t)}{dt} = \alpha_1 u_1(t)(h_1 - a_1 u_1(t)) - \dfrac{d_1 u_1(t) u_2(t)}{1 + b u_1(t)} + k[u_1(t-\delta) - u_1(t)], \\ \dfrac{du_2(t)}{dt} = u_2(t)[h_2 - a_2 u_2(t-\delta)] + \dfrac{d d_1 u_1(t) u_2(t)}{1 + b u_1(t)}, \end{cases} \tag{7.2}$$

where $h_1 = 0.5, h_2 = 0.5, a_1 = 2, a_2 = 2, d_1 = 0.4, b = 0.1, d = 0.45$. Let $\alpha_1 = 0.6, k = 0.5$. Clearly, model (7.2) admits a unique positive equilibrium point $E(0.1975, 0.2674)$. One can easily derive that the conditions $(\mathcal{A}_5)$–$(\mathcal{A}_7)$ of Theorem 4.1 hold. Making use of computer software, one can obtain that $\delta_\star \approx 2.85$. To verify the correctness of the gained outcomes of Theorem 4.1, we choose two nonidentical values of delay. One is $\delta = 2.83$ and the other is $\delta = 3.0$. If $\delta = 2.83 < \delta_* \approx 2.85$, we gain computer simulation diagrams that are given in Figure 3. From Figure 3, we can easily understand that $u_1 \to 0.1975, u_2 \to 0.2674$ when $t \to +\infty$. Namely, unique positive equilibrium point $E(0.1975, 0.2674)$ of model (7.2) maintains locally asymptotically stable status. If $\delta = 3.0 > \delta_* \approx 2.85$, we gain computer simulation diagrams that are given in Figure 4. From Figure 4, we are able to see that u_1 is to keep a periodic quavering level around the value 0.1975, u_2 is to keep a periodic quavering level around the value 0.2674. In other words, a cluster of periodic solutions (namely, Hopf bifurcations) arise near the positive equilibrium point $E(0.1975, 0.2674)$.

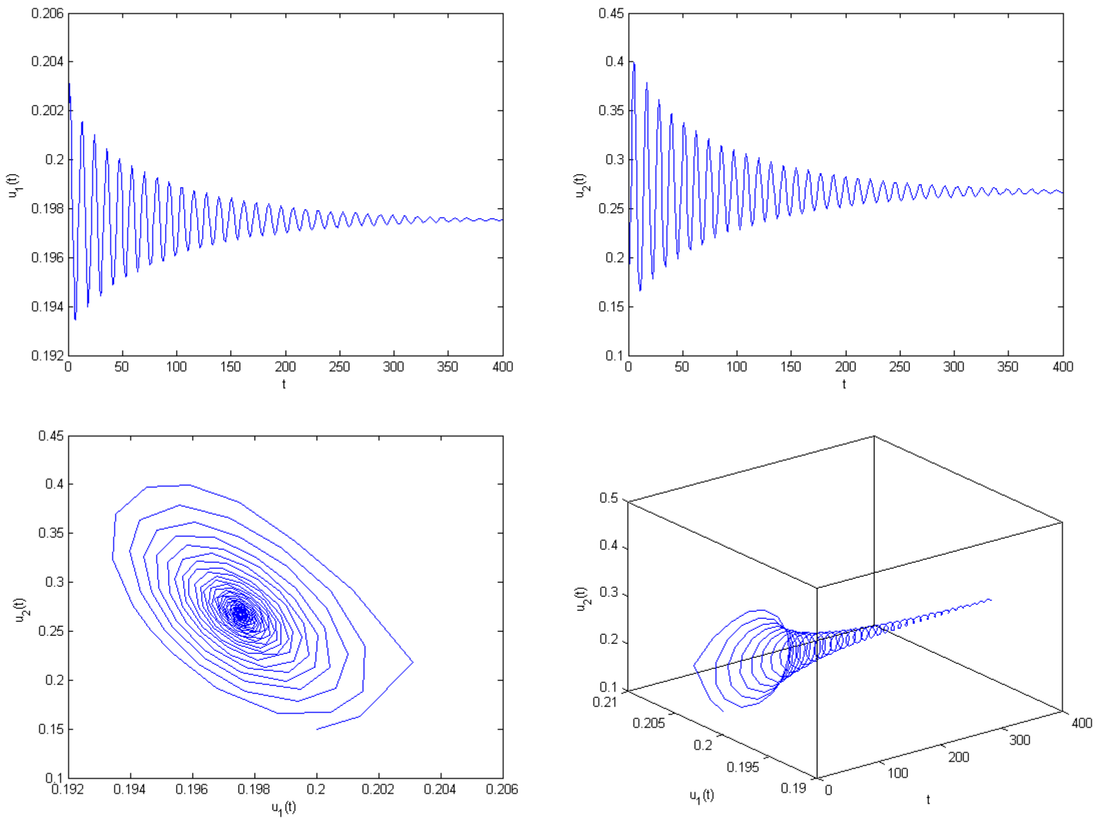

Figure 3. Computer experiment results of model (7.2) including the delay $\delta = 2.83 < \delta_* = 2.85$. The positive equilibrium point $E(0.1975, 0.2674)$ keeps locally asymptotically stable status.

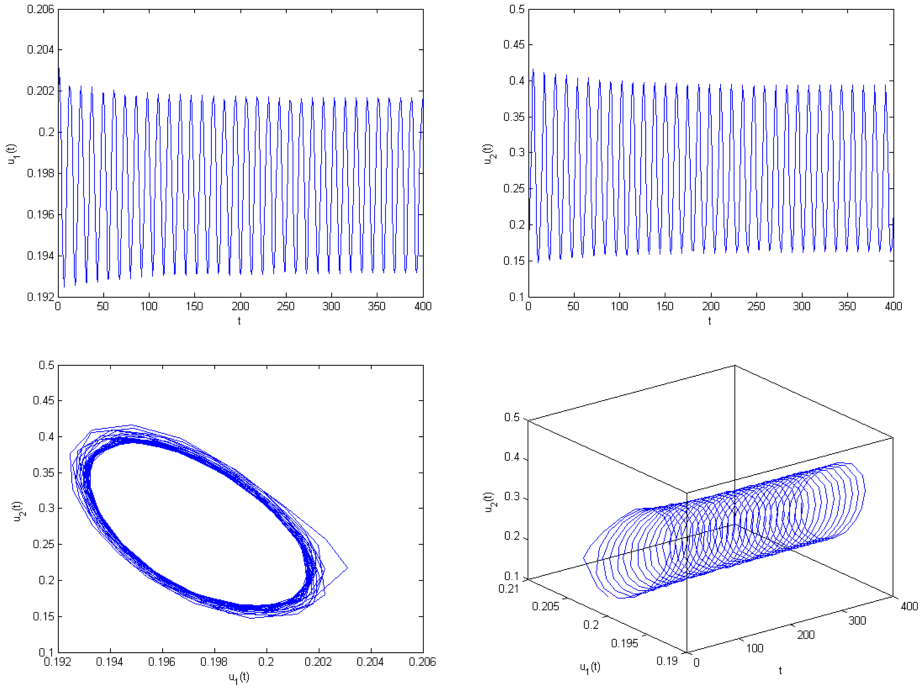

Figure 4. Computer experiment results of model (7.2) including the delay $\delta = 3.0 > \delta_* = 2.85$. A cluster of periodic solutions (i.e., Hopf bifurcations) arise near the positive equilibrium point $E(0.1975, 0.2674)$.

Example 7.3. Consider the following predator-prey system incorporating delay:

$$\begin{cases} \dfrac{du_1(t)}{dt} = u_1(t)(h_1 - a_1u_1(t-\theta)) - \dfrac{d_1u_1(t)u_2(t)}{1+bu_1(t)}, \\ \dfrac{du_2(t)}{dt} = u_2(t)[h_2 - a_2u_2(t-\theta)] + \dfrac{dd_1u_1(t)u_2(t)}{1+bu_1(t)}, \end{cases} \tag{7.3}$$

where $h_1 = 0.5, h_2 = 0.5, a_1 = 2, a_2 = 2, d_1 = 0.4, b = 0.1, d = 0.45$. Clearly, model (7.3) admits a unique positive equilibrium point $E(0.1975, 0.2674)$. One can easily derive that the conditions $(\mathcal{A}_8)$–$(\mathcal{A}_{10})$ of Theorem 5.1 hold. Making use of computer software, one can obtain that $\delta_\star \approx 2.8$. To verify the correctness of the gained outcomes of Theorem 5.1, we choose two nonidentical values of delay. One is $\delta = 2.7$ and the other is $\delta = 2.88$. If $\delta = 2.7 < \delta_\star \approx 2.8$, we gain computer simulation diagrams that are given in Figure 5. From Figure 5, we can easily understand that $u_1 \to 0.1975, u_2 \to 0.2674$ when $t \to +\infty$. Namely, unique positive equilibrium point $E(0.1975, 0.2674)$ of model (7.3) maintains locally asymptotically stable status. If $\delta = 2.88 > \delta_\star \approx 2.8$, we gain computer simulation diagrams that are given in Figure 6. From Figure 6, we are able to see that u_1 is to keep a periodic quavering level around the value 0.1975, u_2 is to keep a periodic quavering level around the value 0.2674. In other words, a cluster of periodic solutions (namely, Hopf bifurcations) arise near the positive equilibrium point $E(0.1975, 0.2674)$.

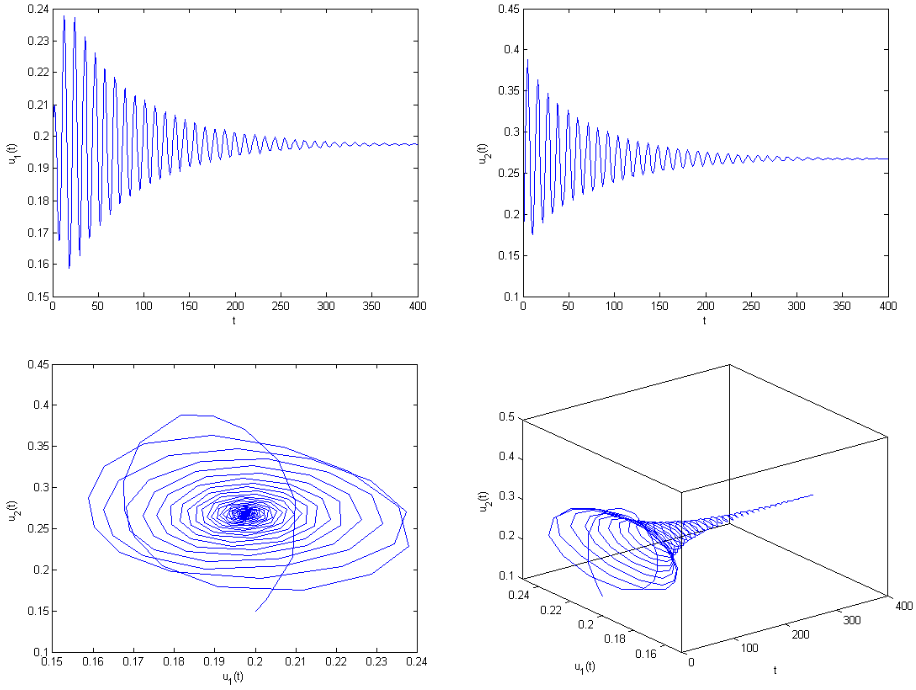

Figure 5. Computer experiment results of model (7.3) including the delay $\delta = 2.7 < \delta_\star = 2.8$. The positive equilibrium point $E(0.1975, 0.2674)$ keeps locally asymptotically stable status.

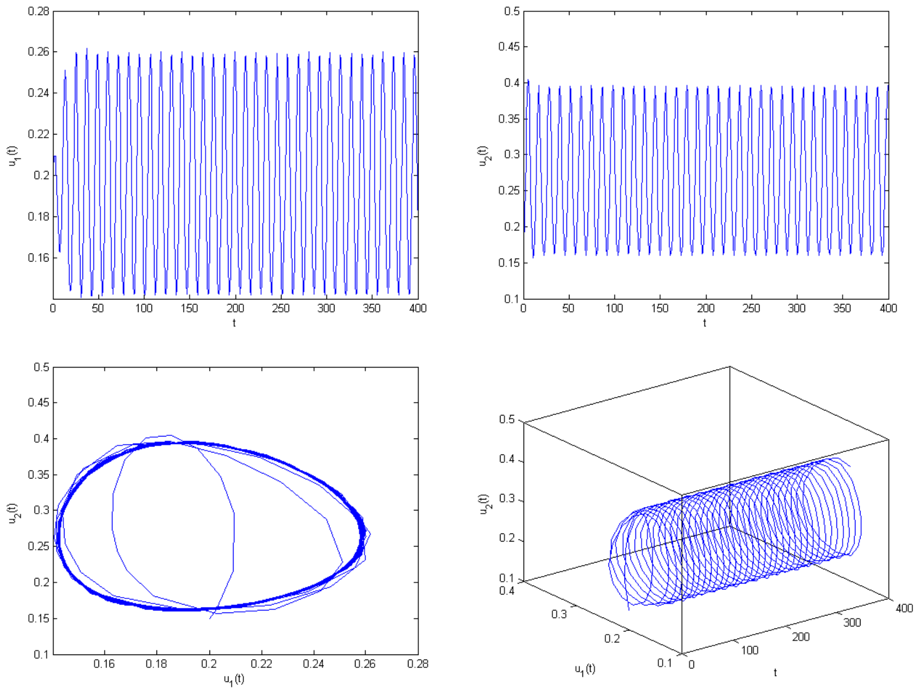

Figure 6. Computer experiment results of model (7.3) including the delay $\delta = 2.7 > \delta_\star = 2.8$. A cluster of periodic solutions (i.e., Hopf bifurcations) arise near the positive equilibrium point $E(0.1975, 0.2674)$.

Example 7.4. Consider the following controlled predator-prey system incorporating delay:

$$\begin{cases} \dfrac{du_1(t)}{dt} = u_1(t)(h_1 - a_1u_1(t-\delta)) - \dfrac{d_1u_1(t)u_2(t)}{1+bu_1(t)} + k_1[u_1(t-\delta) - u_1(t)], \\ \dfrac{du_2(t)}{dt} = u_2(t)[h_2 - a_2u_2(t-\delta)] + \dfrac{dd_1u_1(t)u_2(t)}{1+bu_1(t)} + k_2[u_2(t-\delta) - u_2(t)], \end{cases} \tag{7.4}$$

where $h_1 = 0.5, h_2 = 0.5, a_1 = 2, a_2 = 2, d_1 = 0.4, b = 0.1, d = 0.45$. Let $k_1 = 0.3, k_2 = 0.1$. Clearly, model (7.4) admits a unique positive equilibrium point $E(0.1975, 0.2674)$. One can easily derive that the conditions $(\mathcal{A}_8)$–$(\mathcal{A}_{10})$ of Theorem 6.1 hold. Making use of computer software, one can obtain that $\delta_{0*} \approx 4.1$. To verify the correctness of the gained outcomes of Theorem 6.1, we choose two nonidentical values of delay. One is $\delta = 3.8$ and the other is $\delta = 4.4$. If $\delta = 3.8 < \delta_{0*} \approx 4.1$, we gain computer simulation diagrams that are given in Figure 7. From Figure 7, we can easily understand that $u_1 \to 0.1975, u_2 \to 0.2674$ when $t \to +\infty$. Namely, unique positive equilibrium point $E(0.1975, 0.2674)$ of model (7.4) maintains locally asymptotically stable status. If $\delta = 4.4 > \delta_{0*} \approx 4.1$, we gain computer simulation diagrams that are given in Figure 8. From Figure 8, we are able to see that u_1 is to keep a periodic quavering level around the value 0.1975, u_2 is to keep a periodic quavering level around the value 0.2674. In other words, a cluster of periodic solutions (namely, Hopf bifurcations) arise near the positive equilibrium point $E(0.1975, 0.2674)$.

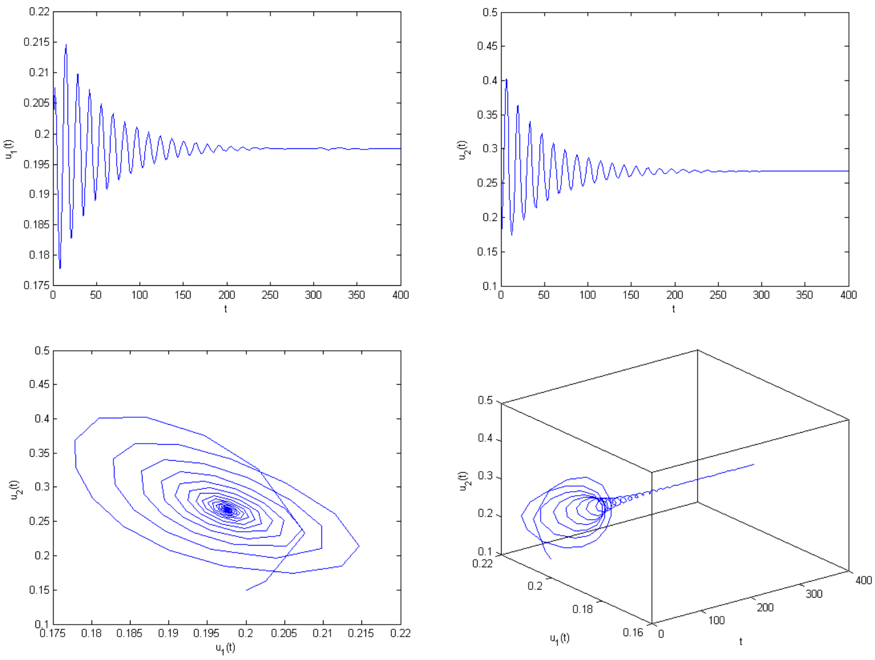

Figure 7. Computer experiment results of model (7.4) including the delay $\delta = 3.8 < \delta_{0*} = 4.1$. The positive equilibrium point $E(0.1975, 0.2674)$ keeps locally asymptotically stable status.

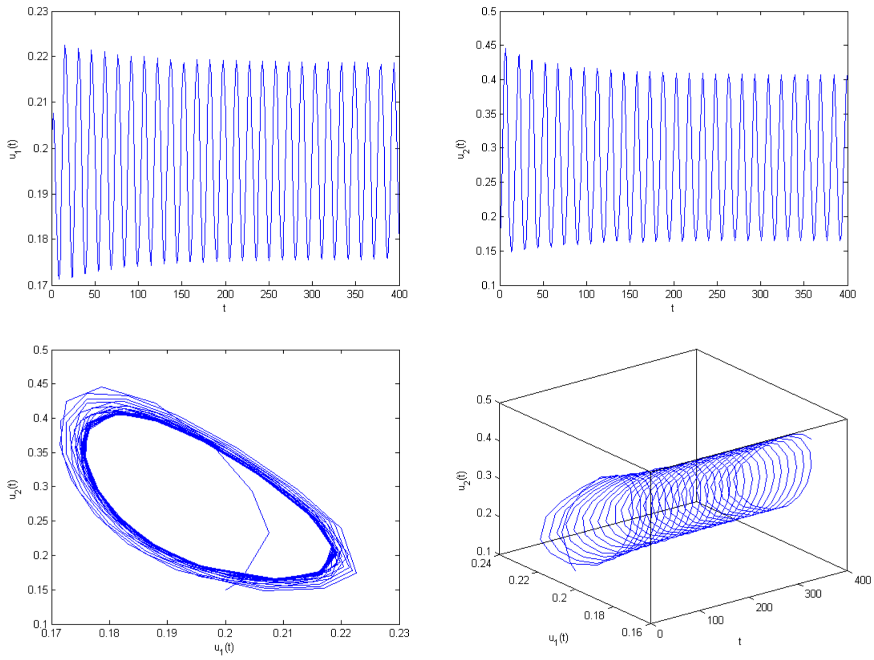

Figure 8. Computer experiment results of model (7.4) including the delay $\delta = 4.4 > \delta_{0*} = 4.1$. A cluster of periodic solutions (i.e., Hopf bifurcations) arise near positive equilibrium point $E(0.1975, 0.2674)$.

Remark 7.1. *Based on the computer simulation figures in Examples 7.1 and 7.2, one can easily know that the bifurcation values of model (7.1) and model (7.2) are* $\delta_0 \approx 2.9$ *and* $\delta_* \approx 2.85$*, which implies that we can reduce the domain of stability and shorten the time of emergence of bifurcation of model (7.1) via the designed hybrid controller. Based on the computer simulation figures in Examples 7.3 and 7.4, one can easily know that the bifurcation values of model (7.3) and model (7.3) are* $\delta_\star \approx 2.8$ *and* $\delta_{0*} \approx 4.1$*, which implies that we can enlarge the domain of stability and delay the time of emergence of bifurcation of model (7.3) via the designed extended delayed feedback controller.*

8. Conclusions

Nowadays, the investigation of predator-prey models has attracted much interest from mathematical and biological circles. From a mathematical point of view, revealing the effect of time delay on the many dynamical peculiarities of predator-prey models is a very significant topic. In this article, two new delayed predator-prey models are formulated. The non-negativeness, existence and uniqueness, and boundedness of solution of the established delayed predator-prey models are detailedly analyzed. By regarding the delay as the parameter of bifurcation, we gain two delay-independent criteria to guarantee the emergence of bifurcation and stability of the established two delayed predator-prey models. Making use of two different controllers, we have availably adjusted the region of stability and the time of onset of the bifurcation phenomenon of the two delayed predator-prey models. The fruits of this article have immense theoretical significance in taking control of the balance of the concentrations of predator and prey. Furthermore, the exploration idea can be applied to explore the control problem of bifurcation in many other differential models. In the near future, we will adopt other controllers to deal with the bifurcation control of these two delayed predator-prey models. Recently, there have many studies on Hopf bifurcation of fractional-order dynamical models [26–31]. We will also focus on Hopf bifurcation of fractional-order predator-prey models in the near future.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflict of interest.

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