AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 9(1): 1595–1621.
DOI: 10.3934/math.2024079
Received: 28 October 2023
Revised: 25 November 2023
Accepted: 30 November 2023
Published: 13 December 2023

Research article

On multi-bump solutions for a class of Schrödinger-Poisson systems with p-Laplacian in $\mathbb{R}^3$

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Abstract: In this article, we consider the following a class of Schrödinger-Poisson systems with p-Laplacian in $\mathbb{R}^3$ of the form:

$$\begin{cases} -\Delta_p u + (\lambda b(x) + 1)|u|^{p-2}u + \phi|u|^{s-2}u = g(u) & \text{in } \mathbb{R}^3, \\ -\Delta\phi = |u|^s & \text{in } \mathbb{R}^3, \end{cases}$$

where $1 < p < 3$, $\frac{p}{2} < s < p$, $\Delta_p u := div(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator, λ is a positive parameter. Assume that the nonnegative function b possesses a potential well $\text{int}(b^{-1}(\{0\}))$, which is composed of k disjoint components $\Omega_1, \Omega_2, \cdots, \Omega_k$ and consider the nonlinearity g with subcritical growth. Using the variational methods and Morse iteration technique, the existence of positive multi-bump solutions are obtained.

Keywords: Schrödinger-Poisson system; p-Laplacian; multi-bump solution; variational methods
Mathematics Subject Classification: 35J20, 35J60, 35J62

1. Introduction

In this article, we deal with the following Schrödinger-Poisson system with p-Laplacian in $\mathbb{R}^3$:

$$\begin{cases} -\Delta_p u + (\lambda b(x) + 1)|u|^{p-2}u + \phi|u|^{s-2}u = g(u) & \text{in } \mathbb{R}^3, \\ -\Delta\phi = |u|^s & \text{in } \mathbb{R}^3, \end{cases} \tag{P_λ}$$

where $1 < p < 3$, $\frac{p}{2} < s < p$, $\Delta_p u := div(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator, and λ is a positive parameter. Furthermore, we give fundamental assumptions regarding the nonnegative function b:

(B) The set $\text{int}(b^{-1}(\{0\}))$ is nonempty, and there exist mutually exclusive open components Ω_1, $\Omega_2, \cdots, \Omega_k$ such that

$$\text{int}(b^{-1}(\{0\})) = \cup_{j=1}^{k}\Omega_j \tag{1.1}$$

and

$$\mathrm{dist}(\Omega_i, \Omega_j) > 0 \ \ for \ \ i \neq j, \ \ i, j = 1, 2, \cdots, k. \tag{1.2}$$

Obviously, condition (B) implies that

$$b^{-1}(\{0\}) = \cup_{j=1}^{k} \overline{\Omega}_j. \tag{1.3}$$

Furthermore, the nonlinear term g fulfills the following assumptions:
(G_1) $\lim\limits_{s\to 0} \frac{g(s)}{s^{p-1}} = 0$ and $g(s) = 0$ for $s \leq 0$;
(G_2) $\lim\limits_{|s|\to+\infty} \frac{g(s)}{s^{p^*-1}} = 0$, where $p^* := \frac{3p}{3-p}$ is the critical Sobolev exponent;
(G_3) there exists $\theta > 2p$ such that

$$0 < \theta G(s) \leq s g(s), \ \forall s \in \mathbb{R} \setminus \{0\};$$

(G_4) $\frac{G(s)}{s^{2p-1}}$ is increasing in $|s| > 0$.

Our research into problem (P_λ) is founded upon the necessity of incorporating mathematical theory and practical applications. First of all, for the case $p = 2$, problem (P_λ) is reduced to a special form of the stationary Schrödinger-Poisson system

$$\begin{cases} -\Delta u + V(x)u + \phi u = g(u) & \text{in } \mathbb{R}^3, \\ -\Delta \phi = |u|^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{1.4}$$

which explores the interaction between electromagnetic fields generated by movement and quantum particles, and it is widely studied due to its strong physical background. For further background on the modelsat hand, we refer readers to [1–5], and in particular to the seminal work of Brézis and Nirenberg [6] for the Laplacian, which serves as a springboard for the development of various quasilinear extensions. To gain a deeper understanding of the physical underpinnings behind problem (1.4), we cite the papers of Ruiz [7], Sénchez and Soler [8] and Zhang and Zhang [9].

On the other hand, extensive research has been conducted by numerous scholars for problem (1.4), who focused on establishing the existence and non-existence of solutions, ground state solutions, multiplicity of solutions, semiclassical limit and concentrations of solutions and radial and non-radial solutions. We cite the papers of Azzollini and Pomponio [10], Cerami and Vaira [11], Coclite [12], D'Aprile and Mugnai [13], d'Avenia [14], Ianni and Vaira [15], Kikuchi [16], Siciliano [17] and Zhao and Zhao [18]. We need to point out in particular that if there is no Poisson term in problem (1.4). Ding and Tanaka [19] considered the existence of positive multi-bump solution for the problem

$$\begin{cases} -\Delta u + (\lambda b(x) + Z(x))u = u^q & \text{in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{1.5}$$

where $q \in (1, \frac{N+2}{N-2})$, they have demonstrated that problem (1.5) possesses at least $2^k - 1$ solutions u_λ for sufficiently large values of λ. Specifically, it has been proven that for every non-empty subset Υ of $\{1, \cdots, k\}$, given any sequence $\lambda_n \to \infty$, we can extract a subsequence (λ_{n_i}) such that $(u_{\lambda_{n_i}})$ converges strongly in $H^1(\mathbb{R}^N)$ to a function u. This function u satisfies the condition $u = 0$ outside $\Omega_\Upsilon = \bigcup_{j\in\Upsilon} \Omega_j$ and on each domain $u|_{\Omega_j}$, where $j \in \Upsilon$, it represents the least energy solution for

$$\begin{cases} -\Delta u + Z(x)u = u^q & \text{in } \Omega_j, \\ u \in H_0^1(\Omega_j), \ \ u > 0 & \text{in } \Omega_j. \end{cases} \tag{1.6}$$

Alves and Yang [20] studied problem (1.4) using a similar method and proved the existence of positive multi-bump solutions by variational methods. Alves and Figueiredo [21] considered a class of Kirchhoff problems with subcritical growth, and the existence of positive multi-bump solutions are obtained using variational methods. Recently, Liang and Shi [22] studied a class of the (p, q) Kirchhoff type problems with a convolution term in $\mathbb{R}^N$. With the appropriate assumptions, together with the penalization techniques, the Morse iterative method and variational method, the existence and multiplicity of multi-bump solutions are obtained for this problem.

For the case $p \neq 2$, Du et al. [23] studied the results of the existence of Kirchhoff-Poisson systems with p-Laplacian under the subcritical case through application of the Mountain Pass Theorem. Later, Du et al. in [24] conducted a comprehensive investigation on quasilinear Schrödinger-Poisson systems. For critical case, Du et al. in [25] also successfully established the existence of ground state solutions using the variational approach. However, when we shift our focus towards exploring positive multi-bump solutions for the Schrödinger-Poisson system with p-Laplacian, it becomes evident that there is a relative scarcity of literature in this area.

Inspired by the above achievements, we aim to prove the existence of positive multi-bump solutions for the Schrödinger-Poisson system with p-Laplacian (P_λ). The primary challenge in addressing problem (1.1) resides in its non-local term and the entirety of space, which significantly complicates the study of this issue. We also have to demonstrate the existence of the least energy solution for the corresponding problem (see Section 2). To some extent, we generalize the previous results [19, 20].

Before presenting our main conclusions, we first consider the following Poisson problem

$$-\Delta\phi = |u|^s \ \text{ in } \mathbb{R}^3. \tag{1.7}$$

According to the Lax-Milgram Theorem, given $u \in W^{1,p}(\mathbb{R}^3)$, we know that there exists an unique $\phi = \phi_u \in D^{1,2}(\mathbb{R}^3)$ such that $-\Delta\phi = |u|^s$ in $\mathbb{R}^3$. By employing conventional arguments, it can be deduced that ϕ_u verifies the subsequent properties (see [7, 13, 18, 26]).

Lemma 1.1. *For any $u \in W^{1,p}(\mathbb{R}^3)$, we have*

i) $\phi_u(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{|u(y)|^s}{|x-y|}dy$ *for all* $x \in \mathbb{R}^3$.

ii) There exists a positive constant $C > 0$ such that

$$\int_{\mathbb{R}^3}|\nabla\phi_u|^2dx = \int_{\mathbb{R}^3}\phi_u|u|^sdx \le C\|u\|^s_{W^{1,p}(\mathbb{R}^3)}, \ \ \forall\, u \in W^{1,p}(\mathbb{R}^3),$$

where $\|u\|_{W^{1,p}(\mathbb{R}^3)} = (\int_{\mathbb{R}^3}(|\nabla u|^p + |u|^p)dx)^{\frac{1}{p}}$.

iii) $\phi_u \ge 0, \ \forall\, u \in W^{1,p}(\mathbb{R}^3)$.

iv) $\phi_{tu} = t^s\phi_u, \ \forall\, t > 0$ *and* $u \in W^{1,p}(\mathbb{R}^3)$.

v) If $u_n \rightharpoonup u$ *in* $W^{1,p}(\mathbb{R}^3)$, *then* $\phi_{u_n} \rightharpoonup \phi_u$ *in* $D^{1,2}(\mathbb{R}^3)$ *and*

$$\lim_{n\to+\infty}\int_{\mathbb{R}^3}\phi_{u_n}|u_n|^sdx \ge \int_{\mathbb{R}^3}\phi_u|u|^sdx.$$

vi) If $u_n \to u$ *in* $W^{1,p}(\mathbb{R}^3)$, *then* $\phi_{u_n} \to \phi_u$ *in* $D^{1,2}(\mathbb{R}^3)$. *Hence,*

$$\lim_{n\to+\infty}\int_{\mathbb{R}^3}\phi_{u_n}|u_n|^sdx = \int_{\mathbb{R}^3}\phi_u|u|^sdx.$$

Therefore, $(u,\phi)\in W^{1,p}(\mathbb{R}^3)\times D^{1,2}(\mathbb{R}^3)$ is a solution of problem (P_λ) if and only if, $u\in W^{1,p}(\mathbb{R}^3)$ is a solution to the nonlocal problem

$$\begin{cases} -\Delta_p u + b(x)|u|^{p-2}u + \phi_u|u|^{s-2}u = g(u) & \text{in } \mathbb{R}^3,\\ u\in W^{1,p}(\mathbb{R}^3) & \text{in } \mathbb{R}^3, \end{cases} \tag{P}$$

where $\phi_u=\phi\in D^{1,2}(\mathbb{R}^3)$.

Now, we present our primary findings as follows:

Theorem 1.1. *Assume that (B) and (G_1)–(G_4) hold. Then, there exists a positive value $\lambda_0>0$ with the following characteristic: For any non-empty subset Υ of $\{1,2,\cdots k\}$ and $\lambda\geq\lambda_0$, problem (P_λ) has a solution u_λ. Furthermore, if we fix the subset Υ, then for any sequence $\lambda_n\to\infty$, we can extract a subsequence (λ_{n_i}) such that $u_{\lambda_{n_i}}$ converges strongly in $W^{1,p}(\mathbb{R}^3)$ to a function u, which satisfies $u=0$ outside $\Omega_\Upsilon=\cup_{j\in\Upsilon}\Omega_j$, and $u|_{\Omega_\Upsilon}$ is a least energy solution for nonlocal problem*

$$\begin{cases} -\Delta_p u + |u|^{p-2}u + \phi|u|^{s-2}u = g(u) \ \ in \ \ \Omega_\Upsilon,\\ -\Delta\phi = |u|^s \ \ in \ \ \Omega_\Upsilon. \end{cases} \tag{$(P)_{\infty,\Upsilon}$}$$

The paper is organized as follows. In Section 2, we will prove the existence of least energy solution for problem $(P)_{\infty,\Upsilon}$. In Section 3, we consider the auxiliary problem by adapting the concepts explored by del Pino and Felmer in their seminal work [27], and then prove the (PS) condition holds. In Section 4, we first prove the boundedness of certain solutions to problem $(P)_\infty$ outside Ω'_Υ, and then we mainly give a special minimax value for J_λ. Finally, we will prove Theorem 2.1.

2. Least energy solution for the problem $(P)_{\infty,\Upsilon}$

In this section, our primary objective is to prove the existence of least energy solution for the problem $(P)_{\infty,\Upsilon}$. For simplicity, let $\Upsilon=\{1,2\}$ and $\Omega_\Upsilon=\Omega_1\cup\Omega_2$. The following energy function J associated with nonlocal problem $(P)_{\infty,\Upsilon}$ can be expressed as follows

$$J(u)=\frac{1}{p}\int_{\Omega_\Upsilon}(|\nabla u|^p+|u|^p)dx+\frac{1}{2s}\int_{\Omega_\Upsilon}\phi_u|u|^s dx-\int_{\Omega_\Upsilon}G(u)dx.$$

We will demonstrate the existence of $w\in\mathcal{M}_\Upsilon$ such that

$$J(w)=\inf_{u\in\mathcal{M}_\Upsilon}J(u),$$

where

$$\mathcal{M}_\Upsilon=\{u\in\mathcal{N}_\Upsilon : J'(u)u_j=0 \ \ and \ \ u_j\neq0 \ \ \forall j\in\Upsilon\},$$

and $u_j=u|_{\Omega_j}$, $\mathcal{N}_\Upsilon$ the corresponding Nehari manifold defined by

$$\mathcal{N}_\Upsilon=\{u\in W_0^{1,p}(\Omega_\Upsilon)\backslash\{0\} : J'(u)u=0\}.$$

Afterwards, we employ a deformation lemma to establish that w serves as a critical point of J. Consequently, w emerges as the least energy solution for problem $(P)_{\infty,\Upsilon}$. The key characteristic of the least energy solution w is that it satisfies $w(x) > 0$, $\forall x \in \Omega_j$ and $\forall j \in \Upsilon$, which will be employed to describe the existence of multi-bump solutions.

Since our objective is to prove the existence of a least energy solution for problem $(P)_{\infty,\Upsilon}$, it is crucial to establish the existence of a critical point for J within the set $\mathcal{M}_\Upsilon$. To this end, we need to prove the properties of set $\mathcal{M}_\Upsilon$.

Lemma 2.1. *We have the following conclusions:*

(i) The set $\mathcal{M}_\Upsilon$ is not empty.
(ii) $\|w_j\|_j \geq \rho$, $\forall\, w \in \mathcal{M}_\Upsilon$, where $w_j = w|_{\Omega_j}$, $j = 1, 2$.
(iii) If (w_n) is a bounded sequence in $\mathcal{M}_\Upsilon$ and $q \in (p, p^*)$, we have

$$\liminf_n \int_\Omega |w_{n,j}|^q dx > 0,$$

where $w_{n,j} = w_j|_{w_j}$ for $j = 1, 2$.

Proof. To prove the conclusion (i), take $v \in W_0^{1,p}(\Omega)$ with $v_j \neq 0$ for $j = 1, 2$, we claim that there are $t, m > 0$ such that $J'(tv_1 + mv_2)v_1 = 0$ and $J'(tv_1 + mv_2)v_2 = 0$.

In fact, let

$$\mathcal{H}(t, m) = (J'(tv_1 + mv_2)(tv_1), J'(tv_1 + mv_2)(mv_2)).$$

From (G_1)–(G_3), a simple calculation reveals that there is $0 < r < R$ such that

$$J'(rv_1 + mv_2)(rv_1), J'(tv_1 + rv_2)(rv_2) > 0, \quad \forall t, m \in [r, R]$$

and

$$J'(Rv_1 + mv_2)(Rv_1), J'(tv_1 + Rv_2)(Rv_2) < 0, \quad \forall t, m \in [r, R].$$

Then the conclusion (i) of Lemma 2.1 follows by applying the Miranda theorem [28].

In order to obtain the conclusion (ii) of Lemma 2.1, we first claim that there exists $\rho > 0$ such that

$$J(u) \geq \frac{\|u\|^2}{2p} \text{ and } \|u\| \geq \rho, \forall u \in \mathcal{N}_\Upsilon. \tag{2.1}$$

In fact, from (G_4), for any $u \in \mathcal{N}_\Upsilon$, we have

$$2pJ(u) = 2pJ(u) - J'(u)u = \|u\|^p + (\frac{p}{s} - 1)\int_\Omega \phi_u |u|^s dx + \int_\Omega [ug(u) - 2pG(u)]dx \geq \|u\|^p$$

and so

$$J(u) \geq \frac{\|u\|^p}{2p}, \quad \forall u \in \mathcal{N}_\Upsilon.$$

On the other hand, by (G_1) and (G_2), there is $C > 0$ such that

$$g(s)s \leq \varepsilon s^p + C_\varepsilon s^{p^*} \quad for \ all \ s \in \mathbb{R}.$$

Since $J'(u)u = 0$, thus

$$\|u\|^p \leq \|u\|^p + \int_\Omega \phi_u |u|^s dx = \int_\Omega ug(u)dx \leq \varepsilon \int_\Omega |u|^p dx + C_\varepsilon \int_\Omega |u|^{p^*} dx.$$

Then, by the Sobolev embeddings, one has

$$\|u\|^p \leq \frac{1}{2}\varepsilon C\|u\|^p + \hat{C}_\varepsilon\|u\|^{p^*}.$$

Let $\varepsilon \in (0, \frac{1}{C})$ and take $\rho = (\frac{1}{2\hat{C}_\varepsilon})^{\frac{3-p}{p^2}}$, we can see that (2.1) is true.

If $w \in \mathcal{M}_\Upsilon$, we have that $J'(w)w_1 = J'(w)w_2 = 0$.

$$\|w_j\|_j < \|w_j\|_j + \int_\Omega \phi_{w_j}|w_j|^s dx = \int_\Omega g(w_j)w_j dx \ \ for \ \ j = 1, 2.$$

From the previous discussion, it can be inferred that $\|w_j\|_j \geq \rho$ for $j = 1, 2$. The conclusion (*ii*) of Lemma 2.1 is obtained.

Finally, by (G_1) and (G_2), given $\varepsilon > 0$ there exists $C > 0$ such that

$$g(s)s \leq \varepsilon s^p + C_\varepsilon|s|^q + \varepsilon s^{p^*} \ \ for \ \ all \ \ s \in \mathbb{R}.$$

Since $w_n \in \mathcal{M}_\Upsilon$, thus

$$\rho^p \leq \|w_{n,j}\|_j^p < \int_{\Omega_j} w_{n,j}g(w_{n,j})dx \leq \varepsilon \int_{\Omega_j} |w_{n,j}|^p dx + C_\varepsilon \int_{\Omega_j} |w_{n,j}|^q dx + \varepsilon \int_{\Omega_j} |w_{n,j}|^{p^*} dx,$$

that is,

$$\rho^p \leq \varepsilon(\int_{\Omega_j} |w_{n,j}|^p dx + \int_{\Omega_j} |w_{n,j}|^{p^*} dx) + C\int_{\Omega_j} |w_{n,j}|^q dx.$$

Using the boundedness of (w_n), there is C_1 such that

$$\rho^p \leq \varepsilon C_1 + \int_{\Omega_j} |w_{n,j}|^q dx.$$

Fixing $\varepsilon = \frac{\rho^p}{2C_1}$, we get

$$\int_{\Omega_j} |w_{n,j}|^q dx \geq \frac{\rho^p}{2C},$$

showing that

$$\liminf_n \int_\Omega |w_{n,j}|^q dx \geq \frac{\rho^p}{2C} > 0.$$

The proof of conclusion (*iii*) of Lemma 2.1 is thus complete. □

Now, our primary objective is to demonstrate the following theorem.

Theorem 2.1. *Assume that* (G_1)–(G_4) *hold. Then there exists a positive least energy solution for problem* $(P)_{\infty,\Upsilon}$.

Proof. In the following, we represent c_0 as the infimum of J on $\mathcal{M}_\Upsilon$, that is,

$$c_0 = \inf_{v \in \mathcal{M}_\Upsilon} J(v).$$

According to Lemma 2.1, we conclude that $c_0 > 0$.

On the other hand, since $\mathcal{M}_\Upsilon$ is non-empty, we know that there is a sequence $(w_n) \subset \mathcal{M}_\Upsilon$ satisfying

$$\lim_n J(w_n) = c_0.$$

It can be demonstrated that the sequence (w_n) is bounded. Therefore, without loss of generality, we may suppose that there is $w \in W_0^{1,p}(\Omega)$ verifying

$$w_n \rightharpoonup w \ \ in \ \ W_0^{1,p}(\Omega),$$

$$w_n \to w \ \ in \ \ L^q(\Omega), \ \forall\, q \in [1, p^*)$$

and

$$w_n(x) \to w(x) \text{ a.e. in } \Omega.$$

Then, together (G_2) with the Strauss' compactness lemma [29], we have

$$\lim_n \int_{\Omega_j} |w_{n,j}|^q dx = \int_{\Omega_j} |w_n|^q dx,$$

$$\lim_n \int_{\Omega_j} w_{n,j} g(w_{n,j}) dx = \int_{\Omega_j} w_n g(w_j) dx$$

and

$$\lim_n \int_{\Omega_j} G(w_{n,j}) dx = \int_{\Omega_j} G(w_n) dx.$$

It can be deduced from Lemma 2.1 that $w_j \neq 0$ for $j = 1, 2$. Subsequently, according to Lemma 2.1, there are $t, m > 0$ verfying

$$J'(tw_1 + mw_2)w_1 = 0 \ \ and \ \ J'(tw_1 + mw_2)w_2 = 0.$$

Next, we shall establish that $t, m \leq 1$. In fact, since $J'(w_{n,j})w_{n,j} = 0$ for $j = 1, 2$, we can get

$$\|w_{n,1}\|_1^p + \int_{\Omega_1} \phi_{w_{n,1}} |w_{n,1}|^s dx + \int_{\Omega_1} \phi_{w_{n,2}} |w_{n,1}|^s dx = \int_{\Omega_1} g(w_{n,1}) w_{n,1} dx$$

and

$$\|w_{n,2}\|_2^p + \int_{\Omega_2} \phi_{w_{n,2}} |w_{n,2}|^s dx + \int_{\Omega_2} \phi_{w_{n,1}} |w_{n,2}|^s dx = \int_{\Omega_2} g(w_{n,2}) w_{n,2} dx.$$

Taking the limit in the above equalities. Since $\|w\|_1^p \leq \lim_{n\to\infty} \|w_{n_1}\|^p$, we obtain

$$\|w_1\|_1^p + \int_{\Omega_1} \phi_{w_1} |w_1|^s dx + \int_{\Omega_1} \phi_{w_2} |w_1|^s dx \leq \int_{\Omega_1} g(w_1) w_1 dx$$

and

$$\|w_2\|_2^p + \int_{\Omega_2} \phi_{w_2} |w_2|^s dx + \int_{\Omega_2} \phi_{w_1} |w_2|^s dx \leq \int_{\Omega_2} g(w_2) w_2 dx.$$

Recalling that

$$J'(tw_1 + mw_2)(tw_1) = J'(tw_1 + mw_2)(mw_2) = 0.$$

It follows that

$$t^p\|w_1\|_1^p + t^{2s}\int_{\Omega_1}\phi_{w_1}|w_1|^s dx + t^s m^s\int_{\Omega_1}\phi_{w_2}|w_1|^s dx = \int_{\Omega_1} g(tw_1)tw_1 dx$$

and

$$m^p\|w_2\|_2^p + m^{2s}\int_{\Omega_2}\phi_{w_2}|w_2|^s dx + t^s m^s\int_{\Omega_2}\phi_{w_1}|w_2|^s dx = \int_{\Omega_2} g(mw_2)mw_2 dx.$$

Now, without loss of generality, we shall assume that $m \geq t$. Thus

$$m^p\|w_2\|_2^p + m^{2s}\int_{\Omega_2}\phi_{w_2}|w_2|^s dx + m^{2s}\int_{\Omega_2}\phi_{w_1}|w_2|^s dx \geq \int_{\Omega_2} g(mw_2)mw_2 dx$$

and then

$$(\frac{1}{m^{2s-p}} - 1)\|w_2\|_2^p \geq \int_{\Omega_2}(\frac{g(mw_2)mw_2}{(mw_2)^{2s}} - \frac{g(w_2)w_2}{(w_2)^{2s}})(w_2)^{2s}dx.$$

If $m > 1$, the left side in this inequality is negative, but from (f_4), the right side is positive, thus we must have $m \leq 1$, which also implies that $t \leq 1$.

Our next step is to show that $J(tw_1 + mw_2) = c_0$. Recalling that $tw_1 + mw_2 \in \mathcal{M}_\Upsilon$, we derive that

$$c_0 \leq J(tw_1 + mw_2) = J(tw_1 + mw_2) - \frac{1}{2p}J'(tw_1 + mw_2)(tw_1 + mw_2).$$

Hence

$$c_0 \leq (J(tw_1) - \frac{1}{2p}J'(tw_1)(tw_1)) + (J(mw_2) - \frac{1}{2p}J'(mw_2)(mw_2)).$$

By the direct computation, we have

$$J(tw_1) - \frac{1}{2p}J'(tw_1)(tw_1) = \frac{t^p}{2p}\|w_1\|_1^p + t^{2s}(\frac{1}{2s} - \frac{1}{2p})\int_{\Omega_1}\phi_w|w_1|^s + \int_{\Omega_1}[\frac{1}{2p}g(tw_1)tw_1 - G(tw_1)]dx$$

and

$$J(w_1) - \frac{1}{2p}J'(w_1)(w_1) = \frac{1}{2p}\|w_1\|_1^p + (\frac{1}{2s} - \frac{1}{2p})\int_{\Omega_1}\phi_w|w_1|^s + \int_{\Omega_1}[\frac{1}{2p}g(w_1)w_1 - G(w_1)]dx.$$

From (G_4) and $t \leq 1$, we get

$$J(tw_1) - \frac{1}{2p}J'(tw_1)(tw_1) \leq J(w_1) - \frac{1}{2p}J'(w_1)(w_1)$$

and

$$J(mw_2) - \frac{1}{2p}J'(mw_2)(mw_2) \leq J(w_2) - \frac{1}{2p}J'(w_2)(w_2)$$

leading to

$$c_0 \leq (J(w_1) - \frac{1}{2p}J'(w_1)(w_1)) + (J(w_2) - \frac{1}{2p}J'(w_2)(w_2)).$$

By using the Fatou's lemma and (G_4), we see that

$$c_0 \leq J(tw_1 + mw_2) = \liminf_n (J(w_n) - \frac{1}{2p}J'(w_n)(w_n)) = \lim_n J(w_n) = c_0,$$

which means that

$$c_0 = J(tw_1 + mw_2).$$

Until now, we have demonstrated the existence of a $w_0 = tw_1 + mw_2 \in \mathcal{M}_\Upsilon$ such that $J(w_0) = c_0$. Moving forward, let us refer to w_0 as w, consequently

$$J(w_0) = c_0 \ \ and \ \ w \in \mathcal{M}_\Upsilon.$$

To establish the proof of Theorem 1.2, we claim that w serves as a critical point for functional J. In order to prove this claim, for each $\varphi \in W_0^{1,p}(\Omega)$, we introduce the function $Q^i : \mathbb{R}^3 \to \mathbb{R}$, $i = 1, 2$ given by

$$\begin{aligned} Q^1(r,z,l) = &\int_{\Omega_1} |\nabla(w_1 + r\varphi_1 + zw_1)|^p dx + \int_{\Omega_1} \phi(_{w+r\varphi+zw_1+lw_2})|w_1 + r\varphi_1 + zw_1|^s dx \\ &- \int_{\Omega_1} g(w_1 + r\varphi_1 + zw_1)(w_1 + r\varphi_1 + zw_1) dx \end{aligned}$$

and

$$\begin{aligned} Q^2(r,z,l) = &\int_{\Omega_2} |\nabla(w_2 + r\varphi_2 + zw_2)|^p dx + \int_{\Omega_2} \phi(_{w+r\varphi+zw_1+lw_2})|w_2 + r\varphi_2 + zw_2|^s dx \\ &- \int_{\Omega_2} g(w_2 + r\varphi_2 + zw_2)(w_2 + r\varphi_2 + zw_2) dx. \end{aligned}$$

By direct computation, we have

$$\frac{\partial Q^1}{\partial z}(0,0,0) = p\int_{\Omega_1} |\nabla w_1|^p dx + 2s\int_{\Omega_1} \phi_w |w_1|^s dx - \int_{\Omega_1} (g'(w_1)w_1^2 + g(w_1)w_1) dx$$

and so,

$$\begin{aligned} \frac{\partial Q^1}{\partial z}(0,0,0) &< \int_{\Omega_1} [(p-1)g(w_1)w_1 - g'(w_1)w_1^2] dx + (2s-p)\int_{\Omega_1} \phi_w |w_1|^s dx \\ &< \int_{\Omega_1} [(p-1)g(w_1)w_1 - g'(w_1)w_1^2] dx + p\int_{\Omega_1} \phi_w |w_1|^s dx. \end{aligned}$$

By (G_4), we know that $g'(s)s^2 \geq (2p-1)g(s)s$ for all $s \geq 0$, thus,

$$\frac{\partial Q^1}{\partial z}(0,0,0) < -p\left(\int_{\Omega_1} g(w_1)w_1 dx - \int_{\Omega_1} \phi_w |w_1|^s dx\right).$$

Now, recalling that $J'(w)w_1 = 0$, we have

$$\|w_1\|^p + \int_{\Omega_1} \phi_w |w_1|^s dx = \int_{\Omega_1} g(w_1)w_1 dx.$$

Then

$$\frac{\partial Q^1}{\partial z}(0,0,0) \leq -p\|w_1\|^p.$$

The same line of reasoning gives

$$\frac{\partial Q^2}{\partial z}(0,0,0) \leq -p\|w_2\|^p.$$

Hence, by employing the implicit function theorem, we can establish the existence of C^1-class functions $z(s)$, $l(s)$, defined on an interval $(-\xi, \xi)$, where $\xi > 0$. These functions satisfy the initial conditions $z(0) = l(0) = 0$ and

$$Q^i(r, z(s), l(s)) = 0, \quad s \in (-\xi, \xi), \quad i = 1, 2.$$

This shows that for any $r \in (-\xi, \xi)$, one has

$$v(r) = w + s\varphi + z(s)w_1 + l(s)w_2 \in \mathcal{M}_\Upsilon.$$

Since

$$J(w) = c_0 = \inf_{v \in \mathcal{M}_\Upsilon} J(v),$$

we derive that

$$J(v(s)) \geq J(w), \quad \forall s \in (-\xi, \xi),$$

that is,

$$J(w + s\varphi + z(s)w_1 + l(s)w_2) \geq J(w), \quad \forall s \in (-\xi, \xi).$$

From this, we have

$$\frac{J(w + s\varphi + z(s)w_1 + l(s)w_2) - J(w)}{s} \geq 0, \quad \forall s \in (0, \xi).$$

Taking the limit as $s \to 0$, we get

$$J'(w)(\varphi + z'(0)w_1 + l'(0)w_2) \geq 0.$$

Given that $J'(w)w_1 = J'(w)w_2 = 0$, the aforementioned inequality leads to

$$J'(w)\varphi \geq 0, \quad \forall \varphi \in W_0^{1,p}(\Omega).$$

The same line of reasoning gives

$$J'(w)\varphi \leq 0, \quad \forall \varphi \in W_0^{1,p}(\Omega)$$

and so,

$$J'(w)\varphi = 0, \quad \forall \varphi \in W_0^{1,p}(\Omega),$$

showing that w is a critical point for J. This completes the proof of Theorem 2.1. □

3. Compactness condition

In this section, we first give the recollection of the energy functional $I_\lambda : E_\lambda \to \mathbb{R}$ that is associated with problem $(P)_\lambda$ given by

$$I_\lambda(u) := \frac{1}{p} \int_{\mathbb{R}^3} (|\nabla u|^p + (\lambda b(x) + 1)|u|^p) dx + \frac{1}{2s} \int_{\mathbb{R}^3} \phi_u |u|^s dx - \int_{\mathbb{R}^3} G(u) dx,$$

where $E_\lambda = (E, \|\cdot\|_\lambda)$ with

$$E = \left\{ u \in W^{1,p}(\mathbb{R}^3); \int_{\mathbb{R}^3} b(x)|u|^p dx < \infty \right\}$$

and

$$\|u\|_{\lambda}=\left(\int_{\mathbb{R}^3}(|\nabla u|^p+(\lambda b(x)+1)|u|^p)dx\right)^{\frac{1}{p}}.$$

Thus $E_\lambda \hookrightarrow W^{1,p}(\mathbb{R}^3)$ continuously for $\lambda \geq 0$ and E_λ is compactly embedded in $L^{\varsigma}_{\text{loc}}(\mathbb{R}^3)$ for all $1 \leq \varsigma < p^* = \frac{Np}{N-p}$ for $N \geq 3$. Furthermore, taking open set $O \subset \mathbb{R}^3$, we know that

$$\int_O(|\nabla u|^p+(\lambda b(x)+1)|u|^p)dx \geq \int_O |u|^p dx \tag{3.1}$$

for all $u \in E_\lambda$ with $\lambda \geq 0$, fixed $\xi \in (0,1)$, there are $\nu > 0$ such that

$$\|u\|^p_{\lambda,O}-\nu|u|^p_{p,O} \geq \xi\|u\|^p_{\lambda,O},\ \forall u \in E_\lambda,\ \lambda \geq 0. \tag{3.2}$$

Hereafter

$$\|u\|_{\lambda,O}=\left(\int_O(|\nabla u|^p+(\lambda b(x)+1)|u|^p)dx\right)^{\frac{1}{p}}$$

and

$$|u|_{p,O}=\left(\int_O |u|^p dx\right)^{\frac{1}{p}}.$$

We note that for any $\epsilon > 0$, the hypotheses (G_1) and (G_2) yield

$$g(s) \leq \epsilon|s|^{p-1}+C_\epsilon|s|^{p^*-1},\ \forall\ x \in \mathbb{R}^3 \ \text{ and } s \in \mathbb{R}. \tag{3.3}$$

Consequently

$$G(s) \leq \epsilon|s|^p+C_\epsilon|s|^{p^*},\ \forall x \in \mathbb{R}^3 \ \text{ and } s \in \mathbb{R}, \tag{3.4}$$

where C_ϵ depends on ϵ. Inspired by del Pino and Felmer in their seminal work [27] (also refer to Ding and Tanaka [19], Alves [30]), let $\nu > 0$ fixed in (3.2), the Assumptions (G_1) and (G_4) imply that there is an unique $a > 0$ verifying

$$\frac{g(a)}{a^{p-1}}=\nu. \tag{3.5}$$

By utilizing the variables a and ν, we define the function $\tilde{g}:\mathbb{R}^3\times\mathbb{R}\to\mathbb{R}$ as follows

$$\tilde{g}(s)=\begin{cases} g(s), & s \leq a,\\ \nu s^{p-1}, & s \geq a,\end{cases}$$

which fulfills the inequality

$$\tilde{g}(s) \leq \nu|s|^{p-1},\ \ \forall s \in \mathbb{R}. \tag{3.6}$$

Thus

$$\tilde{g}(s)s \leq \nu|s|^p,\ \ \forall s \in \mathbb{R} \tag{3.7}$$

and

$$\tilde{G}(s) \leq \frac{\nu}{p}|t|^p,\ \ \forall s \in \mathbb{R}, \tag{3.8}$$

where $\tilde{G}(s)=\int_0^t \tilde{g}(t)dt$.

Now, considering that $\Omega = \text{int}(b^{-1}(\{0\}))$ consists of k connected components $\Omega_1, \cdots, \Omega_k$ with dist $(\Omega_i, \Omega_j) > 0$ for $i \neq j$, then for each $j \in \{i, \cdots, k\}$, it is possible to select a smooth bounded domain Ω_j such that

$$\overline{\Omega_j} \subset \Omega'_j \text{ and } \overline{\Omega'_i} \cap \overline{\Omega'_j} = \emptyset, \quad \text{for } i \neq j. \tag{3.9}$$

Henceforth, we fix a non-empty subset $\Upsilon \subset \{1, \cdots, k\}$ and

$$\Omega_\Upsilon = \bigcup_{j\in\Upsilon} \Omega_j, \ \Omega'_\Upsilon = \bigcup_{j\in\Upsilon} \Omega'_j \text{ and } \chi_\Upsilon = \begin{cases} 0, & if \ \ x \notin \Omega'_\Upsilon, \\ 1, & if \ \ x \in \Omega'_\Upsilon. \end{cases}$$

Let

$$h(x, s) = \chi_\Upsilon(x) g(s) + (1 - \chi_\Upsilon)\tilde{g}(s), \quad (x, s) \in \mathbb{R}^3 \times \mathbb{R}$$

and

$$H(x, s) = \int_0^s h(x, t)dt, \quad (x, s) \in \mathbb{R}^3 \times \mathbb{R}$$

and the auxiliary nonlocal problem

$$\begin{cases} -\Delta_p u + (\lambda b(x) + 1)|u|^{p-2}u + \phi_u |u|^{s-2} u = h(x, u) \quad \text{in } \mathbb{R}^3, \\ u \in E_\lambda. \end{cases} \tag{A_λ}$$

The problem (A_λ) is connected to (P_λ) in such a way that, if u_λ is a valid solution for problem (A_λ) satisfying

$$u_\lambda \leq a, \quad \forall\, x \in \mathbb{R}^N \setminus \Omega'_\Upsilon,$$

then it is a solution for problem (P_λ).

Problem (A_λ) possesses an advantage over problem (P_λ) in that the energy functional associated with problem (A_λ), namely, $J_\lambda : E_\lambda \to \mathbb{R}$ given by

$$J_\lambda(u) = \frac{1}{p}\int_{\mathbb{R}^3} (|\nabla u|^p + (\lambda b(x) + 1)|u|^p)dx + \frac{1}{2s}\int_{\mathbb{R}^3} \phi_u |u|^s dx - \int_{\mathbb{R}^3} H(x, u)dx.$$

Using the hypothesis for h, we can easily prove that J_λ is continuously differentiable. Now, we mainly prove that the energy functional J_λ satisfies the (PS) condition.

Lemma 3.1. *For given $d \in \mathbb{R}$, J_λ verifies the $(PS)_d$ condition.*

Proof. We divide the proof of this lemma into three claims.
Claim 1. All $(PS)_d$ sequences for J_λ are bounded in E_λ.

In fact, let (u_n) be a $(PS)_d$ sequences for J_λ. So, there is $n_0 \in \mathbb{N}$ such that

$$J_\lambda(u_n) - \frac{1}{\theta} J'_\lambda(u_n)u_n \leq d + 1 + \|u_n\|_\lambda \text{ for } n \geq n_0.$$

On the other hand, by (3.7) and (3.8), we have

$$\tilde{G}(s) - \frac{1}{\theta}\tilde{g}(s)s \leq (\frac{1}{p} - \frac{1}{\theta})\nu|s|^p, \quad \forall x \in \mathbb{R}^3, s \in \mathbb{R}.$$

Together with (3.2), one has

$$J_\lambda(u_n) - \frac{1}{\theta}J'_\lambda(u_n)u_n \geq (\frac{1}{p} - \frac{1}{\theta})\xi\|u_n\|_\lambda^p$$

from which it follows that (u_n) is bounded in E_λ.

Claim 2. If (u_n) is a $(PS)_d$ sequence for J_λ, then given $\varepsilon > 0$, there is $R > 0$ such that

$$\limsup_n \int_{\mathbb{R}^3\setminus B_R(0)} (|\nabla u|^p + (\lambda b(x) + 1)|u|^p)dx < \varepsilon. \tag{3.10}$$

Hence, once that h has a subcritical growth, if $u \in E_\lambda$ is the weak limit of (u_n), then

$$\int_{\mathbb{R}^3} h(x, u_n)u_n dx \to \int_{\mathbb{R}^3} h(x, u)udx \ \ and \ \ \int_{\mathbb{R}^3} h(x, u_n)vdx \to \int_{\mathbb{R}^3} h(x, u)vdx, \ \ \forall v \in E_\lambda.$$

In fact, we take (u_n) be a $(PS)_d$ sequence for J_λ, $R > 0$ large such that $\eta_R \in C^\infty(\mathbb{R}^3)$ with $0 \leq \eta_R \leq 1$ and $\Omega'_\Upsilon \subset B_{\frac{R}{2}}(0)$ satisfying

$$\eta_R(x) = \begin{cases} 1, & x \in \mathbb{R}^3 \setminus B_R(0), \\ 0, & x \in B_{\frac{R}{2}}(0) \end{cases}$$

and $|\nabla\eta_R| \leq \frac{C}{R}$, where $C > 0$ does not depend on R. This way

$$\begin{aligned}&\int_{\mathbb{R}^3} (|\nabla u|^p + (\lambda b(x) + 1)|u|^p)\eta_R dx + \int_{\mathbb{R}^3} \phi_{u_n}|u_n|^s\eta_R dx \\ =&J'_\lambda(u_n)(u_n\eta_R) - \int_{\mathbb{R}^3} u_n|\nabla u_n|^{p-2}\nabla u_n\nabla\eta_R dx + \int_{\mathbb{R}^3\setminus\Omega'_\Upsilon} \tilde{g}(u_n)u_n\eta_R dx.\end{aligned}$$

Denoting

$$L = \int_{\mathbb{R}^3} (|\nabla u|^p + (\lambda b(x) + 1)|u|^p)\eta_R dx.$$

From (3.7), we can infer,

$$L \leq J'_\lambda(u_n)(u_n\eta_R) + \frac{C}{R}\int_{\mathbb{R}^3} |u_n||\nabla u_n|^{p-2}|\nabla u_n|dx + \nu\int_{\mathbb{R}^3} |u_n|^p\eta_R dx.$$

Using the Hölder's inequality, we derive

$$L \leq J'_\lambda(u_n)(u_n\eta_R) + \frac{C}{R}|u_n|_p|\nabla u_n|_p + \nu L.$$

Since (u_n) and (∇u_n) are bounded in $L^p(\mathbb{R}^3)$, we obtain

$$L \leq o_n(1) + \frac{C}{(1-\nu)R}.$$

Given $\epsilon > 0$, we can choose a sufficiently large $R > 0$ such that $\frac{C}{(1-\nu)R} < \epsilon$, which proves (3.10).

Now, we will show that

$$\int_{\mathbb{R}^3} h(x, u_n)u_n dx \to \int_{\mathbb{R}^3} h(x, u)udx.$$

By utilizing the property that $h(x,u)u \in L^1(\mathbb{R}^3)$, along with (3.10) and Sobolev embeddings given $\varepsilon > 0$, we can choose $R > 0$ such that

$$\limsup_{n\to\infty} \int_{\mathbb{R}^3\setminus B_R(0)} |h(x,u_n)u_n| dx \leq \frac{\epsilon}{4} \ \ and \ \ \limsup_{n\to\infty} \int_{\mathbb{R}^3\setminus B_R(0)} |h(x,u)u| dx \leq \frac{\epsilon}{4}.$$

On the other hand, due to the subcritical growth of h, we can deduce from compact embedding that

$$\int_{B_R(0)} h(x,u_n)u_n dx \to \int_{B_R(0)} h(x,u)u dx.$$

Based on the information provided above, we conclude that

$$\int_{\mathbb{R}^3} h(x,u_n)u_n dx \to \int_{\mathbb{R}^3} h(x,u)u dx.$$

The same type of arguments can be used to establish that

$$\int_{\mathbb{R}^3} h(x,u_n)v dx \to \int_{\mathbb{R}^3} h(x,u)v dx, \ \ \forall\ v \in E_\lambda.$$

This completes the proof of Claim 2.

Claim 3. $u_n \to u$ in E_λ.

In fact, according to Claim 2, it follows that

$$\int_{\mathbb{R}^3} h(x,u_n)u_n dx \to \int_{\mathbb{R}^3} h(x,u)u dx \ \ and \ \ \int_{\mathbb{R}^3} h(x,u_n)v dx \to \int_{\mathbb{R}^3} h(x,u)v dx, \ \ \forall v \in E_\lambda.$$

Moreover, the limit also gives

$$\int_{\mathbb{R}^3} |\nabla u|^{p-2}\nabla u(\nabla u - u) dx \to 0$$

and

$$\int_{\mathbb{R}^3} (\lambda b(x) + 1)|u|^{p-2}u(u_n - u) dx \to 0.$$

Now, if

$$P_n^1 = (|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u)(\nabla u_n - \nabla u)$$

and

$$P_n^2 = (|u_n|^{p-2}u_n - |u|^{p-2}u)(u_n - u).$$

We derive

$$\begin{aligned} \int_{\mathbb{R}^3} (P_n^1(x) + (\lambda b(x) + 1)P_n^2(x)) dx = J'_\lambda(u_n)u_n &+ \int_{\mathbb{R}^3} h(x,u_n)u_n dx - J'_\lambda(u_n)u - \int_{\mathbb{R}^3} h(x,u_n)u dx \\ &- \int_{\mathbb{R}^3} [(|\nabla u|^{p-2}\nabla u \nabla(u_n - u) + (\lambda b(x) + 1)|u|^{p-2}u(u_n - u)] dx \\ &- \int_{\mathbb{R}^3} \phi_{u_n}|u_n|^s dx - \int_{\mathbb{R}^3} \phi_{u_n}|u_n|^{s-2}u_n u dx. \end{aligned}$$

Recalling the fact that $J'_\lambda(u_n)u_n = o_n(1)$ and $J'_\lambda(u_n)u = o_n(1)$, the aforementioned limits result in

$$\|u_n - u\|_\lambda^p \to 0.$$

This completes the proof of Claim 3. □

Now, let's recall the definition of the $(PS)_\infty$ sequence. A sequence $(u_n) \subset W^{1,p}(\mathbb{R}^3)$ is called a $(PS)_\infty$ sequence for the family $(J_\lambda)_{\lambda\geq 1}$, if there is a sequence $(\lambda_n) \subset [1, \infty)$ with $\lambda_n \to \infty$ as $n \to \infty$, verifying

$$J_{\lambda_n}(u_n) \to c \ \ and \ \ \|J'_\lambda(u_n)\|_{E*\lambda} \to 0 \ \ as \ \ n \to \infty$$

for some $c \in \mathbb{R}$.

Lemma 3.2. *Let $(u_n) \subset W^{1,p}(\mathbb{R}^3)$ be a $(PS)_\infty$ sequence for the family $(J_\lambda)_{\lambda\geq 1}$. Then, up to a subsequence, there are $u \in W^{1,p}(\mathbb{R}^3)$ such that $u_n \rightharpoonup u$ in $W^{1,p}(\mathbb{R}^3)$. Furthermore,*

i) $u_n \to u$ in $W^{1,p}(\mathbb{R}^3)$;
ii) $u_n = 0$ in $\mathbb{R}^3 \setminus \Omega_\Upsilon$, $u|_{\Omega_j} \geq 0$ for all $j \in \Upsilon$, and u is a solution for

$$\begin{cases} -\Delta_p u + |u|^{p-2}u + \phi_u|u|^{s-2}u = g(u) \ \ in \ \Omega_\Upsilon, \\ u \in W^{1,p}(\Omega_\Upsilon); \end{cases} \tag{$(P)_{\infty,\Upsilon}$}$$

iii) $\lambda_n \int_{\mathbb{R}^3} b(x)|u_n|^p dx \to 0$;
iv) $\|u_n - u\|^p_{\lambda,\Omega'_\Upsilon} \to 0$;
v) $\|u_n\|^p_{\lambda,\mathbb{R}^3\setminus\Omega'_\Upsilon} \to 0$;
vi) $J_\lambda(u_n) \to \frac{1}{p}\int_{\Omega_\Upsilon}(|\nabla u|^p + |u|^p)dx + \frac{1}{2s}\int_{\Omega_\Upsilon}\phi_u|u|^s dx - \int_{\Omega_\Upsilon} G(u)dx$.

Proof. According to Lemma 3.1, it can be inferred that $(\|u_n\|_{\lambda_n})$ is bounded in $\mathbb{R}$ and (u_n) is bounded in $W^{1,p}(\mathbb{R}^3)$. So, up to a subsequence, there exists $u \in W^{1,p}(\mathbb{R}^3)$ such that

$$u_n \rightharpoonup u \ \ in \ \ W^{1,p}(\mathbb{R}^3) \ \ and \ \ u_n(x) \to u(x) \ \ for \ \ a.e. \ \ x \in \mathbb{R}^3.$$

Now, for each $m \in \mathbb{N}$, we define $C_m = \{x \in \mathbb{R}^3 : b(x) \geq \frac{1}{m}\}$. Without loss of generality, we can assume $\lambda_n < 2(\lambda_n - 1)$, $\forall n \in \mathbb{N}$. Thus

$$\int_{C_m} |u_n|^p dx \leq \frac{2m}{\lambda_n}\int_{C_m}(\lambda b(x) + 1)|u|^p dx \leq \frac{2m}{\lambda_n}\|u_n\|^p_{\lambda_n} \leq \frac{C}{\lambda_n}.$$

By the Fatou's lemma, we derive

$$\int_{C_m} |u|^p dx = 0, \ \ \forall m \in \mathbb{N}.$$

This observation suggests that $u = 0$ in C_m and, consequently, $u = 0$ in $\mathbb{R}^3 \setminus \overline{\Omega}$. As a result, we can establish the validity of $i) - vi)$.

i) Since $u = 0$ in $\mathbb{R}^3 \setminus \overline{\Omega}$, by employing the reasoning discussed in Lemma 3.1, we obtain

$$\int_{\mathbb{R}^3}(P^1_n(x) + (\lambda b(x) + 1)P^2_n(x))dx \to 0,$$

where

$$P^1_n = (|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u)(\nabla u_n - \nabla u)$$

and

$$P^2_n = (|u_n|^{p-2}u_n - |u|^{p-2}u)(u_n - u),$$

which implies $u_n \to u$ in $W^{1,p}(\mathbb{R}^3)$.

ii) Given that $u \in W^{1,p}(\mathbb{R}^3)$ and $u = 0$ in $\mathbb{R}^3 \setminus \overline{\Omega}$, it follows that $u \in W_0^{1,p}(\mathbb{R}^3)$ or equivalently, $u|_{\Omega_j} \in W_0^{1,p}(\Omega_j)$, for $j \in \{1, \cdots, k\}$. Additionally, the limit $u_n \to u$ in $W^{1,p}(\mathbb{R}^3)$ combined with $J'_{\lambda_n}(u_n)\varphi \to 0$ for $\varphi \in C_0^\infty(\Omega_\Upsilon)$ implies that

$$\int_{\Omega_\Upsilon} (|\nabla u|^{p-2}\nabla u \nabla\varphi + |u|^{p-2}u\varphi)dx + \int_{\Omega_\Upsilon} \phi_u |u|^{s-2}u\varphi dx - \int_{\Omega_\Upsilon} g(u)\varphi dx = 0 \tag{3.11}$$

showing that $u|_{\Omega_j}$ is a solution for the following nonlocal problem $(P)_{\infty,\Upsilon}$. Alternatively, in the case where $j \notin \Upsilon$, it follows that

$$\int_{\Omega_j} (|\nabla u|^p + |u|^p)dx + \int_{\Omega_j} \phi_u |u|^s dx - \int_{\Omega_j} \tilde{g}(u)u dx = 0.$$

The above equality together with (3.2) and (3.7) gives

$$0 \geq \|u\|^p_{\lambda,\Omega_j} - \nu|u|^p_{p,\Omega_j} \geq \|u\|^p_{\lambda,\Omega_j} \geq 0,$$

from which it follows that $u|_{\Omega_j} = 0$ for $j \notin \Upsilon$. This demonstrates $u = 0$ outside Ω_Υ and $u \geq 0$ in $\mathbb{R}^3$.

iii) Let $j \in \Upsilon$. From *i*)

$$\lambda_n \int_{\mathbb{R}^3} b(x)|u_n|^p dx = \int_{\mathbb{R}^3} |u_n - u|^p dx \leq \|u_n - u\|^{\lambda_n}_p.$$

This fact implies that

$$\lambda_n \int_{\mathbb{R}^3} b(x)|u_n|^p dx \to 0.$$

iv) Let $j \in \Upsilon$. From *i*)

$$|u_n - u|^p_{p,\Omega'_j}, |\nabla u_n - \nabla u|^p_{p,\Omega'_j} \to 0.$$

Then

$$\int_{\Omega'_\Upsilon} (|\nabla u_n|^p - |\nabla u|^p)dx \to 0 \ \ and \ \ \int_{\Omega'_\Upsilon} (|u_n|^p - |u|^p)dx \to 0.$$

From *iii*), we have

$$\int_{\Omega'_\Upsilon} \lambda_n b(x)|u_n|^p dx \to 0.$$

This way

$$\|u_n\|^p_{\lambda_n,\Omega'_\Upsilon} \to \int_{\Omega_\Upsilon} (|\nabla u_n|^p + |u|^p)dx.$$

v) By *i*), $\|u_n - u\|^p_{\lambda_n} \to 0$, and so,

$$\|u_n\|^p_{\lambda_n,\mathbb{R}^3\setminus\Omega_\Upsilon} \to 0.$$

vi) We have the option to express the functional J_{λ_n} in the subsequent manner

$$J_{\lambda_n}(u_n) = \sum_{j\in\Upsilon}[\frac{1}{p}\int_{\Omega'_j} (|\nabla u_n|^p + (\lambda_n b(x) + 1)|u|^p)dx + \frac{1}{2s}\int_{\Omega'_j} \phi_{u_n}|u_n|^s dx]$$
$$+ \frac{1}{p}\int_{\mathbb{R}^3\setminus\Omega'_\Upsilon} (|\nabla u_n|^p + (\lambda_n b(x) + 1)|u|^p)dx + \frac{1}{2s}\int_{\mathbb{R}^3\setminus\Omega'_\Upsilon} \phi_{u_n}|u_n|^s dx - \int_{\mathbb{R}^3} H(x, u_n)dx.$$

By $i) - v)$,

$$\frac{1}{p}\int_{\Omega'_j}(|\nabla u_n|^p + (\lambda_n b(x)+1)|u|^p)dx \to \frac{1}{p}\int_{\Omega'_j}(|u_n|^p + u|^p)dx,$$

$$\frac{1}{p}\int_{\mathbb{R}^3\setminus\Omega'_\Upsilon}(|\nabla u_n|^p + (\lambda_n b(x)+1)|u|^p)dx \to 0,$$

$$\int_{\Omega'_j}\phi_{u_n}|u_n|^s dx \to \int_{\Omega'_j}\phi_u|u|^s dx,$$

$$\int_{\mathbb{R}^3\setminus\Omega'_\Upsilon}\phi_{u_n}|u_n|^s dx \to 0$$

and

$$\int_{\mathbb{R}^3} H(x,u_n)dx \to \int_{\Omega_\Upsilon} G(u)dx.$$

Therefore

$$J_\lambda(u_n) \to \frac{1}{p}\int_{\Omega_\Upsilon}(|\nabla u|^p + |u|^p)dx + \frac{1}{2s}\int_{\Omega_\Upsilon}\phi_u|u|^s dx - \int_{\Omega_\Upsilon} G(u)dx.$$

This completes the proof of Lemma 3.2. □

4. Proof of Theorem 2.1

In order to get the solution of the original problem (P_λ), we need to establish the L^∞ estimate of certain solutions to problem $(A)_\lambda$ outside Ω'_Υ.

Lemma 4.1. *Let (u_λ) be a family of positive solutions of problem $(A)_\lambda$ with $u_\lambda \to 0$ in $W^{1,p}(\mathbb{R}^3 \setminus \Omega_\Upsilon)$ as $\lambda \to \infty$. Then, there exists $\lambda^* > 0$ such that*

$$|u_\lambda|_{\infty,\mathbb{R}^3\setminus\Omega'_\Upsilon} \le a,\ \forall\lambda \ge \lambda^*.$$

Moreover, u_λ is a positive solution of problem $(P)_\lambda$.

Proof. In this proof, we adopt some arguments developed in Li [31] (see also Alves and Figueiredo [32]). In the sequel, u denotes u_λ. Now, fixing $\Omega'_j \subset \tilde{\Omega}_j$, let $\eta \in C^\infty(\mathbb{R}^3)$, and η verifying

$$0 \le \eta(x) \le 1,\ \forall x \in \mathbb{R}^3,$$

$$\eta(x) = 1,\ \forall x \in \cup_{j\in\Upsilon}\tilde{\Omega}_j,$$

$$\eta(x) = 0,\ \forall x \in \Omega'_j.$$

Let

$$v = \eta^p u u_L^{p(\beta-1)} \ \text{and}\ W_L = \eta u u_L(\beta-1),$$

where $u_L = \min\{u, L\}$. By Sobolev embedding theorem and the definition of v, we have

$$|W_L|^p_{p^*} \le C\int_{\mathbb{R}^3}|\nabla W_L|^p dx \le C\beta^p(\int_{\mathbb{R}^3}|\nabla\eta|^p u^p u_L^{p(\beta-1)}dx).$$

This fact implies that

$$|W_L|_{p^*,\mathcal{S}}^p \le C_1\beta^p\left(\int_\Gamma u^p u_L^{p(\beta-1)}dx\right),$$

where $\Gamma = \cup_{j\in\Upsilon}\tilde{\Omega}_j \setminus \Omega'_j$ and $\mathcal{S} = \mathbb{R}^3 \setminus \cup_{j\in\Upsilon}\tilde{\Omega}_j$.

Fixing $\beta = \frac{p^*}{p}$, the final inequality implies that

$$u \in L^{\frac{p^{*2}}{p}}(\mathcal{S}).$$

Now, if $\beta = \frac{p^*(t-1)}{t}$ with $t = \frac{p^{*2}}{(p^*-p)p}$, then $\beta > 1$ and $\frac{pt}{t-1} \in (p, p^*)$. Thus, from the Holder's inequality, we have

$$|W_L|_{p^*p,\mathcal{S}} \le C_2\beta^p(\int_\Gamma u^{\frac{p\beta t}{t-1}}dx)^{\frac{t-1}{t}}.$$

Hence, letting $L \to \infty$, we obtain that

$$|u|_{\beta p^*,\mathcal{S}}^{p\beta} \le C_2\beta^p|u|_{\frac{p\beta t}{t-1},\Upsilon}^{p\beta}.$$

Defining $\xi = \frac{p^*(t-1)}{pt}$, $s = \frac{pt}{t-1}$ and $\beta = \xi^m (m = 1, 2, 3, \cdots)$, it is easy to prove that there exists $C_3 > 0$ such that

$$|u|_{\xi^{m+1}s,\mathcal{S}} \le C_3|u|_{\xi s,\Gamma}, \ \forall\, m \in \{1, 2, 3, \cdots\}.$$

Letting $m \to +\infty$, we get

$$|u|_{\infty,\mathcal{S}} \le C_3|u|_{p^*,\Gamma}.$$

Note that

$$u_\lambda \to 0 \ \ in \ \ W^{1,p}(\mathbb{R}^3 \setminus \Omega_\Upsilon) \ \ as \ \ \lambda \to +\infty.$$

From this fact, it can be inferred that there exists $\lambda^* > 0$ such that

$$|u_\lambda|_{\infty,\mathbb{R}^3\setminus\Omega'_\Upsilon} \le a, \ \forall\, \lambda \ge \lambda^*.$$

This completes the proof of Lemma 4.1. □

Now, let fixed non-empty subset $\Upsilon \subset \{1, \cdots, k\}$, we consider the energy functional related to problem $(P)_{\infty,\Upsilon}$ as follows:

$$I_\Upsilon(u) = \frac{1}{p}\int_{\Omega_\Upsilon}(|\nabla u|^p + |u|^p)dx + \frac{1}{2s}\int_{\Omega_\Upsilon}\phi_u|u|^s dx - \int_{\Omega_\Upsilon}G(u)dx, \ u \in W_0^{1,p}(\Omega_\Upsilon)$$

and $J_{\lambda,\Upsilon} : W^{1,p}(\Omega'_\Upsilon) \to \mathbb{R}$ given by

$$J_{\lambda,\Upsilon}(u) = \frac{1}{p}\int_{\Omega'_\Upsilon}(|\nabla u|^p + (\lambda b(x) + 1)|u|^p)dx + \frac{1}{2s}\int_{\Omega'_\Upsilon}\phi_{\tilde{u}}|u|^s dx - \int_{\Omega'_\Upsilon}G(u)dx,$$

the energy functional $J_{\lambda,\Upsilon}$ related to the nonlocal problem

$$\begin{cases} -\Delta_p u + (\lambda b(x) + 1)|u|^{p-2}u + \phi_{\tilde{u}}|u|^{s-2}u = g(u) \ \text{ in } \mathbb{R}^3, \\ \frac{\partial u}{\partial \eta} = 0 \text{ on } \partial\Omega'_\Upsilon. \end{cases}$$

In the following, let

$$c_\Upsilon = \inf_{u\in\mathcal{M}_\Upsilon} I_\Upsilon(u) \quad \text{and} \quad c_{\lambda,\Upsilon} = \inf_{u\in\mathcal{M}'_\Upsilon} J_{\lambda,\Upsilon}(u),$$

where

$$\mathcal{M}_\Upsilon = \{u \in \mathcal{N}_\Upsilon : I'_\Upsilon(u)u_j = 0 \text{ and } u_j \neq 0 \ \ \forall j \in \Upsilon\},$$

$$\mathcal{N}_\Upsilon = \{u \in W_0^{1,p}(\Omega_\Upsilon) \setminus \{0\} : I'_\Upsilon(u)u = 0\}$$

and

$$\mathcal{M}'_\Upsilon = \{u \in \mathcal{N}'_\Upsilon : J'_{\lambda,\Upsilon}(u)u_j = 0 \text{ and } u_j \neq 0 \ \ \forall j \in \Upsilon\},$$

$$\mathcal{N}'_\Upsilon = \{u \in W^{1,p}(\Omega'_\Upsilon) \setminus \{0\} : J'_{\lambda,\Upsilon}(u)u = 0\}.$$

Repeating the same approach used in Section 2, we ensure that there exist $w_\Upsilon \in W_0^{1,p}(\Omega_\Upsilon)$ and $w_{\lambda,\Upsilon} \in W^{1,p}(\Omega'_\Upsilon)$ such that

$$I_\Upsilon(w_\Upsilon) = c_\Upsilon \ \ \textit{and} \ \ I'_\Upsilon(w_\Upsilon) = 0$$

and

$$J_{\lambda,\Upsilon}(w_{\lambda,\Upsilon}) = c_{\lambda,\Upsilon} \ \ \textit{and} \ \ J'_{\lambda,\Upsilon}(w_{\lambda,\Upsilon}) = 0.$$

Inspired by references [20, 29, 30], we can give the relationship between $c_{\lambda,\Upsilon}$ and c_Υ.

Lemma 4.2. *There holds that*

i) $0 < c_{\lambda,\Upsilon} \le c_\Upsilon, \ \ \forall\, \lambda \ge 0;$
ii) $c_{\lambda,\Upsilon} \to c_\Upsilon$ *as* $\lambda \to \infty$.

Next, we fix $T > 1$ verifying

$$0 < I'_j(\frac{1}{T}w_j)(\frac{1}{T}w_j) \text{ and } I'_j(Tw_j)(Tw_j) < 0 \text{ for } j \in \Upsilon, \tag{4.1}$$

where I_j denotes the energy functional

$$I_j(u) = \frac{1}{p}\int_{\Omega_j}(|\nabla u|^p + |u|^p)dx + \frac{1}{2s}\int_{\Omega_j}\phi_u|u|^s dx - \int_{\Omega_j}G(u)dx, \ \ u \in W_0^{1,p}(\Omega_j).$$

Let $\Upsilon = \{1, 2, \cdots, l\}$, where $1 \le l \le k$. We define

$$\psi_0(\mathbf{z})(x) = \sum_{j=1}^{l} z_j T w_j(x) \in W_0^{1,p}(\Omega_\Upsilon), \ \ \forall \mathbf{z} = (z_1, \cdots, z_l) \in [1/T^2, 1]^l,$$

$$\Gamma_* = \{\psi \in C([1/T^2, 1]^l, E_\lambda \setminus \{0\}); \psi(t)|_{\Omega_j} \neq 0, \ \ \forall j \in \Upsilon; \psi = \psi_0 \ \ \textit{on} \ \ \partial[1/T^2, 1]^l\}$$

and

$$b_{\lambda,\Upsilon} = \inf_{\psi\in\Gamma_*} \max_{t\in[1/T^2,1]^l} J_\lambda(\psi(\mathbf{z})).$$

Similarly, using the topological degree theory, as the arguments employed in references [20, 29, 30], we have the following key lemmas.

Lemma 4.3. *For all $\psi \in \Gamma_*$, there exists $(z_1, \ldots, z_l) \in [1/T^2, 1]^l$ such that*

$$J'_{\lambda,j}(\psi(z_1, \cdots, z_l))(\psi(z_1, \cdots, z_l)) = 0, \quad \forall j \in \Upsilon,$$

where

$$J_{\lambda,j} = \frac{1}{p}\int_{\Omega'_j}(|\nabla u|^p + (\lambda b(x) + 1)|u|^p)dx + \frac{1}{2s}\int_{\Omega'_j}\phi_{\tilde{u}}|u|^s dx - \int_{\Omega'_j} G(u)dx, \quad u \in W^{1,p}(\Omega'_\Upsilon).$$

Lemma 4.4. *We have the following conclusions:*

(a) $c_{\lambda,\Upsilon} \le b_{\lambda,\Upsilon} \le c_\Upsilon, \quad \forall \lambda \ge 1;$
(b) $b_{\lambda,\Upsilon} \to c_\Upsilon$ *as* $\lambda \to \infty;$
(c) $J_\lambda(\psi(\mathbf{z})) < c_\Upsilon, \quad \forall\ \lambda \ge 1, \quad \psi \in \Gamma_*$ *and* $\mathbf{z} = (z_1, \cdots, z_l) \in \partial[1/T^2, 1]^l$.

Proof. (a) Since $\psi_0 \in \Gamma_*$, we have

$$b_{\lambda,\Upsilon} \le \max_{(z_1,\ldots,z_l)\in[1/T^2,1]^l} J_\lambda(\gamma_0(z_1, \ldots, z_l)) \le \max_{(z_1,\ldots,z_l)\in\mathbb{R}^l} I_\Upsilon(\sum_{j=1}^{l} z_j T w_j) = c_\Upsilon.$$

Now, fixing $\mathbf{z} = (z_1, \ldots, z_l) \in [1/T^2, 1]^l$ given by Lemma 4.3, it follows that

$$J_{\lambda,\Upsilon}(\psi(\mathbf{z})) \ge c_{\lambda,\Upsilon}.$$

On the other hand, by (3.8), we have

$$J_{\lambda,\mathbb{R}^3\setminus\Omega'_\Upsilon}(u) \ge 0, \quad \forall u \in W^{1,p}(\mathbb{R}^3 \setminus \Omega'_\Upsilon),$$

which leads to

$$J_\lambda(\psi(\mathbf{z})) \ge J_{\lambda,\Upsilon}(\psi(\mathbf{z})), \quad \forall \mathbf{t} \in (z_1, \cdots, z_l) \in [1/T^2, 1]^l.$$

Thus

$$\max_{(z_1,\cdots,z_l)\in[1/T^2,1]^l} J_\lambda(\psi(z_1, \cdots, z_l)) \ge J_{\lambda,\Upsilon}(\psi(\mathbf{z})) \ge c_{\lambda,\Upsilon},$$

showing that

$$b_{\lambda,\Upsilon} \ge c_{\lambda,\Upsilon};$$

(b) the previous item establishes the limit in question, since we already know $c_{\lambda,\Upsilon} \to c_\Upsilon$, as $\lambda \to \infty$;
(c) for $\mathbf{z} = (z_1, \cdots, z_l) \in \partial[\frac{1}{T^2}, 1]^l$, it hold $\psi(\mathbf{z}) = \psi_0(\mathbf{z})$. From this

$$J_\lambda(\psi(\mathbf{z})) = I_\Upsilon(\psi_0(\mathbf{z})).$$

By (4.1), we derive

$$J_\lambda(\psi(\mathbf{z})) \le c_\Upsilon - \epsilon \quad \text{for some } \epsilon > 0.$$

This completes the proof of Lemma 4.4. □

Now, let's start to prove Theorem 2.1, to this end, it is necessary to identify nonnegative solutions u_λ for large values of λ that converge towards the least energy solution of problem $(P)_{\infty,\Upsilon}$ as $\lambda \to \infty$. To achieve this objective, two lemmas will be established.

Lemma 4.5. *For each $\zeta > 0$, there exist $\Lambda_* \geq 1$ and d_0 independent of λ such that*

$$\|J'_\lambda(u)\|_{E^*_\lambda} \geq d_0 \ for \ \lambda \geq \Lambda_* \ \text{and all} \ u \in (\mathcal{D}^\lambda_{2\zeta} \setminus \mathcal{D}^\lambda_\zeta) \cap J^{c_\Upsilon}_\lambda, \tag{4.2}$$

where

$$\mathcal{D}^\lambda_\zeta = \{u \in \Xi_{2\delta} : |J_\lambda(u) - c_\Upsilon| \leq \zeta\}, \ \ \delta = \frac{\tau}{48R}, \tag{4.3}$$

$$\Xi = \{u \in E_\lambda : \|u\|_{\lambda,\Omega'_j} > \frac{\tau}{8T} \ \ \forall j \in \Upsilon\}$$

and

$$J^{c_\Upsilon}_\lambda = \{u \in E_\lambda; J_\lambda(u) \leq c_\Upsilon\}.$$

Proof. Since $w_\Upsilon \in \mathcal{D}^\lambda_\zeta \cap J^{c_\Upsilon}_\lambda$, thus $\mathcal{D}^\lambda_\zeta \cap J^{c_\Upsilon}_\lambda \neq \emptyset$. We make the assumption that there exist $\lambda_n \to \infty$ and $u_n \in (\mathcal{D}^\lambda_{2\zeta} \setminus \mathcal{D}^\lambda_\zeta) \cap J^{c_\Upsilon}_\lambda$ such that $\|J'_{\lambda_n}(u_n)\|_{E^*\lambda_n} \to 0$. Given that $u_n \in \mathcal{D}^{\lambda_n}_{2\zeta}$, this implies that $(\|u_n\|_{\lambda_n})$ is a sequence with bounded values. Consequently, it can be inferred that $(J_{\lambda_n}(u_n))$ is also a bounded sequence. By considering a subsequence if necessary, we can assume convergence of $(J_{\lambda_n}(u_n))$. Therefore, according to Lemma 3.2, there exists $0 \leq u \in W^{1,p}_0(\Omega_\Upsilon)$ which serves as a solution for problem $(SP)_\Upsilon$, one has

$$u_n \to u \ \ in \ \ W^{1,p}(\mathbb{R}^3), \ \ \|u_n\|_{\lambda_n,\mathbb{R}^3\setminus\Omega_\Upsilon} \to 0 \ \ and \ \ J_{\lambda_n}(u_n) \to I_\Upsilon(u).$$

Recalling the inclusion relationship $(u_n) \subset \Theta_{2\delta}$, it can be inferred that

$$\|u_n\|_{\lambda_n,\Omega'_j} > \frac{\tau}{12T}, \ \ \forall j \in \Upsilon.$$

Subsequently, by considering the limit as $n \to \infty$, we have

$$\|u\|_j \geq \frac{\tau}{12T}, \ \ \forall j \in \Upsilon,$$

yields $u|_{\Omega_j} \neq 0$ for all $j \in \Upsilon$ and $I'_\Upsilon(u) = 0$. Obviously, we can get

$$\|u\|_j > \frac{\tau}{8T}, \ \ \forall j \in \Upsilon.$$

This way, $I_\Upsilon(u) \geq c_\Upsilon$. Since $J_{\lambda_n}(u_n) \leq c_\Upsilon$ and $J_{\lambda_n}(u_n) \to I_\Upsilon(u)$, for n large, it holds

$$\|u_n\|_j > \frac{\tau}{8T} \ \ and \ \ |J_{\lambda_n}(u_n) - c_\Upsilon| \leq \zeta, \ \ \forall j \in \Upsilon.$$

So $u_n \in \mathcal{D}^{\lambda_n}_\zeta$, which leads to a contradiction. Therefore, this completes the proof of Lemma 4.5. □

Now, we give the definition of two positive numbers ζ_1 and ζ^* as follows:

$$\min_{\mathbf{z}\in\partial[1/T^2,1]^l} |I_\Upsilon(\psi_0(\mathbf{z})) - c_\Upsilon| = \zeta_1 > 0 \quad \text{and} \quad \zeta^* = \min\{\zeta_1, \delta, \frac{r}{2}\},$$

where δ is given (4.3) and

$$r = R^p \sum_{j=1}^{l} (\frac{1}{p} - \frac{1}{\theta})^{-1} c_\Upsilon.$$

Lemma 4.6. *Let $\zeta > 0$ small enough and $\Lambda_* \geq 1$ as stated in the previous proposition. Then, for $\lambda \geq \Lambda_*$, there exists a solution u_λ of problem (A_λ) such that $u_\lambda \in \mathcal{D}_\zeta^\lambda \cap J_\lambda^{c_\Upsilon} \cap \mathcal{B}_{r+1}$, where $\mathcal{B}_r = \{u \in E_\lambda; \|u\|_\lambda \leq r\}$ for $r \geq 0$.*

Proof. First, set $\lambda \geq \Lambda_*$. If J_λ does not have any critical points in $\mathcal{D}_\zeta^\lambda \cap J_\lambda^{c_\Upsilon} \cap \mathcal{B}_{r+1}$. We note that J_λ verifies the (PS) condition, this fact implies that there exists a constant $\omega_\lambda > 0$ such that

$$\|J'_\lambda(u)\|_{E^*_\lambda} \geq \omega_\lambda \text{ for all } u \in \mathcal{D}_\zeta^\lambda \cap J_\lambda^{c_\Upsilon} \cap \mathcal{B}_{r+1}.$$

From Lemma 4.5, we have

$$\|J'_\lambda(u)\|_{E^*_\lambda} \geq d_0 \text{ for all } u \in (\mathcal{D}_{2\zeta}^\lambda \setminus \mathcal{D}_\zeta^\lambda) \cap J_\lambda^{c_\Upsilon},$$

where $d_0 > 0$ independent on λ.

Next, we can define a continuous functional $\Phi : E_\lambda \to \mathbb{R}$ such that

$$\Phi(u) = 1 \ \text{ for } u \in \mathcal{D}_{\frac{3}{2}\zeta}^\lambda \cap \Xi_\delta \cap \mathcal{B}_r^\lambda,$$

$$\Phi(u) = 0 \ \text{ for } u \in \mathcal{D}_{2\zeta}^\lambda \cap \Xi_{2\delta} \cap \mathcal{B}_{r+1}^\lambda$$

and

$$0 \leq \Phi(u) \leq 1, \ \ \forall\, u \in E_\lambda.$$

Let $\mathcal{L} : J_\lambda^{c_\Upsilon} \to E_\lambda$ defined by

$$\mathcal{L}(u) = \begin{cases} -\Phi(u)\|P(u)\|^{-1}P(u) \ for\ all\ u \in \mathcal{D}_{2\zeta}^\lambda \cap \mathcal{B}_{r+1}, \\ 0\ for\ all\ u \in \mathcal{D}_{2\zeta}^\lambda \cap \mathcal{B}_{r+1}, \end{cases}$$

where P is a pseudo-gradient vector field for J_λ on $\mathcal{K} = \{u \in E_\lambda; J'_\lambda(u) \neq 0\}$. Obviously, the functional P is well defined. In case $J'_\lambda(u) \neq 0$ for $u \in \mathcal{D}_{2\zeta}^\lambda \cap J_\lambda^{c_\Upsilon}$. We immediately get the following inequality:

$$\|P(u)\| \leq 1, \ \ \forall \lambda \geq \Lambda_* \text{ and } u \in J_\lambda^{c_\Upsilon}.$$

This fact gives us there exists an deformation flow $\phi : [0, \infty) \times J_\lambda^{c_\Upsilon} \to J_\lambda^{c_\Upsilon}$ defined by

$$\frac{d\phi}{dt} = P(\phi), \ \ \phi(0, u) = u \in J_\lambda^{c_\Upsilon}$$

verifies

$$\frac{d}{dt} J_\lambda(\phi(t, u)) \leq -\frac{1}{2}\Phi(\phi(t, u))\|J'_\lambda(\phi(t, u))\| \leq 0, \tag{4.4}$$

$$\|\frac{d\phi}{dt}\|_\lambda = \|P(\eta)\|_\lambda \leq 1 \tag{4.5}$$

and

$$\phi(t, u) = u \ \text{ for all } t \geq 0 \text{ and } u \in J_\lambda^{c_\Upsilon} \setminus \mathcal{D}_{2\zeta}^\lambda \cap \mathcal{B}_{r+1}. \tag{4.6}$$

Now, we consider the following two paths:

Path I. The path $\mathbf{z} \mapsto \eta(t, \psi_0(\mathbf{z}))$, where $\mathbf{z} = (z_1, \cdots, z_l) \in [1/T^2, 1]^l$.
Path II. The path $\mathbf{z} \mapsto \psi_0(\mathbf{z})$, where $\mathbf{z} = (z_1, \cdots, z_l) \in [1/T^2, 1]^l$.

For Path I, let $\mu \in (0, \mu^*)$, one has

$$\psi_0(\mathbf{z}) \notin \mathcal{D}^\lambda_{2\zeta}, \ \ \forall \mathbf{z} \in \partial[1/T^2, 1]^l.$$

Since

$$J_\lambda(\psi_0(\mathbf{z})) < c_\Upsilon, \ \ \forall \mathbf{z} \in \partial[1/T^2, 1]^l,$$

from (4.6), it follows that

$$\eta(t, \psi_0(\mathbf{t})) = \psi_0(\mathbf{z}), \ \ \forall \mathbf{z} \in \partial[1/T^2, 1]^l.$$

So, $\phi(t, \psi_0(\mathbf{t})) \in \Gamma_*, \forall\ t \geq 0$.

For Path II, we note that

$$\text{supp}(\psi_0(\mathbf{z})) \subset \overline{\Omega}_\Upsilon$$

and

$$J_\lambda(\psi_0(\mathbf{t})) \ \text{ does not depend on } \lambda \geq 1$$

for all $\mathbf{z} \in [1/T^2, 1]^l$. Moreover,

$$J_\lambda(\psi_0(\mathbf{t})) \leq c_\Upsilon, \ \ \forall \mathbf{t} \in [1/T^2, 1]^l$$

and

$$J_\lambda(\psi_0(\mathbf{t})) = c_\Upsilon \ \Leftrightarrow \ t_j = \frac{1}{T}, \ \forall j \in \Upsilon.$$

So, we have

$$q_0 = \sup\{J_\lambda(u); u \in \gamma_0([1/T^2, 1]^l) \setminus \mathcal{D}^\lambda_\zeta\}$$

is independent of λ and $q_0 < c_\Upsilon$. Then, there exists $Q^* > 0$ such that

$$|J_\lambda(u) - J_\lambda(v)| \leq Q^* \|u - v\|_\lambda, \ \ \forall u, v \in \mathcal{B}_r.$$

Moreover, we have

$$\max_{\mathbf{z} \in [1/T^2, 1]^l} J_\lambda(\eta(L, \gamma_0(\mathbf{t}))) \leq \max\{q_0, c_\Upsilon - \frac{1}{2Q^*} d_0 \mu\} \tag{4.7}$$

for $L > 0$ large.

Indeed, setting $u = \psi_0(\mathbf{z})$, $\mathbf{z} \in [1/T^2, 1]^l$, if $u \notin \mathcal{D}^\lambda_\zeta$, from (4.4), one has

$$J_\lambda(\phi(t, u)) \leq J_\lambda(u) \leq q_0, \ \ \forall t \geq 0.$$

Let $u \in \mathcal{D}^\lambda_\zeta$ and

$$\tilde{\phi}(t) = \phi(t, u), \ \ \widetilde{\omega}_\lambda = \min\{\omega_\lambda, d_0\} \ \textit{and} \ \ L = \frac{d_0 \zeta}{Q^* \widetilde{\omega}_\lambda}.$$

Next, let's consider it in two steps:

Step I. $\widetilde{\phi}(t) \in \mathcal{D}_{\frac{3}{2}\zeta} \cap \Xi_\delta \cap \mathcal{B}_r$, $\forall t \in [0, L]$.

Step II. $\widetilde{\phi}(t_0) \notin \mathcal{D}_{\frac{3}{2}\zeta} \cap \Xi_\delta \cap \mathcal{B}_r$ for some $t_0 \in [0, L]$.

For Step I, we have $\Phi(\widetilde{\phi}(t)) = 1$ and $\|J'_\lambda(\widetilde{\phi}(t))\| \geq \widetilde{\omega}_\lambda$ for all $t \in [0, L]$. By (4.4), we have

$$J_\lambda(\widetilde{\phi}(L)) = J_\lambda(u) + \int_0^L \frac{d}{ds} J_\lambda(\widetilde{\phi}(s)) ds \leq c_\Upsilon - \frac{1}{2} \int_0^L \widetilde{\omega}_\lambda ds.$$

This fact implies that

$$J_\lambda(\widetilde{\eta}(L)) \leq c_\Upsilon - \frac{1}{2}\widetilde{\omega}_\lambda L = c_\Upsilon - \frac{1}{2Q^*}d_0\zeta,$$

showing (4.7).

For Step II, we consider three cases:

Case 1, let $t_2 \in [0, L]$ satisfies $\widetilde{\phi}(t_2) \notin \Xi_\delta$, and thus, for $z_1 = 0$ it follows that

$$\|\widetilde{\phi}(z_2) - \widetilde{\phi}(z_1)\| \geq \delta > \zeta,$$

due to $\widetilde{\phi}(z_1) = u \in \Xi$.

Case 2, let $t_2 \in [0, T]$ satisfies $\widetilde{\phi}(z_2) \notin \mathcal{B}_r$. Therefore, for $t_1 = 0$, one has

$$\|\widetilde{\phi}(z_2) - \widetilde{\phi}(z_1)\| \geq r > \zeta,$$

since $\widetilde{\phi}(z_1) = u \in \mathcal{B}_r$.

Case 3, let $\widetilde{\phi}(t) \in \Xi_\delta \cap \mathcal{B}_r^\lambda$ for all $t \in [0, L]$ and there are $0 \leq z_1 \leq z_2 \leq L$ satisfy $\widetilde{\phi}(t) \in \mathcal{D}^\lambda_{\frac{3}{2}\zeta} \setminus \mathcal{D}^\lambda_\zeta$ for all $z \in [z_1, z_2]$ with

$$|J_\lambda(\widetilde{\phi}(z_1)) - c_\Upsilon| = \zeta \ \text{ and } |J_\lambda(\widetilde{\phi}(z_2)) - c_\Upsilon| = \frac{3\zeta}{2}.$$

Together with the definition of Q^*, we have

$$\|w_2 - w_1\| \geq \frac{1}{Q^*}|J_\lambda(w_2) - J_\lambda(w_1)| \geq \frac{1}{2Q^*}\zeta.$$

By means of the mean value theorem, we know that $z_2 - z_1 \geq \frac{1}{2Q^*}\zeta$ and

$$J_\lambda(\widetilde{\phi}(L)) \leq J_\lambda(u) - \int_0^L \Phi(\widetilde{\phi}(s))\|J'_\lambda(\widetilde{\phi}(s))\|ds.$$

This fact implies that

$$J_\lambda(\widetilde{\phi}(T)) \leq c_\Upsilon - \int_{z_1}^{z_2} d_0 ds = c_\Upsilon - d_0(z_2 - z_1) \leq c_\Upsilon - \frac{1}{2Q^*}d_0\zeta,$$

which proves (4.7). Define $\widehat{\phi}(z_1, \cdots, z_l) = \phi(L, \phi_0(z_1, \cdots, z_l))$, we have that $\widehat{\phi}(z_1, \cdots, z_l) \in \Xi_{2\delta}$, and so, $\widehat{\phi}(z_1, \cdots, z_l)|_{\Omega'_j} \neq 0$ fora ll $j \in \Upsilon$. Thus, $\widehat{\phi} \in \Gamma_*$. Moreover, we have

$$b_{\lambda,\Gamma} \leq \max_{(z_1,\cdots,z_l)\in[1/T^2,1]} J_\lambda(\widehat{\phi}(z_1, \cdots, z_l)) \leq \max\{q_0, c_\Upsilon - \frac{1}{2Q^*}d_0\zeta\} < c_\Upsilon,$$

which contradicts the fact $b_{\lambda,\Upsilon} \to c_\Upsilon$. This completes the proof of Lemma 4.6. □

Proof of Theorem 1.1. First, by Lemma 4.6, for $\zeta \in (0, \zeta^*)$ and $\Lambda_* \geq 1$, we know that there exists a solution u_λ for problem (P_λ) such that $u_\lambda \in \mathcal{D}^\lambda_\zeta \cap J_\lambda^{c_\Upsilon}$, for all $\lambda \geq \Lambda_*$.

Next, let $\zeta \in (0, \zeta_0)$, assuming that there exists a sequence $\lambda_n \to \infty$ such that (u_{λ_n}) is not a solution for problem $(SP)_{\lambda_n}$. According to Lemma 4.6, we know that (u_{λ_n}) satisfies $\|u_n\|^p_{\lambda_n,\mathbb{R}^3\setminus\Omega_\Upsilon}(u_{\lambda_n}) \to 0$. However, from Lemma 4.1, we can deduce that u_{λ_n} is a solution for problem $(SP)_{\lambda_n}$ as $n \to \infty$, which is a contradiction. This means that there are $\Lambda_0 \geq \Lambda_*$ and $\zeta_0 > 0$ small enough, such that u_λ is a solution for problem (P_λ) for $\lambda \geq \Lambda_0$, and $\zeta \in (0, \zeta_0)$.

Finally, we are devoted to proving the second part of the Theorem 2.1. To this end, we consider the sequence (u_{λ_n}) which satisfies the conclusions of Lemma 3.2. This means that (u_{λ_n}) converges to u in $W^{1,p}(\mathbb{R}^3)$, and u satisfies conditions such that $u = 0$ outside Ω_Υ and $u|_{\Omega_j} \neq 0$, $j \in \Upsilon$, while also serving as a positive solution for problem $(P)_{\infty,\Upsilon}$ and so, $I_\Upsilon(u) \geq c_\Upsilon$. On the other hand, we have $J_{\lambda_n}(u) \to I_\Upsilon(u)$. This fact implies that $I_\Upsilon(u) = d \ \ and \ \ d \geq c_\Upsilon$. Since $d \leq c_\Upsilon$, we infer that $I_\Upsilon(u) = c_\Upsilon$. Thus, u is a least energy solution for problem $(P)_{\infty,\Upsilon}$. This completes the proof of Theorem 2.1.

5. Conclusions

This paper studies a class of Schrödinger-Poisson systems with p-Laplacian in $\mathbb{R}^3$, and the existence of multi-bump solutions are discussed. First, we show that the existence of least energy solution to the energy function. Then, the auxiliary nonlocal problem is constructed, and the solution of the problem is proved to be the solution of the original system. Finally, we prove that the system has a multi-bump solutions.

Use of AI tools declaration

The authors declare that have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgements

The authors were supported by the Science and Technology Development Plan Project of Jilin Province, China (No. 20230101287JC), the National Natural Science Foundation of China (No. 12001061), and Innovation and Entrepreneurship Talent Funding Project of Jilin Province (No. 2023QN21).

Conflict of interest

The authors declare that they have no competing interests.

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