AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 9(1): 1576–1594.
DOI: 10.3934/math.2024078
Received: 16 October 2023
Revised: 24 November 2023
Accepted: 04 December 2023
Published: 13 December 2023

Research article

A study of coupled nonlinear generalized fractional differential equations with coupled nonlocal multipoint Riemann-Stieltjes and generalized fractional integral boundary conditions

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Abstract: This paper was concerned with the existence and uniqueness results for a coupled system of nonlinear generalized fractional differential equations supplemented with a new class of nonlocal coupled multipoint boundary conditions containing Riemann-Stieltjes and generalized fractional integrals. The nonlinearities in the given system depend on the unknown functions as well as their lower order generalized fractional derivatives. We made use of the Leray-Schauder alternative and Banach contraction mapping principle to obtain the desired results. An illustrative example was also discussed. The paper concluded with some interesting observations.

Keywords: generalized fractional integral and derivative operators; multipoint; integral boundary conditions; existence; fixed point
Mathematics Subject Classification: 34A08, 26A33, 34B15

1. Introduction

Fractional calculus gained a great interest in view of its applications in a variety of disciplines such as mathematical sciences, dynamical systems, engineering, finance, control theory, etc.; for details and explanation, see [1–4] and the references cited therein. Examples of fractional differential systems include distributed-order dynamical systems [5], quantum evolution of complex systems [6], Chua circuit [7], Lorenz system [8], Duffing system [9], synchronization of coupled fractional-order chaotic systems [10–12], systems of nonlocal thermoelasticity [13, 14], anomalous diffusion [15, 16], etc.

There has also been witnessed a great surge in developing the theoretical aspects (existence, uniqueness and stability of solutions) of fractional order boundary value problems. In [17], the authors studied a coupled Riemann-Stieltjes type integro-multipoint boundary value problem of Caputo-type sequential fractional differential equations by using the standard fixed point theorems. The existence of solutions for a nonlinear fractional system involving both Caputo and Riemann-Liouville generalized fractional derivatives with coupled integral boundary conditions was investigated in [18]. One can find some more interesting results on the topic in the articles [19–27] and the references cited therein. In a recent work [28], the authors studied a system of generalized coupled fractional differential equations equipped with uncoupled Riemann-Stieltjes and generalized fractional integral boundary conditions.

Keeping in mind that the concept of the coupled boundary data is more general and important, we introduce a class of nonlocal coupled multipoint integral boundary conditions containing Riemann-Stieltjes and generalized fractional integrals and solve the system considered in [28] with these boundary conditions. In precise terms, we apply the fixed point approach to develop the existence criteria for solutions to the following system of nonlinear generalized coupled fractional differential equations complemented with nonlocal coupled multipoint Riemann-Stieltjes and generalized fractional integral boundary conditions:

$$\begin{cases} {}^{\rho}D^{\alpha}_{0^+}u(t) = f(t, u(t), v(t), {}^{\rho}D^{\gamma_1}_{0^+}v(t)), & t \in [0, T], \\ {}^{\rho}D^{\beta}_{0^+}v(t) = g(t, u(t), {}^{\rho}D^{\gamma_2}_{0^+}u(t), v(t)), & t \in [0, T], \\ u(0) = v(0) = 0 \\ \displaystyle\int_0^T u(s)dH_1(s) = \lambda_1 \, {}^{\rho}I^{\delta_1}_{0+}v(\xi_1) + \sum_{p=1}^{m} a_p \, v(\eta_p), & \xi_1 \in (0, T), \\ \displaystyle\int_0^T v(s)dH_2(s) = \lambda_2 \, {}^{\rho}I^{\delta_2}_{0+}u(\xi_2) + \sum_{p=1}^{m} b_p \, u(\eta_p), & \xi_2 \in (0, T), \end{cases} \tag{1.1}$$

where ${}^{\rho}D^{\alpha}_{0^+}$ and ${}^{\rho}D^{\beta}_{0^+}$ are the generalized fractional derivative operators of order $1 < \alpha,\beta \leq 2$, respectively, $0 < \gamma_1, \gamma_2 < 1$, ${}^{\rho}I^{\delta_1}_{0^+}$ and ${}^{\rho}I^{\delta_2}_{0^+}$ are the generalized fractional integral operators of order $\delta_1, \delta_2 > 0$, respectively, $\int_0^T u(s)dH_i(s)$ $(i = 1, 2)$ are the Riemann-Stieltjes integrals with respect to the functions $H_i : [0, T] \to \mathbb{R}$, $f, g \in C([0, T] \times \mathbb{R}^3, \mathbb{R})$, $\lambda_1, \lambda_2, a_p, b_p \in \mathbb{R}$ and $\eta_p \in (0, T)$, $p = 1, 2, \ldots, m$.

Here, we emphasize that the system of fractional differential equations in (1.1) reduces to the one with Hadamard and Riemann-Liouville fractional differential and integral operators, respectively, for $\rho \to 0^+$ and $\rho = 1$. Thus, the results obtained in this paper will correspond to the Hadamard and Riemann-Liouville fractional differential systems equipped with coupled multipoint Riemann-Stieltjes and Hadamard/Riemann-Liouville fractional integral boundary conditions as special cases.

We arrange the rest of the article as follows. Some preliminary concepts related to our work are outlined in Section 2. The main results for the given problem will be derived in Section 3. An illustrative example is also included in Section 3. Section 4 contains some concluding remarks.

2. Preliminaries

Here, we recall some basic concepts of fractional calculus related to our work. For $1 \leq p \leq \infty$, $c \in \mathbb{R}$, define

$$X^p_c(a, b) = \{\phi : (a, b) \to \mathbb{R};\ \ \phi \text{ is Lebesgue measurable function},\ \ \|\phi\|_{X^p_c} < \infty\},$$

where

$$\|\phi\|_{X_c^p} = \left(\int_a^b |x^s \phi(x)|^p \frac{dx}{x} \right)^{\frac{1}{p}}.$$

Definition 2.1. *[29] The generalized fractional integral of order $\alpha > 0$ of function $f \in X_c^p(a,b)$ is defined as*

$${}^\rho I_{0^+}^\alpha f(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{1-\alpha}} f(s) ds,$$

where $\rho > 0$, $-\infty < a < t < b < \infty$ and $\Gamma(\cdot)$ is the Gamma function.

Definition 2.2. *[30] The generalized fractional derivative of order $\alpha > 0$ associated with the generalized fractional integral is defined for $0 \leq a < x < b < \infty$ as*

$$\begin{aligned} {}^\rho D_{0^+}^\alpha f(t) &= \left(t^{1-\rho} \frac{d}{dt} \right)^n {}^\rho I_{0^+}^{n-\alpha} f(t) \\ &= \frac{\rho^{\alpha-n+1}}{\Gamma(n-\alpha)} \left(t^{1-\rho} \frac{d}{dt} \right)^n \int_a^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{\alpha-n+1}} f(s) ds, \end{aligned}$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of real number α.

For example, we have

$${}^\rho D_{0+}^\gamma t^{\rho(\alpha-1)} = \frac{\Gamma(\alpha)}{\Gamma(\alpha+1-\gamma)} \rho^\gamma (\alpha-\gamma) t^{\rho(\alpha-1-\gamma)}.$$

Lemma 2.3. *[31] The equality ${}^\rho D_{0^+}^\alpha {}^\rho I_{0^+}^\alpha g(t) = g(t)$, $\rho > 0$ holds for $g \in X_c^p(a,b)$, $a > 0$.*

Lemma 2.4. *[30] Let $q_1, q_2 \in \mathbb{C}$, $1 \leq p \leq \infty$ and $0 < a < b < \infty$, then, for $f \in X_c^p(a,b)$, $\rho > 0$,*

$${}^\rho I_{0^+}^{q_1} {}^\rho I_{0^+}^{q_2} f = {}^\rho I_{0^+}^{q_1+q_2} f \quad \text{and} \quad {}^\rho D_{0^+}^{q_1} {}^\rho D_{0^+}^{q_2} f = {}^\rho D_{0^+}^{q_1+q_2} f.$$

Let $C([0,T],\mathbb{R})$ denote the set of all continuous functions from $[0,T]$ to $\mathbb{R}$. Set

$$X = \{\Phi |\ \Phi \in C([0,T],\mathbb{R}) \text{ and } {}^\rho D_{0^+}^{\gamma_2} \Phi \in C([0,T],\mathbb{R})\},$$

endowed with the norm

$$\|\Phi\|_X = \sup_{t\in[0,T]} |\Phi(t)| + \sup_{t\in[0,T]} |{}^\rho D_{0^+}^{\gamma_2} \Phi(t)| := \|\Phi\| + \|{}^\rho D_{0^+}^{\gamma_2} \Phi\|.$$

As argued in [32], $(X, \|\cdot\|_X)$ is a Banach space. Also, we define

$$Y = \{\Psi |\ \Psi \in C([0,T],\mathbb{R}) \text{ and } {}^\rho D_{0^+}^{\gamma_1} \Psi \in C([0,T],\mathbb{R})\}$$

endowed with the norm

$$\|\Psi\|_Y = \sup_{t\in[0,T]} |\Psi(t)| + \sup_{t\in[0,T]} |{}^\rho D_{0^+}^{\gamma_1} \Psi(t)| := \|\Psi\| + \|{}^\rho D_{0^+}^{\gamma_1} \Psi\|.$$

Likewise, $(Y, \|\cdot\|_Y)$ is a Banach space.

We know from [32] that the space $(X \times Y, \|\cdot\|_{X\times Y})$ with the norm $\|(\Phi,\Psi)\|_{X\times Y} = \|\Phi\|_X + \|\Psi\|_Y$ for any $(\Phi,\Psi) \in X \times Y$ is a Banach space.

Let $AC[0,T]$ denote the space of absolutely continuous functions on $[0,T]$.

Lemma 2.5. *[31] Let $\rho > 0$, $1 < \alpha \leq 2$, $u \in X_c^p(0,T)$ and ${}^{\rho}I_{0^+}^{2-\alpha}u \in AC_\rho^2[0,T]$, where*

$$AC_\rho^2[0,T] = \left\{ g : [0,T] \to \mathbb{R} : \ \left(t^{1-\rho}\frac{d}{dt}\right)g(t) \in AC[0,T] \right\},$$

then the solution of the equation ${}^{\rho}D_{0^+}^{\alpha}u(t) = 0$ is

$$u(t) = c_1 t^{\rho(\alpha-1)} + c_2 t^{\rho(\alpha-2)},$$

where $c_i \in \mathbb{R}$, $i = 1, 2$ are constants. Moreover,

$${}^{\rho}I_{0^+}^{\alpha}\,{}^{\rho}D_{0^+}^{\alpha}u(t) = u(t) + c_1 t^{\rho(\alpha-1)} + c_2 t^{\rho(\alpha-2)}.$$

In the following lemma, we solve the linear variant of the system (1.1), which facilitates the conversion of the given nonlinear problem into an equivalent fixed point problem.

Lemma 2.6. *Assume that $\Theta_1, \Theta_2 \in C([0,T],\mathbb{R})$ with ${}^{\rho}I_{0^+}^{2-\alpha}u, {}^{\rho}I_{0^+}^{2-\beta}v \in AC_\rho^2[0,T]$ and $\Lambda \neq 0$. Then, the following system*

$$\begin{cases} {}^{\rho}D_{0^+}^{\alpha}u(t) = \Theta_1(t), & t \in [0,T], \\ {}^{\rho}D_{0^+}^{\beta}v(t) = \Theta_2(t), & t \in [0,T], \\ u(0) = v(0) = 0 \\ \displaystyle\int_0^T u(s)dH_1(s) = \lambda_1\, {}^{\rho}I_{0+}^{\delta_1}v(\xi_1) + \sum_{p=1}^{m} a_p\, v(\eta_p), & \xi_1 \in (0,T) \\ \displaystyle\int_0^T v(s)dH_2(s) = \lambda_2\, {}^{\rho}I_{0+}^{\delta_2}u(\xi_2) + \sum_{p=1}^{m} b_p\, u(\eta_p), & \xi_2 \in (0,T), \end{cases} \tag{2.1}$$

has a solution (u, v) given by

$$\begin{aligned} u(t) \;=\; & {}^{\rho}I_{0+}^{\alpha}\Theta_1(t) + \frac{t^{\rho(\alpha-1)}}{\Lambda}\Big[A_2\Big\{\lambda_1\, {}^{\rho}I_{0+}^{\beta+\delta_1}\Theta_2(\xi_1) - \int_0^T {}^{\rho}I_{0+}^{\alpha}\Theta_1(s)dH_1(s) + \sum_{p=1}^{m} a_p\, {}^{\rho}I_{0+}^{\beta}\Theta_2(\eta_p)\Big\} \\ & + B_1\Big\{\lambda_2\, {}^{\rho}I_{0+}^{\alpha+\delta_2}\Theta_1(\xi_2) - \int_0^T {}^{\rho}I_{0+}^{\beta}\Theta_2(s)dH_2(s) + \sum_{p=1}^{m} b_p\, {}^{\rho}I_{0+}^{\alpha}\Theta_1(\eta_p)\Big\}\Big], \end{aligned} \tag{2.2}$$

$$\begin{aligned} v(t) \;=\; & {}^{\rho}I_{0+}^{\beta}\Theta_2(t) + \frac{t^{\rho(\beta-1)}}{\Lambda}\Big[B_2\Big\{\lambda_1\, {}^{\rho}I_{0+}^{\beta+\delta_1}\Theta_2(\xi_1) - \int_0^T {}^{\rho}I_{0+}^{\alpha}\Theta_1(s)dH_1(s) + \sum_{p=1}^{m} a_p\, {}^{\rho}I_{0+}^{\beta}\Theta_2(\eta_p)\Big\} \\ & + A_1\Big\{\lambda_2\, {}^{\rho}I_{0+}^{\alpha+\delta_2}\Theta_1(\xi_2) - \int_0^T {}^{\rho}I_{0+}^{\beta}\Theta_2(s)dH_2(s) + \sum_{p=1}^{m} b_p\, {}^{\rho}I_{0+}^{\alpha}\Theta_1(\eta_p)\Big\}\Big], \end{aligned} \tag{2.3}$$

where

$$\begin{aligned} \Lambda &= A_1A_2 - B_1B_2, \\ A_1 &= \int_0^T s^{\rho(\alpha-1)}dH_1(s), \quad A_2 = \int_0^T s^{\rho(\beta-1)}dH_2(s), \end{aligned}$$

$$
\begin{aligned}
B_1 &= \lambda_1 \frac{\Gamma(\beta)}{\rho^{\delta_1}(\beta+\delta_1)} \xi_1^{\rho(\beta+\delta_1-1)} + \sum_{p=1}^{m} a_p \eta_p^{\rho(\beta-1)}, \\
B_2 &= \lambda_2 \frac{\Gamma(\alpha)}{\rho^{\delta_2}(\alpha+\delta_2)} \xi_2^{\rho(\alpha+\delta_2-1)} + \sum_{p=1}^{m} b_p \eta_p^{\rho(\alpha-1)}.
\end{aligned} \tag{2.4}
$$

Proof. Solving the system of fractional differential equations in (1.1), we get

$$
\begin{aligned}
u(t) &= {}^{\rho}I_{0+}^{\alpha}\, \Theta_1(t) + c_1 t^{\rho(\alpha-1)} + c_2 t^{\rho(\alpha-2)}, \\
v(t) &= {}^{\rho}I_{0+}^{\beta}\, \Theta_2(t) + c_3 t^{\rho(\beta-1)} + c_4 t^{\rho(\beta-2)}.
\end{aligned} \tag{2.5}
$$

Making use of the condition $u(0) = v(0) = 0$ in (2.5), we get $c_2 = c_4 = 0$, and then applying the generalized integral operators ${}^{\rho}I_{0+}^{\beta}$ and ${}^{\rho}I_{0+}^{\alpha}$ to the first and second equations in (2.5), respectively, we obtain

$$
\begin{aligned}
&{}^{\rho}I_{0+}^{\delta_2} u(t) = {}^{\rho}I_{0+}^{\delta_2+\alpha}\Theta_1(t) + c_1 \frac{\Gamma(\alpha)}{\rho^{\delta_2}\Gamma(\alpha+\delta_2)} t^{\rho(\alpha+\delta_2-1)}, \\
&{}^{\rho}I_{0+}^{\delta_1} v(t) = {}^{\rho}I_{0+}^{\delta_1+\beta}\Theta_2(t) + c_3 \frac{\Gamma(\beta)}{\rho^{\delta_1}\Gamma(\beta+\delta_1)} t^{\rho(\beta+\delta_1-1)}.
\end{aligned} \tag{2.6}
$$

Using (2.5) and (2.6) in the conditions:

$$
\int_0^T u(s)dH_1(s) = \lambda_1\, {}^{\rho}I_{0+}^{\delta_1} v(\xi_1) + \sum_{p=1}^{m} a_p\, v(\eta_p), \quad \int_0^T v(s)dH_2(s) = \lambda_2\, {}^{\rho}I_{0+}^{\delta_2} u(\xi_2) + \sum_{p=1}^{m} b_p\, u(\eta_p),
$$

we obtain a system of equations in the unknown constants c_1 and c_3 given by

$$
c_1 A_1 - c_3 B_1 = \lambda_1\, {}^{\rho}I_{0+}^{\delta_1+\beta}\Theta_2(\xi_1) - \int_0^T {}^{\rho}I_{0+}^{\alpha}\Theta_1(s)dH_1(s) + \sum_{p=1}^{m} a_p\, {}^{\rho}I_{0+}^{\beta}\Theta_2(\eta_p), \tag{2.7}
$$

$$
-c_1 B_2 + c_3 A_2 = \lambda_2\, {}^{\rho}I_{0+}^{\alpha+\delta_1}\Theta_1(\xi_2) - \int_0^T {}^{\rho}I_{0+}^{\beta}\Theta_2(s)dH_2(s) + \sum_{p=1}^{m} b_p\, {}^{\rho}I_{0+}^{\alpha}\Theta_1(\eta_p), \tag{2.8}
$$

where A_1, A_2, B_1 and B_2 are defined in (2.4). Solving the systems (2.7) and (2.8) for c_1 and c_3, we find that

$$
\begin{aligned}
c_1 &= \frac{1}{\Lambda}\Big[B_1\Big\{\lambda_2 {}^{\rho}I_{0+}^{\alpha+\delta_2}\Theta_1(\xi_2) - \int_0^T {}^{\rho}I_{0+}^{\beta}\Theta_2(s)dH_2(s) + \sum_{p=1}^{m} b_p\, {}^{\rho}I_{0+}^{\alpha}\Theta_1(\eta_p)\Big\} \\
&\quad + A_2\Big\{\lambda_1 {}^{\rho}I_{0+}^{\delta_1+\beta}\Theta_2(\xi_1) - \int_0^T {}^{\rho}I_{0+}^{\alpha}\Theta_1(s)dH_1(s) + \sum_{p=1}^{m} a_p\, {}^{\rho}I_{0+}^{\beta}\Theta_2(\eta_p)\Big\}\Big], \\
c_3 &= \frac{1}{\Lambda}\Big[B_2\Big\{\lambda_1\, {}^{\rho}I_{0+}^{\beta+\delta_1}\Theta_2(\xi_1) - \int_0^T {}^{\rho}I_{0+}^{\alpha}\Theta_1(s)dH_1(s) + \sum_{p=1}^{m} a_p\, {}^{\rho}I_{0+}^{\beta}\Theta_2(\eta_p)\Big\} \\
&\quad + A_1\Big\{\lambda_2\, {}^{\rho}I_{0+}^{\alpha+\delta_2}\Theta_1(\xi_2) - \int_0^T {}^{\rho}I_{0+}^{\beta}\Theta_2(s)dH_2(s) + \sum_{p=1}^{m} b_p\, {}^{\rho}I_{0+}^{\alpha}\Theta_1(\eta_p)\Big\}\Big],
\end{aligned}
$$

where Λ is given in (2.4). Inserting the above values and $c_2 = c_4 = 0$ in (2.5), we get the solutions (2.2) and (2.3), respectively. The converse of the lemma can be established by direct computation. The proof is finished. □

Relative to the problem (1.1), in view of Lemma 2.6, we define an operator $\mathcal{G} : X \times Y \to X \times Y$ as

$$\mathcal{G}(u, v)(t) = (\mathcal{G}_1(u.v), \mathcal{G}_2(u, v)), \tag{2.9}$$

where

$$\begin{aligned}
&\mathcal{G}_1(u, v)(t) \\
= \; & {}^\rho I^{\alpha}_{0+} f(t, u(t), v(t), {}^\rho D^{\gamma_1}_{0+} v(t)) + \frac{t^{\rho(\alpha-1)}}{\Lambda}\Big[A_2\Big\{\lambda_1\, {}^\rho I^{\beta+\delta_1}_{0+} g(\xi_1, u(\xi_1), {}^\rho D^{\gamma_2}_{0+} u(\xi_1), v(\xi_1)) \\
& -\int_0^T {}^\rho I^{\alpha}_{0+} f(s, u(s), v(s), {}^\rho D^{\gamma_1}_{0+} v(s)) dH_1(s) + \sum_{p=1}^{m} a_p\, {}^\rho I^{\beta}_{0+} g(\eta_p, u(\eta_p), {}^\rho D^{\gamma_2}_{0+} u(\eta_p), v(\eta_p))\Big\} \\
& +B_1\Big\{\lambda_2\, {}^\rho I^{\alpha+\delta_2}_{0+} f(\xi_2, u(\xi_2), v(\xi_2), {}^\rho D^{\gamma_1}_{0+} v(\xi_2)) - \int_0^T {}^\rho I^{\beta}_{0+} g(s, u(s), {}^\rho D^{\gamma_2}_{0+} u(s), v(s)) dH_2(s) \\
& + \sum_{p=1}^{m} b_p {}^\rho I^{\alpha}_{0+} f(\eta_p, u(\eta_p), v(\eta_p), {}^\rho D^{\gamma_1}_{0+} v(\eta_p))\Big\}\Big], \quad t \in [0, T],
\end{aligned}$$

and

$$\begin{aligned}
&\mathcal{G}_2(u, v)(t) \\
= \; & {}^\rho I^{\beta}_{0+} g(t, u(t), {}^\rho D^{\gamma_2}_{0+} u(t), v(t)) + \frac{t^{\rho(\beta-1)}}{\Lambda}\Big[B_2\Big\{\lambda_1\, {}^\rho I^{\beta+\delta_1}_{0+} g(\xi_1, u(\xi_1), {}^\rho D^{\gamma_2}_{0+} u(\xi_1), v(\xi_1)) \\
- \; & \int_0^T {}^\rho I^{\alpha}_{0+} f(s, u(s), v(s), {}^\rho D^{\gamma_1}_{0+} v(s)) dH_1(s) + \sum_{p=1}^{m} a_p\, {}^\rho I^{\beta}_{0+} g(\eta_p, u(\eta_p), {}^\rho D^{\gamma_2}_{0+} u(\eta_p), v(\eta_p))\Big\} \\
& +A_1\Big\{\lambda_2\, {}^\rho I^{\alpha+\delta_2}_{0+} f(\xi_2, u(\xi_2), v(\xi_2), {}^\rho D^{\gamma_1}_{0+} v(\xi_2)) - \int_0^T {}^\rho I^{\beta}_{0+} g(s, u(s), {}^\rho D^{\gamma_2}_{0+} u(s), v(s)) dH_2(s) \\
& + \sum_{p=1}^{m} b_p\, {}^\rho I^{\alpha}_{0+} f(\eta_p, u(\eta_p), v(\eta_p), {}^\rho D^{\gamma_1}_{0+} v(\eta_p))\Big\}\Big], \quad t \in [0, T].
\end{aligned}$$

Lemma 2.7. *[31] If $\vartheta : [0, T] \to \mathbb{R}$ is a continuous function, $w : [0, T] \to \mathbb{R}$ is a function of bounded variation on $[0, T]$ and $M = \max_{t \in [0,T]} |\vartheta(t)|$, then*

$$\left|\int_0^T \vartheta(s) dw(s)\right| \leq M V_0^T w,$$

where $V_0^T w$ denotes the variation of function w defined by

$$V_0^T w = \sup_P \sum_{j=0}^{n} |w(s_i) - w(s_{i-1})|,$$

and $P : 0 = s_0 < s_1 < \ldots < s_n = T$ is an arbitrary partition of $[0, T]$.

Recall that w is called a bounded variation function on $[0, T]$ if $V_0^T w < \infty$.

3. Main results

Before proceeding for our main results, we set our notation as follows:

$$\begin{aligned}
M_1 &= \frac{T^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)} + \frac{T^{\rho(\alpha-1)}}{|\Lambda|}\Big[|A_2\lambda_1|\frac{\xi_1^{\rho(\beta+\delta_1)}}{\rho^{\beta+\delta_1}\Gamma(\beta+\delta_1+1)} + |A_2|\frac{T^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)}V_0^T H_1 \\
&\quad + |A_2|\sum_{p=1}^{m} a_p\,\frac{\eta_p^{\rho\beta}}{\rho^{\beta}\Gamma(\beta+1)} + |B_1\lambda_2|\frac{\xi_2^{\rho(\alpha+\delta_2)}}{\rho^{\alpha+\delta_2}\Gamma(\alpha+\delta_2+1)} + |B_1|\frac{T^{\rho\beta}}{\rho^{\beta}\Gamma(\beta+1)}V_0^T H_2 \\
&\quad + |B_1|\sum_{p=1}^{m} b_p\,\frac{\eta_p^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)}\Big], \\
M_1' &= \frac{T^{\rho\beta}}{\rho^{\beta}\Gamma(\beta+1)} + \frac{T^{\rho(\beta-1)}}{|\Lambda|}\Big[|B_2\lambda_1|\frac{\xi_1^{\rho(\beta+\delta_1)}}{\rho^{\beta+\delta_1}\Gamma(\beta+\delta_1+1)} + |B_2|\frac{T^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)}V_0^T H_1 \\
&\quad + |B_2|\sum_{p=1}^{m} a_p\,\frac{\eta_p^{\rho\beta}}{\rho^{\beta}\Gamma(\beta+1)} + |A_1\lambda_2|\frac{\xi_2^{\rho(\alpha+\delta_2)}}{\rho^{\alpha+\delta_2}\Gamma(\alpha+\delta_2+1)} + |A_1|\frac{T^{\rho\beta}}{\rho^{\beta}\Gamma(\beta+1)}V_0^T H_2 \\
&\quad + |A_1|\sum_{p=1}^{m} b_p\,\frac{\eta_p^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)}\Big], \\
M_2 &= \frac{T^{\rho(\alpha-\gamma_2)}}{\rho^{(\alpha-\gamma_2)}\Gamma(\alpha-\gamma_2+1)} + \frac{\Gamma(\alpha)}{|\Lambda|\Gamma(\alpha+1-\gamma_2)}\rho^{\gamma_2}(\alpha-\gamma_2)T^{\rho(\alpha-\gamma_2-1)} \\
&\quad \times\Big[|A_2\lambda_1|\frac{\xi_1^{\rho(\beta+\delta_1)}}{\rho^{\beta+\delta_1}\Gamma(\beta+\delta_1+1)} + |A_2|\frac{T^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)}V_0^T H_1 + |A_2|\sum_{p=1}^{m} a_p\,\frac{\eta_p^{\rho\beta}}{\rho^{\beta}\Gamma(\beta+1)} \\
&\quad + |B_1\lambda_2|\frac{\xi_2^{\rho(\alpha+\delta_2)}}{\rho^{\alpha+\delta_2}\Gamma(\alpha+\delta_2+1)} + |B_1|\frac{T^{\rho\beta}}{\rho^{\beta}\Gamma(\beta+1)}V_0^T H_2 + |B_1|\sum_{p=1}^{m} b_p\,\frac{\eta_p^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)}\Big], \\
M_2' &= \frac{T^{\rho(\beta-\gamma_1)}}{\rho^{(\beta-\gamma_1)}\Gamma(\beta-\gamma_1+1)} + \frac{\Gamma(\beta)}{|\Lambda|\Gamma(\beta+1-\gamma_1)}\rho^{\gamma_1}(\beta-\gamma_1)T^{\rho(\beta-\gamma_1-1)} \\
&\quad \times\Big[|B_2\lambda_1|\frac{\xi_1^{\rho(\beta+\delta_1)}}{\rho^{\beta+\delta_1}\Gamma(\beta+\delta_1+1)} + |B_2|\frac{T^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)}V_0^T H_1 + |B_2|\sum_{p=1}^{m} a_p\,\frac{\eta_p^{\rho\beta}}{\rho^{\beta}\Gamma(\beta+1)} \\
&\quad + |A_1\lambda_2|\frac{\xi_2^{\rho(\alpha+\delta_2)}}{\rho^{\alpha+\delta_2}\Gamma(\alpha+\delta_2+1)} + |A_1|\frac{T^{\rho\beta}}{\rho^{\beta}\Gamma(\beta+1)}V_0^T H_2 + |A_1|\sum_{p=1}^{m} b_p\,\frac{\eta_p^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)}\Big]. \qquad (3.1)
\end{aligned}$$

In our first result, we show the existence of at least one solution to the system (1.1) by applying the Leray-Schauder alternative [33].

Theorem 3.1. *Suppose that* $f, g \in C\left([0,T]\times\mathbb{R}^3, \mathbb{R}\right)$ *and there exist constants* $b_{0f}, b_{0g} > 0$ *and* $b_{if}, b_{ig} \geq 0, i = 1,2,3$ *such that the following condition holds:*

$(H1)$ $|f(t,\varpi_1,\varpi_2,\varpi_3)| \leq b_{0f} + \sum_{i=1}^{3} b_{if}|\varpi_i|$, *and* $|g(t,\varpi_1,\varpi_2,\varpi_3)| \leq b_{0g} + \sum_{i=1}^{3} b_{ig}|\varpi_i|$,

for all $\varpi_i \in \mathbb{R}, i = 1,2,3.$

If $[M_1 + M_2 + M_1' + M_2'](b_{1f} + \max\{b_{1g}, b_{2g}\}) < 1$ *and* $[M_1 + M_2 + M_1' + M_2'](\max\{b_{2f}, b_{3f}\} + b_{3g}) < 1$, *then the system* (1.1) *has at least one solution on* $[0, T]$.

Proof. We complete the proof in three steps.

Step 1. We claim that $\mathcal{G}$ is uniformly bounded. Observe that continuity of f and g implies that $\mathcal{G}$ is a continuous operator. Assume that $B_q = \{(u, v) \in X \times Y : \|(u, v)\|_{X\times Y} \le q\}$ is a bounded subset of $X \times Y$. By $(H1)$, for any $(\zeta, \sigma) \in B_q$, we have

$$\begin{aligned} |f(t, \zeta(t), \sigma(t), {}^{\rho}D^{\gamma_1}_{0+}\sigma(t))| &\le b_{0f} + b_{1f}|\zeta| + b_{2f}|\sigma| + b_{3f}|{}^{\rho}D^{\gamma_1}_{0+}\sigma| \\ &\le b_{0f} + b_{1f}\|\zeta\|_X + \max\{b_{2f}, b_{3f}\}\|\sigma\|_Y \\ &\le b_{0f} + \Big[b_{1f} + \max\{b_{2f}, b_{3f}\}\Big]\|(\zeta, \sigma)\|_{X\times Y} \\ &\le b_{0f} + \Big[b_{1f} + \max\{b_{2f}, b_{3f}\}\Big]q := \Upsilon_f. \end{aligned}$$

Similarly, we obtain

$$\left|g\left(t, \zeta(t), {}^{\rho}D^{\gamma_2}_{0^+}\zeta(t), \sigma(t)\right)\right| \le b_{0g} + \Big[\max\{b_{1g}, b_{2g}\} + b_{3g}\Big]q := \Upsilon_g.$$

Hence, for any $(\zeta, \sigma) \in B_q$, one can get

$$\begin{aligned} &|\mathcal{G}_1(\zeta, \sigma)(t)| \\ \le\ & (\Upsilon_f + \Upsilon_g)\Big[\frac{T^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha + 1)} + \frac{T^{\rho(\alpha-1)}}{|\Lambda|}\Big\{|A_2\lambda_1|\frac{\xi_1^{\rho(\beta+\delta_1)}}{\rho^{\beta+\delta_1}\Gamma(\beta + \delta_1 + 1)} + |A_2|\frac{T^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha + 1)}V_0^T H_1 \\ & +|A_2|\sum_{p=1}^{m} a_p\frac{\eta^{\rho\beta}}{\rho^{\beta}\Gamma(\beta + 1)} + |B_1\lambda_2|\frac{\xi_2^{\rho(\alpha+\delta_2)}}{\rho^{\alpha+\delta_2}\Gamma(\alpha + \delta_2 + 1)} + |B_1|\frac{T^{\rho\beta}}{\rho^{\beta}\Gamma(\beta + 1)}V_0^T H_2 \\ & +|B_1|\sum_{p=1}^{m} b_p\frac{\eta^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha + 1)}\Big\}\Big] \\ \le\ & (\Upsilon_f + \Upsilon_g)M_1 \end{aligned}$$

and

$$\begin{aligned} &|{}^{\rho}D^{\gamma_2}_{0^+}\mathcal{G}_1(\zeta, \sigma)(t)| \\ \le\ & (\Upsilon_f + \Upsilon_g)\Big[\frac{T^{\rho(\alpha-\gamma_2)}}{\rho^{\alpha-\gamma_2}\Gamma(\alpha - \gamma_2 + 1)} + \frac{\Gamma(\alpha)}{|\Lambda|\Gamma(\alpha + 1 - \gamma_2)}\rho^{\gamma_2}(\alpha - \gamma_2)T^{\rho(\alpha-\gamma_2-1)} \\ & \times\Big\{|A_2\lambda_1|\frac{\xi_1^{\rho(\beta+\delta_1)}}{\rho^{\beta+\delta_1}\Gamma(\beta + \delta_1 + 1)} + |A_2|\frac{T^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha + 1)}V_0^T H_1 + |A_2|\sum_{p=1}^{m} a_p\frac{\eta^{\rho\beta}}{\rho^{\beta}\Gamma(\beta + 1)} \\ & +|B_1\lambda_2|\frac{\xi_2^{\rho(\alpha+\delta_2)}}{\rho^{\alpha+\delta_2}\Gamma(\alpha + \delta_2 + 1)} + |B_1|\frac{T^{\rho\beta}}{\rho^{\beta}\Gamma(\beta + 1)}V_0^T H_2 + |B_1|\sum_{p=1}^{m} b_p\frac{\eta^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha + 1)}\Big\}\Big] \\ \le\ & (\Upsilon_f + \Upsilon_g)M_2. \end{aligned}$$

Thus, we have

$$\|\mathcal{G}_1(\zeta, \sigma)\|_X \le (\Upsilon_f + \Upsilon_g)(M_1 + M_2). \tag{3.2}$$

Similarly, one can obtain that

$$\|\mathcal{G}_2(\zeta,\sigma)\|_Y \leq (\Upsilon_f + \Upsilon_g)(M_1' + M_2'). \tag{3.3}$$

Consequently, it follows from (3.2) and (3.3) that $\|\mathcal{G}(\zeta,\sigma)\|_{X\times Y} \leq (\Upsilon_f + \Upsilon_g)[(M_1 + M_2) + (M_1' + M_2')]$, which implies that $\mathcal{G}$ is a uniformly bounded operator.

Step 2. We claim that $\mathcal{G}$ is completely continuous. For that, let $0 \leq \tau_1 < \tau_2 \leq T$, then we have

$$\begin{aligned}
&|\mathcal{G}_1(\zeta,\sigma)(\tau_2) - \mathcal{G}_1(\zeta,\sigma)(\tau_1)|\\
\leq\ & \Big|\int_0^{\tau_1} \frac{\rho^{1-\alpha}s^{\rho-1}[(\tau_2^\rho - s^\rho)^{\alpha-1} - (\tau_1^\rho - s^\rho)^{\alpha-1}]}{\Gamma(\alpha)} f(s,\zeta(s),\sigma(s),{}^\rho D_{0^+}^{\gamma_1}\sigma(s))ds\Big|\\
&+\Big|\int_{\tau_1}^{\tau_2} \frac{\rho^{1-\alpha}s^{\rho-1}(\tau_2^\rho - s^\rho)^{\alpha-1}}{\Gamma(\alpha)} f(s,\zeta(s),\sigma(s),{}^\rho D_{0^+}^{\gamma_1}\sigma(s))ds\Big|\\
&+\frac{|\tau_2^{\rho(\alpha-1)} - \tau_1^{\rho(\alpha-1)}|}{|\Lambda|}\Big[|A_2\lambda_1|{}^\rho I_{0^+}^{\beta+\delta_1}|g(\xi_1,\zeta(\xi_1),{}^\rho D_{0^+}^{\gamma_2}\zeta(\xi_1),\sigma(\xi_1))|\\
&+|A_2|\int_0^T {}^\rho I_{0^+}^{\alpha}|f(s,\zeta(s),\sigma(s),{}^\rho D_{0^+}^{\gamma_1}\sigma(s))|dH_1(s)\\
&+|A_2|\sum_{p=1}^m a_p {}^\rho I_{0^+}^{\beta}|g\left(\eta_p,\zeta\left(\eta_p\right),{}^\rho D_{0^+}^{\gamma_2}\zeta\left(\eta_p\right),\sigma(\eta_p)\right)|\\
&+|B_1\lambda_2|{}^\rho I_{0^+}^{\alpha+\delta_2}|f(\xi_2,\zeta(\xi_2),\sigma(\xi_2),{}^\rho D_{0^+}^{\gamma_1}\sigma(\xi_2))|\\
&+|B_1|\int_0^T {}^\rho I_{0^+}^{\beta}|g(s,\zeta(s),{}^\rho D_{0^+}^{\gamma_2}\zeta(s),\sigma(s))|dH_2(s)\\
&+|B_1|\sum_{p=1}^m b_p {}^\rho I_{0^+}^{\alpha}|f\left(\eta_p,\zeta\left(\eta_p\right),\sigma\left(\eta_p\right),{}^\rho D_{0^+}^{\gamma_1}\sigma\left(\eta_P\right)\right)|\Big]\\
\leq\ & \frac{\Upsilon_f}{\rho^\alpha\Gamma(\alpha+1)}[2(\tau_2^\rho - \tau_1^\rho)^\alpha + |\tau_2^{\rho\alpha} - \tau_1^{\rho\alpha}|]\\
&+\frac{(\Upsilon_f + \Upsilon_g)[\tau_2^{\rho(\alpha-1)} - \tau_1^{\rho(\alpha-1)}]}{|\Lambda|}\Big[|A_2\lambda_1|\frac{\xi_1^{\rho(\beta+\delta_1)}}{\rho^{\beta+\delta_1}\Gamma(\beta+\delta_1+1)} + |A_2|\frac{T^{\rho\alpha}}{\rho^\alpha\Gamma(\alpha+1)}V_0^T H_1\\
+\ & |A_2|\sum_{p=1}^m a_p\frac{\eta^{\rho\beta}}{\rho^\beta\Gamma(\beta+1)} + |B_1\lambda_2|\frac{\xi_2^{\rho(\alpha+\delta_2)}}{\rho^{\alpha+\delta_2}\Gamma(\alpha+\delta_2+1)} + |B_1|\frac{T^{\rho\beta}}{\rho^\beta\Gamma(\beta+1)}V_0^T H_2\\
&+|B_1|\sum_{p=1}^m b_p\frac{\eta^{\rho\alpha}}{\rho^\alpha\Gamma(\alpha+1)}\Big]
\end{aligned}$$

and

$$\begin{aligned}
&|{}^\rho D_{0^+}^{\gamma_2}\mathcal{G}_1(\zeta,\sigma)(\tau_2) - {}^\rho D_{0^+}^{\gamma_2}\mathcal{G}_1(\zeta,\sigma)(\tau_1)|\\
\leq\ & \Big|\int_0^{\tau_1} \frac{\rho^{1-\alpha+\gamma_2}s^{\rho-1}[(\tau_2^\rho - s^\rho)^{\alpha-\gamma_2-1} - (\tau_1^\rho - s^\rho)^{\alpha-\gamma_2-1}]}{\Gamma(\alpha-\gamma_2)} f(s,\zeta(s),\sigma(s),{}^\rho D_{0^+}^{\gamma_1}\sigma(s))ds\Big|\\
&+\Big|\int_{\tau_1}^{\tau_2} \frac{\rho^{1-\alpha+\gamma_2}s^{\rho-1}(\tau_2^\rho - s^\rho)^{\alpha-\gamma_2-1}}{\Gamma(\alpha-\gamma_2)} f(s,\zeta(s),\sigma(s),{}^\rho D_{0^+}^{\gamma_1}\sigma(s))ds\Big|
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\Gamma(\alpha)}{|\Lambda|\Gamma(\alpha+1-\gamma_2)}\rho^{\gamma_2}(\alpha-\gamma_2)[\tau_2^{\rho(\alpha-\gamma_2-1)}-\tau_1^{\rho(\alpha-\gamma_2-1)}]\\
&\times\Big[|A_2\lambda_1|^\rho I_{0^+}^{\beta+\delta_1}|g(\xi_1,\zeta(\xi_1),{}^\rho D_{0^+}^{\gamma_2}\zeta(\xi_1),\sigma(\xi_1))\\
&+|A_2|\int_0^T {}^\rho I_{0^+}^{\alpha}|f(s,\zeta(s),\sigma(s),{}^\rho D_{0^+}^{\gamma_1}\sigma(s))|dH_1(s)\\
&+|A_2|\sum_{p=1}^m a_p{}^\rho I_{0^+}^{\beta}|g\left(\eta_p,\zeta\left(\eta_p\right),{}^\rho D_{0^+}^{\gamma_2}\zeta\left(\eta_p\right),\sigma(\eta_p)\right)|\\
&+|B_1\lambda_2|^\rho I_{0^+}^{\alpha+\delta_2}|f(\xi_2,\zeta(\xi_2),\sigma(\xi_2),{}^\rho D_{0^+}^{\gamma_1}\sigma(\xi_2))\\
&+|B_1|\int_0^T {}^\rho I_{0^+}^{\beta}|g(s,\zeta(s),{}^\rho D_{0^+}^{\gamma_2}\zeta(s),\sigma(s))|dH_2(s)\\
&+|B_1|\sum_{p=1}^m b_p{}^\rho I_{0^+}^{\alpha}|f\left(\eta_p,\zeta\left(\eta_p\right),\sigma\left(\eta_p\right),{}^\rho D_{0^+}^{\gamma_1}\sigma\left(\eta_P\right)\right)\Big]\\
\leq\ &\frac{\left(\Upsilon_f+\Upsilon_g\right)}{\rho^{\alpha-\gamma_2}\Gamma(\alpha-\gamma_2+1)}\Big[2(\tau_2^\rho-\tau_1^\rho)^{\alpha-\gamma_2}+|\tau_2^{\rho(\alpha-\gamma_2)}-\tau_1^{\rho(\alpha-\gamma_2)}|\Big]\\
&+\frac{\left(\Upsilon_f+\Upsilon_g\right)\Gamma(\alpha)}{|\Lambda|\Gamma(\alpha-\gamma_2)}\rho^{\gamma_2}\Big[\tau_2^{\rho(\alpha-\gamma_2-1)}-\tau_1^{\rho(\alpha-\gamma_2-1)}\Big]\times\Big[|A_2\lambda_1|\frac{\xi_1^{\rho(\beta+\delta_1)}}{\rho^{\beta+\delta_1}\Gamma\left(\beta+\delta_1+1\right)}\\
&+|A_2|\frac{T^{\rho\alpha}}{\rho^\alpha\Gamma(\alpha+1)}V_0^TH_1+|A_2|\sum_{p=1}^m a_p\frac{\eta^{\rho\beta}}{\rho^\beta\Gamma(\beta+1)}+|B_1\lambda_2|\frac{\xi_2^{\rho(\alpha+\delta_2)}}{\rho^{\alpha+\delta_2}\Gamma\left(\alpha+\delta_2+1\right)}\\
&+|B_1|\frac{T^{\rho\beta}}{\rho^\beta\Gamma(\beta+1)}V_0^TH_2+|B_1|\sum_{p=1}^m b_p\frac{\eta^{\rho\alpha}}{\rho^\alpha\Gamma(\alpha+1)}\Big].
\end{aligned}
$$

In consequence, we have that $\|\mathcal{G}_1(\zeta,\sigma)(\tau_2)-\mathcal{G}_1(\zeta,\sigma)(\tau_1)\|_X\to 0$ as $\tau_2\to\tau_1$, independent of $(\zeta,\sigma)\in B_q$. Similarly, we can find that $\|\mathcal{G}_2(\zeta,\sigma)(\tau_2)-\mathcal{G}_2(\zeta,\sigma)(\tau_1)\|_Y\to 0$ as $\tau_2\to\tau_1$, independent of $(\zeta,\sigma)\in B_q$. Thus, the operator $\mathcal{G}$ is equicontinuous. Hence, by the Arzelá-Ascoli Theorem, $\mathcal{G}$ is completely continuous.

Step 3. We define $\mathcal{E}=\{(u,v)\in X\times Y|(u,v)=\mu\mathcal{G}(u,v),0\leq\mu\leq 1\}$ and show that it is bounded. Let $(u,v)\in\mathcal{E}$, then $(u,v)=\mu\mathcal{G}(u,v)$ and $u(t)=\mu\mathcal{G}_1(u,v),\ v(t)=\mu\mathcal{G}_2(u,v),\ \forall t\in[0,T]$. Hence, we get

$$
\begin{aligned}
&|u(t)|\leq{}^\rho I_{0^+}^{\alpha}|f(t,u,v,{}^\rho D_{0^+}^{\gamma_1}v)|+\frac{t^{\rho(\alpha-1)}}{|\Lambda|}\Big[|A_2\lambda_1|^\rho I_{0^+}^{\beta+\delta_1}|g(\xi_1,u(\xi_1),{}^\rho D_{0^+}^{\gamma_2}u(\xi_1),v(\xi_1))|\\
&+|A_2|\int_0^T {}^\rho I_{0^+}^{\alpha}|f(s,u,v,{}^\rho D_{0^+}^{\gamma_1}v)|dH_1(s)+|A_2|\sum_{p=1}^m a_p{}^\rho I_{0^+}^{\beta}|g\left(\eta_p,u\left(\eta_p\right),{}^\rho D_{0^+}^{\gamma_2}u\left(\eta_p\right),v(\eta_p)\right)|\\
&+|B_1\lambda_2|^\rho I_{0^+}^{\alpha+\delta_2}|f(\xi_2,u(\xi_2),v(\xi_2),{}^\rho D_{0^+}^{\gamma_1}v(\xi_2))+|B_1|\int_0^T {}^\rho I_{0^+}^{\beta}|g(s,u,{}^\rho D_{0^+}^{\gamma_2}u,v)|dH_2(s)\\
&+|B_1|\sum_{p=1}^m b_p{}^\rho I_{0^+}^{\alpha}|f\left(\eta_p,u\left(\eta_p\right),v\left(\eta_p\right),{}^\rho D_{0^+}^{\gamma_1}v\left(\eta_P\right)\right)\Big]\\
\leq\ &(b_{0f}+b_{1f}\|u\|_X+\max\{b_{2f},b_{3f}\}\|v\|_Y)+\left(b_{0g}+\max\left\{b_{1g},b_{2g}\right\}\|u\|_X+b_{3g}\|v\|_Y\right)
\end{aligned}
$$

$$
\begin{aligned}
& \times\Big[\frac{T^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)}+\frac{T^{\rho(\alpha-1)}}{|\Lambda|}\Big\{|A_2\lambda_1|\frac{\xi_1^{\rho(\beta+\delta_1)}}{\rho^{\beta+\delta_1}\Gamma(\beta+\delta_1+1)}+|A_2|\frac{T^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)}V_0^T H_1 \\
& +|A_2|\sum_{p=1}^{m}a_p\frac{\eta^{\rho\beta}}{\rho^{\beta}\Gamma(\beta+1)}+|B_1\lambda_2|\frac{\xi_2^{\rho(\alpha+\delta_2)}}{\rho^{\alpha+\delta_2}\Gamma(\alpha+\delta_2+1)}+|B_1\lambda_2|\frac{\xi_1^{\rho(\alpha+\delta_2)}}{\rho^{\alpha+\delta_2}\Gamma(\alpha+\delta_2+1)} \\
& +|B_1|\frac{T^{\rho\beta}}{\rho^{\beta}\Gamma(\beta+1)}V_0^T H_2+|B_1|\sum_{p=1}^{m}b_p\frac{\eta^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)}\Big\}\Big] \\
= \; & \Big[(b_{0f}+b_{1f}\|u\|_X+\max\{b_{2f},b_{3f}\}\|v\|_Y)+\Big(b_{0g}+\max\Big\{b_{1g},b_{2g}\Big\}\|u\|_X+b_{3g}\|v\|_Y\Big)\Big]M_1,
\end{aligned}
$$

and

$$
\begin{aligned}
& |{}^{\rho}D_{0^+}^{\gamma_2}\mathcal{G}_1(u,v)(t)| \\
\leq \; & {}^{\rho}I_{0^+}^{\alpha-\gamma_2}\left|f\left(t,u,v,{}^{\rho}D_{0^+}^{\gamma_1}v\right)\right|+\frac{\Gamma(\alpha)}{|\Lambda|\,\Gamma(\alpha+1-\gamma_2)}\rho^{\gamma_2}(\alpha-\gamma_2)\,t^{\rho(\alpha-\gamma_2-1)} \\
& \times\Bigg[|A_2\lambda_1|{}^{\rho}I_{0^+}^{\beta+\delta_1}\left|g\left(\xi_1,u(\xi_1),{}^{\rho}D_{0^+}^{\gamma_2}u(\xi_1),v(\xi_1)\right)\right|+|A_2|\int_0^T{}^{\rho}I_{0^+}^{\alpha}\left|f\left(s,u,v,{}^{\rho}D_{0^+}^{\gamma_1}v\right)\right|dH_1(s) \\
& +|A_2|\sum_{p=1}^{m}a_p{}^{\rho}I_{0^+}^{\beta}|g\left(\eta_p,u\left(\eta_p\right),{}^{\rho}D_{0^+}^{\gamma_2}u\left(\eta_p\right),v\left(\eta_p\right)\right)| \\
& +|B_1\lambda_2|{}^{\rho}I_{0^+}^{\alpha+\delta_2}|f\left(\xi_2,u(\xi_2),v(\xi_2),{}^{\rho}D_{0^+}^{\gamma_1}v(\xi_2)\right)| \\
& +|B_1|\int_0^T{}^{\rho}I_{0^+}^{\beta}\left|g\left(s,u,{}^{\rho}D_{0^+}^{\gamma_2}u,v\right)\right|dH_2(s)+|B_1|\sum_{p=1}^{m}{}^{\rho}I_{0^+}^{\alpha}\left|f\left(\eta_p,u\left(\eta_p\right),v\left(\eta_p\right),{}^{\rho}D_{0^+}^{\gamma_1}v\left(\eta_p\right)\right)\right|\Bigg] \\
\leq \; & \Big[\Big(b_{0f}+b_{1f}\|u\|_X+\max\Big\{b_{2f},b_{3f}\Big\}\|v\|_Y\Big)+\Big(b_{0g}+\max\Big\{b_{1g},b_{2g}\Big\}\|u\|_X+b_{3g}\|v\|_Y\Big)\Big] \\
& \times\Bigg\{\frac{T^{\rho(\alpha-\gamma_2)}}{\rho^{\alpha-\gamma_2}\Gamma(\alpha-\gamma_2+1)}+\frac{\Gamma(\alpha)}{|\Lambda|\,\Gamma(\alpha+1-\gamma_2)}\rho^{\gamma_2}(\alpha-\gamma_2)\,T^{\rho(\alpha-\gamma_2-1)} \\
& \times\Bigg[|A_2\lambda_1|\frac{\xi_1^{\rho(\beta+\delta_1)}}{\rho^{\beta+\delta_1}\Gamma(\beta+\delta_1+1)}+|A_2|\frac{T^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)}V_0^T H_1+|A_2|\sum_{p=1}^{m}a_p\frac{\eta^{\rho\beta}}{\rho^{\beta}\Gamma(\beta+1)} \\
& +|B_1\lambda_2|\frac{\xi_2^{\rho(\alpha+\delta_2)}}{\rho^{\alpha+\delta_2}\Gamma(\alpha+\delta_2+1)}+|B_1|\frac{T^{\rho\beta}}{\rho^{\beta}\Gamma(\beta+1)}V_0^T H_2+|B_1|\sum_{p=1}^{m}b_p\frac{\eta^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)}\Bigg]\Bigg\} \\
= \; & \Big[\Big(b_{0f}+b_{1f}\|u\|_X+\max\Big\{b_{2f},b_{3f}\Big\}\|v\|_Y\Big)+\Big(b_{0g}+\max\Big\{b_{1g},b_{2g}\Big\}\|u\|_X+b_{3g}\|v\|_Y\Big)\Big]M_2.
\end{aligned}
$$

In view of the foregoing inequalities, we obtain

$$
\begin{aligned}
\|u\|_X \leq \; & \Big(b_{0f}+b_{1f}\|u\|_X+\max\Big\{b_{2f},b_{3f}\Big\}\|v\|_Y+b_{0g}+\max\Big\{b_{1g},b_{2g}\Big\}\|u\|_X \\
& +|b_{3g}|\|v\|_Y\Big)(M_1+M_2).
\end{aligned} \tag{3.4}
$$

Similarly, we can find that

$$
\|v\|_Y \leq \Big(b_{0f}+b_{1f}\|u\|_X+\max\Big\{b_{2f},b_{3f}\Big\}\|v\|_Y+b_{0g}+\max\Big\{b_{1g},b_{2g}\Big\}\|u\|_X
$$

$$+b_{3g}\|v\|_Y\Big)\left(M_1' + M_2'\right). \tag{3.5}$$

Therefore, from (3.4) and (3.5), we have

$$\begin{aligned}\|u\|_X + \|v\|_Y \;\; \leq \;\; & (b_{0f} + b_{0g})(M_1 + M_2 + M_1' + M_2') + \Big[b_{1f}\|u\|_X + \max\left\{b_{1g}, b_{2g}\right\}\|u\|_X \\ & + \max\left\{b_{2f}, b_{3f}\right\}\|v\|_Y + b_{3g}\|v\|_Y\Big](M_1 + M_2 + M_1' + M_2').\end{aligned}$$

By choosing

$$\begin{aligned}M_0 \;\; = \;\; & \min\Big\{1 - (b_{1f} + \max\left\{b_{1g}, b_{2g}\right\})(M_1 + M_2 + M_1' + M_2'), \\ & 1 - (\max\left\{b_{2f}, b_{3f}\right\} + b_{3g})(M_1 + M_2 + M_1' + M_2')\Big\},\end{aligned}$$

we obtain the inequality

$$\|(u, v)\|_{X\times Y} \leq \frac{(b_{0f} + b_{0g})(M_1 + M_2 + M_1' + M_2')}{M_0}.$$

Thus, $\mathcal{E}$ is bounded and the conclusion of the Leray-Schauder alternative applies, and, hence, the operator $\mathcal{G}$ has at least one fixed point, which corresponds to at least one solution of system (1.1). □

In our next result, we establish the existence of a unique solution to the system (1.1) by means of the Banach's fixed point theorem.

Theorem 3.2. *Suppose that the following conditions hold:*

(H2) $H_i : [0, T] \times \mathbb{R} \to \mathbb{R}$ *(*$i = 1, 2$*) are functions of bounded variations on* $[0, T]$.

(H3) For $f, g : [0, T] \times \mathbb{R}^3 \to \mathbb{R}$*, there exist constants* $L_f > 0$ *and* $L_g > 0$ *such that, for any* $t \in [0, T]$ *and* $\varpi_i, \varrho_i \in \mathbb{R}$, $i = 1, 2, 3,$ *we have*

$$|f(t, \varpi_1, \varpi_2, \varpi_3) - f(t, \varrho_1, \varrho_2, \varrho_3)| \leq L_f(|\varpi_1 - \varrho_1| + |\varpi_2 - \varrho_2| + |\varpi_3 - \varrho_3|),$$

and

$$|g(t, \varpi_1, \varpi_2, \varpi_3) - g(t, \varrho_1, \varrho_2, \varrho_3)| \leq L_g(|\varpi_1 - \varrho_1| + |\varpi_2 - \varrho_2| + |\varpi_3 - \varrho_3|).$$

If

$$(L_f + L_g)(M_1 + M_2 + M_1' + M_2') < 1, \tag{3.6}$$

then the system (1.1) *has a unique solution on* $[0, T]$*, where* M_1, M_2, M_1' *and* M_2' *are given in (3.1).*

Proof. Set $\max_{t\in[0,T]} f(t, 0, 0, 0) = f_0 < \infty$, $\max_{t\in[0,T]} g(t, 0, 0, 0) = g_0 < \infty$ and define $B_r = \{(u, v) \in X \times Y : \|(u, v)\|_{X\times Y} \leq r\}$ with

$$r > \frac{(f_0 + g_0)(M_1 + M_2 + M_1' + M_2')}{1 - (L_f + L_g)(M_1 + M_2 + M_1' + M_2')}. \tag{3.7}$$

In the first step, it will be shown that $\mathcal{G}B_r \subset B_r$. In view of the assumptions ($H2$) and ($H3$), for $(u, v) \in X \times Y$, we have

$$|f(t, u(t), v(t), {}^\rho D^{\gamma_1}_{0+} v(t))| \leq |f(t, 0, 0, 0)| + |f(t, u(t), v(t), {}^\rho D^{\gamma_1}_{0+} v(t)) - f(t, 0, 0, 0)|$$

$$
\begin{aligned}
&\le f_0 + L_f(|u| + |v| + |{}^\rho D^{\gamma_1}_{0+}v|)\\
&\le f_0 + L_f(\|u\|_X + \|v\|_Y) = f_0 + L_f\|(u,v)\|_{X\times Y}\\
&\le f_0 + L_f r.
\end{aligned}
$$

Similarly, one can find that

$$
|g(t,u(t),{}^\rho D^{\gamma_2}_{0+}u(t),v(t))| \le g_0 + L_g\|(u,v)\|_{X\times Y} \le g_0 + L_g r.
$$

Using the foregoing estimates, we obtain

$$
\begin{aligned}
&|\mathcal{G}_1(u,v)(t)|\\
\le\ & {}^\rho I^\alpha_{0+}|f(t,u(t),v(t),{}^\rho D^{\gamma_1}_{0+}v(t))| + \frac{t^{\rho(\alpha-1)}}{|\Lambda|}\Big[|A_2|\Big\{|\lambda_1|{}^\rho I^{\beta+\delta_1}_{0+}|g(\xi_1,u(\xi_1),{}^\rho D^{\gamma_2}_{0+}u(\xi_1),v(\xi_1))|\\
&+\int_0^T {}^\rho I^\alpha_{0+}|f(s,u(s),v(s),{}^\rho D^{\gamma_1}_{0+}v(s))|dH_1(s) + \sum_{p=1}^m a_p I^\beta_{0^+}|g\left(\eta_p,u\left(\eta_p\right),{}^\rho D^{\gamma_2}_{0^+}u\left(\eta_p\right),v\left(\eta_p\right)\right)|\Big\}\\
&+|B_1|\Big\{|\lambda_2|{}^\rho I^{\alpha+\delta_2}_{0+}|f(\xi_2,u(\xi_2),v(\xi_2),{}^\rho D^{\gamma_1}_{0+}v(\xi_2))| + \int_0^T {}^\rho I^\beta_{0+}|g(s,u(s),{}^\rho D^{\gamma_2}_{0+}u(s),v(s))|dH_2(s)\\
&+\sum_{p=1}^m b_p{}^\rho I^\alpha_{0^+}\left|f\left(\eta_p,u\left(\eta_p\right),v\left(\eta_p\right),{}^\rho D^{\gamma_1}_{0^+}v\left(\eta_p\right)\right)\right|\Big\}\Big]\\
\le\ & [(L_f r + f_0) + (L_g r + g_0)]\Big[\frac{T^{\rho\alpha}}{\rho^\alpha\Gamma(\alpha+1)}\\
&+\frac{T^{\rho(\alpha-1)}}{|\Lambda|}\Big\{|A_2\lambda_1|\frac{\xi_1^{\rho(\beta+\delta_1)}}{\rho^{\beta+\delta_1}\Gamma(\beta+\delta_1+1)} + |A_2|\frac{T^{\rho\alpha}}{\rho^\alpha\Gamma(\alpha+1)}V_0^T H_1 + |A_2|\sum_{p=1}^m a_p\frac{\eta^{\rho\beta}}{\rho^\beta\Gamma(\beta+1)}\\
&+|B_1\lambda_2|\frac{\xi_2^{\rho(\alpha+\delta_2)}}{\rho^{\alpha+\delta_2}\Gamma(\alpha+\delta_2+1)} + |B_1|\frac{T^{\rho\beta}}{\rho^\beta\Gamma(\beta+1)}V_0^T H_2 + |B_1|\sum_{p=1}^m b_p\ \frac{\eta_p^{\rho\alpha}}{\rho^\alpha\Gamma(\alpha+1)}\Big\}\Big]\\
\le\ & [(L_f + L_g)r + (f_0+g_0)]M_1.
\end{aligned}
$$

In view of the relation: ${}^\rho D^{\gamma_2}_{0^+}{}^\rho I^\alpha_{0^+} = {}^\rho D^{\gamma_2}_{0^+}{}^\rho I^{\gamma_2}_{0^+}{}^\rho I^{\alpha-\gamma_2}_{0^+} = {}^\rho I^{\alpha-\gamma_2}_{0^+}$ for $\gamma_2 < \alpha$, we get

$$
\begin{aligned}
&|{}^\rho D^{\gamma_2}_{0^+}\mathcal{G}_1(u,v)(t)|\\
\le\ & {}^\rho I^{\alpha-\gamma_2}_{0+}|f(t,u(t),v(t),{}^\rho D^{\gamma_1}_{0+}v(t))| + \frac{\Gamma(\alpha)\rho^{\gamma_2}t^{\rho(\alpha-\gamma_2-1)}}{|\Lambda|\Gamma(\alpha-\gamma_2)}\\
&\times\Big[|A_2|\Big\{|\lambda_1|{}^\rho I^{\beta+\delta_1}_{0+}|g(\xi_1,u(\xi_1),{}^\rho D^{\gamma_2}_{0+}u(\xi_1),v(\xi_1))|\\
&+\int_0^T {}^\rho I^\alpha_{0+}|f(s,u(s),v(s),{}^\rho D^{\gamma_1}_{0+}v(s))|dH_1(s) + \sum_{p=1}^m a_p{}^\rho I^\beta_{0^+}\left|g\left(\eta_p,u\left(\eta_p\right),{}^\rho D^{\gamma_2}_{0^+}u\left(\eta_p\right),v\left(\eta_p\right)\right)\right|\Big\}\\
&+|B_1|\Big\{|\lambda_2|{}^\rho I^{\alpha+\delta_2}_{0+}|f(\xi_2,u(\xi_2),v(\xi_2),{}^\rho D^{\gamma_1}_{0+}v(\xi_2))| + \int_0^T {}^\rho I^\beta_{0+}|g(s,u(s),{}^\rho D^{\gamma_2}_{0+}u(s),v(s))|dH_2(s)\\
&+\sum_{p=1}^m b_p{}^\rho I^\alpha_{0^+}\left|f\left(\eta_p,u\left(\eta_p\right),v\left(\eta_p\right),{}^\rho D^{\gamma_1}_{0^+}v\left(\eta_p\right)\right)\right|\Big\}\Big]
\end{aligned}
$$

$$
\begin{aligned}
\leq\ & [(L_f r + f_0) + (L_g r + g_0)]\Big[\frac{T^{\rho(\alpha-\gamma_2)}}{\rho^{\alpha-\gamma_2}\Gamma(\alpha-\gamma_2+1)} \\
& + \frac{\Gamma(\alpha)}{|\Lambda|\Gamma(\alpha-\gamma_2)}\rho^{\gamma_2}T^{\rho(\alpha-\gamma_2-1)}\Big\{|A_2\lambda_1|\frac{\xi_1^{\rho(\beta+\delta_1)}}{\rho^{\beta+\delta_1}\Gamma(\beta+\delta_1+1)} + |A_2|\frac{T^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)}V_0^T H_1 \\
& + |A_2|\sum_{p=1}^{m} a_p \frac{\eta^{\rho\beta}}{\rho^{\beta}\Gamma(\beta+1)} + |B_1\lambda_2|\frac{\xi_2^{\rho(\alpha+\delta_2)}}{\rho^{\alpha+\delta_2}\Gamma(\alpha+\delta_2+1)} + |B_1|\frac{T^{\rho\beta}}{\rho^{\beta}\Gamma(\beta+1)}V_0^T H_2 \\
& + |B_1|\sum_{p=1}^{m} b_p \frac{\eta^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)}\Big\}\Big] \\
\leq\ & [(L_f + L_g)r + (f_0 + g_0)]M_2.
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
\|\mathcal{G}_1(u,v)\|_X &= \|\mathcal{G}_1(u,v)\| + \|{}^{\rho}D_{0^+}^{\gamma_2}\mathcal{G}_1(u,v)\| \\
&\leq ((L_f + L_g)r + (f_0 + g_0))M_1 + ((L_f + L_g)r + (f_0 + g_0))M_2 \\
&\leq [(L_f + L_g)r + (f_0 + g_0)](M_1 + M_2). \qquad (3.8)
\end{aligned}
$$

In a similar manner, we can obtain that

$$
\begin{aligned}
\|\mathcal{G}_2(u,v)\|_Y &= \|\mathcal{G}_2(u,v)\| + \|{}^{\rho}D_{0^+}^{\gamma_1}\mathcal{G}_2(u,v)\| \\
&\leq [(L_f + L_g)r + (f_0 + g_0)](M_1' + M_2'). \qquad (3.9)
\end{aligned}
$$

In view of (3.7), it follows from (3.8) and (3.9) that

$$
\begin{aligned}
\|\mathcal{G}(u,v)\|_{X\times Y} &= \|\mathcal{G}_1(u,v)\|_X + \|\mathcal{G}_2(u,v)\|_Y \\
&\leq [(L_f + L_g)r + (f_0 + g_0)](M_1 + M_2 + M_1' + M_2') \leq r.
\end{aligned}
$$

In the second step, we show that the operator $\mathcal{G}$ is a contraction. To this end, let $\zeta_1, \zeta_2 \in X$ and $\varsigma_1, \varsigma_2 \in Y$, then we have

$$
\begin{aligned}
& |\mathcal{G}_1(\zeta_1,\varsigma_1)(t) - \mathcal{G}_1(\zeta_2,\varsigma_2)(t)| \\
\leq\ & {}^{\rho}I_{0+}^{\alpha}\left|f\left(t,\zeta_1(t),\varsigma_1(t),{}^{\rho}D_{0^+}^{\gamma_1}\varsigma_1(t)\right) - f\left(t,\zeta_2(t),\varsigma_2(t),{}^{\rho}D_{0^+}^{\gamma_1}\varsigma_2(t)\right)\right| \\
& + \frac{t^{\rho(\alpha-1)}}{|\Lambda|}\Big[|A_2|\Big\{|\lambda_1|{}^{\rho}I_{0+}^{\beta+\delta_1}\left|g\left(\xi_1,\zeta_1(\xi_1),{}^{\rho}D_{0^+}^{\gamma_2}\zeta_1(\xi_1),\varsigma_1(\xi_1)\right) - g\left(\xi_1,\zeta_2(\xi_1),{}^{\rho}D_{0^+}^{\gamma_2}\zeta_2(\xi_1),\varsigma_2(\xi_1)\right)\right| \\
& + \int_0^T {}^{\rho}I_{0+}^{\alpha}\left|f\left(s,\zeta_1(s),\varsigma_1(s),{}^{\rho}D_{0^+}^{\gamma_1}\varsigma_1(s)\right) - f\left(s,\zeta_2(s),\varsigma_2(s),{}^{\rho}D_{0^+}^{\gamma_1}\varsigma_2(s)\right)\right| dH_1(s) \\
& + \sum_{p=1}^{m} a_p\, {}^{\rho}I_{0+}^{\beta}\left|g\left(\eta_p,\zeta_1(\eta_p),{}^{\rho}D_{0^+}^{\gamma_2}\zeta_1(\eta_p),\varsigma_1(\eta_p)\right) - g\left(\eta_p,\zeta_2(\eta_p),{}^{\rho}D_{0^+}^{\gamma_2}\zeta_2(\eta_p),\varsigma_2(\eta_p)\right)\right|\Big\} \\
& + |B_1|\Big\{|\lambda_2|{}^{\rho}I_{0+}^{\alpha+\delta_2}\left|f\left(\xi_2,\zeta_1(\xi_2),\varsigma_1(\xi_2),{}^{\rho}D_{0^+}^{\gamma_1}\varsigma_1(\xi_2)\right) - f\left(\xi_2,\zeta_2(\xi_2),\varsigma_2(\xi_2),{}^{\rho}D_{0^+}^{\gamma_1}\varsigma_2(\xi_2)\right)\right| \\
& + \int_0^T {}^{\rho}I_{0+}^{\beta}\left|g\left(s,\zeta_1(s),{}^{\rho}D_{0^+}^{\gamma_2}\zeta_1(s),\varsigma_1(s)\right) - |g\left(s,\zeta_2(s),{}^{\rho}D_{0^+}^{\gamma_2}\zeta_2(s),\varsigma_2(s)\right)\right| dH_2(s)
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{p=1}^{m} b_p{}^{\rho}I_{0^+}^{\alpha}\left|f\left(\eta_p,\zeta_1\left(\eta_p\right),\varsigma_1\left(\eta_p\right),{}^{\rho}D_{0^+}^{\gamma_1}\varsigma_1\left(\eta_p\right)\right)-f\left(\eta_p,\zeta_2\left(\eta_p\right),\varsigma_2\left(\eta_p\right),{}^{\rho}D_{0^+}^{\gamma_1}\varsigma_2\left(\eta_p\right)\right)\right|\Big\}\Big]\\
\leq\;& \Big\{\left(L_f+L_g\right)\Big(\|\zeta_1-\zeta_2\|+\|\varsigma_1-\varsigma_2\|\Big)+L_f\left\|{}^{\rho}D_{0^+}^{\gamma_1}\varsigma_1-{}^{\rho}D_{0^+}^{\gamma_1}\varsigma_2\right\|+L_g\left\|{}^{\rho}D_{0^+}^{\gamma_2}\zeta_1-{}^{\rho}D_{0^+}^{\gamma_2}\zeta_2\right\|\Big\}\\
\times\;& \Big[\frac{T^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)}+\frac{T^{\rho(\alpha-1)}}{|\Lambda|}\Big\{|A_2\lambda_1|\frac{\xi_1^{\rho(\beta+\delta_1)}}{\rho^{\beta+\delta_1}\Gamma(\beta+\delta_1+1)}+|A_2|\frac{T^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)}V_0^TH_1\\
&+|A_2|\sum_{p=1}^{m}a_p\frac{\eta^{\rho\beta}}{\rho^{\beta}\Gamma(\beta+1)}+|B_1\lambda_2|\frac{\xi_2^{\rho(\alpha+\delta_2)}}{\rho^{\alpha+\delta_2}\Gamma(\alpha+\delta_2+1)}\\
&+|B_1|\frac{T^{\rho\beta}}{\rho^{\beta}\Gamma(\beta+1)}V_0^TH_2+|B_1|\sum_{p=1}^{m}b_p\frac{\eta^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)}\Big\}\Big]\\
\leq\;& \Big[(L_f+L_g)(\|\zeta_1-\zeta_2\|+\|\varsigma_1-\varsigma_2\|)+L_f\|{}^{\rho}D_{0^+}^{\gamma_1}\varsigma_1-{}^{\rho}D_{0^+}^{\gamma_1}\varsigma_2\|+L_g\|{}^{\rho}D_{0^+}^{\gamma_2}\zeta_1-{}^{\rho}D_{0^+}^{\gamma_2}\zeta_2\|\Big]M_1\\
\leq\;& (L_f+L_g)(\|\zeta_1-\zeta_2\|_X+\|\varsigma_1-\varsigma_2\|_Y)M_1,\quad \forall t\in[0,T].
\end{aligned}
$$

Also, we obtain

$$
\begin{aligned}
&|{}^{\rho}D_{0^+}^{\gamma_2}\mathcal{G}_1(\zeta_1,\varsigma_1)(t)-{}^{\rho}D_{0^+}^{\gamma_2}\mathcal{G}_1(\zeta_2,\varsigma_2)(t)|\\
\leq\;& {}^{\rho}I_{0+}^{\alpha-\gamma_2}|f(t,\zeta_1(t),\varsigma_1(t),{}^{\rho}D_{0^+}^{\gamma_1}\varsigma_1(t))-f(t,\zeta_2(t),\varsigma_2(t),{}^{\rho}D_{0^+}^{\gamma_1}\varsigma_2(t))|+\frac{\Gamma(\alpha)\rho^{\gamma_2}t^{\rho(\alpha-\gamma_2-1)}}{|\Lambda|\Gamma(\alpha-\gamma_2)}\\
&\times\Big[|A_2|\Big\{|\lambda_1|{}^{\rho}I_{0+}^{\beta+\delta_1}|g(\xi_1,\zeta_1(\xi_1),{}^{\rho}D_{0^+}^{\gamma_2}\zeta_1(\xi_1),\varsigma_1(\xi_1))-g(\xi_1,\zeta_2(\xi_1),{}^{\rho}D_{0^+}^{\gamma_2}\zeta_2(\xi_1),\varsigma_2(\xi_1))|\\
&+\int_0^T{}^{\rho}I_{0+}^{\alpha}|f(s,\zeta_1(s),\varsigma_1(s),{}^{\rho}D_{0^+}^{\gamma_1}\varsigma_1(s))-f(s,\zeta_2(s),\varsigma_2(s),{}^{\rho}D_{0^+}^{\gamma_1}\varsigma_2(s))|dH_1(s)\\
&+\sum_{p=1}^{m}a_p\,{}^{\rho}I_{0^+}^{\beta}\left|g\left(\eta_p,\zeta_1\left(\eta_p\right),{}^{\rho}D_{0^+}^{\gamma_2}\zeta_1\left(\eta_p\right),\varsigma_1\left(\eta_p\right)\right)-g\left(\eta_p,\zeta_2\left(\eta_p\right),{}^{\rho}D_{0^+}^{\gamma_2}\zeta_2\left(\eta_p\right),\varsigma_2\left(\eta_p\right)\right)\right|\Big\}\\
&+|B_1|\Big\{|\lambda_2|{}^{\rho}I_{0+}^{\alpha+\delta_2}|f(\xi_2,\zeta_1(\xi_2),\varsigma_1(\xi_2),{}^{\rho}D_{0^+}^{\gamma_1}\varsigma_1(\xi_2))-f(\xi_2,\zeta_2(\xi_2),\varsigma_2(\xi_2),{}^{\rho}D_{0^+}^{\gamma_1}\varsigma_2(\xi_2))|\\
&+\int_0^T{}^{\rho}I_{0+}^{\beta}|g(s,\zeta_1(s),{}^{\rho}D_{0^+}^{\gamma_2}\zeta_1(s),\varsigma_1(s))-g(s,\zeta_2(s),{}^{\rho}D_{0^+}^{\gamma_2}\zeta_2(s),\varsigma_2(s))|dH_2(s)\\
&+\sum_{p=1}^{m}b_p{}^{\rho}I_{0^+}^{\alpha}\left|f\left(\eta_p,\zeta_1\left(\eta_p\right),\varsigma_1\left(\eta_p\right),{}^{\rho}D_{0^+}^{\gamma_1}\varsigma_1\left(\eta_p\right)\right)-f\left(\eta_p,\zeta_2\left(\eta_p\right),\varsigma_2\left(\eta_p\right),{}^{\rho}D_{0^+}^{\gamma_1}\varsigma_2\left(\eta_p\right)\right)\right|\Big\}\Big]\\
\leq\;& \Big\{(L_f+L_g)(\|\zeta_1-\zeta_2\|+\|\varsigma_1-\varsigma_2\|)+L_f\|{}^{\rho}D_{0^+}^{\gamma_1}\varsigma_1-{}^{\rho}D_{0^+}^{\gamma_1}\varsigma_2\|+L_g\|{}^{\rho}D_{0^+}^{\gamma_2}\zeta_1-{}^{\rho}D_{0^+}^{\gamma_2}\zeta_2\|\Big\}\\
&\times\Big[\frac{T^{\rho(\alpha-\gamma_2)}}{\rho^{\alpha-\gamma_2}\Gamma(\alpha-\gamma_2+1)}+\frac{\Gamma(\alpha)}{|\Lambda|\Gamma(\alpha-\gamma_2)}\rho^{\gamma_2}T^{\rho(\alpha-\gamma_2-1)}\times\Big\{|A_2\lambda_1|\frac{\xi_1^{\rho(\beta+\delta_1)}}{\rho^{\beta+\delta_1}\Gamma(\beta+\delta_1+1)}\\
&+|A_2|\frac{T^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)}V_0^TH_1+|A_2|\sum_{p=1}^{m}a_p\frac{\eta^{\rho\beta}}{\rho^{\beta}\Gamma(\beta+1)}+|B_1\lambda_2|\frac{\xi_2^{\rho(\alpha+\delta_2)}}{\rho^{\alpha+\delta_2}\Gamma(\alpha+\delta_2+1)}\\
&+|B_1|\frac{T^{\rho\beta}}{\rho^{\beta}\Gamma(\beta+1)}V_0^TH_2+|B_1|\sum_{p=1}^{m}b_p\frac{\eta^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)}\Big\}\Big]\\
\leq\;& \Big[(L_f+L_g)(\|\zeta_1-\zeta_2\|+\|\varsigma_1-\varsigma_2\|)+L_f\|{}^{\rho}D_{0^+}^{\gamma_1}\varsigma_1-{}^{\rho}D_{0^+}^{\gamma_1}\varsigma_2\|+L_g\|{}^{\rho}D_{0^+}^{\gamma_2}\zeta_1-{}^{\rho}D_{0^+}^{\gamma_2}\zeta_2\|\Big]M_2
\end{aligned}
$$

$$\leq \quad (L_f + L_g)(\|\zeta_1 - \zeta_2\|_X + \|\varsigma_1 - \varsigma_2\|_Y)M_2.$$

Consequently, we have

$$\begin{aligned} & \|\mathcal{G}_1(\zeta_1, \varsigma_1) - \mathcal{G}_1(\zeta_2, \varsigma_2)\|_X \\ = \quad & \|{}^\rho D_{0^+}^{\gamma_2}\mathcal{G}_1(\zeta_1, \varsigma_1) - {}^\rho D_{0^+}^{\gamma_2}\mathcal{G}_1(\zeta_2, \varsigma_2)\| + \|\mathcal{G}_1(\zeta_1, \varsigma_1) - \mathcal{G}_1(\zeta_2, \varsigma_2)\| \\ \leq \quad & (L_f + L_g)(\|\zeta_1 - \zeta_2\|_X + \|\varsigma_1 - \varsigma_2\|_Y)(M_1 + M_2). \end{aligned} \tag{3.10}$$

Using a similar procedure, one can obtain

$$\begin{aligned} & \|\mathcal{G}_2(\zeta_1, \varsigma_1) - \mathcal{G}_2(\zeta_2, \varsigma_2)\|_Y \\ = \quad & \left\|{}^\rho D_{0^+}^{\gamma_1}\mathcal{G}_2(\zeta_1, \zeta_1) - {}^\rho D_{0^+}^{\gamma_1}\mathcal{G}_2(\zeta_2, \varsigma_2)\right\| + \|\mathcal{G}_2(\zeta_1, \varsigma_1) - \mathcal{G}_2(\zeta_2, \varsigma_2)\| \\ \leq \quad & \left(L_f + L_g\right)(\|\zeta_1 - \zeta_2\|_X + \|\varsigma_1 - \varsigma_2\|_Y)(M_1' + M_2'). \end{aligned} \tag{3.11}$$

From (3.10) and (3.11), we deduce that

$$\|\mathcal{G}(\zeta_1, \varsigma_1) - \mathcal{G}(\zeta_2, \varsigma_2)\|_{X\times Y} \leq [M_1 + M_2 + M_1' + M_2']\left(L_f + L_g\right)(\|\zeta_1 - \zeta_2\|_X + \|\varsigma_1 - \varsigma_2\|_Y),$$

which, by the conditions (3.6), implies that the operator $\mathcal{G}$ is a contraction. Thus, by Banach's fixed point theorem, $\mathcal{G}$ has a unique fixed point. Therefore, the system (1.1) has a unique solution on $[0, T]$ and, hence, we have the conclusion. □

Example

Consider the following fractional differential system:

$$\begin{cases} {}^{\frac{1}{4}}D_{0^+}^{\frac{6}{5}}u(t) = f\left(t, u, v, {}^\rho D_{0^+}^{\gamma_1}v\right), & t \in [0, 2], \\ {}^{\frac{1}{4}}D_{0^+}^{\frac{7}{5}}v(t) = g\left(t, u, {}^\rho D_{0^+}^{\gamma_2}u, v\right), & t \in [0, 2], \\ u(0) = 0, \ \displaystyle\int_0^T u(s)dH_1(s) = \frac{3}{2}^{\frac{1}{4}} I_{0^+}^{\frac{3}{5}} v(\xi_1) + \sum_{p=1}^{2} a_p v(\eta_p), \\ v(0) = 0, \ \displaystyle\int_0^T v(s)dH_2(s) = \frac{3}{2}^{\frac{1}{4}} I_{0^+}^{\frac{4}{5}} u(\xi_2) + \sum_{p=1}^{2} b_p u(\eta_p). \end{cases} \tag{3.12}$$

Here, $T = 2$, $\xi_1 = \xi_2 = \frac{3}{2}$, $\lambda_1 = \lambda_2 = \frac{3}{2}$, $\rho = \frac{1}{4}, \alpha = \frac{6}{5}, \beta = \frac{7}{5}, \gamma_1 = \frac{1}{5}$, $\gamma_2 = \frac{1}{3}$, $\delta_1 = \frac{3}{5}$, $\delta_2 = \frac{4}{5}$, $m = 2$, $\eta_1 = \frac{13}{8}$, $\eta_2 = \frac{7}{4}$, $a_1 = \frac{1}{2}$, $a_2 = \frac{3}{4}$, $b_1 = 1$, $b_2 = \frac{3}{2}$,

$$f\left(t, u, v, {}^\rho D_{0^+}^{\gamma_1}v\right) = \frac{e^{-3t}}{18\sqrt{900 + t^2}}\left(\frac{1}{1+t}\sin(u(t)) + e^{-2t}\cos(v(t)) + \cos(\tfrac{\pi}{2}t)|{}^{\frac{1}{4}}D_{0^+}^{\frac{1}{5}}v(t)| + e^{-t}\right) \text{ and}$$

$$g\left(t, u, {}^\rho D_{0^+}^{\gamma_2}u, v\right) = \frac{1}{2\sqrt{3600 + t^2}}\left(\tan^{-1}(u(t)) + \frac{e^{-3t}}{2(1+t^4)}\frac{|v(t)|}{2 + |v(t)|} + \frac{|{}^{\frac{1}{4}}D_{0^+}^{\frac{1}{3}}u(t)|}{4 + 4|{}^{\frac{1}{4}}D_{0^+}^{\frac{1}{3}}u(t)|}\right) + \frac{1}{16}.$$

Letting $H_1(t) = 2 + 3t$ and $H_2(t) = 5 + 4t^2$ and using the given values, we find that $L_f = \dfrac{1}{540}$, $L_g = \dfrac{1}{120}$, $M_1 = 29.930973$, $M_1' = 26.5686$, $M_2 = 16.023896$ and $M_2' = 19.637255$. Moreover,

$$\left(L_f + L_g\right)(M_1 + M_2 + M_1' + M_2') \approx 0.938673948 < 1.$$

Clearly, the hypotheses of Theorem 3.2 are satisfied and, hence, we deduce from its conclusion that the problem (3.12) has a unique solution on $[0, 2]$.

4. Conclusions

We have presented the criteria ensuring the existence and uniqueness of solutions for a coupled system of nonlinear generalized fractional differential equations equipped with nonlocal coupled multipoint boundary conditions involving Riemann-Stieltjes and generalized fractional integrals. The nonlinearities in the given system were assumed to be dependent on the unknown functions together with their lower order generalized fractional derivatives. We made use of the standard tools of the fixed point theory to accomplish the desired work. Our results were new in the given configuration and produced some new results by specializing the parameters involved in the problem at hand. For example, our results relate to the given coupled system of nonlinear generalized fractional differential equations supplemented with nonlocal multipoint and Riemann-Stieltjes type integral boundary conditions for $\lambda_1 = 0 = \lambda_2$. In case we take $\rho \to 0^+$ and $\rho = 1$, the results established in this paper correspond to the Hadamard and Riemann-Liouville fractional differential systems equipped with coupled multipoint Riemann-Stieltjes and Hadamard/Riemann-Liouville fractional integral boundary conditions, respectively.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

We thank the reviewers for their constructive remarks on our paper.

Conflict of interest

The authors declare that they have no conflict of interest.

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