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Research article

Soft weakly connected sets and soft weakly connected components

Samer Al-Ghour[1,*] **and Hanan Al-Saadi**[2,*]

[1] Department of Mathematics and Statistics, Jordan University of Science and Technology, Irbid 22110, Jordan

[2] Department of Mathematics, Faculty of Applied Sciences, Umm Al-Qura University, Makkah 24225, Saudi Arabia

* **Correspondence:** Email: algore@just.edu.jo, hsssaadi@uqu.edu.sa.

Abstract: Although the concept of connectedness may seem simple, it holds profound implications for topology and its applications. The concept of connectedness serves as a fundamental component in the Intermediate Value Theorem. Connectedness is significant in various applications, including geographic information systems, population modeling and robotics motion planning. Furthermore, connectedness plays a crucial role in distinguishing between different topological spaces. In this paper, we define soft weakly connected sets as a new class of soft sets that strictly contains the class of soft connected sets. We characterize this new class of sets by several methods. We explore various results related to soft subsets, supersets, unions, intersections and subspaces within the context of soft weakly connected sets. Additionally, we provide characterizations for soft weakly connected sets classified as soft pre-open, semi-open or α-open sets. Furthermore, we introduce the concept of a soft weakly connected component as follows: Given a soft point a_x in a soft topological space (X, Δ, A), we define the soft weakly component of (X, Δ, A) determined by a_x as the largest soft weakly connected set, with respect to the soft inclusion ($\widetilde{\subseteq}$) relation, that contains a_x. We demonstrate that the family of soft weakly components within a soft topological space comprises soft closed sets, forming a soft partition of the space. Lastly, we establish that soft weak connectedness is preserved under soft α-continuity.

Keywords: weakly connected sets; soft connectedness; soft α-open sets; soft connected components
Mathematics Subject Classification: 54A40, 54D05

1. Introduction and preliminaries

Some mathematical concepts, such as the theory of fuzzy sets, the theory of intuitionistic fuzzy sets, the theory of vague sets, the theory of rough sets and the theory of probability, might be regarded as mathematical instruments for dealing with uncertainties. Some applications of these mathematical concepts appear in [17–19,39,40]. However, each of these theories has its own set of problems. Molodtsov [36] invented the notion of soft sets in 1999 in order to deal with uncertainties while modeling issues with inadequate information. He effectively utilized soft set theory in game theory, smoothness of functions, operations research, Riemann integration, Perron integration, probability, and theory of measurement in another study [37]. The properties and uses of soft sets have been investigated in [25,27,30,31,33–35,37,42,47,48,51] and others. More information on the algebraic structure of soft sets may be found in [1,3,21,26,28,29]. Shabir and Naz [43] began researching soft topological spaces as a generalization of topological spaces in 2011. Many soft topological notions, including soft separation axioms [8, 9, 20, 22, 41], soft covering properties [6,7,11,12,15,38], soft connectedness [10,23,24,32,44,45,49], and different weak and strong types of soft continuity [13], have been developed and investigated in recent years.

Connectedness is a key topic of topology that can provide numerous links between other scientific fields and mathematical models. The concept of connectedness conveys the impression of picture elements hanging together in an object by giving connectedness strength to every potential path between every possible pair of image elements. It is a useful tool for creating picture segmentation algorithms. In the present paper, we will introduce and investigate the concept of soft weak connectedness in soft topological spaces. This research not only gives a theoretical basis for future soft topology applications but can also contribute to the development of information systems.

This article is organized as follows:

In Section 1, after the introduction, we give some definitions which will be used in this paper.

Section 2 defines the concept of a soft weakly connected set, which is a weaker form of a soft weakly connected set. We will obtain various characterizations of soft weakly connected sets. Within the setting of soft weakly connected sets, we will investigate several results related to soft subsets, supersets, unions, intersections, and subspaces.

Section 3 defines soft weakly connected components in a given soft topological space. We will show that this class of soft sets consists of soft closed sets and forms a soft partition of the space. In addition, we will discuss the behavior of soft weak connected sets under soft α-continuity.

Section 4 contains some findings and potential future studies.

We will now go over several significant concepts and terminologies that will be used in the sequel.

Let Y be an initial universe and A be a set of parameters. A soft set over Y relative to A is a function $G : A \longrightarrow \mathcal{P}(Y)$, where $\mathcal{P}(Y)$ denotes the powerset of Y. $SS(Y, A)$ denotes the family of all soft sets over Y relative to A. Let $G \in SS(Y, A)$. If $G(a) = \emptyset$ for every $a \in A$, then G is called the null soft set over Y relative to A and denoted by 0_A. If $G(a) = Y$ for every $a \in A$, then G is called the absolute soft set over Y relative to A and denoted by 1_A. G is called a soft point over Y relative to A and denoted by a_y if there exist $a \in A$ and $y \in Y$ such that $G(a) = \{y\}$ and $G(b) = \emptyset$ for all $b \in A - \{a\}$. $SP(Y, A)$ denotes the family of all soft points over Y relative to A. If for some $a \in A$ and $Z \subseteq Y$, $G(a) = Z$ and $G(b) = \emptyset$ for all $b \in A - \{a\}$, then G will be denoted by a_Z. If for some $Z \subseteq Y$, $G(a) = Z$ for all $a \in A$, then G will be denoted by C_Z. If $G \in SS(Y, A)$ and $a_y \in SP(Y, A)$, then a_y is said to belong to G (notation: $a_y \widetilde{\in} G$) if $y \in G(a)$.

Definition 1.1. [43] Let Y be an initial universe and A be a set of parameters. Let $\Psi \subseteq SS(Y, A)$. Then Ψ is called a soft topology on Y relative to A if

(1) $0_A, 1_A \in \Psi$,

(2) Ψ is closed under arbitrary soft union,

(3) Ψ is closed under finite soft intersection.

The triplet (Y, Ψ, A) is called a soft topological space. The members of Ψ are called soft open sets in (Y, Ψ, A) and their complements are called soft closed sets in (Y, Ψ, A).

This study adheres to the terminology and concepts utilized in [4,5]. STS will be used in the present research to refer to soft topological space. Let (R, Ψ, M) be an STS and $K \in SS(R, M)$. The soft interior of K in (R, Ψ, M) and the soft closure of K in (R, Ψ, M), respectively, shall be referred to by the terms $Int_\Psi(K)$ and $Cl_\Psi(K)$, while Ψ^c and $CO(R, \Psi, M)$ stand for the family of the family of soft closed sets on (R, Ψ, M) and the family of clopen sets on (R, Ψ, M), respectively.

Definition 1.2. An STS (X, Ψ, A) is called

(a) [32] soft connected if $CO(X, \Psi, A) = \{0_A, 1_A\}$.

(b) [32] soft disconnected if it is not soft connected.

(c) [14] soft locally indiscrete if $\Psi = \Psi^c$.

Definition 1.3. Let (X, Ψ, A) be an STS and let $G \in SS(X, A)$. Then G is called.

(a) [46] soft pre-open if $G \widetilde{\subseteq} Int_\Psi(Cl_\Psi(G))$.

(b) [16] soft semi-open if $G \widetilde{\subseteq} Cl_\Psi(Int_\Psi(G))$.

(c) [2] soft α-open if $G \widetilde{\subseteq} Int_\Psi(Cl_\Psi(Int_\Psi(G)))$.

(d) [50] soft dense if $Cl_\Psi(G) = 1_A$.

We will denote the family of soft α-open in (X, Ψ, A) by Ψ^α. It is proved in [2] that Ψ^α forms a soft topology that is finer than Ψ.

Definition 1.4. [2] A soft function $f_{pu} : (R, \Psi, M) \longrightarrow (L, \Theta, N)$ is called soft α-continuous if $f_{pu}^{-1}(G) \in \Psi^\alpha$ for all $G \in \Theta$.

2. Soft weak connectedness

In this section, we define the concept of a soft weakly connected set, which is a weaker form of a soft weakly connected set. We obtain various characterizations of soft weakly connected sets. Within the setting of soft weakly connected sets, we investigate several results related to soft subsets, supersets, unions, intersections and subspaces.

Definition 2.1. Let (L, Δ, R) be a STS and let $G \in SS(L, R)$. Then

(a) G is called soft weakly connected in (L, Δ, R) if there are no $K, H \in CO(L, \Delta, R)$ such that $1_R = K \widetilde{\cup} H$, $K \widetilde{\cap} H = 0_R$ and $K \widetilde{\cap} G \neq 0_R \neq G \widetilde{\cap} H$.

(b) G is called soft weakly disconnected in (L, Δ, R) if G is not soft weakly connected in (L, Δ, R).

Theorem 2.2. Let (L, Δ, R) be a STS and let Y be a non-empty subset of L. If C_Y is soft weakly disconnected in (L, Δ, R), then (Y, Δ_Y, R) is soft disconnected.

Proof. Suppose that C_Y is soft weakly disconnected in (L, Δ, R). Then there are $K, H \in CO(L, \Delta, R)$ such that $1_R = K \widetilde{\cup} H$, $K \widetilde{\cap} H = 0_R$, and $K \widetilde{\cap} C_Y \neq 0_R \neq H \widetilde{\cap} C_Y$. Since $K, H \in CO(L, \Delta, R)$, $K \widetilde{\cap} C_Y$, $H \widetilde{\cap} C_Y \in CO(Y, \Delta_Y, R)$. Since $1_R = K \widetilde{\cup} H$ and $K \widetilde{\cap} H = 0_R$, $C_Y = \left(K \widetilde{\cup} H\right) \widetilde{\cap} C_Y = \left(K \widetilde{\cap} C_Y\right) \widetilde{\cup} \left(H \widetilde{\cap} C_Y\right)$ and $\left(K \widetilde{\cap} H\right) \widetilde{\cap} C_Y = 0_R \widetilde{\cap} C_Y = 0_R$. Therefore, (Y, Δ_Y, R) is soft disconnected.

Corollary 2.3. Let (L, Δ, R) be a STS and let Y be a non-empty subset of L. If (Y, Δ_Y, R) is soft connected, then C_Y is soft weakly connected in (L, Δ, R).

Theorem 2.4. A STS (L, Δ, R) is soft connected if and only if G is soft weakly connected for every $G \in SS(L, R) - \{0_R\}$.

Proof. Necessity. Suppose that (L, Δ, R) is soft connected and suppose to the contrary that there exists $G \in SS(L, R) - \{0_R\}$ such that G is soft weakly disconnected. Then there are $K, H \in CO(L, \Delta, R)$ such that $1_R = K\widetilde{\cup}H$, $K\widetilde{\cap}H = 0_R$, and $K\widetilde{\cap}G \neq 0_R \neq G\widetilde{\cap}H$. Since $K\widetilde{\cap}G \neq 0_R \neq G\widetilde{\cap}H$, then $K \neq 0_R \neq H$. This shows that (L, Δ, R) is soft disconnected, a contradiction.

Sufficiency. Suppose that G is soft weakly connected for every $G \in SS(L, R) - \{0_R\}$, and suppose to the contrary that (L, Δ, R) is soft disconnected. Then there are $K, H \in CO(L, \Delta, R) - \{0_R\}$ such that $1_R = K\widetilde{\cup}H$, $K\widetilde{\cap}H = 0_R$. Since $K = K\widetilde{\cap}1_R$ and $H = H\widetilde{\cap}1_R$, $K\widetilde{\cap}1_R \neq 0_R \neq H\widetilde{\cap}1_R$. Thus, we have $1_R \in SS(L, R) - \{0_R\}$ while 1_R is soft weakly disconnected, a contradiction.

The converse of Theorem 2.2 does not have to be true in general, as demonstrated by the following two examples:

Example 2.5. Let $L = \{2, 3, 4\}$, $R = \{a\}$, and $\Delta = \{0_R, 1_R, a_2, a_3, a_{\{2,3\}}\}$. Let $Y = \{2, 3\}$. Then $\Delta_Y = \{0_R, C_Y, a_2, a_3\}$. Since $a_2, a_3 \in CO(Y, \Delta_Y, R) - \{0_R\}$, $a_2\widetilde{\cup}a_3 = C_Y$, and $a_2\widetilde{\cap}a_3 = 0_R$, then (Y, Δ_Y, R) is soft disconnected. On the other hand, if C_Y is soft weakly disconnected in (L, Δ, R), then by Theorem 2.4, (L, Δ, R) is soft disconnected, but (L, Δ, R) is soft connected.

Example 2.6. Let $L = \mathbb{R}$ and $A = \{a, b, c\}$. Define $K, G \in SS(L, A)$ by $K = \{(a, \{1\}), (b, \{1, 2\}), (c, \mathbb{N})\}$ and $G = \{(a, \mathbb{N} - \{1\}), (b, \mathbb{N} - \{1, 2\}), (c, \emptyset)\}$. Let $\Delta = \{0_A, 1_A, K, G, C_{\mathbb{N}}\}$ and $Y = \mathbb{N}$. Then $\Delta_Y = \{0_R, C_Y, G, K, C_{\mathbb{N}}\}$. Since $G, K \in CO(Y, \Delta_Y, A) - \{0_A\}$, $G\widetilde{\cup}K = C_Y$, and $G\widetilde{\cap}K = 0_A$, then (Y, Δ_Y, A) is soft disconnected. On the other hand, if C_Y is soft weakly disconnected in (L, Δ, A), then by Theorem 2.4, (L, Δ, A) is soft disconnected, but (L, Δ, A) is soft connected.

Theorem 2.7. Let (L, Δ, R) and (L, Γ, R) be soft disconnected STSs such that $\Delta \subseteq \Gamma$. If G is soft weakly connected in (L, Γ, R), then G is soft weakly connected in (L, Δ, R).

Proof. Suppose that G is soft weakly connected in (L, Γ, R). Suppose to the contrary that G is soft weakly disconnected in (L, Δ, R). Then there are $K, H \in CO(L, \Delta, R)$ such that $1_R = K\widetilde{\cup}H$, $K\widetilde{\cap}H = 0_R$, and $K\widetilde{\cap}G \neq 0_R \neq G\widetilde{\cap}H$. Since $\Delta \subseteq \Gamma$, $K, H \in CO(L, \Gamma, R)$. This implies that G is soft weakly disconnected in (L, Δ, R), a contradiction.

Theorem 2.8. Let (L, Δ, R) be soft disconnected and let $G \in SS(L, R)$. Then G is soft weakly connected in (L, Δ, R) if and only if for any $K, H \in SS(L, R)$ such that $Int_\Delta(A) = A = Cl_\Delta(A)$, $Int_\Delta(B) = B = Cl_\Delta(B)$, $1_R = K\widetilde{\cup}H$, and $K\widetilde{\cap}H = 0_R$; we have $G\widetilde{\subseteq}K$ or $G\widetilde{\subseteq}H$.

Proof. Necessity. Suppose that G is soft weakly connected in (L, Δ, R). Let $K, H \in SS(L, R)$ such that $Int_\Delta(K) = K = Cl_\Delta(K)$, $Int_\Delta(H) = H = Cl_\Delta(H)$, $1_R = K\widetilde{\cup}H$, and $K\widetilde{\cap}H = 0_R$. Since $Int_\Delta(K) = K = Cl_\Delta(K)$ and $Int_\Delta(H) = H = Cl_\Delta(H)$, $K, H \in CO(L, \Delta, R)$. Since G is soft weakly connected in (L, Δ, R), then we have $K\widetilde{\cap}G = 0_R$ or $G\widetilde{\cap}H = 0_R$. Since $1_R = K\widetilde{\cup}H$ and $K\widetilde{\cap}H = 0_R$, then $K = 1_R - H$ and $H = 1_R - K$. Since we have $K\widetilde{\cap}G = 0_R$ or $G\widetilde{\cap}H = 0_R$, then we have $G\widetilde{\subseteq}1_R - K\widetilde{\subseteq}H$ or $G\widetilde{\subseteq}1_R - H\widetilde{\subseteq}K$.

Sufficiency. Suppose to the contrary that G is soft weakly connected in (L, Δ, R). Then there are $K, H \in CO(L, \Delta, R)$ such that $1_R = K\widetilde{\cup}H$, $K\widetilde{\cap}H = 0_R$, and $K\widetilde{\cap}G \neq 0_R \neq G\widetilde{\cap}H$. Since $K, H \in CO(L, \Delta, R)$, then $Int_\Delta(K) = K = Cl_\Delta(K)$ and $Int_\Delta(H) = H = Cl_\Delta(H)$. Thus, by assumption, we must have $G\widetilde{\subseteq}K$ or $G\widetilde{\subseteq}H$. Without loss of generality, we may assume that $G\widetilde{\subseteq}K$. Since $1_R = K\widetilde{\cup}H$ and $K\widetilde{\cap}H = 0_R$, then $K = 1_R - H$. Therefore, we have $G\widetilde{\subseteq}1_R - H$ and hence $0_R = G\widetilde{\cap}H$, a contradiction.

Theorem 2.9. Let (L, Δ, R) be soft disconnected and let $G \in SS(L, R)$. Then the following are equivalent:

(a) G is soft weakly connected in (L, Δ, R).

(b) For each $K \in CO(L, \Delta, R) - \{0_R\}$, we have $G\widetilde{\subseteq}K$ or $G\widetilde{\subseteq}1_R - K$.

(c) For each $K \in SS(L, R) - \{0_R\}$ with $Bd_\Delta(K) = 0_R$, $G\widetilde{\subseteq}K$ or $G\widetilde{\subseteq}1_R - K$.

Proof. (a) $\longrightarrow$ (b): Let $K \in CO(L, \Delta, R) - \{0_R\}$. Then we have $Int_\Delta(K) = K = Cl_\Delta(K)$, $Int_\Delta(1_R - K) = 1_R - K = Cl_\Delta(1_R - K)$, $1_R = K \widetilde{\cup} (1_R - K)$, and $K \widetilde{\cap} (1_R - K) = 0_R$. So, by (a) and Theorem 2.8, we have $G \widetilde{\subseteq} K$ or $G \widetilde{\subseteq} 1_R - K$.

(b) $\longrightarrow$ (c): Since $Bd_\Delta(K) = 0_R$, $K \in CO(L, \Delta, R)$. Thus, by (b), $G \widetilde{\subseteq} K$ or $G \widetilde{\subseteq} 1_R - K$.

(c) $\longrightarrow$ (a): Suppose to the contrary that G is soft weakly disconnected in (L, Δ, R). Then there are $K, H \in CO(L, \Delta, R)$ such that $1_R = K \widetilde{\cup} H$, $K \widetilde{\cap} H = 0_R$, and $K \widetilde{\cap} G \neq 0_R \neq G \widetilde{\cap} H$. Since $1_R = K \widetilde{\cup} H$ and $K \widetilde{\cap} H = 0_R$, then $H = 1_R - K$. Since $K \widetilde{\cap} G \neq 0_R$, then $K \neq 0_R$. So, we have $K \in SS(L, R) - \{0_R\}$ with $Bd_\Delta(K) = Cl_\Delta(K) \widetilde{\cap} Cl_\Delta(1_R - K) = K \widetilde{\cap} (1_R - K) = 0_R$ and by (c), $G \widetilde{\subseteq} K$ or $G \widetilde{\subseteq} 1_R - K$. Thus, we have $G \widetilde{\cap} H = (1_R - K) \widetilde{\cap} G = 0_R$ or $K \widetilde{\cap} G = 0_R$. However, $K \widetilde{\cap} G \neq 0_R \neq G \widetilde{\cap} H$, a contradiction.

Theorem 2.10. Let (L, Δ, R) be soft disconnected and let $G \in SS(L, R)$. Then the following are equivalent:

(a) G is soft weakly connected in (L, Δ, R).

(b) For every pair $K, H \in SS(L, R) - \{0_R\}$ of soft separated sets in (L, Δ, R) such that $1_R = K \widetilde{\cup} H$ we have $G \widetilde{\subseteq} K$ or $G \widetilde{\subseteq} H$.

Proof. (a) $\longrightarrow$ (b): Suppose to the contrary that there are soft separated sets $K, H \in SS(L, R) - \{0_R\}$ in (L, Δ, R) such that $1_R = K \widetilde{\cup} H$ and $(1_R - K) \widetilde{\cap} G \neq 0_R \neq G \widetilde{\cap} (1_R - H)$. Since K and H are soft separated sets in (L, Δ, R), $K \widetilde{\cap} H = 0_R$. Since $1_R = K \widetilde{\cup} H$ and $K \widetilde{\cap} H = 0_R$, $K = 1_R - H$ and $H = 1_R - K$. Thus, we have $K \widetilde{\cap} G \neq 0_R \neq G \widetilde{\cap} H$. Since $1_R = K \widetilde{\cup} H$, $1_R = K \widetilde{\cup} Cl_\Delta(H)$ and $1_R = H \widetilde{\cup} Cl_\Delta(K)$. Since K and H are soft separated sets in (L, Δ, R), $K \widetilde{\cap} Cl_\Delta(H) = 0_R$ and $Cl_\Delta(K) \widetilde{\cap} H = 0_R$. Thus, we have $K = 1_R - Cl_\Delta(H)$ and $H = 1_R - Cl_\Delta(K)$ and hence $K, H \in \Delta$. Therefore by (a), $A \notin \Delta^c$ or $B \notin \Delta^c$, say $A \notin \Delta^c$. Then there exists $r_x \widetilde{\in} Cl_\Delta(K) - K = Cl_\Delta(K) \widetilde{\cap} (1_R - K) = Cl_\Delta(K) \widetilde{\cap} H$, a contradiction.

(b) $\longrightarrow$ (a): Suppose to the contrary that G is soft weakly disconnected in (L, Δ, R). Then there are $K, H \in CO(L, \Delta, R)$ such that $1_R = K \widetilde{\cup} H$, $K \widetilde{\cap} H = 0_R$, and $K \widetilde{\cap} G \neq 0_R \neq G \widetilde{\cap} H$. Since $K, H \in \Delta^c$ and $K \widetilde{\cap} H = 0_R$, $K \widetilde{\cap} Cl_\Delta(B) = 0_R$ and $Cl_\Delta(A) \widetilde{\cap} H = 0_R$. Thus, K, K are soft separated sets in (L, Δ, R). Since $K \widetilde{\cap} G \neq 0_R \neq G \widetilde{\cap} H$, $K \neq 0_R \neq H$. Since $1_R = K \widetilde{\cup} H$, $K \widetilde{\cap} H = 0_R$, $K = 1_R - H$ and $H = 1_R - K$. Therefore, by (b), $G \widetilde{\subseteq} K$ or $G \widetilde{\subseteq} H$ and so, $G \widetilde{\cap} H = G \widetilde{\cap} (1_R - K) = 0_R$ or $G \widetilde{\cap} K = G \widetilde{\cap} (1_R - H) = 0_R$, a contradiction.

Theorem 2.11. Let (L, Δ, R) be soft disconnected, Y a non-empty subset of L, and $G \in SS(Y, R)$. If G is soft weakly connected in (Y, Δ_Y, R), then G is soft weakly connected in (L, Δ, R).

Proof. Suppose that G is soft weakly connected in (Y, Δ_Y, R). Suppose to the contrary that G is soft weakly disconnected in (L, Δ, R). Then Then there are $K, H \in CO(L, \Delta, R)$ such that $1_R = K \widetilde{\cup} H$, $K \widetilde{\cap} H = 0_R$, and $K \widetilde{\cap} G \neq 0_R \neq G \widetilde{\cap} H$. Since $K, H \in CO(L, \Delta, R)$, $K \widetilde{\cap} C_Y$, $H \widetilde{\cap} C_Y \in (Y, \Delta_Y, R)$. Since $1_R = K \widetilde{\cup} H$, then $C_Y \widetilde{\cap} 1_R = C_Y \widetilde{\cap} \left(K \widetilde{\cup} H\right) = \left(C_Y \widetilde{\cap} K\right) \widetilde{\cup} \left(C_Y \widetilde{\cap} H\right)$. Since $K \widetilde{\cap} H = 0_R$, then $\left(C_Y \widetilde{\cap} K\right) \widetilde{\cap} \left(C_Y \widetilde{\cap} H\right) = C_Y \widetilde{\cap} \left(K \widetilde{\cap} H\right) = C_Y \widetilde{\cap} (0_R) = 0_R$. Since $G \in SS(Y, R)$, then $G \widetilde{\cap} C_Y = G$. Therefore, we have $\left(C_Y \widetilde{\cap} K\right) \widetilde{\cap} G = \left(G \widetilde{\cap} C_Y\right) \widetilde{\cap} K = G \widetilde{\cap} K \neq 0_R$ and $\left(C_Y \widetilde{\cap} H\right) \widetilde{\cap} G = \left(G \widetilde{\cap} C_Y\right) \widetilde{\cap} H = G \widetilde{\cap} H \neq 0_R$. This shows that G is soft weakly disconnected in (Y, Δ_Y, R), a contradiction.

The following question is natural:

Let (L, Δ, R) be a STS, Y a non-empty subset of L, and $G \in SS(Y, R)$ such that G is soft weakly connected in (L, Δ, R). Is it true that G is soft weakly connected in (Y, Δ_Y, R).

Each of the following two examples gives a negative answer to the above question:

Example 2.12. Let $L = \{2, 3, 4, 5\}$, $R = \{a\}$, $\Delta = \{0_R, 1_R, a_{\{2\}}, a_{\{3,4\}}, a_{\{2,3,4\}}\}$, $Y = \{2, 3, 4\}$, and $G = a_{\{2,3\}}$. Then $\Delta_Y = \{0_R, C_Y, a_{\{2\}}, a_{\{3,4\}}\}$. Let $K = a_{\{2\}}$ and $H = a_{\{3,4\}}$. Then $K, H \in CO(Y, \Delta_Y, R)$ such that $C_Y = K \widetilde{\cup} H$, $K \widetilde{\cap} H = 0_R$, $K \widetilde{\cap} G = K \neq 0_R$, and $G \widetilde{\cap} H = a_{\{3\}} \neq 0_R$. This shows that G is soft weakly disconnected in (Y, Δ_Y, R). Since $CO(L, \Delta, R) = \{0_R, 1_R\}$, then (L, Δ, R) is soft connected, and by Theorem 2.4, G is soft weakly connected in (L, Δ, R).

Example 2.13. Let $L = \mathbb{N}$ and $A = \{a, b\}$. Define $S, T \in SS(L, A)$ by $S = \{(a, \{1, 3\}), (b, \{1, 2\})\}$ and $T = \{(a, \{2, 4\}), (b, \{3, 4\})\}$. Let $Y = \{1, 2, 3, 4\}$ and $G \in SS(Y, A)$ defined by $G = \{(a, \{3\}), (b, \{4\})\}$. Let $\Delta = \{0_A, 1_A, S, T, C_Y\}$. Then $\Delta_Y = \{0_A, C_Y, S, T\}$. Then we have $S, T \in CO(Y, \Delta_Y, A)$ such that $C_Y = S \widetilde{\cup} T$, $S \widetilde{\cap} T = 0_A$, $S \widetilde{\cap} G \neq 0_A$, and $T \widetilde{\cap} G \neq 0_A$. This shows that G is soft weakly disconnected in (Y, Δ_Y, A). Since $CO(L, \Delta, A) = \{0_A, 1_A\}$, then (L, Δ, A) is soft connected, and by Theorem 2.4, G is soft weakly connected in (L, Δ, A).

Theorem 2.14. Let (L, Δ, R) be soft locally indiscrete, Y a non-empty subset of L, and $G \in SS(Y, R)$. If G is soft weakly connected in (L, Δ, R), then G is soft weakly connected in (Y, Δ_Y, R).

Proof. Suppose to the contrary that G is soft weakly disconnected in (Y, Δ_Y, R). Then there are $S, T \in CO(Y, \Delta_Y, R)$ such that $C_Y = S \widetilde{\cup} T$, $S \widetilde{\cap} T = 0_R$, and $S \widetilde{\cap} G \neq 0_R \neq G \widetilde{\cap} T$. Since $S, T \in \Delta_Y$, there exist $K, H \in \Delta$ such that $S = K \widetilde{\cap} C_Y$ and $T = H \widetilde{\cap} C_Y$. Put $M = K - H$ and $N = H \widetilde{\cup} \left(1_R - \left(K \widetilde{\cup} H\right)\right)$. Since (L, Δ, R) is soft locally indiscrete, $M, N \in CO(L, \Delta, R)$. Also, it is not difficult to see that $1_R = M \widetilde{\cup} N$, $\left(M \widetilde{\cap} N\right) = 0_R$, and $M \widetilde{\cap} G \neq 0_R \neq N \widetilde{\cap} G$. This shows that G is soft weakly disconnected in (L, Δ, R), a contradiction.

Corollary 2.15. Let (L, Δ, R) be soft locally indiscrete, Y a non-empty subset of L, and $G \in SS(Y, R)$. Then G is soft weakly connected in (L, Δ, R) if and only if G is soft weakly connected in (Y, Δ_Y, R).

Proof. The proof follows from Theorems 2.11 and 2.14.

Theorem 2.16. Let (L, Δ, R) be soft disconnected, Y a non-empty subset of L such that $C_Y \in CO(L, \Delta, R)$ and (Y, Δ_Y, R) is soft disconnected, and $G \in SS(Y, R)$. If G is soft weakly connected in (L, Δ, R), then G is soft weakly connected in (Y, Δ_Y, R).

Proof. Suppose to the contrary that G is soft weakly disconnected in (Y, Δ_Y, R). Then there are $S, T \in CO(Y, \Delta_Y, R)$ such that $C_Y = S \widetilde{\cup} T$, $S \widetilde{\cap} T = 0_R$, and $S \widetilde{\cap} G \neq 0_R \neq G \widetilde{\cap} T$. Since $S, T \in \Delta_Y$, there exist $K, H \in \Delta$ such that $S = K \widetilde{\cap} C_Y$ and $T = H \widetilde{\cap} C_Y$. Since $C_Y \in \Delta$, $S, T \in \Delta$. Since $C_Y \in \Delta^c$, $1_R - C_Y = C_{L-Y} \in \Delta$. Put $N = T \widetilde{\cup} (C_{L-Y})$. Then we have $S, N \in \Delta$, $1_R = S \widetilde{\cup} N$, $\left(S \widetilde{\cap} N\right) = 0_R$, and $S \widetilde{\cap} G \neq 0_R \neq N \widetilde{\cap} G$. Moreover, since $S, N \in \Delta$, $1_R = S \widetilde{\cup} N$, and $\left(S \widetilde{\cap} N\right) = 0_R$, then $S = 1_R - N \in \Delta^c$ and $N = 1_R - S \in \Delta^c$ and hence $S, N \in CO(L, \Delta, R)$. This shows that G is soft weakly disconnected in (L, Δ, R), a contradiction.

Theorem 2.17. Let (L, Δ, R) be soft disconnected and let $K, H \in SS(L, R) - \{0_R\}$ be soft separated sets in (L, Δ, R) such that $K \widetilde{\cup} H = C_Y$ for some $Y \subseteq L$. If $G \widetilde{\subseteq} C_Y$ such that G is soft weakly connected in (Y, Δ_Y, R), then $G \widetilde{\subseteq} K$ or $G \widetilde{\subseteq} H$.

Proof. Since K and H are soft separated sets in (L, Δ, R), then $K \widetilde{\cap} Cl_\Delta(H) = 0_R$ and $Cl_\Delta(K) \widetilde{\cap} H = 0_R$. Thus, $K \widetilde{\cap} Cl_{\Delta_Y}(H) = K \widetilde{\cap} \left(Cl_\Delta(H) \widetilde{\cap} C_Y\right) = \left(Cl_\Delta(K) \widetilde{\cap} H\right) \widetilde{\cap} C_Y = 0_R \widetilde{\cap} C_Y = 0_R$ and $H \widetilde{\cap} Cl_{\Delta_Y}(K) = H \widetilde{\cap} \left(Cl_\Delta(K) \widetilde{\cap} C_Y\right) = \left(Cl_\Delta(H) \widetilde{\cap} K\right) \widetilde{\cap} C_Y = 0_R \widetilde{\cap} C_Y = 0_R$. Hence, since K and H are soft separated sets in (Y, Δ_Y, R). Therefore, by Theorem 2.10, $G \widetilde{\subseteq} K$ or $G \widetilde{\subseteq} H$.

Theorem 2.18. A STS (L, Δ, R) is soft connected if and only if for each $a_x, b_y \in SP(L, R)$ with $a_x \neq b_y$ there exists a soft weakly connected set G in (L, Δ, R) such that $a_x, b_y \widetilde{\in} G$.

Proof. Necessity. Suppose to the contrary that (L, Δ, R) is soft connected. Take $G = 1_R$. Then by Theorem 2.4, G is soft weakly connected in (L, Δ, R) such that $a_x, b_y \widetilde{\in} G$.

Sufficiency. Suppose the sufficiency condition holds but (L, Δ, R) is soft disconnected. Then there exist $K, H \in \Delta - \{0_R\}$ such that $1_R = K \widetilde{\cup} H$, $K \widetilde{\cap} H = 0_R$. Choose $a_x \widetilde{\in} K$ and $b_y \widetilde{\in} H$. Then by assumption, there exists a soft weakly connected set G in (L,Δ,R) such that $a_x, b_y \widetilde{\in} G$. Since $1_R = K\widetilde{\cup}H$, $K\widetilde{\cap}H = 0_R$, then $K = 1_R - H$ and $H = 1_R - K$. Hence, $K, H \in CO(L,\Delta,R)$. Since $a_x \widetilde{\in} G\widetilde{\cap}K$ and $b_y \widetilde{\in} G\widetilde{\cap}H$, then we have $K\widetilde{\cap}G \neq 0_R \neq G\widetilde{\cap}H$. This implies that G soft weakly disconnected in (L,Δ,R), a contradiction.

Theorem 2.19. Let (L,Δ,R) be a STS. If there is $G \in SS(L,R)$ such that G is soft weakly connected and soft dense in (L,Δ,R), then (L,Δ,R) is soft connected.

Proof. Suppose to the contrary that (L,Δ,R) is soft disconnected. Then there exist $K, H \in \Delta - \{0_R\}$ such that $1_R = K\widetilde{\cup}H$, $K\widetilde{\cap}H = 0_R$. Since G is soft dense in (L,Δ,R), then $K\widetilde{\cap}G \neq 0_R \neq G\widetilde{\cap}H$. Since $1_R = K\widetilde{\cup}H$, $K\widetilde{\cap}H = 0_R$, then $K = 1_R - H$ and $H = 1_R - K$. Hence, $K, H \in CO(L,\Delta,R)$. This implies that G soft weakly disconnected in (L,Δ,R), a contradiction.

Theorem 2.20. Let (L,Δ,R) be soft disconnected and let $G, N \in SS(L,R)$ such that $G\widetilde{\subseteq}N$. If N is soft weakly connected in (L,Δ,R), then G is soft weakly connected in (L,Δ,R).

Proof. Suppose to the contrary that G is soft weakly disconnected in (L,Δ,R). Then there are $K, H \in CO(L,\Delta,R)$ such that $1_R = K\widetilde{\cup}H$, $K\widetilde{\cap}H = 0_R$, and $K\widetilde{\cap}G \neq 0_R \neq G\widetilde{\cap}H$. Since $G\widetilde{\subseteq}N$, then $0_R \neq K\widetilde{\cap}G\widetilde{\subseteq}K\widetilde{\cap}N$ and $0_R \neq H\widetilde{\cap}G\widetilde{\subseteq}H\widetilde{\cap}N$. This shows that N is soft weakly disconnected in (L,Δ,R), a contradiction.

Corollary 2.21. Let (L,Δ,R) be soft disconnected and let $G_\alpha \in SS(L,R)$ for all $\alpha \in F$. If for some $\beta \in F$, G_β is soft weakly connected in (L,Δ,R), then $\widetilde{\cap}_{\alpha\in F} G_\alpha$ is soft weakly connected in (L,Δ,R).

Proof. Suppose that G_β is soft weakly connected in (L,Δ,R) for some $\beta \in F$. Since $\widetilde{\cap}_{\alpha\in F} G_\alpha \widetilde{\subseteq} G_\beta$, then by Theorem 2.20, $\widetilde{\cap}_{\alpha\in F} G_\alpha$ is soft weakly connected in (L,Δ,R).

Corollary 2.22. Let (L,Δ,R) be soft disconnected and let $S, T, G \in SS(L,R)$ such that $S\widetilde{\subseteq}T$ and $G\widetilde{\subseteq}T - S$. If T is soft weakly connected in (L,Δ,R), then $S\widetilde{\cup}G$ is soft weakly connected in (L,Δ,R).

Proof. Suppose that T is soft weakly connected in (L,Δ,R). Since $S\widetilde{\subseteq}T$ and $G\widetilde{\subseteq}T - S$, $S\widetilde{\cup}G\widetilde{\subseteq}T\widetilde{\cup}(T - S)\widetilde{\subseteq}T$. Thus, by Theorem 2.20, $S\widetilde{\cup}G$ is soft weakly connected in (L,Δ,R).

Theorem 2.23. Let (L,Δ,R) be soft disconnected. If S and T are soft weakly connected in (L,Δ,R) such that $S\widetilde{\cap}T \neq 0_R$, then $S\widetilde{\cup}T$ is soft weakly connected in (L,Δ,R).

Proof. Suppose to the contrary that $S\widetilde{\cup}T$ is soft weakly disconnected in (L,Δ,R). Then there are $K, H \in CO(L,\Delta,R)$ such that $1_R = K\widetilde{\cup}H$, $K\widetilde{\cap}H = 0_R$, and $K\widetilde{\cap}\left(S\widetilde{\cup}T\right) \neq 0_R \neq \left(S\widetilde{\cup}T\right)\widetilde{\cap}H$. By Theorem 2.9 (b), $\left(S\widetilde{\subseteq}K \text{ and } T\widetilde{\subseteq}K\right)$, $\left(S\widetilde{\subseteq}K \text{ and } T\widetilde{\subseteq}H\right)$,$\left(S\widetilde{\subseteq}H \text{ and } T\widetilde{\subseteq}K\right)$, or $\left(S\widetilde{\subseteq}H \text{ and } T\widetilde{\subseteq}H\right)$. If $\left(S\widetilde{\subseteq}K \text{ and } T\widetilde{\subseteq}H\right)$ or $\left(S\widetilde{\subseteq}H \text{ and } T\widetilde{\subseteq}K\right)$, then $0_R \neq S\widetilde{\cap}T\widetilde{\subseteq}H\widetilde{\cap}K = 0_R$. Therefore, $\left(S\widetilde{\subseteq}K \text{ and } T\widetilde{\subseteq}K\right)$ or $\left(S\widetilde{\subseteq}H \text{ and } T\widetilde{\subseteq}H\right)$ and hence $S\widetilde{\cup}T\widetilde{\subseteq}K$ or $S\widetilde{\cup}T\widetilde{\subseteq}H$. Thus, we have $\left(S\widetilde{\cup}T\right)\widetilde{\cap}H = 0_R$ or $\left(S\widetilde{\cup}T\right)\widetilde{\cap}K = 0_R$, a contradiction.

Theorem 2.24. Let (L,Δ,R) be soft disconnected and let $\{G_\alpha : \alpha \in F\} \subseteq SS(L,R)$ such that $G_\alpha\widetilde{\cap}G_\beta \neq 0_R$ for all $\alpha, \beta \in F$. If G_α is soft weakly connected in (L,Δ,R) for all $\alpha \in F$, then $\widetilde{\cup}_{\alpha\in F} G_\alpha$ is soft weakly connected in (L,Δ,R).

Proof. Suppose to the contrary that $\widetilde{\cup}_{\alpha\in F} G_\alpha$ is soft weakly disconnected in (L,Δ,R). Then there are $K, H \in CO(L,\Delta,R)$ such that $1_R = K\widetilde{\cup}H$, $K\widetilde{\cap}H = 0_R$, and $K\widetilde{\cap}\left(\widetilde{\cup}_{\alpha\in F} G_\alpha\right) \neq 0_R \neq \left(\widetilde{\cup}_{\alpha\in F} G_\alpha\right)\widetilde{\cap}H$. By Theorem 2.9 (b), for all $\alpha \in F$ either $G_\alpha\widetilde{\subseteq}K$ or $G_\alpha\widetilde{\subseteq}H$. Since $G_\alpha\widetilde{\cap}G_\beta \neq 0_R$ for all $\alpha, \beta \in F$, then either $\widetilde{\cup}_{\alpha\in F} G_\alpha\widetilde{\subseteq}K$ or $\widetilde{\cup}_{\alpha\in F} G_\alpha\widetilde{\subseteq}H$. Thus, we have $\left(\widetilde{\cup}_{\alpha\in F} G_\alpha\right)\widetilde{\cap}H = 0_R$ or $\left(\widetilde{\cup}_{\alpha\in F} G_\alpha\right)\widetilde{\cap}K = 0_R$, a contradiction.

Corollary 2.25. Let (L,Δ,R) be soft disconnected and let $\{G_\alpha : \alpha \in F\} \subseteq SS(L,R)$ such that $\widetilde{\cap}_{\alpha\in F} G_\alpha \neq 0_R$. If G_α is soft weakly connected in (L,Δ,R) for all $\alpha \in F$, then $\widetilde{\cup}_{\alpha\in F} G_\alpha$ is soft weakly connected in (L,Δ,R).

Theorem 2.26. Let (L,Δ,R) be soft disconnected. If G is soft weakly connected in (L,Δ,R), then $Cl_\Delta(G)$ is soft weakly connected in (L,Δ,R).

Proof. Suppose to the contrary that $Cl_\Delta(G)$ is soft weakly disconnected in (L,Δ,R). Then there are $K,H \in CO(L,\Delta,R)$ such that $1_R = K\widetilde{\cup}H$, $K\widetilde{\cap}H = 0_R$, and $K\widetilde{\cap}Cl_\Delta(G) \neq 0_R \neq Cl_\Delta(G)\widetilde{\cap}H$. Since $K,H \in \Delta$ and $K\widetilde{\cap}Cl_\Delta(G) \neq 0_R \neq Cl_\Delta(G)\widetilde{\cap}H$, then $K\widetilde{\cap}G \neq 0_R \neq G\widetilde{\cap}H$. This shows that G is soft weakly disconnected in (L,Δ,R), a contradiction.

Corollary 2.27. Let (L,Δ,R) be soft disconnected and let $G \in SS(L,R)$. Then G is soft weakly connected in (L,Δ,R) if and only if $Cl_\Delta(G)$ is soft weakly connected in (L,Δ,R).

Proof. The proof follows from Theorems 2.20 and 2.26.

Corollary 2.28. Let (L,Δ,R) be soft disconnected and let $G \in SS(L,R)$. Then G is soft weakly connected in (L,Δ,R) if and only if each $T \in SS(L,R)$ such that $G\widetilde{\subseteq}T\widetilde{\subseteq} Cl_\Delta(G)$ is soft weakly connected in (L,Δ,R).

Proof. The proof follows from Theorems 2.20 and 2.26.

Theorem 2.29. Let (L,Δ,R) be soft disconnected and let $\{G_i\}_{i\in I} \subseteq SS(L,R)$. If $\widetilde{\cup}_{i\in I}G_i\widetilde{\subseteq}T$ and T is soft weakly connected in (L,Δ,R), then $\widetilde{\cup}_{i\in I}Cl_\Delta(G_i)$ is soft weakly connected in (L,Δ,R).

Proof. Since T is soft weakly connected in (L,Δ,R), by Theorem 2.26 we have $Cl_\Delta(T)$ is soft weakly connected in (L,Δ,R). Since $\widetilde{\cup}_{i\in I}Cl_\Delta(G_i)\widetilde{\subseteq}Cl_\Delta\left(\widetilde{\cup}_{i\in I}G_i\right)\widetilde{\subseteq}Cl_\Delta(T)$, by Theorem 2.20, $\widetilde{\cup}_{i\in I}Cl_\Delta(G_i)$ is soft weakly connected in (L,Δ,R).

Theorem 2.30. Let (L,Δ,R) be soft disconnected and let $G \in SS(L,R)$. If G is soft weakly connected in (L,Δ,R), then $Cl_\Delta(Int_\Delta(Cl_\Delta(G)))$ is soft weakly connected in (L,Δ,R).

Proof. Suppose that G is soft weakly connected in (L,Δ,R). Then by Theorem 2.26, $Cl_\Delta(G)$ is soft weakly connected in (L,Δ,R). Since $Int_\Delta(Cl_\Delta(G))\widetilde{\subseteq}Cl_\Delta(G)$, by Theorem 2.20, $Int_\Delta(Cl_\Delta(G))$ is soft weakly connected in (L,Δ,R). Again, by Theorem 2.26, $Cl_\Delta(Int_\Delta(Cl_\Delta(G)))$ is soft weakly connected in (L,Δ,R).

Theorem 2.31. Let (L,Δ,R) be soft disconnected. If S and T are soft weakly connected in (L,Δ,R) such that $Cl_\Delta(S)\widetilde{\cap}Cl_\Delta(T) \neq 0_R$, then $Cl_\Delta(Int_\Delta(Cl_\Delta(S)))\widetilde{\cup}Cl_\Delta(Int_\Delta(Cl_\Delta(T)))$ is soft weakly connected in (L,Δ,R).

Proof. Since S and T are soft weakly connected in (L,Δ,R), by Theorem 2.26, $Cl_\Delta(S)$ and $Cl_\Delta(T)$ are soft weakly connected in (L,Δ,R). Hence, by Theorem 2.23, $Cl_\Delta(S)\widetilde{\cup}Cl_\Delta(T)$ is soft weakly connected in (L,Δ,R). Since $Cl_\Delta(Int_\Delta(Cl_\Delta(S)))\widetilde{\cup}Cl_\Delta(Int_\Delta(Cl_\Delta(T))) =$

$$\begin{aligned} & Cl_\Delta\left(Int_\Delta(Cl_\Delta(S))\widetilde{\cup}Int_\Delta(Cl_\Delta(T))\right) \\ \widetilde{\subseteq}\ & Cl_\Delta\left(Cl_\Delta(S)\widetilde{\cup}Cl_\Delta(T)\right) \\ =\ & Cl_\Delta(S)\widetilde{\cup}Cl_\Delta(T), \end{aligned}$$

by Theorem 2.20, $Cl_\Delta(Int_\Delta(Cl_\Delta(S)))\widetilde{\cup}Cl_\Delta(Int_\Delta(Cl_\Delta(T)))$ is soft weakly connected in (L,Δ,R).

Theorem 2.32. Let (L,Δ,R) be soft disconnected. If G is soft weakly connected in (L,Δ,R), then

(a) $Cl_\Delta(Int_\Delta(G))$ is soft weakly connected in (L,Δ,R).

(b) $Int_\Delta(Cl_\Delta(G))$ is soft weakly connected in (L,Δ,R).

(c) $Int_\Delta(Cl_\Delta(Int_\Delta(G)))$ is soft weakly connected in (L,Δ,R).

Proof. (a) Since G is soft weakly connected in (L,Δ,R), by Theorem 2.26, $Cl_\Delta(G)$ is soft weakly connected in (L,Δ,R). Since $Cl_\Delta(Int_\Delta(G))\widetilde{\subseteq}Cl_\Delta(G)$, by Theorem 2.20, $Cl_\Delta(Int_\Delta(G))$ is soft weakly connected in (L,Δ,R).

(b) Since G is soft weakly connected in (L,Δ,R), by Theorem 2.26, $Cl_\Delta(G)$ is soft weakly connected in (L,Δ,R). Since $Int_\Delta(Cl_\Delta(G))\widetilde{\subseteq}Cl_\Delta(G)$, by Theorem 2.20, $Int_\Delta(Cl_\Delta(G))$ is soft weakly connected in (L,Δ,R).

(c) By (a), $Cl_{\Delta}(Int_{\Delta}(G))$ is soft weakly connected in (L, Δ, R). Since $Int_{\Delta}(Cl_{\Delta}(Int_{\Delta}(G))) \widetilde{\subseteq} Cl_{\Delta}(Int_{\Delta}(G))$, by Theorem 2.20, $Int_{\Delta}(Cl_{\Delta}(Int_{\Delta}(G)))$ is soft weakly connected in (L, Δ, R).

Theorem 2.33. Let (L, Δ, R) be soft disconnected and let G be soft pre-open in (L, Δ, R). Then G is soft weakly connected in (L, Δ, R) if and only if $Cl_{\Delta}(Int_{\Delta}(G))$ is soft weakly connected in (L, Δ, R).

Proof. Necessity. Follows from Theorem 2.32 (a).

Sufficiency. Suppose that $Cl_{\Delta}(Int_{\Delta}(G))$ is soft weakly connected in (L, Δ, R). Since G is soft semi-open in (L, Δ, R), $G \widetilde{\subseteq} Cl_{\Delta}(Int_{\Delta}(G))$. Thus, by Theorem 2.20, G is soft weakly connected in (L, Δ, R).

Theorem 2.34. Let (L, Δ, R) be soft disconnected and let G be soft semi-open in (L, Δ, R). Then G is soft weakly connected in (L, Δ, R) if and only if $Int_{\Delta}(Cl_{\Delta}(G))$ is soft weakly connected in (L, Δ, R).

Proof. Necessity. Follows from Theorem 2.32 (b).

Sufficiency. Suppose that $Int_{\Delta}(Cl_{\Delta}(G))$ is soft weakly connected in (L, Δ, R). Since G is soft pre-open in (L, Δ, R), $G \widetilde{\subseteq} Int_{\Delta}(Cl_{\Delta}(G))$. Thus, by Theorem 2.20, G is soft weakly connected in (L, Δ, R).

Theorem 2.35. Let (L, Δ, R) be soft disconnected and let G be soft α-open in (L, Δ, R). Then G is soft weakly connected in (L, Δ, R) if and only if $Int_{\Delta}(Cl_{\Delta}(Int_{\Delta}(G)))$ is soft weakly connected in (L, Δ, R).

Proof. Necessity. Follows from Theorem 2.32 (c).

Sufficiency. Suppose that $Int_{\Delta}(Cl_{\Delta}(Int_{\Delta}(G)))$ is soft weakly connected in (L, Δ, R). Since G is soft α-open in (L, Δ, R), $G \widetilde{\subseteq} Int_{\Delta}(Cl_{\Delta}(Int_{\Delta}(G)))$. Thus, by Theorem 2.20, G is soft weakly connected in (L, Δ, R).

Theorem 2.36. Let (L, Δ, R) be soft disconnected and let $G \in SS(L, R)$. If $Int_{\Delta}(G)$ is soft weakly connected in (L, Δ, R) and G is soft semi-open in (L, Δ, R), then G is soft weakly connected in (L, Δ, R).

Proof. Since $Int_{\Delta}(G)$ is soft weakly connected in (L, Δ, R), by Theorem 2.26, $Cl_{\Delta}(Int_{\Delta}(G))$ is soft weakly connected in (L, Δ, R). Since G is soft semi-open in (L, Δ, R), $G \widetilde{\subseteq} Cl_{\Delta}(Int_{\Delta}(G))$. Thus, by Theorem 2.20, G is soft weakly connected in (L, Δ, R).

3. Soft weakly connected components and a soft mapping theorem

In this section, we define soft weakly connected components in a given soft topological space. We show that this class of soft sets consists of soft closed sets and forms a soft partition of the space. In addition, we discuss the behavior of soft weak connected sets under soft α-continuity.

Theorem 3.1. Let (L, Δ, R) be soft disconnected. For any $a_x, b_y \in SP(L, R)$, define $a_x C b_y$ if and only if there exists a soft weakly connected set G in (L, Δ, R) such that $a_x, b_y \widetilde{\in} G$. Then C is an equivalence relation on $SP(L, R)$.

Proof. To see that C is reflexive, let $a_x \in SP(L, R)$.

Claim. a_x is soft weakly connected in (L, Δ, R).

Proof of Claim. Suppose to the contrary that there are $K, H \in CO(L, \Delta, R)$ such that $1_R = K \widetilde{\cup} H$, $K \widetilde{\cap} H = 0_R$, and $K \widetilde{\cap} a_x \neq 0_R \neq a_x \widetilde{\cap} H$. Thus, we have $a_x \widetilde{\in} K \widetilde{\cap} H = 0_R$, a contradiction.

Therefore, by the above claim, $a_x C a_x$. This shows that C is reflexive.

To see that C is transitive, suppose that $a_x C b_y$ and $b_y C d_z$. Then there are soft weakly connected sets S, T in (L, Δ, R) such that $a_x, b_y \widetilde{\in} S$ and $b_y, d_z \widetilde{\in} T$. Since $b_y \widetilde{\in} S \widetilde{\cap} T$, by Theorem 2.21, $S \widetilde{\cup} T$ is soft weakly connected in (L, Δ, R). Since $a_x, d_z \widetilde{\in} S \widetilde{\cup} T$, $a_x C d_z$. This shows that C is transitive.

Finally, it is clear from the definition that C is symmetric.

Definition 3.2. Let (L, Δ, R) be soft disconnected and let $a_x \in SP(L, R)$. Let C be the equivalence relation described in Theorem 3.1. The equivalence class determined by a_x relative to the equivalence relation C will be denoted by $C(a_x)$ and the soft set $\widetilde{\cup}_{b_y \in C(a_x)} b_y$ will be denoted by C_{a_x} and will be called the soft weakly-component (w-component, for short) of (L, Δ, R) determined by a_x.

Theorem 3.3. Let (L, Δ, R) be soft disconnected $a_x \in SP(L, R)$. Then

(a) $a_x \widetilde{\in} C_{a_x}$.

(b) C_{a_x} is soft weakly connected in (L, Δ, R).

(c) If K is a soft weakly connected in (L, Δ, R) and $C_{a_x} \widetilde{\subseteq} K$, then $G = K$.

(d) C_{a_x} is soft closed in (L, Δ, R).

Proof. (a) Since C is reflexive, then $a_x \in C(a_x)$ and so, $a_x \widetilde{\in} \widetilde{\cup}_{b_y \in C(a_x)} b_y = C_{a_x}$.

(b) Let $G = \widetilde{\cup}\left\{T : a_x \widetilde{\in} T \text{ and } T \text{ is soft weakly connected in } (L, \Delta, R)\right\}$. Since $a_x \widetilde{\in} \widetilde{\cap}\left\{T : a_x \widetilde{\in} T \text{ and } T \text{ is soft weakly connected in } (L, \Delta, R)\right\}$, by Corollary 2.23, G is soft weakly connected in (L, Δ, R).

Claim. $G = C_{a_x}$ and hence C_{a_x} is soft weakly connected in (L, Δ, R).

Proof of Claim. To see that $G \widetilde{\subseteq} C_{a_x}$, let $b_y \widetilde{\in} G$. Then there exists a soft weakly connected set T in (L, Δ, R) such that $a_x, b_y \widetilde{\in} T$. Hence, $b_y \widetilde{\in} C_{a_x}$.

To see that $C_{a_x} \widetilde{\subseteq} G$, let $b_y \widetilde{\in} C_{a_x}$. Then there exists a soft weakly connected set T in (L, Δ, R) such that $a_x, b_y \widetilde{\in} T$. Hence, $b_y \widetilde{\in} G$.

(c) Let K be soft weakly connected in (L, Δ, R) such that $C_{a_x} \widetilde{\subseteq} K$. To see that $K \widetilde{\subseteq} C_{a_x}$, let $b_y \widetilde{\in} K$. Since $a_x \widetilde{\in} C_{a_x} \widetilde{\subseteq} K$, then we have $a_x, b_y \widetilde{\in} K$ where K is soft weakly connected in (L, Δ, R). Hence, $b_y \widetilde{\in} C_{a_x}$.

(d) By (b), C_{a_x} is soft weakly connected in (L, Δ, R). So, by Theorem 2.24, $Cl_\Delta(C_{a_x})$ is soft weakly connected in (L, Δ, R). Since $C_{a_x} \widetilde{\subseteq} Cl_\Delta(C_{a_x})$, then by (c), $C_{a_x} = Cl_\Delta(C_{a_x})$. This shows that C_{a_x} is soft closed in (L, Δ, R).

Example 3.4. Let $L = \{1, 2, 3, 4\}$ and $A = \{a, b\}$. Define $S, T \in SS(L, A)$ by $S = \{(a, \{1, 2\}), (b, \{3, 4\})\}$ and $T = \{(a, \{3, 4\}), (b, \{1, 2\})\}$. Let Let $\Delta = \{0_A, 1_A, S, T\}$. Then $C_{a_1} = C_{a_2} = C_{b_3} = C_{b_4} = S$ and $C_{a_3} = C_{a_4} = C_{b_1} = C_{b_2} = T$.

Theorem 3.5. Let (L, Δ, R) be soft disconnected and let $G \in SS(L, R)$. Then G is soft weakly disconnected in (L, Δ, R) if and only if G is soft weakly disconnected in (L, Δ^α, R).

Proof. Necessity. Suppose that G is soft weakly disconnected in (L, Δ, R). Then there exist $K, H \in CO(L, \Delta, R)$ such that $1_R = K \widetilde{\cup} H$, $K \widetilde{\cap} H = 0_R$, and $K \widetilde{\cap} G \neq 0_R \neq G \widetilde{\cap} H$. Since $CO(L, \Delta, R) \subseteq CO(L, \Delta^\alpha, R)$, then $K, H \in CO(L, \Delta^\alpha, R)$. This shows that G is soft weakly disconnected in (L, Δ^α, R).

Sufficiency. Suppose that G is soft weakly disconnected in (L, Δ^α, R). Then there exist $K, H \in \Delta^\alpha$ such that $1_R = K \widetilde{\cup} H$, $K \widetilde{\cap} H = 0_R$, and $K \widetilde{\cap} G \neq 0_R \neq G \widetilde{\cap} H$. Since $K \widetilde{\cap} H = 0_R$, then $Int_\Delta(K) \widetilde{\cap} Int_\Delta(H) = 0_R$. So, $Int_\Delta(K) \widetilde{\cap} Cl_\Delta(Int_\Delta(H)) = 0_R$ and hence, $Int_\Delta(K) \widetilde{\cap} Int_\Delta(Cl_\Delta(Int_\Delta(H))) = 0_R$. Therefore, $Cl_\Delta(Int_\Delta(K)) \widetilde{\cap} Int_\Delta(Cl_\Delta(Int_\Delta(H))) = 0_R$ and thus,

$Int_\Delta(Cl_\Delta(Int_\Delta(K))) \widetilde{\cap} Int_\Delta(Cl_\Delta(Int_\Delta(H))) = 0_R$. Since $K, H \in \Delta^\alpha$, then $K \widetilde{\subseteq} Int_\Delta(Cl_\Delta(Int_\Delta(K)))$ and $H \widetilde{\subseteq} Int_\Delta(Cl_\Delta(Int_\Delta(H)))$. Put $S = Int_\Delta(Cl_\Delta(Int_\Delta(K)))$ and $T = Int_\Delta(Cl_\Delta(Int_\Delta(H)))$. Then $S, T \in \Delta$, $K \widetilde{\subseteq} S$, $H \widetilde{\subseteq} T$, and $S \widetilde{\cap} T = 0_R$. Since $1_R = K \widetilde{\cup} H \widetilde{\subseteq} S \widetilde{\cup} T$, then $1_R = S \widetilde{\cup} T$. Finally, since $K \widetilde{\cap} G \neq 0_R \neq G \widetilde{\cap} H$, $K \widetilde{\subseteq} S$, and $H \widetilde{\subseteq} T$, then we have $S \widetilde{\cap} G \neq 0_R \neq G \widetilde{\cap} T$. This shows that G is soft weakly disconnected in (L, Δ, R).

Theorem 3.6. Let (L, Δ, R) and (M, Π, B) be soft disconnected and $f_{pu} : (L, \Delta, R) \longrightarrow (M, \Pi, B)$ be soft α-continuous. If G is soft weakly connected in (L, Δ, R), then $f_{pu}(G)$ is soft weakly connected in (M, Π, B).

Proof. Suppose to the contrary that $f_{pu}(G)$ is soft weakly disconnected in (M,Π,B). Then there exist $K,H \in CO(M,\Pi,B)$ such that $1_B = K\widetilde{\cup}H$, $K\widetilde{\cap}H = 0_B$, and $K\widetilde{\cap}f_{pu}(G) \neq 0_B \neq f_{pu}(G)\widetilde{\cap}H$. Hence, $f_{pu}^{-1}(K)\widetilde{\cup}f_{pu}^{-1}(H) = f_{pu}^{-1}\left(K\widetilde{\cup}H\right) = f_{pu}^{-1}(1_B) = 1_R$, $f_{pu}^{-1}(K)\widetilde{\cap}f_{pu}^{-1}(H) = f_{pu}^{-1}\left(K\widetilde{\cap}H\right) = f_{pu}^{-1}(0_B) = 0_R$, and $f_{pu}^{-1}(K)\widetilde{\cap}G \neq 0_R \neq G\widetilde{\cap}f_{pu}^{-1}(H)$. Since f_{pu} is soft α-continuous, $f_{pu}^{-1}(K), f_{pu}^{-1}(H) \in \Delta^{\alpha}$. This implies that G is soft weakly disconnected in (L,Δ^{α},R). Therefore, by Theorem 3.5, G is soft weakly disconnected in (L,Δ,R), a contradiction.

4. Conclusions

The study of soft sets and soft topology is particularly significant during the investigation of possible applications in classical and non-classical logic. Soft topological spaces, which are a collection of information granules based on soft set theory, are the mathematical descriptions of approximate reasoning about information systems. Here, the concept of soft weak connectedness as a weaker type of soft connectedness is defined. Several properties and characterizations regarding soft weakly connected sets are introduced. Furthermore, using soft weakly connected sets, soft weakly connected components, it is proven that the family of soft weakly components within a soft topological space comprises soft closed sets, forming a soft partition of the space. In addition, the behavior of soft weak connected sets under soft α-continuity is discussed.

These two new concepts will also serve to strengthen the foundations of the soft topology toolbox. The findings of this paper can be applied to problems with uncertainties in many disciplines, and they will motivate future research into soft topology in order to carry out a generic framework for practical applications.

In the future, we may look at the following topics: (1) Defining soft weakly pre-connected sets, and (2) finding an application for our new two conceptions in the "decision-making problem" or "information systems" or "expert systems".

Use of AI tools declaration

The authors declare that they have not used Artificial Intelligence tools in the creation of this article.

Conflict of interest

The authors declare that they have no conflicts of interest.

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