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Research article

A note on closed vector fields

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Abstract: Special vector fields, such as conformal vector fields and Killing vector fields, are commonly used in studying the geometry of a Riemannian manifold. Though there are Riemannian manifolds, which do not admit certain conformal vector fields or certain Killing vector fields, respectively. Closed vector fields exist in abundance on each Riemannian manifold. In this paper, we used closed vector fields to study the geometry of the Riemannian manifold. In the first result, we showed that a compact Riemannian manifold (M^n, g) admits a closed vector field ω with divω non-constant and an eigenvector of the rough Laplace operator, the integral of the Ricci curvature $Ric(\omega,\omega)$ has a suitable lower bound that is necessarily isometric to $S^n(c)$ and that the converse holds. In the other result, we found a characterization of an Euclidean space using a closed vector field ω with non-constant length that annihilates the rough Laplace operator and squared length of its covariant derivative that has a suitable upper bound. Finally, we used the closed vector field provided by the gradient of the non-trivial solution of the Fischer-Marsden equation on a complete and simply connected Riemannian manifold (M, g) and showed that it is necessary and sufficient for (M, g) to be isometric to a sphere and that the squared length of the covariant derivative of this closed vector field has a suitable upper bound.

Keywords: closed vector field; n-sphere; Euclidean space; Fischer-Marsden equation
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1. Introduction

In classical differential geometry, in order to study the geometry of an n-dimensional Riemannian manifold (M^n, g) the main used tools were curvature, Ricci curvature, scalar curvature and also the extrinsic tools obtained through the isometric immersion of (M^n, g) into some known Riemannian manifold, such as second fundamental form and the fundamental equations of submanifolds [2]. Apart from these tools, the trend of using special vector fields on a Riemannian manifold (M^n, g) such as a conformal vector field, a Killing vector field, a Jacobi-type vector field and a Torse-forming vector field facilitated the study of the geometry of the host manifolds on which these vector fields are defined [3–11, 13–15]. A conformal field ω on (M^n, g) gives rise to a smooth function $\lambda : M^n \to R$ called the conformal factor, such that the Lie derivative $£_{\omega}g$ of the metric satisfies

$$£_{\omega}g = 2\lambda g,$$

and this conformal factor plays an important role in shaping the geometry of the host manifold [3–9, 11, 13–15]. Similarly, a Killing field $\mathbf{u}$ on (M^n, g) is the one whose local one-parameter groups of local transformations consists of local isometries of (M^n, g) or, equivalently, the Lie derivative $£_{\mathbf{u}}g$ satisfies

$$£_{\mathbf{u}}g = 0,$$

and a Killing vector field restricts topology as well as the geometry of the host manifold [10]. The next important special field on a Riemannian manifold (M^n, g) is the torse-forming vector field ξ that satisfies

$$D_X\xi = \lambda X + \alpha(X)\xi,$$

where D_X is the covariant derivative with respect to smooth vector field X, α is a one-form and λ is a smooth function on (M^n, g) [18]. Torse-forming vector fields are neither conformal nor Killing and they influence geometry of the host manifolds as well as have physical applications [2, 18]. Given a Killing vector field $\mathbf{u}$ of constant length on (M^n, g), it follows that its integral curves are geodesics. However, if we remove the restriction on $\mathbf{u}$ of being a constant length, then the integral curves of the Killing vector field $\mathbf{u}$ are not unnecessarily geodesics. This leads to the definition of a geodesic vector field on a Riemannian manifold (M^n, g) [7]. Geodesic vector fields, which are non-Killing, are in abundance owing to the presence of non-Sasakian structures such as the Kenmotsu structure or a trans-Sasakian structure, as well as Eikonal equations on complete manifolds [7, 19].

All vector fields mentioned earlier put severe restrictions on the host manifolds and, therefore, there are Riemannian manifolds in which some of them do not exist. For instance, on a compact even dimensional Riemannian manifold of positive curvature, a unit Killing vector field does not exist. Similarly, a non-trivial closed conformal vector field on a compact Riemannian manifold of non-positive Ricci curvature does not exist. However, there is a very large class of some special vector field that exists on each Riemannian manifold (M^n, g) and these are closed vector fields. For instance, each non-constant smooth function $\sigma : M^n \to R$ provides the gradient $\nabla\sigma$ on (M^n, g), which is a closed vector field. Apart from gradients, there are non-gradient closed vector fields on (M^n, g). We shall abbreviate a closed vector field $\mathbf{u}$ on a Riemannian manifold (M^n, g) as CLVF $\mathbf{u}$. Geometry of semi-Riemannian manifolds admitting a CLVF have been studied in [21–23]. Among these, Hicks in [22] has studied the submanifolds of a semi-Riemannian manifold determined by a CLVF and obtained many interesting results, where as in [21, 23], authors have studied closed conformal vector fields on a Riemannian manifold. Note that closed vector fields are related to foliations with singularities in this sense, if ξ is a CLVF on a Riemannian manifold (M, g), then on the open subset $U = \left\{p \in M : \xi_p \neq 0\right\}$, the orthogonal distribution is integrable and defines a codimension-one foliation. Here, in this paper, we are interested in closed vector fields on a Riemannian manifold (M^n, g), which are eigenvectors of rough Laplace operator $\boxdot$. Given (M^n, g), we denote by $\Gamma(TM^n)$ the space of smooth sections of the tangent bundle TM^n, and the rough Laplace operator $\boxdot$: $\Gamma(TM^n) \to \Gamma(TM^n)$ is defined by

$$\boxdot X = \sum_{i=1}^{n} \left(D_{E_i} D_{E_i} X - D_{D_{E_I} E_i} X \right), \quad X \in \Gamma(TM^n),$$

where $\{E_i\}_1^n$ is a local frame on (M^n, g). A vector field $\xi \in \Gamma(TM^n)$ is said to be an eigenvector of the operator $\boxdot$ if $\boxdot\xi = a\xi$ for a constant a. For example, consider the Sasakian structure (ϕ, ξ, η, g) on the unit sphere S^{2n+1} [10]. We have

$$D_X \xi = \phi X, \quad (D_X \phi)(Y) = \eta(Y) X - g(X, Y) \xi, \quad X, Y \in \Gamma\left(TS^{2n+1}\right),$$

then, it follows that

$$\boxdot\xi = \sum_{i=1}^{2n+1} \left(D_{E_i} D_{E_i} \xi - D_{D_{E_I} E_i} \xi \right) = \sum_{i=1}^{2n+1} (D_{E_i} \phi)(E_i) = \xi - (2n+1)\xi = -2n\xi;$$

that is, the vector field ξ is an eigenvector of the operator $\boxdot$ on the sphere S^{2n+1}. Note that the unit vector field ξ is a non-trivial Killing vector field and, therefore, it is not closed.

We shall show that there are abundantly many closed vector fields on the n-sphere $S^n(c)$ of constant curvature c, which are eigenvectors of the rough Laplace operator $\boxdot$. To realize it, we treat $S^n(c)$ as surface in the Euclidean space R^{n+1} with unit normal ζ and shape operator $B = -\sqrt{c}I$. Denoting the Euclidean metric on R^{n+1} by $\overline{g}$ and that induced on $S^n(c)$ by g and corresponding covariant derivative operators by $\overline{D}_X$ and D_X, $X \in \Gamma(TS^n(c))$ respectively, we have the fundamental equations for the surface S^n

$$\overline{D}_X Y = D_X Y - \sqrt{c} g(X, Y) \zeta, \quad \overline{D}_X \zeta = \sqrt{c} X, \quad X, Y \in \Gamma(TS^n(c)). \tag{1.1}$$

Now, choosing a non-zero constant vector $\mathbf{a} \in \Gamma\left(TR^{n+1}\right)$ and denoting its tangential component to $S^n(c)$ by ω, we ascertain that $\mathbf{a} = \omega + \rho\zeta$, where $\rho = \overline{g}(\mathbf{a}, \zeta)$ is a smooth function defined on $S^n(c)$. Differentiating the equation $\mathbf{a} = \omega + \rho\zeta$ while using Eq (1.1), we arrive at

$$D_X \omega = -\sqrt{c}\rho X, \quad \nabla\rho = \sqrt{c}\omega, \tag{1.2}$$

where $\nabla\rho$ is the gradient of ρ. Thus, it follows that ω is a CLVF on $S^n(c)$. Using a local frame $\{E_i\}_1^n$ and Eq (1.2), we conclude that

$$\boxdot\omega = -c\omega; \tag{1.3}$$

that is, ω is a CLVF that is an eigenvector of the rough Laplace operator $\boxdot$. Note that each constant vector $\mathbf{a}$ on the Euclidean space R^n gives a CLVF ω on the n-sphere $S^n(c)$ that satisfies $\boxdot\omega = -c\omega$. This raises a question: Under what condition is an n-dimensional compact Riemannian manifold (M^n, g) admitting a CLVF ω that satisfies $\boxdot\omega = -c\omega$ for a positive constant c that is isometric to n-sphere $S^n(c)$? We answer this question by proving the following using the bound on the integral of the Ricci curvature $Ric(\omega, \omega)$ in the direction of a CLVF ω that satisfies $\boxdot\omega = -c\omega$ for a positive constant c on a compact (M^n, g):

Theorem 1. *An n-dimensional compact Riemannian manifold* (M^n, g) *admits a CLVF* $\boldsymbol{\omega}$ *with non-constant* $\operatorname{div}\boldsymbol{\omega}$*, satisfying*

(i) $\int\limits_{M^n} Ric(\boldsymbol{\omega}, \boldsymbol{\omega}) \geq \frac{n-1}{n} \int\limits_{M^n} (\operatorname{div}\boldsymbol{\omega})^2$ *(ii)* $\boxdot\boldsymbol{\omega} = -c\boldsymbol{\omega}$,

for a positive constant c if, and only if, (M^n, g) *is isometric to n-sphere* $S^n(c)$.

Similarly, for the Euclidean space R^n, we prove the following:

Theorem 2. *An n-dimensional complete and connected Riemannian manifold* (M^n, g) *admits a non-parallel CLVF* $\boldsymbol{\omega}$ *with non-constant length satisfying*

(i) $\|D\boldsymbol{\omega}\|^2 \leq \frac{1}{n}(\operatorname{div}\boldsymbol{\omega})^2$ *(ii)* $\boxdot\boldsymbol{\omega} = 0$,

if, and only if, (M^n, g) *is isometric to the Euclidean space* R^n.

Naturally, each smooth function $\sigma : M^n \to R$ on a Riemannian manifold (M^n, g) gives a CLVF $\boldsymbol{\omega} = \nabla\sigma$ the gradient of σ. However, there are many closed fields on a Riemannian manifold (M^n, g), which are not gradient fields. For instance, a (M^n, g) with de-Rham cohomology group $H^1(M^n, R) \neq 0$ has many closed fields that are not gradient. For example, take the warped product $M^n = S^1 \times_\rho S^{n-1}(c)$ for smooth function ρ defined on the unit one-sphere S^1 with metric $g = d\theta^2 + \rho^2\overline{g}$, where θ is the coordinate function on S^1 and $\overline{g}$ is the canonical metric on the sphere $S^{n-1}(c)$; then the vector field $\mathbf{u} = \rho\frac{\partial}{\partial\theta}$ defined on (M^n, g) has covariant derivative [16]

$$D_X\mathbf{u} = \rho' X, \quad X \in \Gamma(TM^n).$$

Consequently, we see that $\mathbf{u}$ is a CLVF and it is not a gradient field. Moreover, the Ricci curvature $Ric(\mathbf{u}, \mathbf{u})$ of the compact Riemannian manifold (M^n, g) and $\operatorname{div}\mathbf{u}$ is given by [16]

$$Ric(\mathbf{u}, \mathbf{u}) = -\frac{n-1}{2}\left(\rho^2\right)'', \quad \operatorname{div}\mathbf{u} = n\rho'.$$

Finally, in this paper we consider the differential equation introduced by Fischer and Marsden on a Riemannian manifold (M^n, g), namely [12],

$$(\Delta\sigma)g + \sigma Ric = Hes(\sigma),$$

which we shall refer to as an F-M equation (here and hereafter F-M means Fischer and Marsden), where Δ is the Laplace operator on (M^n, g), $Hes(\sigma)$ is the Hessian operator of the function σ defined by

$$Hes(\sigma)(X, Y) = g(D_X\nabla\sigma, Y), \quad X, Y \in \Gamma(TM^n) \tag{1.4}$$

and use the non-trivial solution σ of the F-M equation to get a CLVF $\xi = \nabla\sigma$ to study the geometry of the host Riemannian manifold (M^n, g). It is known that if the Riemannian manifold (M^n, g) admits a non-trivial solution of the F-M equation, then scalar curvature τ is a constant [12]. It is worth mentioning that a Fischer-Marsden conjectured in [12] that a compact (M^n, g) admitting a non-trivial solution of the F-M equation must be an Einstein space, which is known as a Fischer-Marsden conjecture. However, this conjecture is not true, as in [1], authors have shown that the product Riemannian manifold $M^n = S^m \times N^{n-m}$, where S^m is the unit sphere and N^{n-m} is a compact Einstein space of constant scalar curvature $\tau \neq m(m-1)$, admits a non-trivial solution of the F-M equation and, thus, provides a counter example to the Fischer-Marsden conjecture. Using a non-trivial solution of the F-M equation on a complete and simply connected (M^n, g), we prove the following:

Theorem 3. *A non-trivial solution σ of the F-M equation on an n-dimensional complete and simply connected Riemannian manifold (M^n, g) of positive scalar curvature τ with CLVF $\omega = \nabla\sigma$ satisfies*

$$\|D\omega\|^2 \leq \frac{1}{n(n-1)^2}\tau^2\sigma^2,$$

if, and only if, (M^n, g) is isometric to n-sphere $S^n(c)$.

2. Preliminaries

Let ω be a CLVF on an n-dimensional Riemannian manifold (M^n, g). We denote by α smooth one-form dual to ω; that is, $\alpha(X) = g(\omega, X)$, $X \in \Gamma(TM^n)$. As ω is a CLVF, we have $d\alpha = 0$ and we define a symmetric operator $B : \Gamma(TM^n) \to \Gamma(TM^n)$ by

$$\frac{1}{2}(\pounds_{\omega} g)(X, Y) = g(BX, Y), \quad X, Y \in \Gamma(TM^n), \tag{2.1}$$

then

$$\begin{aligned} 2g(D_X\omega, Y) &= g(D_X\omega, Y) + g(D_Y\omega, X) + g(D_X\omega, Y) - g(D_Y\omega, X) \\ &= (\pounds_{\omega} g)(X, Y) = 2g(BX, Y), \quad X, Y \in \Gamma(TM^n). \end{aligned}$$

Thus, for the CLVF ω defined on (M^n, g), we have

$$D_X\omega = BX, \quad X \in \Gamma(TM^n), \tag{2.2}$$

where B is a symmetric operator satisfying Eq (2.1). Using the expression

$$R(X, Y)Z = D_X D_Y Z - D_Y D_X Z - D_{[X,Y]}Z$$

for the curvature tensor of (M^n, g) and Eq (2.2), we have

$$R(X, Y)\omega = (D_X B)(Y) - (D_Y B)(X), \quad X, Y \in \Gamma(TM^n). \tag{2.3}$$

Using a local orthonormal frame $\{E_i\}_1^n$ on (M^n, g) and the expression of the Ricci tensor

$$Ric(X.Y) = \sum_{i=1}^{n} g(R(E_i, X)Y, E_i)$$

and Eq (2.3), we conclude

$$Ric(Y, \omega) = \sum_{i=1}^{n} g((D_{E_i}B)(Y), E_i) - \sum_{i=1}^{n} g((D_Y B)(E_i), E_i).$$

Now, letting the trace $TrB = h$ and employing the symmetry of the operator B in the above equation yields

$$Ric(Y, \omega) = g\left(Y, \sum_{i=1}^{n} (D_{E_i}B)(E_i)\right) - Y(h). \tag{2.4}$$

The symmetric operator Q corresponding to the Ricci tensor is called the Ricci operator of (M^n, g) given by

$$g(QX, Y) = Ric(X, Y), \quad X, Y \in \Gamma(TM^n)$$

and, consequently, by virtue of Eq (2.4), we get the following

$$Q(\omega) = \sum_{i=1}^{n} (D_{E_i}B)(E_i) - \nabla h,$$

where ∇h is the gradient of $h = TrB$.

Note that for a frame $\{E_i\}_1^n$ on (M^n, g), we have the squared length of operator B given by

$$\|B\|^2 = \sum_{i=1}^{n} g(BE_i, BE_i),$$

and we get

$$\left\|B - \frac{h}{n}I\right\|^2 = \sum_{i=1}^{n} g\left(BE_i - \frac{h}{n}E_i, BE_i - \frac{h}{n}E_i\right) = \|B\|^2 + \frac{h^2}{n} - 2\frac{h}{n}\sum_{i=1}^{n} g(BE_i, E_i);$$

that is,

$$\left\|B - \frac{h}{n}I\right\|^2 = \|B\|^2 - \frac{h^2}{n}. \tag{2.5}$$

3. Proof of Theorem-1

Suppose the compact Riemannian manifold (M^n, g) admits a CLVF ω with non-constant divω and satisfies

$$\int_{M^n} Ric(\omega, \omega) \geq \frac{n-1}{n}\int_{M^n} (\mathrm{div}\omega)^2 \tag{3.1}$$

and

$$\boxdot\omega = -c\omega, \tag{3.2}$$

where c is a positive constant. Note that owing to Eq (2.2), we have div$\omega = h = TrB$ and, therefore, by the above assumption the smooth function h is non-constant. In view of Eq (2.2), we have

$$BX - \frac{h}{n}X = D_X\omega - \frac{h}{n}X, \quad X \in \Gamma(TM^n)$$

and, with a frame $\{E_i\}_1^n$ on (M^n, g) by the above equation, we have

$$\begin{aligned}\left\|B - \frac{h}{n}I\right\|^2 &= \sum_{i=1}^{n} g\left(BE_i - \frac{h}{n}E_i, BE_i - \frac{h}{n}E_i\right) = \sum_{i=1}^{n} g\left(D_{E_i}\omega - \frac{h}{n}E_i, D_{E_i}\omega - \frac{h}{n}E_i\right)\\ &= \|D\omega\|^2 + \frac{h^2}{n} - 2\frac{h}{n}\sum_{i=1}^{n} g(D_{E_i}\omega, E_i) = \|D\omega\|^2 + \frac{h^2}{n} - 2\frac{h}{n}\mathrm{div}\omega,\end{aligned}$$

that is,

$$\left\|B-\frac{h}{n}I\right\|^{2}=\|D\omega\|^{2}-\frac{h^{2}}{n}. \tag{3.3}$$

Also, by Eq (2.1), we get

$$\frac{1}{4}\left|\pounds_{\omega}g\right|^{2}=\|B\|^{2}. \tag{3.4}$$

On compact (M^{n},g), we have the following integral formula [20]

$$\int_{M}\left\{Ric\left(\omega,\omega\right)+\frac{1}{2}\left|\pounds_{\omega}g\right|^{2}-\|D\omega\|^{2}-(\mathrm{div}\omega)^{2}\right\}=0.$$

In view of the above formula, on integrating Eq (3.3) and by using (3.4) and $\mathrm{div}\omega=h$, we arrive at

$$\int_{M}\left\|B-\frac{h}{n}I\right\|^{2}=\int_{M}\left\{Ric\left(\omega,\omega\right)+2\|B\|^{2}-h^{2}-\frac{h^{2}}{n}\right\};$$

that is,

$$\int_{M}\left\|B-\frac{h}{n}I\right\|^{2}=\int_{M}\left\{Ric\left(\omega,\omega\right)+2\left(\|B\|^{2}-\frac{1}{n}h^{2}\right)-\left(\frac{n-1}{n}\right)h^{2}\right\}.$$

Now, using Eq (2.5) with the above equation, we get

$$\int_{M}\left\|B-\frac{h}{n}I\right\|^{2}=\left(\frac{n-1}{n}\right)\int_{M}(\mathrm{div}\omega)^{2}-\int_{M}Ric\left(\omega,\omega\right)$$

and combining it with Eq (3.1), we conclude

$$\left\|B-\frac{h}{n}I\right\|^{2}=0.$$

Thus, we have $B=\frac{h}{n}I$, and Eq (2.2) takes the form

$$D_{X}\omega=\frac{h}{n}X, \tag{3.5}$$

which together with a frame $\{E_i\}_1^n$ on (M^n,g) implies

$$D_{E_i}D_{E_i}\omega=\frac{1}{n}E_i(h)E_i+\frac{h}{n}D_{E_i}E_i,$$

and we conclude

$$\boxdot\omega=\sum_{i=1}^{n}\left(D_{E_i}D_{E_i}\omega-D_{D_{E_I}E_I}\omega\right)=\frac{1}{n}\nabla h.$$

On combining the above equation with Eq (3.2), we conclude

$$-c\omega=\frac{1}{n}\nabla h.$$

Thus, we have

$$-cD_X\omega = \frac{1}{n}D_X\nabla h$$

and utilizing Eq (3.5), we have

$$D_X\nabla h = -chX, \quad X \in \Gamma(TM^n),$$

where c is a positive constant and h is a non-constant function. Hence, by the above equation, we conclude that (M^n, g) is isometric to the n-sphere $S^n(c)$ [14, 15].

Conversely, suppose that the compact (M^n, g) is isometric to the n-sphere $S^n(c)$, then we know that $S^n(c)$ admits a CLVF ω that satisfies Eqs (1.2) and (1.3); that is,

$$D_X\omega = -\sqrt{c}\rho X, \quad \nabla\rho = \sqrt{c}\omega, \quad \boxdot\,\omega = -c\omega. \tag{3.6}$$

First, we claim that divω is non-constant. Note that by Eq (3.6), we have div$\omega = -n\sqrt{c}\rho$, and if divω were a constant, by Stokes' Theorem on compact $S^n(c)$, it will imply $\rho = 0$. Now, $\rho = 0$ in Eq (1.2) will imply $\omega = 0$ and in sequel it will imply the constant vector $\mathbf{a} = 0$, which is a contradiction as $\mathbf{a}$ is a non-zero constant vector on the Euclidean space R^{n+1}. Thus, divω is non-constant. Now, it remains to show that the inequality (3.1) holds. The Ricci curvature $Ric(\omega, \omega)$ of the sphere $S^n(c)$ is $Ric(\omega, \omega) = (n-1)c\|\omega\|^2$, which on using Eq (3.6) implies

$$\int_{S^n(c)} Ric(\omega, \omega) = (n-1)\int_{S^n(c)} \|\nabla\rho\|^2. \tag{3.7}$$

Also, Eq (3.6) implies div$\omega = -n\sqrt{c}\rho$; that is,

$$(\mathrm{div}\omega)^2 = n^2 c\rho^2.$$

Using div$\omega = -n\sqrt{c}\rho$ with the equation $\nabla\rho = \sqrt{c}\omega$ of Eq (3.6), we get the Laplacian $\Delta\rho = \mathrm{div}(\nabla\rho) = \sqrt{c}\left(-n\sqrt{c}\rho\right) = -nc\rho$; that is, $\rho\Delta\rho = -nc\rho^2$. Integrating this last equation by parts, we conclude

$$\int_{S^n(c)} \|\nabla\rho\|^2 = nc\int_{S^n(c)} \rho^2,$$

and substituting in the above equation div$\omega = -n\sqrt{c}\rho$, we get

$$\int_{S^n(c)} \|\nabla\rho\|^2 = \frac{1}{n}\int_{S^n(c)} (\mathrm{div}\omega)^2.$$

Combining it with Eq (3.7), we conclude

$$\int_{S^n(c)} Ric(\omega, \omega) = \frac{(n-1)}{n}\int_{S^n(c)} (\mathrm{div}\omega)^2.$$

This completes the proof.

4. Proof of Theorem-2

Suppose an n-dimensional complete and connected Riemannian manifold (M^n, g) admits a non-parallel CLVF $\boldsymbol{\omega}$ with a non-constant length satisfying

$$\|D\boldsymbol{\omega}\|^2 \leq \frac{1}{n}\left(\operatorname{div}\boldsymbol{\omega}\right)^2 \tag{4.1}$$

and

$$\boxdot\boldsymbol{\omega} = 0. \tag{4.2}$$

Using Eq (2.2), we have

$$BX - \frac{h}{n}X = D_X\boldsymbol{\omega} - \frac{h}{n}X, \quad X \in \Gamma\left(TM^n\right),$$

which with a frame $\{E_i\}_1^n$ on (M^n, g) and $\operatorname{div}\boldsymbol{\omega} = h$ gives

$$\begin{aligned}\left\|B - \frac{h}{n}I\right\|^2 &= \sum_{i=1}^n g\left(BE_i - \frac{h}{n}E_i, BE_i - \frac{h}{n}E_i\right) = \sum_{i=1}^n g\left(D_{E_i}\boldsymbol{\omega} - \frac{h}{n}E_i, D_{E_i}\boldsymbol{\omega} - \frac{h}{n}E_i\right)\\ &= \|D\boldsymbol{\omega}\|^2 + \frac{h^2}{n} - 2\frac{h}{n}\sum_{i=1}^n g\left(D_{E_i}\boldsymbol{\omega}, E_i\right) = \|D\boldsymbol{\omega}\|^2 + \frac{1}{n}\left(\operatorname{div}\boldsymbol{\omega}\right)^2 - 2\frac{h}{n}\operatorname{div}\boldsymbol{\omega};\end{aligned}$$

that is,

$$\left\|B - \frac{h}{n}I\right\|^2 = \|D\boldsymbol{\omega}\|^2 - \frac{1}{n}\left(\operatorname{div}\boldsymbol{\omega}\right)^2.$$

Using inequality (4.1) in the above equation, it confirms that $B = \frac{h}{n}I$ and Eq (2.2) now takes the form

$$D_X\omega = \frac{h}{n}X, \quad X \in \Gamma\left(TM^n\right). \tag{4.3}$$

Taking a frame $\{E_i\}_1^n$ on (M^n, g) and using Eq (4.3), we have

$$\boxdot\boldsymbol{\omega} = \sum_{i=1}^n \left(D_{E_i}D_{E_i}\boldsymbol{\omega} - D_{D_{E_I}E_I}\boldsymbol{\omega}\right) = \frac{1}{n}\nabla h,$$

and combining it with Eq (4.2), it confirms that the function h is a constant. Moreover, as $\boldsymbol{\omega}$ is non-parallel, Eq (4.3) implies that the constant $h \neq 0$. Since, $\boldsymbol{\omega}$ has a non-constant length, we have a non-constant function σ defined by

$$\sigma = \frac{1}{2}g\left(\boldsymbol{\omega}, \boldsymbol{\omega}\right)$$

and utilizing Eq (4.3), we get the gradient $\nabla\sigma$ of σ given by

$$\nabla\sigma = \frac{h}{n}\boldsymbol{\omega}.$$

Thus, again using Eq (4.3), we have

$$D_X\nabla\sigma = \frac{h^2}{n^2}X.$$

Consequently, the above equation together with Eq (1.4) implies

$$Hes\,(\sigma) = cg, \quad c = \frac{h^2}{n^2}, \tag{4.4}$$

where σ is a non-constant function and c is a non-zero constant. Hence, Eq (4.4) confirms that (M^n, g) is isometric to the Euclidean space R^n [17].

Conversely, suppose that (M^n, g) is isometric to the Euclidean space R^n. We consider the vector field $\boldsymbol{\omega} \in \Gamma(TR^n)$ defined by

$$\boldsymbol{\omega} = \sum_{i=1}^{n} z^i \frac{\partial}{\partial z^i},$$

where $z^1, ..., z^n$ are Euclidean coordinate functions on R^n. Let D_X be the covariant derivative operator on R^n with respect to the Euclidean connection, then we get

$$D_X \boldsymbol{\omega} = X, \quad X \in \Gamma(TR^n). \tag{4.5}$$

Equation (4.5) implies that $\boldsymbol{\omega}$ is a CLVF and it has a non-constant length. Also, Eq (4.5) gives $\operatorname{div}\boldsymbol{\omega} = n$ and

$$\|D\boldsymbol{\omega}\|^2 = n.$$

Thus, we conclude

$$\|D\boldsymbol{\omega}\|^2 = \frac{1}{n}(\operatorname{div}\boldsymbol{\omega})^2;$$

that is, the equality in (4.1) holds. Finally, we use Eq (4.5) to compute $\boxdot\boldsymbol{\omega}$ with a frame $\{E_i\}_1^n$ on (M^n, g) and get

$$\boxdot\boldsymbol{\omega} = \sum_{i=1}^{n}\left(D_{E_i}D_{E_i}\boldsymbol{\omega} - D_{D_{E_I}E_I}\boldsymbol{\omega}\right) = \sum_{i=1}^{n}\left(D_{E_i}E_i - D_{E_i}E_i\right) = 0,$$

which is Eq (4.2), and this completes the proof.

5. Proof of Theorem-3

Suppose σ is a non-trivial solution of F-M equation on an n -dimensional complete and simply connected Riemannian manifold (M^n, g) of positive scalar curvature τ and the CLVF $\boldsymbol{\omega} = \nabla\sigma$ satisfies

$$\|D\boldsymbol{\omega}\|^2 \le \frac{1}{n(n-1)^2}\tau^2\sigma^2. \tag{5.1}$$

Since σ is a solution of the F-M equation, we have

$$(\Delta\sigma)\,g + \sigma Ric = Hes(\sigma), \tag{5.2}$$

and taking trace in the above equation implies

$$\Delta\sigma = -\frac{\tau}{n-1}\sigma, \tag{5.3}$$

where τ is the scalar curvature of (M^n, g), which is a constant [12]. By hypothesis, we see that τ is a positive constant. Using Eqs (5.2) and $\boldsymbol{\omega} = \nabla\sigma$, we have

$$D_X\boldsymbol{\omega} = (\Delta\sigma)\,X + \sigma QX, \quad X \in \Gamma\,(TM^n)\,. \tag{5.4}$$

Using Eq (5.3) in the above equation, we have

$$\mathrm{div}\boldsymbol{\omega} = -\frac{\tau}{n-1}\sigma. \tag{5.5}$$

Combining the above Eq (5.5) with Eq (5.3), we have

$$\sigma\left(QX - \frac{\tau}{n}X\right) = D_X\boldsymbol{\omega} + \frac{\tau}{n(n-1)}\sigma X.$$

Choosing a frame $\{E_i\}_1^n$ on (M^n, g) with the above equation, we compute

$$\begin{aligned}\sigma^2\left\|Q - \frac{\tau}{n}I\right\|^2 &= \sum_{i=1}^{n} g\left(D_{E_i}\boldsymbol{\omega} + \frac{\tau}{n(n-1)}\sigma E_i, D_{E_i}\boldsymbol{\omega} + \frac{\tau}{n(n-1)}\sigma E_i\right)\\ &= \|D\boldsymbol{\omega}\|^2 + \frac{1}{n(n-1)^2}\tau^2\sigma^2 + 2\frac{\tau}{n(n-1)}\sigma\mathrm{div}\boldsymbol{\omega},\end{aligned}$$

which in view of Eq (5.3), yields

$$\sigma^2\left\|Q - \frac{\tau}{n}I\right\|^2 = \|D\boldsymbol{\omega}\|^2 - \frac{1}{n(n-1)^2}\tau^2\sigma^2.$$

Using inequality (5.1) in the above equation arrives at

$$\sigma^2\left\|Q - \frac{\tau}{n}I\right\|^2 = 0.$$

However, as σ is a non-trivial solution of the F-M equation on connected M^n, the above equation yields

$$Q = \frac{\tau}{n}I.$$

Thus, Eqs (5.3) and (5.4) in view of the above equation imply

$$D_X\boldsymbol{\omega} = \left(-\frac{\tau}{n-1}\sigma\right)X + \frac{\tau}{n}\sigma X = -\frac{\tau}{n(n-1)}\sigma X, \quad X \in \Gamma\,(TM^n)\,;$$

that is, on taking the constant $\tau = n(n-1)c$ as $\tau > 0$ and $c > 0$, we have

$$D_X\nabla\sigma = -c\sigma X, \quad X \in \Gamma\,(TM^n)\,,$$

where σ being a non-trivial solution of the F-M equation is a non-constant function. Hence, by the above equation, we see that the complete simply connected (M^n, g) is isometric to $S^n(c)$ [14, 15].

Conversely, suppose (M^n, g) is isometric to $S^n(c)$, then certainly the scalar curvature is $\tau > 0$ and there is a non-constant function ρ defined on $S^n(c)$ that satisfies Eq (1.2). It follows that $\Delta\rho = -nc\rho$ and

$$Hes(\rho)(X, Y) = g\,(D_X\nabla\rho, Y) = g\left(\sqrt{c}D_X\boldsymbol{\omega}, Y\right) = \sqrt{c}g\left(-\sqrt{c}\rho X, Y\right);$$

that is,

$$Hes(\rho)(X,Y) = -cg\,(X,Y)\,, \quad X,Y \in \Gamma\,(TM^n)\,. \tag{5.6}$$

Also, using the expression for the Ricci tensor of $S^n(c)$, $Ric\,(X,Y) = (n-1)cg\,(X,Y)$ and $\Delta\rho = -nc\rho$, we get

$$(\Delta\rho)\,g + \rho Ric = -nc\rho g + (n-1)c\rho g = -cg$$

and combining it with Eq (5.6), we conclude

$$(\Delta\rho)\,g + \rho Ric = Hes(\rho).$$

Hence, $S^n(c)$ admits the solution ρ of the F-M equation. As seen in the proof of Theorem-1, ρ is a non-constant function, and we conclude the function ρ as a non-trivial solution of the F-M equation. Finally, Eq (1.2) for $\mathbf{u} = \nabla\rho$ implies that

$$D_X\mathbf{u} = \sqrt{c}\left(-\sqrt{c}\rho X\right) = -c\rho X;$$

that is,

$$D_X\mathbf{u} = -c\rho X, \quad X \in \Gamma\,(TM^n)\,.$$

Thus,

$$\|D\mathbf{u}\|^2 = nc^2\rho^2,$$

and using the expression for the scalar curvature $\tau = n(n-1)c$, we get

$$\|D\mathbf{u}\|^2 = \frac{1}{n(n-1)^2}\tau^2\rho^2,$$

which is the condition required in the statement. This completes the proof.

6. Conclusions

We have seen Theorems 1 and 2, we used a CLVF ω, which need not be exact, to find characterizations of a sphere $S^n(c)$ and the Euclidean space R^n, respectively, where as in Theorem 3, we used a CLVF $\boldsymbol{\omega}$ that is exact namely, $\boldsymbol{\omega} = \nabla\sigma$ such that σ is a non-trivial solution of the F-M equation to find yet another characterization of the sphere $S^n(c)$. Note, that in Theorem 1, we required a CLVF $\boldsymbol{\omega}$ on a compact Riemannian manifold (M^n, g) to satisfy two conditions

$$\int_{M^n} Ric\,(\omega,\omega) \geq \frac{n-1}{n}\int_{M^n} (\mathrm{div}\boldsymbol{\omega})^2\,, \quad \boxdot\omega = -c\omega \tag{6.1}$$

for a positive constant c, in order that (M^n, g) is isometric to the sphere $S^n(c)$. A natural question arises, if we ask a CLVF $\boldsymbol{\omega}$ in Theorem 1 to be exact, that is, $\boldsymbol{\omega} = \nabla\sigma$ for some smooth function σ with $\boldsymbol{\omega} \neq 0$ or, equivalently, σ a non-constant function, can we relax conditions in Eq (6.1) in order to reach the same conclusion? Natural substitute in this situation to condition $\boxdot\omega = -c\omega$ could be $\Delta\sigma = -nc\sigma$. Precisely, can we prove the following?

An n-dimensional compact Riemannian manifold (M^n, g) admits a non-zero CLVF ω with $\omega = \nabla\sigma$ satisfying

$$\int_{M^n} Ric\,(\omega,\omega) \geq \frac{n-1}{n}\int_{M^n} (\mathrm{div}\omega)^2\,, \quad \Delta\sigma = -nc\sigma$$

for a positive constant c, if and only if (M^n, g) is isometric to $S^n(c)$. This will be an interesting question for future study.

Use of AI tools declaration

The authours declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authours declare no conflicts of interest.

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