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Research article

Consistent pairs of $\mathfrak{s}$-torsion pairs in extriangulated categories with negative first extensions

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Abstract: As a generalization of a consistent pair of t-structures on triangulated categories, we introduced the notion of a consistent pair of $\mathfrak{s}$-torsion pairs in the extriangulated setup. Let $(\mathcal{T}_i, \mathcal{F}_i)$ be an $\mathfrak{s}$-torsion pair in an extriangulated category with a negative first extension for any $i = 1, 2$. By using the consistent pair, we gave a criterion for $(\mathcal{T}_1 * \mathcal{T}_2, \mathcal{F}_1 \cap \mathcal{F}_2)$ to be an $\mathfrak{s}$-torsion pair. Our results were then applied to the torsion theory induced by τ-rigid modules.

Keywords: extriangulated categories; $\mathfrak{s}$-torsion pairs; consistent pairs; τ-rigid modules
Mathematics Subject Classification: 18G25, 18G80

1. Introduction

Let $\mathcal{K}$ be an abelian category or a triangulated category and $\mathcal{X}, \mathcal{Y}$ be two subcategories. We denote by $\mathcal{X} * \mathcal{Y}$ the subcategory which consists of objects Z in $\mathcal{K}$ such that there is a short exact sequence

$$0 \to X \to Z \to Y \to 0$$

with $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$ or there is a triangle

$$X \to Z \to Y \to \Sigma X$$

with $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$, where Σ is the shift functor. It is called the extension subcategory of $\mathcal{Y}$ by $\mathcal{X}$. This is a classic research object and has been used extensively in the representation theory of algebra. Gentle and Todorov [6] proved that in an abelian category with enough projective objects, the extension subcategory of two covariantly finite subcategories is covariantly finite. Chen [5] proved a triangulated version of Gentle-Todorov's result. Jorgensen and Kato [10] gave some necessary and sufficient conditions for $\mathcal{X} * \mathcal{Y}$ to be triangulated for a pair of triangulated subcategories $\mathcal{X}$ and $\mathcal{Y}$. Yoshizawa considered the extension subcategory when the two given subcategories are Serre in [14].

In order to discuss cotorsion pairs in a more general case, Nakaoka and Palu [12] introduced the notion of extriangulated categories. In particular, exact categories and extension-closed subcategories of triangulated categories are typical examples of extriangulated categories. Hence, many results told on exact categories and triangulated categories can be unified in the same framework [7, 11, 13, 15]. However, it is worth mentioning that there exist numerous examples of extriangulated categories that are neither triangulated nor exact categories (see [8, 12, 15]). Recently, Adachi, Enomoto and Tsukamoto [1] introduced the notion of extriangulated categories with negative first extensions (that is, an additive bifunctors $\mathbb{E}^{-1}$ satisfying certain conditions) and $\mathfrak{s}$-torsion pairs as a general framework for the studies of t-structures on triangulated categories and torsion pairs in abelian categories. Let $\mathcal{K}$ be an extriangulated category with a negative first extension. According to [1], a pair $(\mathcal{T}, \mathcal{F})$ of subcategories in $\mathcal{K}$ is called an *$\mathfrak{s}$-torsion pair* if it is a torsion pair in the usual sense and if $\mathbb{E}^{-1}(\mathcal{T}, \mathcal{F}) = 0$ holds. In this case, we also call $\mathcal{T}$ (respectively, $\mathcal{F}$) a *torsion class* (respectively, *torsion-free class*), and they are mutually determined. Exact categories and triangulated categories naturally admit negative first extension structures, then torsion pairs in exact categories and t-structures on triangulated categories are exactly $\mathfrak{s}$-torsion pairs [1]. Let $s_i = (\mathcal{T}_i, \mathcal{F}_i)$ be an $\mathfrak{s}$-torsion pair in $\mathcal{K}$ for any $i = 1, 2$. We have the following natural question: Is the extension subcategory $\mathcal{T}_1 * \mathcal{T}_2$ of $\mathcal{T}_2$ by $\mathcal{T}_1$ a torsion class? Bondal provided some sufficient condition for the intersection of two t-structures on a triangulated category to be a t-structure in terms of consistent pairs [4]. In this paper, we extend the notion of the consistent pair to the extriangulated setup, then give a necessary and sufficient condition for $\mathcal{T}_1 * \mathcal{T}_2$ to be a torsion class. More specifically, we show that $(\mathcal{T}_1 * \mathcal{T}_2, \mathcal{F}_1 \cap \mathcal{F}_2)$ is an $\mathfrak{s}$-torsion pair if and only if (s_1, s_2) is an upper consistent pair, which generalizes Bondal's result [4, Proposition 6]. We also give some conditions under which the operations of $*$ and intersection satisfy the distributive laws. For a τ-rigid module $M = M_1 \oplus M_2$ over a finite-dimensional algebra, we denote by Fac$M$ the subcategory of the module category which consists of all factor modules of finite direct sums of copies of M. As an application of the main theorem, we show that Fac$M$ is equal to the extension subcategory of FacM_2 by FacM_1 (see Proposition 4.4 for details).

We include some notations here. Throughout this paper, we assume that every category is skeletally small that is, the isomorphism classes of objects form a set, and $\mathcal{K}$ denotes an additive category. The symbol $\mathcal{K}^{\mathrm{op}}$ denotes the opposite category of $\mathcal{K}$. When we say that $\mathcal{D}$ is a subcategory of $\mathcal{K}$, we always mean that $\mathcal{D}$ is a full subcategory which is closed under isomorphisms. For a collection $\mathcal{X}$ of objects in $\mathcal{K}$, we define $\mathcal{X}^{\perp} := \{C \in \mathcal{K} | \mathrm{Hom}_{\mathcal{K}}(\mathcal{X}, C) = 0\}$ and ${}^{\perp}\mathcal{X} := \{C \in \mathcal{K} | \mathrm{Hom}_{\mathcal{K}}(C, \mathcal{X}) = 0\}$.

2. Preliminaries

We briefly recall some definitions and some needed properties of extriangulated categories from [12] and $\mathfrak{s}$-torsion pairs from [1].

Let $\mathcal{K}$ be an additive category equipped with an additive bifunctor

$$\mathbb{E} : \mathcal{K}^{\mathrm{op}} \times \mathcal{K} \to \mathrm{Ab},$$

where Ab is the category of abelian groups. For any objects $A, C \in \mathcal{K}$, an element $\delta \in \mathbb{E}(C, A)$ is called an *$\mathbb{E}$-extension*. Let $\mathfrak{s}$ be a correspondence which associates an equivalence class

$$\mathfrak{s}(\delta) = [A \xrightarrow{f} B \xrightarrow{g} C]$$

to any $\mathbb{E}$-extension $\delta \in \mathbb{E}(C, A)$. Such $\mathfrak{s}$ is called a *realization* of $\mathbb{E}$ if it makes the diagrams in [12, Definition 2.9] commutative. A triplet $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ is called an *extriangulated category* if it satisfies the following conditions.

- (ET1) $\mathbb{E} : \mathcal{K}^{\mathrm{op}} \times \mathcal{K} \to \mathrm{Ab}$ is an additive bifunctor.
- (ET2) $\mathfrak{s}$ is an additive realization of $\mathbb{E}$.
- (ET3) Let $\delta \in \mathbb{E}(C, A)$ and $\delta' \in \mathbb{E}(C', A')$ be any pair of E-extensions, realized as

$$\mathfrak{s}(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C],\ \mathfrak{s}(\delta') = [A' \xrightarrow{x'} B' \xrightarrow{y'} C'].$$

For any commutative square

$$\begin{array}{ccccc} A & \xrightarrow{x} & B & \xrightarrow{y} & C \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} & & \\ A' & \xrightarrow{x'} & B' & \xrightarrow{y'} & C' \end{array}$$

in $\mathcal{K}$, there exists a morphism $(a, c) : \delta \to \delta'$ satisfying $cy = y'b$.
- (ET3)$^{\mathrm{op}}$ Dual of (ET3).
- (ET4) Let $\delta \in \mathbb{E}(C, A)$ and $\delta' \in \mathbb{E}(F, B)$ be $\mathbb{E}$-extensions realized by

$$A \xrightarrow{x} B \xrightarrow{y} C \text{ and } B \xrightarrow{u} D \xrightarrow{v} F$$

respectively, then there exists an object $E \in \mathcal{C}$ and a commutative diagram

$$\begin{array}{ccccc} A & \xrightarrow{x} & B & \xrightarrow{y} & C \\ \| & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle s} \\ A & \xrightarrow{z} & D & \xrightarrow{w} & E \\ & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle t} \\ & & F & = & F \end{array}$$

where all rows and columns are $\mathbb{E}$-extensions in $\mathcal{K}$.
- (ET4)$^{\mathrm{op}}$ Dual of (ET4).

Definition 2.1. ([12]) Let $\mathcal{K}$ be an extriangulated category.

(1) A sequence $A \xrightarrow{f} B \xrightarrow{g} C$ in $\mathcal{K}$ is called a conflation if it realizes some $\mathbb{E}$-extension $\delta \in \mathbb{E}(C, A)$. The pair $(A \xrightarrow{f} B \xrightarrow{g} C, \delta)$ is called an $\mathbb{E}$-*triangle* (or $\mathfrak{s}$-*conflation*), and it is written in the following way:

$$A \xrightarrow{f} B \xrightarrow{g} C \overset{\delta}{\dashrightarrow}.$$

(2) Let $A \overset{x}{\to} B \overset{y}{\to} C \overset{\delta}{\dashrightarrow}$ and $A' \overset{x'}{\to} B' \overset{y'}{\to} C' \overset{\delta'}{\dashrightarrow}$ be any pair of $\mathbb{E}$-triangles. If a triplet (a, b, c) realizes $(a, c) : \delta \to \delta'$, then we write it as

$$\begin{array}{ccccccc} A & \xrightarrow{x} & B & \xrightarrow{y} & C & \overset{\delta}{\dashrightarrow} & \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle c} & & \\ A' & \xrightarrow{x'} & B' & \xrightarrow{y'} & C' & \overset{\delta'}{\dashrightarrow} & \end{array}$$

and call (a, b, c) a *morphism* of $\mathbb{E}$-triangles.

(3) Let $\mathcal{X}$ and $\mathcal{Y}$ be subcategories of $\mathcal{K}$ and let $\mathcal{X} * \mathcal{Y}$ denote the subcategory of $\mathcal{K}$ consisting of $M \in \mathcal{K}$ which admits an $\mathbb{E}$-triangle $X \to M \to Y \dashrightarrow$ in $\mathcal{K}$ with $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$. The subcategory $\mathcal{X}$ is said to be *extension-closed* (or equivalently, *closed under extensions*) if $\mathcal{X} * \mathcal{X} \subseteq \mathcal{X}$. Note that the operation $*$ on subcategories is associative by (ET4) and (ET4)$^{\mathrm{op}}$.

Exact categories and extension-closed subcategories of triangulated categories are extriangulated categories. The extension-closed subcategories of an extriangulated category are again extriangulated categories (see [12, Remark 2.18]). The following proposition gives a criterion for an extension subcategory of two given extension-closed subcategories to be extension-closed.

Proposition 2.2. *Let $\mathcal{X}$ and $\mathcal{Y}$ be extension-closed subcategories of an extriangulated category, then the following conditions are equivalent.*

(1) *$\mathcal{X} * \mathcal{Y}$ is an extension-closed subcategory.*

(2) *$\mathcal{Y} * \mathcal{X} \subseteq \mathcal{X} * \mathcal{Y}$.*

Proof. (1) implies (2). For any $C \in \mathcal{Y} * \mathcal{X}$, there exists an $\mathbb{E}$-triangle

$$Y \to C \to X \dashrightarrow$$

such that $Y \in \mathcal{Y}$ and $X \in \mathcal{X}$. Note that Y is in $\mathcal{Y} \subseteq \mathcal{X} * \mathcal{Y}$ and X is in $\mathcal{X} \subseteq \mathcal{X} * \mathcal{Y}$. Since $\mathcal{X} * \mathcal{Y}$ is closed under extensions by the assumption (1), then the $\mathbb{E}$-triangle above implies that $C \in \mathcal{X} * \mathcal{Y}$.

(2) implies (1). Since the operation $*$ is associative and $\mathcal{Y} * \mathcal{X} \subseteq \mathcal{X} * \mathcal{Y}$, we have the following formulas

$$(\mathcal{X} * \mathcal{Y}) * (\mathcal{X} * \mathcal{Y}) = \mathcal{X} * (\mathcal{Y} * \mathcal{X}) * \mathcal{Y} \subseteq \mathcal{X} * (\mathcal{X} * \mathcal{Y}) * \mathcal{Y} = (\mathcal{X} * \mathcal{X}) * (\mathcal{Y} * \mathcal{Y}) \subseteq \mathcal{X} * \mathcal{Y}.$$

Hence $\mathcal{X} * \mathcal{Y}$ is closed under extensions. □

The following is a basic property of extriangulated categories.

Lemma 2.3. *([12, Proposition 3.3]) Let $\mathcal{K}$ be an extriangulated category. For any $\mathbb{E}$-triangle $A \to B \to C \dashrightarrow$, we have the following exact sequences.*

$$\mathrm{Hom}_{\mathcal{K}}(C, -) \to \mathrm{Hom}_{\mathcal{K}}(B, -) \to \mathrm{Hom}_{\mathcal{K}}(A, -) \to \mathbb{E}(C, -) \to \mathbb{E}(B, -) \to \mathbb{E}(A, -);$$
$$\mathrm{Hom}_{\mathcal{K}}(-, A) \to \mathrm{Hom}_{\mathcal{K}}(-, B) \to \mathrm{Hom}_{\mathcal{K}}(-, C) \to \mathbb{E}(-, A) \to \mathbb{E}(-, B) \to \mathbb{E}(-, C).$$

Definition 2.4. ([1]) Let $\mathcal{K}$ be an extriangulated category. A *negative first extension structure* on $\mathcal{K}$ consists of the following data.

- (NE1) $\mathbb{E}^{-1} : \mathcal{K}^{\mathrm{op}} \times \mathcal{K} \to \mathrm{Ab}$ is an additive bifunctor.
- (NE2) For each $\delta \in \mathbb{E}(C, A)$, there exist two natural transformations

$$\delta_{\sharp}^{-1} : \mathbb{E}^{-1}(-, C) \to \mathrm{Hom}_{\mathcal{K}}(-, A), \text{ and } \delta_{-1}^{\sharp} : \mathbb{E}^{-1}(A, -) \to \mathrm{Hom}_{\mathcal{K}}(C, -)$$

such that for each $\mathbb{E}$-triangle $A \xrightarrow{f} B \xrightarrow{g} C \dashrightarrow$ and each $W \in \mathcal{K}$, two sequences

$$\mathbb{E}^{-1}(W, A) \xrightarrow{\mathbb{E}^{-1}(W,f)} \mathbb{E}^{-1}(W, B) \xrightarrow{\mathbb{E}^{-1}(W,g)} \mathbb{E}^{-1}(W, C) \xrightarrow{(\delta_{\sharp}^{-1})_W} \mathrm{Hom}_{\mathcal{K}}(W, A) \xrightarrow{\mathrm{Hom}_{\mathcal{K}}(W,f)} \mathrm{Hom}_{\mathcal{K}}(W, B),$$

$$\mathbb{E}^{-1}(C, W) \xrightarrow{\mathbb{E}^{-1}(g,W)} \mathbb{E}^{-1}(B, W) \xrightarrow{\mathbb{E}^{-1}(f,W)} \mathbb{E}^{-1}(A, W) \xrightarrow{(\delta_{-1}^{\sharp})_W} \mathrm{Hom}_{\mathcal{K}}(C, W) \xrightarrow{\mathrm{Hom}_{\mathcal{K}}(g,W)} \mathrm{Hom}_{\mathcal{K}}(B, W),$$

are exact.

Thus, $(\mathcal{K}, \mathbb{E}, \mathfrak{s}, \mathbb{E}^{-1})$ is called an *extriangulated category with a negative first extension*.

In what follows, we assume that $\mathcal{K}$ is an extriangulated category with a negative first extension $\mathbb{E}^{-1}$.

Definition 2.5. ([1]) A pair $(\mathcal{T}, \mathcal{F})$ of subcategories of $\mathcal{K}$ is called an $\mathfrak{s}$-*torsion pair* in $\mathcal{K}$ if it satisfies the following three conditions.

- (STP1) $\mathcal{K} = \mathcal{T} * \mathcal{F}$.
- (STP2) $\mathrm{Hom}_{\mathcal{K}}(\mathcal{T}, \mathcal{F}) = 0$.
- (STP3) $\mathbb{E}^{-1}(\mathcal{T}, \mathcal{F}) = 0$.

In this case, $\mathcal{T}$ (respectively, $\mathcal{F}$) is called a *torsion class* (respectively, *torsion-free class*) in $\mathcal{K}$.

Let $\mathcal{D}$ be a triangulated category with the shift functor Σ. By regarding triangulated category $\mathcal{D}$ as the extriangulated category with the negative first extension $\mathbb{E}^{-1}(C, A) = \mathrm{Hom}_{\mathcal{D}}(C, \Sigma^{-1}A)$ for all $A, C \in \mathcal{D}$, then t-structures on $\mathcal{D}$ are exactly $\mathfrak{s}$-torsion pairs in $\mathcal{D}$. By regarding an exact category $\mathcal{E}$ as the extriangulated category with the negative first extension $\mathbb{E}^{-1} = 0$, then torsion pairs in the exact category $\mathcal{E}$ are exactly $\mathfrak{s}$-torsion pairs in $\mathcal{E}$ [1].

The following proposition shows that torsion class and torsion-free class in $\mathcal{K}$ are mutually determined.

Lemma 2.6. *([1, Proposition 3.2]) Let $(\mathcal{T}, \mathcal{F})$ be an $\mathfrak{s}$-torsion pair in $\mathcal{K}$, then the following statements hold.*
(1) $\mathcal{T}^{\perp} = \mathcal{F}$.
(2) ${}^{\perp}\mathcal{F} = \mathcal{T}$.
In particular, $\mathcal{T}$ and $\mathcal{F}$ are extension-closed subcategories which are closed under direct summands.

Lemma 2.7. *([1, Proposition 3.7]) Let $(\mathcal{T}, \mathcal{F})$ be an $\mathfrak{s}$-torsion pair in $\mathcal{K}$. For each $C \in \mathcal{K}$, there uniquely exists an $\mathbb{E}$-triangle*

$$T_C \to C \to F^C \dashrightarrow$$

such that $T_C \in \mathcal{T}$ and $F^C \in \mathcal{F}$ (up to isomorphism of $\mathbb{E}$-triangles).

Let $(\mathcal{T}, \mathcal{F})$ be an $\mathfrak{s}$-torsion pair in $\mathcal{K}$. According to Lemma 2.7, $(\mathcal{T}, \mathcal{F})$ gives functors $\mathrm{t} : \mathcal{K} \to \mathcal{T}$ and $\mathrm{f} : \mathcal{K} \to \mathcal{F}$ and natural transformations $\mathrm{t} \to \mathrm{id}_C \to \mathrm{f}$ such that there uniquely exists an $\mathbb{E}$-triangle

$$\mathrm{t}C \to C \to \mathrm{f}C \dashrightarrow \tag{2.1}$$

for each $C \in \mathcal{K}$. We call the functors (t, f) the *torsion functors* of the $\mathfrak{s}$-torsion pair $(\mathcal{T}, \mathcal{F})$. An $\mathbb{E}$-triangle given as (2.1) is called a *canonical* $\mathbb{E}$-*triangle* with respect to the torsion pair $(\mathcal{T}, \mathcal{F})$.

3. Main results

The concept of the consistent pair of t-structures on a triangulated category has been introduced by Bondal in [4]. We extend this notion to an extriangulated category with a negative first extension.

Definition 3.1. Let $\mathcal{K}$ be an extriangulated category with a negative first extension. Let $s_i := (\mathcal{T}_i, \mathcal{F}_i)$ be an $\mathfrak{s}$-torsion pair in $\mathcal{K}$ with the torsion functors $(\mathrm{t}_i, \mathrm{f}_i)$ for any $i = 1, 2$. If $\mathcal{F}_1$ is stable under the functor f_2 that is, $\mathrm{f}_2\mathcal{F}_1 \subset \mathcal{F}_1$, then the pair (s_1, s_2) is called an *upper consistent* pair of $\mathfrak{s}$-torsion pairs. If $\mathcal{T}_2$ is stable under the functor t_1 that is, $\mathrm{t}_1\mathcal{T}_2 \subset \mathcal{T}_2$, then the pair (s_1, s_2) is called a *lower consistent* pair of $\mathfrak{s}$-torsion pairs. A pair is said to be *consistent* if it is upper or lower consistent.

There exists inconsistent pairs of $\mathfrak{s}$-torsion pairs [4, Subsection 1.2]. The following lemma provides a necessary condition.

Lemma 3.2. *Let $s_i := (\mathcal{T}_i, \mathcal{F}_i)$ be an $\mathfrak{s}$-torsion pairs in $\mathcal{K}$ for any $i = 1, 2$.*
(1) *Suppose that $\mathbb{E}(\mathcal{T}_1, \mathcal{T}_2) = 0$, then (s_1, s_2) is upper consistent.*
(2) *Suppose that $\mathbb{E}(\mathcal{F}_1, \mathcal{F}_2) = 0$, then (s_1, s_2) is lower consistent.*

Proof. We only prove (1); the proof of (2) is similar. Denote by $(\mathrm{t}_i, \mathrm{f}_i)$ the torsion functors of $(\mathcal{T}_i, \mathcal{F}_i)$, $i = 1, 2$. For any $C \in \mathcal{F}_1$, take the canonical $\mathbb{E}$-triangle

$$\mathrm{t}_2C \to C \to \mathrm{f}_2C \dashrightarrow$$

with respect to the $\mathfrak{s}$-torsion pair $(\mathcal{T}_2, \mathcal{F}_2)$. Applying $\mathrm{Hom}_{\mathcal{K}}(\mathcal{T}_1, -)$ to the preceding $\mathbb{E}$-triangle deduces an exact sequence

$$0 = \mathrm{Hom}_{\mathcal{K}}(\mathcal{T}_1, C) \to \mathrm{Hom}_{\mathcal{K}}(\mathcal{T}_1, \mathrm{f}_2C) \to \mathbb{E}(\mathcal{T}_1, \mathrm{t}_2C).$$

By assumption, $\mathbb{E}(\mathcal{T}_1, \mathrm{t}_2C) = 0$, so $\mathrm{Hom}_{\mathcal{K}}(\mathcal{T}_1, \mathrm{f}_2C) = 0$. It follows from Lemma 2.6 that $\mathrm{f}_2C \in \mathcal{F}_1$. Therefore, (s_1, s_2) is upper consistent. □

Lemma 3.3. *Let $A \xrightarrow{f} B \xrightarrow{g} C \dashrightarrow$ be an $\mathbb{E}$-triangle, then the following assertions hold.*
(1) *If $f = 0$, then g is a split monomorphism.*
(2) *If $g = 0$, then f is a split epimorphism.*

Proof. We only prove (1); the proof of (2) is similar. First there exists an $\mathbb{E}$-triangle $0 \to B \xrightarrow{\mathrm{id}_B} B \dashrightarrow$, then we consider the following diagram

$$\begin{array}{ccccc} A & \xrightarrow{0} & B & \xrightarrow{g} & C \dashrightarrow \\ \downarrow & & \| & & \\ 0 & \longrightarrow & B & \xrightarrow{\mathrm{id}_B} & B \dashrightarrow \end{array}$$

where the left square is commutative. By (ET3) there exists a morphism $h : C \to B$ such that the following diagram

$$\begin{array}{ccccc} A & \xrightarrow{0} & B & \xrightarrow{g} & C \dashrightarrow \\ \downarrow & & \| & & \downarrow h \\ 0 & \longrightarrow & B & \xrightarrow{\mathrm{id}_B} & B \dashrightarrow \end{array}$$

commutates. Hence $hg = \mathrm{id}_B$. □

Now we can show the main theorem of this paper.

Theorem 3.4. *Let $\mathcal{K}$ be an extriangulated category with a negative first extension. Let $s_i := (\mathcal{T}_i, \mathcal{F}_i)$ be an $\mathfrak{s}$-torsion pair in $\mathcal{K}$ for any $i = 1, 2$, then*

(1) $(\mathcal{T}_1 * \mathcal{T}_2, \mathcal{F}_1 \cap \mathcal{F}_2)$ is an $\mathfrak{s}$-torsion pair in $\mathcal{K}$ if and only if (s_1, s_2) is upper consistent.

(2) $(\mathcal{T}_1 \cap \mathcal{T}_2, \mathcal{F}_1 * \mathcal{F}_2)$ is an $\mathfrak{s}$-torsion pair in $\mathcal{K}$ if and only if (s_1, s_2) is lower consistent.

Proof. We only prove (1); the proof of (2) is similar. Denote by $(\mathrm{t}_i, \mathrm{f}_i)$ the torsion functors of $(\mathcal{T}_i, \mathcal{F}_i)$, $i = 1, 2$. We first show the necessity. Assume that $(\mathcal{T}_1 * \mathcal{T}_2, \mathcal{F}_1 \cap \mathcal{F}_2)$ is an $\mathfrak{s}$-torsion pair in $\mathcal{K}$, then for any $C \in \mathcal{F}_1$, there exists a canonical $\mathbb{E}$-triangle

$$T \to C \to F \dashrightarrow \tag{3.1}$$

with respect to $(\mathcal{T}_1 * \mathcal{T}_2, \mathcal{F}_1 \cap \mathcal{F}_2)$ such that $T \in \mathcal{T}_1 * \mathcal{T}_2$ and $F \in \mathcal{F}_1 \cap \mathcal{F}_2 \subset \mathcal{F}_2$. It suffices to prove that T is in $\mathcal{T}_2$. Indeed, if it is done, then the $\mathbb{E}$-triangle (3.1) is also a canonical $\mathbb{E}$-triangle with respect to the $\mathfrak{s}$-torsion pair $(\mathcal{T}_2, \mathcal{F}_2)$ by Lemma 2.7. Therefore $\mathrm{f}_2 C = F \in \mathcal{F}_1 \cap \mathcal{F}_2 \subset \mathcal{F}_1$, so (s_1, s_2) is upper consistent. Since $T \in \mathcal{T}_1 * \mathcal{T}_2$, there exists an $\mathbb{E}$-triangle

$$T_1 \xrightarrow{a} T \xrightarrow{b} T_2 \dashrightarrow$$

such that $T_1 \in \mathcal{T}_1$ and $T_2 \in \mathcal{T}_2$. Applying $\mathrm{Hom}_{\mathcal{K}}(T_1, -)$ to the $\mathbb{E}$-triangle (3.1), we have an exact sequence

$$\mathbb{E}^{-1}(T_1, F) \to \mathrm{Hom}_{\mathcal{K}}(T_1, T) \to \mathrm{Hom}_{\mathcal{K}}(T_1, C).$$

Since $(\mathcal{T}_1 * \mathcal{T}_2, \mathcal{F}_1 \cap \mathcal{F}_2)$ is an $\mathfrak{s}$-torsion pair and $T_1 \in \mathcal{T}_1 \subset \mathcal{T}_1 * \mathcal{T}_2$, we have $\mathbb{E}^{-1}(T_1, F) = 0$. Since $(\mathcal{T}_1, \mathcal{F}_1)$ is an $\mathfrak{s}$-torsion pair, we have $\mathrm{Hom}_{\mathcal{K}}(T_1, C) = 0$, so $a \in \mathrm{Hom}_{\mathcal{K}}(T_1, T) = 0$. By Lemma 3.3, b is a split monomorphism and T is a direct summand of T_2. It followes from Lemma 2.6 that $T \in \mathcal{T}_2$. Hence, (s_1, s_2) is an upper consistent pair.

Let (s_1, s_2) be an upper consistent pair. To prove the sufficiency, we verify the conditions (STP1), (STP2) and (STP3) of the definition of $\mathfrak{s}$-torsion pair individually.

(STP1) Let $C \in \mathcal{K}$. There exists a canonical $\mathbb{E}$-triangle

$$\mathrm{t}_1 C \to C \to \mathrm{f}_1 C \dashrightarrow \tag{3.2}$$

with respect to the $\mathfrak{s}$-torsion pair $(\mathcal{T}_1, \mathcal{F}_1)$. Similarly, since $(\mathcal{T}_2, \mathcal{F}_2)$ is an $\mathfrak{s}$-torsion pair in $\mathcal{K}$, there exists a canonical $\mathbb{E}$-triangle

$$\mathrm{t}_2 \mathrm{f}_1 C \to \mathrm{f}_1 C \to \mathrm{f}_2 \mathrm{f}_1 C \dashrightarrow . \tag{3.3}$$

On the basis of the assumption that (s_1, s_2) is upper consistent, we have $\mathrm{f}_2 \mathrm{f}_1 C \in \mathcal{F}_1 \cap \mathcal{F}_2$. Applying $(\mathrm{ET4})^{\mathrm{op}}$ to (3.2) and (3.3) induces a commutative diagram

$$\begin{array}{ccccc}
\mathrm{t}_1 C & \longrightarrow & A & \longrightarrow & \mathrm{t}_2 \mathrm{f}_1 C \dashrightarrow \\
\| & & \downarrow & & \downarrow \\
\mathrm{t}_1 C & \longrightarrow & C & \longrightarrow & \mathrm{f}_1 C \dashrightarrow \\
 & & \downarrow & & \downarrow \\
 & & \mathrm{f}_2 \mathrm{f}_1 C & = & \mathrm{f}_2 \mathrm{f}_1 C \\
 & & \downarrow & & \downarrow \\
\end{array}$$

where all rows and columns are $\mathbb{E}$-triangles. In the top horizontal $\mathbb{E}$-triangle, since $\mathrm{t}_1C \in \mathcal{T}_1$ and $\mathrm{t}_2\mathrm{f}_1C \in \mathcal{T}_2$, then $A \in \mathcal{T}_1 * \mathcal{T}_2$. Thus we obtain $C \in (\mathcal{T}_1 * \mathcal{T}_2) * (\mathcal{F}_1 \cap \mathcal{F}_2)$ by the left vertical $\mathbb{E}$-triangle $A \to C \to \mathrm{f}_2\mathrm{f}_1C \dashrightarrow$, meaning that $\mathcal{K} = (\mathcal{T}_1 * \mathcal{T}_2) * (\mathcal{F}_1 \cap \mathcal{F}_2)$.

(STP2) Since $(\mathcal{T}_1, \mathcal{F}_1)$ and $(\mathcal{T}_2, \mathcal{F}_2)$ are $\mathfrak{s}$-torsion pairs, we obtain that $\mathcal{C}(\mathcal{T}_i, \mathcal{F}_1 \cap \mathcal{F}_2) = 0$, for any $i = 1, 2$. This implies that $\mathcal{C}(\mathcal{T}_1 * \mathcal{T}_2, \mathcal{F}_1 \cap \mathcal{F}_2) = 0$ by Lemma 2.3.

(STP3) Since $(\mathcal{T}_1, \mathcal{F}_1)$ and $(\mathcal{T}_2, \mathcal{F}_2)$ are $\mathfrak{s}$-torsion pairs, we obtain that $\mathbb{E}^{-1}(\mathcal{T}_i, \mathcal{F}_1 \cap \mathcal{F}_2) = 0$, for any $i = 1, 2$. This implies that $\mathbb{E}^{-1}(\mathcal{T}_1 * \mathcal{T}_2, \mathcal{F}_1 \cap \mathcal{F}_2) = 0$ by (NE2). This completes the proof of the sufficiency. $\square$

Remark 3.5. (1) Let $s_1 := (\mathcal{T}, \mathcal{F})$ be an $\mathfrak{s}$-torsion pair in $\mathcal{K}$ and $s_2 := (\mathcal{K}, 0)$ be a trivial $\mathfrak{s}$-torsion pair. Since $(\mathcal{T} * \mathcal{K}, \mathcal{F} \cap 0) = (\mathcal{K}, 0)$, $(\mathcal{T} \cap \mathcal{K}, \mathcal{F} * 0) = (\mathcal{T}, \mathcal{F})$, by Theorem 3.4, (s_1, s_2) is simultaneously lower and upper consistent.

(2) Let $\mathcal{K}$ be a triangulated category, then Theorem 3.4 recovers [4, Proposition 6].

When $\mathcal{K}$ is a triangulated category, the following corollary is an analogue of [9, Proposition 2.4] which plays a crucial role in the study of n-cluster tilting subcategories.

Corollary 3.6. *Let $(\mathcal{T}_i, \mathcal{F}_i)$ be an $\mathfrak{s}$-torsion pair in $\mathcal{K}$ for any $i \geq 1$.*

(1) *Suppose that $\mathbb{E}(\mathcal{T}_i, \mathcal{T}_j) = 0$, for any $i < j$. Put*

$$\mathcal{X}_n := \mathcal{T}_1 * \mathcal{T}_2 * \cdots * \mathcal{T}_n, \quad \mathcal{Y}_n := \bigcap_{i=1}^{n} \mathcal{F}_i,$$

then $(\mathcal{X}_n, \mathcal{Y}_n)$ is an $\mathfrak{s}$-torsion pair in $\mathcal{K}$.

(2) *Suppose that $\mathbb{E}(\mathcal{F}_i, \mathcal{F}_j) = 0$ for any $i < j$. Put*

$$\mathcal{X}_n := \bigcap_{i=1}^{n} \mathcal{T}_i, \quad \mathcal{Y}_n := \mathcal{F}_1 * \mathcal{F}_2 * \cdots * \mathcal{F}_n,$$

then $(\mathcal{X}_n, \mathcal{Y}_n)$ is an $\mathfrak{s}$-torsion pair in $\mathcal{K}$.

Proof. We only prove (1); the proof of (2) is similar. We show by induction on n. The case $n = 1$ is obvious. Assume that the assertion is true for $n-1$ that is, $(\mathcal{X}_{n-1}, \mathcal{Y}_{n-1})$ is an $\mathfrak{s}$-torsion pair in $\mathcal{K}$. Since $\mathbb{E}(\mathcal{T}_i, \mathcal{T}_n) = 0$ for any $i < n$, by Lemma 2.3 we obtain that

$$\mathbb{E}(\mathcal{X}_{n-1}, \mathcal{T}_n) = \mathbb{E}(\mathcal{T}_1 * \mathcal{T}_2 * \cdots * \mathcal{T}_{n-1}, \mathcal{T}_n) = 0.$$

It follows from Lemma 3.2 that $((\mathcal{X}_{n-1}, \mathcal{Y}_{n-1}), (\mathcal{T}_n, \mathcal{F}_n))$ is upper consistent. By Theorem 3.4, $(\mathcal{X}_{n-1} * \mathcal{T}_n, \mathcal{Y}_{n-1} \cap \mathcal{F}_n) = (\mathcal{X}_n, \mathcal{Y}_n)$ is an $\mathfrak{s}$-torsion pair in $\mathcal{K}$. $\square$

The following lemma is useful in constructing consistent pairs.

Lemma 3.7. *Let $(\mathcal{T}, \mathcal{F})$ be an $\mathfrak{s}$-torsion pair in $\mathcal{K}$. Let $A \xrightarrow{f} B \xrightarrow{g} C \dashrightarrow$ be an $\mathbb{E}$-triangle, then the following assertions hold.*

(1) *If $B, C \in \mathcal{F}$, then $A \in \mathcal{F}$.*

(2) *If $A, B \in \mathcal{T}$, then $C \in \mathcal{T}$.*

Proof. We only prove (1); the proof of (2) is similar. Applying $\mathrm{Hom}_{\mathcal{K}}(\mathcal{T}, -)$ to the given $\mathbb{E}$-triangle, we have an exact sequence

$$\mathbb{E}^{-1}(\mathcal{T}, C) \to \mathrm{Hom}_{\mathcal{K}}(\mathcal{T}, A) \to \mathrm{Hom}_{\mathcal{K}}(\mathcal{T}, B).$$

Since the left hand side and right hand side vanish, we obtain $\mathrm{Hom}_{\mathcal{K}}(\mathcal{T}, A) = 0$. It follows from Lemma 2.6 that $A \in \mathcal{F}$. □

Next we will construct some new consistent pairs by using the known ones, which enables us to determine the new $\mathfrak{s}$-torsion pairs. Meanwhile, the conditions under which the operations of $*$ and intersection satisfy the distributive laws are given.

Corollary 3.8. *Let $s_i := (\mathcal{T}_i, \mathcal{F}_i)$ be an $\mathfrak{s}$-torsion pair in $\mathcal{K}$ for any $i = 1, 2, 3$. Suppose that (s_1, s_2), (s_1, s_3) are upper consistent and (s_2, s_3) is lower consistent, then the following assertions hold.*

(1) $((\mathcal{T}_1, \mathcal{F}_1), (\mathcal{T}_2 \cap \mathcal{T}_3, \mathcal{F}_2 * \mathcal{F}_3))$ *is upper consistent.*

(2) $((\mathcal{T}_1 * \mathcal{T}_2, \mathcal{F}_1 \cap \mathcal{F}_2), (\mathcal{T}_1 * \mathcal{T}_3, \mathcal{F}_1 \cap \mathcal{F}_3))$ *is lower consistent.*

(3) $\mathcal{T}_1 * (\mathcal{T}_2 \cap \mathcal{T}_3) = (\mathcal{T}_1 * \mathcal{T}_2) \cap (\mathcal{T}_1 * \mathcal{T}_3)$, *and* $\mathcal{F}_1 \cap (\mathcal{F}_2 * \mathcal{F}_3) = (\mathcal{F}_1 \cap \mathcal{F}_2) * (\mathcal{F}_1 \cap \mathcal{F}_3)$.

Proof. Denote by $(\mathrm{t}_i, \mathrm{f}_i)$ the torsion functors of $(\mathcal{T}_i, \mathcal{F}_i)$ for any $i = 1, 2, 3$.

(1) By Theorem 3.4, $(\mathcal{T}_2 \cap \mathcal{T}_3, \mathcal{F}_2 * \mathcal{F}_3)$ is an $\mathfrak{s}$-torsion pair in $\mathcal{K}$. Denote by $(\mathrm{t}_{23}, \mathrm{f}_{23})$ the torsion functors of $(\mathcal{T}_2 \cap \mathcal{T}_3, \mathcal{F}_2 * \mathcal{F}_3)$. We need to prove that $\mathrm{f}_{23}\mathcal{F}_1 \subset \mathcal{F}_1$. Let $C \in \mathcal{F}_1$. Since $(\mathcal{T}_2, \mathcal{F}_2)$ and $(\mathcal{T}_3, \mathcal{F}_3)$ are $\mathfrak{s}$-torsion pairs, we have the following two canonical $\mathbb{E}$-triangles

$$\mathrm{t}_3 C \to C \to \mathrm{f}_3 C \dashrightarrow, \tag{3.4}$$

$$\mathrm{t}_2\mathrm{t}_3 C \to \mathrm{t}_3 C \to \mathrm{f}_2\mathrm{t}_3 C \dashrightarrow . \tag{3.5}$$

Since (s_2, s_3) is lower consistent and $\mathrm{t}_3 C \in \mathcal{T}_3$, we have $\mathrm{t}_2\mathrm{t}_3 C \in \mathcal{T}_2 \cap \mathcal{T}_3$. Applying (ET4) to (3.4) and (3.5) induces a commutative diagram

$$\begin{array}{ccccccc}
\mathrm{t}_2\mathrm{t}_3 C & \longrightarrow & \mathrm{t}_3 C & \longrightarrow & \mathrm{f}_2\mathrm{t}_3 C & \dashrightarrow & \\
\| & & \downarrow & & \downarrow & & \\
\mathrm{t}_2\mathrm{t}_3 C & \longrightarrow & C & \longrightarrow & A & \dashrightarrow & \\
 & & \downarrow & & \downarrow & & \\
 & & \mathrm{f}_3 C & = & \mathrm{f}_3 C & & \\
 & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & &
\end{array}$$

where all rows and columns are $\mathbb{E}$-triangles. Thus there exists an $\mathbb{E}$-triangle

$$\mathrm{t}_2\mathrm{t}_3 C \to C \to A \dashrightarrow \tag{3.6}$$

such that $A \in \mathcal{F}_2 * \mathcal{F}_3$, so the $\mathbb{E}$-triangle (3.6) is a canonical $\mathbb{E}$-triangle with respect to the $\mathfrak{s}$-torsion pair $(\mathcal{T}_2 \cap \mathcal{T}_3, \mathcal{F}_2 * \mathcal{F}_3)$. Since (s_1, s_3) is upper consistent and $C \in \mathcal{F}_1$, we have $\mathrm{f}_3 C \in \mathcal{F}_1$, then it follows from the $\mathbb{E}$-triangle $\mathrm{t}_3 C \to C \to \mathrm{f}_3 C \dashrightarrow$ and Lemma 3.7 that $\mathrm{t}_3 C \in \mathcal{F}_1$. Since (s_1, s_2) is upper consistent, we have $\mathrm{f}_2\mathrm{t}_3 C \in \mathcal{F}_1$. Since the torsion-free class $\mathcal{F}_1$ is closed under extensions, we obtain $A \in \mathcal{F}_1$ by the $\mathbb{E}$-triangle $\mathrm{f}_2\mathrm{t}_3 C \to A \to \mathrm{f}_3 C \dashrightarrow$ and the canonical $\mathbb{E}$-triangle (3.6) implies that $\mathrm{f}_{23} C = A \in \mathcal{F}_1$. Therefore, $((\mathcal{T}_1, \mathcal{F}_1), (\mathcal{T}_2 \cap \mathcal{T}_3, \mathcal{F}_2 * \mathcal{F}_3))$ is upper consistent.

(2) By Theorem 3.4, $(\mathcal{T}_1 * \mathcal{T}_2, \mathcal{F}_1 \cap \mathcal{F}_2)$ and $(\mathcal{T}_1 * \mathcal{T}_3, \mathcal{F}_1 \cap \mathcal{F}_3)$ are $\mathfrak{s}$-torsion pairs in $\mathcal{K}$. Denote by $(\mathrm{t}_{12}, \mathrm{f}_{12})$ the torsion functors of $(\mathcal{T}_1 * \mathcal{T}_2, \mathcal{F}_1 \cap \mathcal{F}_2)$. To complete the proof, we need to show that $\mathrm{t}_{12}(\mathcal{T}_1 * \mathcal{T}_3) \subset \mathcal{T}_1 * \mathcal{T}_3$. Let $C \in \mathcal{T}_1 * \mathcal{T}_3$, then there exists an $\mathbb{E}$-triangle

$$T_1 \to C \to T_3 \dashrightarrow \tag{3.7}$$

such that $T_1 \in \mathcal{T}_1$ and $T_3 \in \mathcal{T}_3$. Since $(\mathcal{T}_1, \mathcal{F}_1)$ and $(\mathcal{T}_3, \mathcal{F}_3)$ are $\mathfrak{s}$-torsion pairs, we have the following two canonical $\mathbb{E}$-triangles

$$\mathrm{t}_1T_3 \to T_3 \to \mathrm{f}_1T_3 \dashrightarrow, \tag{3.8}$$

$$\mathrm{t}_3\mathrm{f}_1T_3 \xrightarrow{a} \mathrm{f}_1T_3 \xrightarrow{b} \mathrm{f}_3\mathrm{f}_1T_3 \dashrightarrow . \tag{3.9}$$

Applying $\mathrm{Hom}_{\mathcal{K}}(-, \mathrm{f}_3\mathrm{f}_1T_3)$ to the $\mathbb{E}$-triangle (3.8), we have an exact sequence

$$\mathbb{E}^{-1}(\mathrm{t}_1T_3, \mathrm{f}_3\mathrm{f}_1T_3) \to \mathrm{Hom}_{\mathcal{K}}(\mathrm{f}_1T_3, \mathrm{f}_3\mathrm{f}_1T_3) \to \mathrm{Hom}_{\mathcal{K}}(T_3, \mathrm{f}_3\mathrm{f}_1T_3).$$

Notice that (s_1, s_3) is upper consistent and $\mathrm{f}_1T_3 \in \mathcal{F}_1$, and we have $\mathrm{f}_3\mathrm{f}_1T_3 \in \mathcal{F}_1 \cap \mathcal{F}_3$. Since $(\mathcal{T}_1, \mathcal{F}_1)$ is an $\mathfrak{s}$-torsion pair and $\mathrm{t}_1T_3 \in \mathcal{T}_1$, $\mathrm{f}_3\mathrm{f}_1T_3 \in \mathcal{F}_1$, we have $\mathbb{E}^{-1}(\mathrm{t}_1T_3, \mathrm{f}_3\mathrm{f}_1T_3) = 0$. Since $(\mathcal{T}_3, \mathcal{F}_3)$ is an $\mathfrak{s}$-torsion pair and $T_3 \in \mathcal{T}_3$, $\mathrm{f}_3\mathrm{f}_1T_3 \in \mathcal{F}_3$, we have $\mathrm{Hom}_{\mathcal{K}}(T_3, \mathrm{f}_3\mathrm{f}_1T_3) = 0$ and $b \in \mathrm{Hom}_{\mathcal{K}}(\mathrm{f}_1T_3, \mathrm{f}_3\mathrm{f}_1T_3) = 0$. It follows from Lemma 3.3 that a is a split epimorphism, so f_1T_3 is a direct summand of $\mathrm{t}_3\mathrm{f}_1T_3$. By Lemma 2.6 , $\mathrm{f}_1T_3 \in \mathcal{T}_3$. Consider the canonical $\mathbb{E}$-triangle

$$\mathrm{t}_2\mathrm{f}_1T_3 \to \mathrm{f}_1T_3 \to \mathrm{f}_2\mathrm{f}_1T_3 \dashrightarrow \tag{3.10}$$

with respect to the $\mathfrak{s}$-torsion pair $(\mathcal{T}_2, \mathcal{F}_2)$. Since (s_2, s_3) is lower consistent and $\mathrm{f}_1T_3 \in \mathcal{T}_3$, we have $\mathrm{t}_2\mathrm{f}_1T_3 \in \mathcal{T}_2 \cap \mathcal{T}_3$. Since (s_1, s_2) is upper consistent and $\mathrm{f}_1T_3 \in \mathcal{F}_1$, we have

$$\mathrm{f}_2\mathrm{f}_1T_3 \in \mathcal{F}_1 \cap \mathcal{F}_2. \tag{3.11}$$

Applying (ET4)$^{\mathrm{op}}$ to (3.8) and (3.10) induces a commutative diagram

$$\begin{array}{ccccc}
\mathrm{t}_1T_3 & \longrightarrow & A & \longrightarrow & \mathrm{t}_2\mathrm{f}_1T_3 \dashrightarrow \\
\| & & \downarrow & & \downarrow \\
\mathrm{t}_1T_3 & \longrightarrow & T_3 & \longrightarrow & \mathrm{f}_1T_3 \dashrightarrow \\
 & & \downarrow & & \downarrow \\
 & & \mathrm{f}_2\mathrm{f}_1T_3 & = & \mathrm{f}_2\mathrm{f}_1T_3 \\
 & & \vdots & & \vdots
\end{array}$$

where all rows and columns are $\mathbb{E}$-triangles. Thus there exists an $\mathbb{E}$-triangle

$$A \to T_3 \to \mathrm{f}_2\mathrm{f}_1T_3 \dashrightarrow \tag{3.12}$$

such that $A \in \mathcal{T}_1 * (\mathcal{T}_2 \cap \mathcal{T}_3)$. Moreover, applying (ET4)$^{\mathrm{op}}$ to (3.7) and (3.12) induces a commutative diagram

$$\begin{array}{ccccc}
T_1 & \longrightarrow & B & \longrightarrow & A \dashrightarrow \\
\| & & \downarrow & & \downarrow \\
T_1 & \longrightarrow & C & \longrightarrow & T_3 \dashrightarrow \\
 & & \downarrow & & \downarrow \\
 & & \mathrm{f}_2\mathrm{f}_1T_3 & = & \mathrm{f}_2\mathrm{f}_1T_3 \\
 & & \downarrow & & \downarrow
\end{array}$$

where all rows and columns are $\mathbb{E}$-triangles, which gives an $\mathbb{E}$-triangle

$$B \to C \to \mathrm{f}_2\mathrm{f}_1T_3 \dashrightarrow . \tag{3.13}$$

Since the torsion class $\mathcal{T}_1$ is closed under extensions, we obtain $B \in \mathcal{T}_1 * (\mathcal{T}_1 * (\mathcal{T}_2 \cap \mathcal{T}_3)) = \mathcal{T}_1 * (\mathcal{T}_2 \cap \mathcal{T}_3)$ by the $\mathbb{E}$-triangle $T_1 \to B \to A \dashrightarrow$. It follows from Lemma 2.7 that the canonical $\mathbb{E}$-triangle uniquely exists. Since $B \in \mathcal{T}_1 * (\mathcal{T}_2 \cap \mathcal{T}_3) \subset \mathcal{T}_1 * \mathcal{T}_2$ and $\mathrm{f}_2\mathrm{f}_1T_3 \in \mathcal{F}_1 \cap \mathcal{F}_2$ by (3.11), the $\mathbb{E}$-triangle (3.13) is a canonical $\mathbb{E}$-triangle with respect to the $\mathfrak{s}$-torsion pair $(\mathcal{T}_1 * \mathcal{T}_2, \mathcal{F}_1 \cap \mathcal{F}_2)$, so $\mathrm{t}_{12}C = B \in \mathcal{T}_1 * (\mathcal{T}_2 \cap \mathcal{T}_3) \subset \mathcal{T}_1 * \mathcal{T}_3$. Therefore, $((\mathcal{T}_1 * \mathcal{T}_2, \mathcal{F}_1 \cap \mathcal{F}_2), (\mathcal{T}_1 * \mathcal{T}_3, \mathcal{F}_1 \cap \mathcal{F}_3))$ is lower consistent.

(3) By Theorem 3.4 and the results of (1) and (2), we have the following two $\mathfrak{s}$-torsion pairs

$$((\mathcal{T}_1 * (\mathcal{T}_2 \cap \mathcal{T}_3), \mathcal{F}_1 \cap (\mathcal{F}_2 * \mathcal{F}_3)) \text{ and } ((\mathcal{T}_1 * \mathcal{T}_2) \cap (\mathcal{T}_1 * \mathcal{T}_3), (\mathcal{F}_1 \cap \mathcal{F}_2) * (\mathcal{F}_1 \cap \mathcal{F}_3)).$$

Hence it suffices to prove that $\mathcal{F}_1 \cap (\mathcal{F}_2 * \mathcal{F}_3) = (\mathcal{F}_1 \cap \mathcal{F}_2) * (\mathcal{F}_1 \cap \mathcal{F}_3)$. It is easy to check that $(\mathcal{F}_1 \cap \mathcal{F}_2) * (\mathcal{F}_1 \cap \mathcal{F}_3) \subset \mathcal{F}_1 \cap (\mathcal{F}_2 * \mathcal{F}_3)$. Conversely, let $C \in \mathcal{F}_1 \cap (\mathcal{F}_2 * \mathcal{F}_3)$, then we have the following three $\mathbb{E}$-triangles

$$F_2 \to C \to F_3 \dashrightarrow, \tag{3.14}$$

$$\mathrm{t}_3C \to C \to \mathrm{f}_3C \dashrightarrow, \tag{3.15}$$

$$\mathrm{t}_2\mathrm{t}_3C \xrightarrow{a} \mathrm{t}_3C \xrightarrow{b} \mathrm{f}_2\mathrm{t}_3C \dashrightarrow, \tag{3.16}$$

where $F_2 \in \mathcal{F}_2$, $F_3 \in \mathcal{F}_3$, (3.15) and (3.16) are canonical. Since (s_2, s_3) is lower consistent and $\mathrm{t}_3C \in \mathcal{T}_3$, we have $\mathrm{t}_2\mathrm{t}_3C \in \mathcal{T}_2 \cap \mathcal{T}_3$. Applying $\mathrm{Hom}_{\mathcal{K}}(\mathrm{t}_2\mathrm{t}_3C, -)$ to the $\mathbb{E}$-triangle (3.14), we have an exact sequence

$$0 = \mathrm{Hom}_{\mathcal{K}}(\mathrm{t}_2\mathrm{t}_3C, F_2) \to \mathrm{Hom}_{\mathcal{K}}(\mathrm{t}_2\mathrm{t}_3C, C) \to \mathrm{Hom}_{\mathcal{K}}(\mathrm{t}_2\mathrm{t}_3C, F_3) = 0,$$

so $\mathrm{Hom}_{\mathcal{K}}(\mathrm{t}_2\mathrm{t}_3C, C) = 0$. Applying $\mathrm{Hom}_{\mathcal{K}}(\mathrm{t}_2\mathrm{t}_3C, -)$ to the $\mathbb{E}$-triangle (3.15), we have an exact sequence

$$\mathbb{E}^{-1}(\mathrm{t}_2\mathrm{t}_3C, \mathrm{f}_3C) \to \mathrm{Hom}_{\mathcal{K}}(\mathrm{t}_2\mathrm{t}_3C, \mathrm{t}_3C) \to \mathrm{Hom}_{\mathcal{K}}(\mathrm{t}_2\mathrm{t}_3C, C) = 0.$$

Since $(\mathcal{T}_3, \mathcal{F}_3)$ is an $\mathfrak{s}$-torsion pair and $\mathrm{t}_2\mathrm{t}_3C \in \mathcal{T}_3$, $\mathrm{f}_3C \in \mathcal{F}_3$, we have $\mathbb{E}^{-1}(\mathrm{t}_2\mathrm{t}_3C, \mathrm{f}_3C) = 0$ and $a \in \mathrm{Hom}_{\mathcal{K}}(\mathrm{t}_2\mathrm{t}_3C, \mathrm{t}_3C) = 0$. By Lemma 3.3, b is a split monomorphism, then t_3C is a direct summand of $\mathrm{f}_2\mathrm{t}_3C$. It follows from Lemma 2.6 that $\mathrm{t}_3C \in \mathcal{F}_2$. Since (s_1, s_3) is upper consistent and $C \in \mathcal{F}_1$, we have $\mathrm{f}_3C \in \mathcal{F}_1$. By Lemma 3.7, using the $\mathbb{E}$-triangle (3.15), we obtain $\mathrm{t}_3C \in \mathcal{F}_1$. Thus $\mathrm{t}_3C \in \mathcal{F}_1 \cap \mathcal{F}_2$ and $\mathrm{f}_3C \in \mathcal{F}_1 \cap \mathcal{F}_3$, which implies that $C \in (\mathcal{F}_1 \cap \mathcal{F}_2) * (\mathcal{F}_1 \cap \mathcal{F}_3)$. Therefore, $\mathcal{F}_1 \cap (\mathcal{F}_2 * \mathcal{F}_3) \subset (\mathcal{F}_1 \cap \mathcal{F}_2) * (\mathcal{F}_1 \cap \mathcal{F}_3)$. Thus we complete the proof. □

Dually, we have the following conclusion.

Corollary 3.9. *Let $s_i := (\mathcal{T}_i, \mathcal{F}_i)$ be an $\mathfrak{s}$-torsion pair in $\mathcal{K}$ for any $i = 1, 2, 3$. Suppose that (s_1, s_3), (s_2, s_3) are lower consistent and (s_1, s_2) is upper consistent, then the following assertions hold.*

(1) $((\mathcal{T}_1 * \mathcal{T}_2, \mathcal{F}_1 \cap \mathcal{F}_2), (\mathcal{T}_3, \mathcal{F}_3))$ *is lower consistent.*

(2) $((\mathcal{T}_1 \cap \mathcal{T}_3, \mathcal{F}_1 * \mathcal{F}_3), (\mathcal{T}_2 \cap \mathcal{T}_3, \mathcal{F}_2 * \mathcal{F}_3))$ *is upper consistent.*

(3) $(\mathcal{T}_1 * \mathcal{T}_2) \cap \mathcal{T}_3 = (\mathcal{T}_1 \cap \mathcal{T}_3) * (\mathcal{T}_2 \cap \mathcal{T}_3)$, *and* $(\mathcal{F}_1 \cap \mathcal{F}_2) * \mathcal{F}_3 = (\mathcal{F}_1 * \mathcal{F}_3) \cap (\mathcal{F}_2 * \mathcal{F}_3)$.

4. Application

In this section, we apply our main theorem to the τ-tilting theory which was introduced by Adachi, Iyama and Reiten in [2]. Let Λ be a finite-dimensional algebra over an algebraically closed field and modΛ the category of finitely generated left Λ-modules. If M is a Λ-module, we denote by Fac$M$ the subcategory of modΛ which consists of all factor modules of finite direct sums of copies of M; the subcategory Sub$M$ is defined dually. We denote the number of pairwise nonisomorphic indecomposable summands of M by $|M|$, then $|\Lambda|$ equals the rank of the Grothendieck group of modΛ.

Definition 4.1. ([2]) Let M be a Λ-module.

(1) M is called *τ-rigid* if $\mathrm{Hom}_\Lambda(M, \tau M) = 0$, where τ is the Auslander-Reiten translation.

(2) M is called *τ-tilting* if M is τ-rigid and $|M| = |\Lambda|$.

Let M be a τ-rigid Λ-module. It is well known that there exist two distinguished torsion pairs in modΛ, namely

$$(\mathrm{Fac}M, M^{\perp}) \text{ and } ({}^{\perp}(\tau M), \mathrm{Sub}\tau M),$$

which satisfy $\mathrm{Fac}M \subseteq {}^{\perp}(\tau M)$ and $\mathrm{Sub}\tau M \subseteq M^{\perp}$. We have the following characterization of an arbitrary module being τ-tilting.

Lemma 4.2. *Let M be a Λ-module, then M is τ-tilting if and only if* $(\mathrm{Fac}M, \mathrm{Sub}\tau M)$ *is a torsion pair in* modΛ.

Proof. According to [2, Theorem 2.12], a τ-rigid module M is τ-tilting if and only if $\mathrm{Fac}M = {}^{\perp}(\tau M)$.

If M is τ-tilting, then $(\mathrm{Fac}M, \mathrm{Sub}\tau M) = ({}^{\perp}(\tau M), \mathrm{Sub}\tau M)$ is a torsion pair. Conversely, assume that $(\mathrm{Fac}M, \mathrm{Sub}\tau M)$ is a torsion pair. Since $M \in \mathrm{Fac}M$, $\tau M \in \mathrm{Sub}\tau M$, we have $\mathrm{Hom}_\Lambda(M, \tau M) = 0$, which implies that M is τ-rigid. Notice that a torsion-free class and the corresponding torsion class are determined by each other, and $(\mathrm{Fac}M, \mathrm{Sub}\tau M)$, $({}^{\perp}(\tau M), \mathrm{Sub}\tau M)$ are torsion pairs. We have $\mathrm{Fac}M = {}^{\perp}(\tau M)$, so M is τ-tilting. □

Lemma 4.3. *([3, Propositions 5.8, 5.6]) Let M and N be two Λ-modules, then the following conditions are equivalent.*

(1) $\mathrm{Hom}_\Lambda(N, \tau M) = 0$.

(2) $\mathrm{Ext}^1_\Lambda(M, \mathrm{Fac}N) = 0$.

(3) $\mathrm{Ext}^1_\Lambda(\mathrm{Sub}\tau M, \tau N) = 0$.

Proposition 4.4. *Let $M = M_1 \oplus M_2$ be a τ-rigid Λ-module, then the following assertions hold.*

(1) $\mathrm{Fac}M = \mathrm{Fac}M_1 * \mathrm{Fac}M_2$.

(2) $\mathrm{Sub}\tau M = \mathrm{Sub}\tau M_1 * \mathrm{Sub}\tau M_2$.

Proof. We only prove (1); the proof of (2) is similar. First, we prove that $((\mathrm{Fac}M_1, M_1^{\perp}), (\mathrm{Fac}M_2, M_2^{\perp}))$ is lower consistent. Let X be a module in $M_1^{\perp}$, then there exists an exact sequence

$$0 \to P \to X \to Q \to 0$$

with respect to the torsion pair $(\mathrm{Fac}M_2, M_2^{\perp})$ such that $P \in \mathrm{Fac}M_2$ and $Q \in M_2^{\perp}$. Applying the functor $\mathrm{Hom}_{\Lambda}(M_1, -)$ to the exact sequence, we have an exact sequence

$$\mathrm{Hom}_{\Lambda}(M_1, X) \to \mathrm{Hom}_{\Lambda}(M_1, Q) \to \mathrm{Ext}^1_{\Lambda}(M_1, P).$$

Since $X \in M_1^{\perp}$, the left hand side vanishes. Since $\mathrm{Fac}M_2 \subseteq \mathrm{Fac}(M_1 \oplus M_2)$, we have that P also belongs to $\mathrm{Fac}(M_1 \oplus M_2)$. Note that $M_1 \oplus M_2$ is a τ-rigid module, then it follows from Lemma 4.3 that $\mathrm{Ext}^1_{\Lambda}(M_1 \oplus M_2, P) = 0$, so $\mathrm{Ext}^1_{\Lambda}(M_1, P) = 0$. Thus $\mathrm{Hom}_{\Lambda}(M_1, Q) = 0$, which implies that Q is in $M_1^{\perp}$. Hence $((\mathrm{Fac}M_1, M_1^{\perp}), (\mathrm{Fac}M_2, M_2^{\perp}))$ is upper consistent, then by Theorem 3.4, $(\mathrm{Fac}M_1 * \mathrm{Fac}M_2, M_1^{\perp} \cap M_2^{\perp})$ is a torsion pair in $\mathrm{mod}\Lambda$. Moreover, since

$$M^{\perp} = (M_1 \oplus M_2)^{\perp} = M_1^{\perp} \cap M_2^{\perp},$$

the torsion pairs $(\mathrm{Fac}M_1 * \mathrm{Fac}M_2, M_1^{\perp} \cap M_2^{\perp})$ and $(\mathrm{Fac}M, M^{\perp})$ share the same torsion-free class, and we have the assertion. □

Let $\mathcal{X}, \mathcal{Y}$ be two subcategories of $\mathrm{mod}\Lambda$. Generally speaking, $\mathcal{X} * \mathcal{Y} \neq \mathcal{Y} * \mathcal{X}$. Proposition 4.4 tells us that $\mathrm{Fac}M_1 * \mathrm{Fac}M_2 = \mathrm{Fac}M_2 * \mathrm{Fac}M_1$ and $\mathrm{Sub}\tau M_1 * \mathrm{Sub}\tau M_2 = \mathrm{Sub}\tau M_2 * \mathrm{Sub}\tau M_1$ when $M_1 \oplus M_2$ is a τ-rigid Λ-module. By Proposition 4.4, we can easily get the following conclusion.

Corollary 4.5. *Let* $M = \bigoplus_{i=1}^{n} M_i$ *be a* τ*-rigid* Λ*-module, then the following assertions hold.*
(1) $\mathrm{Fac}M = \mathrm{Fac}M_1 * \mathrm{Fac}M_2 * \cdots * \mathrm{Fac}M_n$.
(2) $\mathrm{Sub}\tau M = \mathrm{Sub}\tau M_1 * \mathrm{Sub}\tau M_2 * \cdots * \mathrm{Sub}\tau M_n$.

As a byproduct, we have the following characterization of direct sum of Λ-modules being τ-tilting.

Corollary 4.6. *Let* $M = \bigoplus_{i=1}^{n} M_i$ *be a* Λ*-module, then the following conditions are equivalent.*
(1) *M is τ-tilting.*
(2) $(\mathrm{Fac}M_1 * \mathrm{Fac}M_2 * \cdots * \mathrm{Fac}M_n, \mathrm{Sub}\tau M_1 * \mathrm{Sub}\tau M_2 * \cdots * \mathrm{Sub}\tau M_n)$ *is a torsion pair in* $\mathrm{mod}\Lambda$.

Proof. (1) implies (2). If M is a τ-tilting module, then $(\mathrm{Fac}M, \mathrm{Sub}\tau M)$ is a torsion pair in $\mathrm{mod}\Lambda$ by Lemma 4.2, then the statement follows from Corollary 4.5.

(2) implies (1). Since $(\mathrm{Fac}M_1 * \mathrm{Fac}M_2 * \cdots * \mathrm{Fac}M_n, \mathrm{Sub}\tau M_1 * \mathrm{Sub}\tau M_2 * \cdots * \mathrm{Sub}\tau M_n)$ is a torsion pair, $M_i \in \mathrm{Fac}M_1 * \mathrm{Fac}M_2 * \cdots * \mathrm{Fac}M_n$ and $\tau M_j \in \mathrm{Sub}\tau M_1 * \mathrm{Sub}\tau M_2 * \cdots * \mathrm{Sub}\tau M_n$, and we have $\mathrm{Hom}_{\Lambda}(M_i, \tau M_j) = 0$ for any $1 \leq i, j \leq n$. This implies that M is τ-rigid. By Corollary 4.5, we have that $(\mathrm{Fac}M, \mathrm{Sub}\tau M)$ is a torsion pair in $\mathrm{mod}\Lambda$. Thus M is τ-tilting by Lemma 4.2. □

5. Conclusions

We introduced the notion of a consistent pair of $\mathfrak{s}$-torsion pairs in an extriangulated category with a negative first extension. Let $s_i := (\mathcal{T}_i, \mathcal{F}_i)$ be an $\mathfrak{s}$-torsion pair, for any $i = 1, 2$. We showed that $(\mathcal{T}_1 * \mathcal{T}_2, \mathcal{F}_1 \cap \mathcal{F}_2)$ (respectively, $(\mathcal{T}_1 \cap \mathcal{T}_2, \mathcal{F}_1 * \mathcal{F}_2)$) is an $\mathfrak{s}$-torsion pair if and only if (s_1, s_2) is an upper (respectively, lower) consistent pair, which generalizes [4, Proposition 6]. Let $M = M_1 \oplus M_2$ be a τ-rigid module over a finite-dimensional algebra. As an application of the main theorem, we proved that Fac$M$ = Fac$M_1 *$ FacM_2, where Fac$M$ is the category of all factor modules of finite direct sums of copies of M.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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