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Research article

Lifespan of solutions to second order Cauchy problems with small Gevrey data

John Paolo O. Soto, Jose Ernie C. Lope* and Mark Philip F. Ona

Institute of Mathematics, University of the Philippines Diliman, Quezon City, Philippines

* **Correspondence:** Email: jclope@up.edu.ph.

Abstract: Consider the second order nonlinear partial differential equation:

$$\partial_t^2 u = F(u, \partial_x u), \quad (t, x) \in \mathbb{C} \times \mathbb{R}.$$

Given small analytic data, Yamane was able to obtain the order of the lifespan of the solution with respect to the smallness parameter ε. On the other hand, Gourdin and Mechab studied the lifespan of the solution given small Gevrey data, but under the assumption that F is independent of u. In this paper, we considered non-vanishing Gevrey data and used the method of successive approximations to obtain a solution and constructively estimate its lifespan.

Keywords: Cauchy problem; lifespan of solutions; Gevrey functions; small data
Mathematics Subject Classification: 35B30, 35G25

1. Introduction

The lifespan of solutions to Cauchy problems with small data was studied extensively by many authors. In some works, functional analytic methods were used in estimating the lifespan of solutions to semilinear wave equations [4–6]. D'Ancona and Spagnolo [1] considered a second-order nonlinear Cauchy problem with small data together with a hyperbolicity assumption. They proved that as the data becomes smaller, the lifespan of the solution becomes longer. An explicit bound for the lifespan was also obtained in the case where the right-hand side is independent of t.

Yamane [12] considered a nonlinear second-order Cauchy problem without the hyperbolicity assumption. He showed that if the Cauchy data are small in the sense of a Cauchy type inequality, the order of the lifespan of the solution is one with respect to $1/\varepsilon$. This result has been improved in [13] as the right-hand side of the equation in these works also depends on u and $\partial_t u$. As an extension, Tolentino, Bacani and Tahara [10] considered the general mth-order equation with small analytic data.

One notable similarity in these works is that all of them used the Banach fixed point theorem in proving their results. An alternative approach using the method of successive approximations was presented in [9], which allowed for a constructive approach in obtaining estimates for the lifespan of solutions studied in [12].

Gourdin and Mechab [2] considered generalized Kirchoff equations in the real-analytic category and studied the lifespan of the solutions under some smallness conditions. They proved existence and uniqueness results, which also established estimates for the lifespan of the solution. In another work, they also dealt with Cauchy problems involving Gevrey data of general order [3]. In particular, they considered the equation

$$\begin{cases} \partial_t^m u = F(t, D^B u(t,x)), \\ \partial_t^j u(0,x) = \varphi_j(\varepsilon \cdot x), \ j = 0, 1, \dots, m-1, \end{cases}$$

where $(t,x) \in \mathbb{C}\times\mathbb{R}^n$, $B \subseteq \{(j,\alpha) \in \mathbb{N}\times\mathbb{N}^n : j+|\alpha| \leq m, \ \alpha \neq 0\}$, $D^B u = \{D^\delta u\}_{\delta\in B}$ and the Cauchy data φ_j are of Gevrey index d, with $d = \min\{(m-j)/|\alpha|, \ (j,\alpha) \in B\}$. In their paper, they obtained a solution that is holomorphic in t and of Gevrey class with respect to x. Moreover, they showed that if the data φ_j vanishes at the origin, the obtained solution will be stable; that is, the sequence (u_ε) will tend to the zero function as $\varepsilon \to 0$. Lastly, in the case where F is independent of t, they showed that the order of the lifespan of the solution is of order σ with respect to $1/\varepsilon$, where $\sigma = \inf\{|\alpha|/(m-j), \ (j,\alpha) \in B\}$.

In this paper, we will be dealing with two second-order equations involving small Gevrey data. The first equation deals with the second-order version of the one considered by Gourdin-Mechab in [3], while the other one slightly widens the class of equations being considered by incorporating u on the right-hand side of the equation. The framework to be used involves the function spaces and estimates used in [3, 11, 12] together with the method of successive approximations. Through this constructive approach, we will see the roles that the terms on the right-hand side play in determining the order of the lifespan of the solution.

2. Statement of the problem and main result

Let $(t,x) \in \mathbb{C}\times\mathbb{R}$ and Ω be an open subset of $\mathbb{R}$. We first define the Gevrey function spaces that will be used in this paper.

Definition 2.1. *A C^∞ function $\varphi(x)$ is said to be of Gevrey index two on $\Omega \subseteq \mathbb{R}$ if there exists a positive constant C such that for all $\alpha \in \mathbb{N}$,*

$$\sup_{x\in\Omega} |\partial_x^\alpha \varphi(x)| \leq C^{\alpha+1}(\alpha!)^2.$$

We denote by $\mathbb{G}^2(\Omega)$ the collection of all Gevrey index two functions on Ω. For the next definition, we let $B_T \subseteq \mathbb{C}$ be the open disk of radius T centered at the origin and set $O_T = B_T\times\Omega$. Let $C^{\omega,\infty}(O_T)$ be the collection of functions whose higher order derivatives in x are continuous on O_T, and holomorphic in t for every fixed $x \in \Omega$.

Definition 2.2. *A function $u(t,x)$ is said to belong to $\mathbb{G}^{\omega,2}(O_T)$ if $u \in C^{\omega,\infty}(O_T)$ and there exists a positive constant C, such that for any $\alpha \in \mathbb{N}$ and $t \in B_T$,*

$$\sup_{x\in\Omega} |\partial_x^\alpha u(t,x)| \leq C^{\alpha+1}(\alpha!)^2.$$

We formally state our main problem. Consider the second-order Cauchy problem

$$\begin{cases} \partial_t^2 u = F(\partial_x u), \\ u(0,x) = \varphi(x),\ \partial_t u(0,x) = \psi(x), \end{cases} \tag{CP}$$

where the function $F(v)$ and the initial data satisfy the following:

(A1) $F(0) = 0$
(A2) $F(v)$ is analytic in a neighborhood of $v = 0$
(A3) the Cauchy data $\varphi(x)$ and $\psi(x)$ are of Gevrey index two on Ω.

Specifically, suppose that we can write the right-hand side of (CP) as $F(v) = \lambda v + f(v)$, where $\lambda \in \mathbb{C}$ and f consists of the nonlinear terms in the expansion of F and vanishes of second-order at $v = 0$. Under these assumptions, we now state our main result.

Theorem 2.1. *Supposing that conditions (A1)–(A3) hold, then there exists $\mu > 0$ such that the following holds for all $0 < \varepsilon < 1$: If the Cauchy data satisfies*

$$\sup_{x\in\Omega} |\partial_x^\alpha \varphi(x)| \leq \varepsilon^{\alpha+1}(\alpha!)^2 \quad \textit{and} \quad \sup_{x\in\Omega} |\partial_x^\alpha \psi(x)| \leq \varepsilon^{\alpha+1}(\alpha!)^2$$

for all $\alpha \in \mathbb{N}$, then (CP) *has a unique solution in $\mathbb{G}^{\omega,2}(O_T)$, with $T = \mu\varepsilon^{-1/2}$.*

We also consider the case when the right-hand side depends on u; that is, we have the equation

$$\begin{cases} \partial_t^2 u = G(u, \partial_x u), \\ u(0,x) = \varphi(x),\ \partial_t u(0,x) = \psi(x), \end{cases} \tag{CP2}$$

where the data satisfies assumption (A3) and the function $G(u, v)$ satisfies

(B1) $G(0, 0) = 0$ and $\partial_u G(0, 0) = 0$, and
(B2) $G(u, v)$ is analytic in a neighborhood around $(u, v) = 0$.

Assumptions (B1) and (B2) imply that $G(u, v)$ can be written as $\lambda v + g(u, v)$, where g denotes the collection of nonlinear terms in the expansion of G. Note that g vanishes of second-order at $(u, v) = 0$. It is interesting to note that under the same estimates on the data, the order of the lifespan becomes different in this case, as stated in the following theorem.

Theorem 2.2. *Supposing that conditions (B1), (B2) and (A3) hold, then there exists $\mu > 0$ such that the following holds for all $0 < \varepsilon < 1$: If the Cauchy data satisfies*

$$\sup_{x\in\Omega} |\partial_x^\alpha \varphi(x)| \leq \varepsilon^{\alpha+1}(\alpha!)^2 \quad \textit{and} \quad \sup_{x\in\Omega} |\partial_x^\alpha \psi(x)| \leq \varepsilon^{\alpha+1}(\alpha!)^2$$

for all $\alpha \in \mathbb{N}$, then (CP2) *has a unique solution in $\mathbb{G}^{\omega,2}(O_T)$, with $T = \mu\varepsilon^{-1/3}$.*

As can be seen from both results, the order of the lifespan of the solution depends on the equation and also on the estimates of the Cauchy data. This was also observed in the analytic case (see Theorem 1.9 of [13]). For example, consider the problem

$$\begin{cases} \partial_t^2 u = 2(\partial_x^2 u - u\partial_x u), \\ u(0,x) = \dfrac{\varepsilon}{1-\varepsilon x},\ \partial_t u(0,x) = \dfrac{\varepsilon^2}{(1-\varepsilon x)^2}. \end{cases}$$

The solution is given by $u(t, x) = \varepsilon(1 - \varepsilon t - \varepsilon x)^{-1}$, whose lifespan can be seen to be $T_\varepsilon = (1 - \frac{\varepsilon}{2})\varepsilon^{-1}$. This is of order one with respect to ε as $\varepsilon \to 0$, which agrees with results given by the authors in [9] in the analytic case. Other examples may be seen in [3], but also in the analytic case. Work is under way to obtain an example where the initial data satisfies Gevrey estimates.

3. Preliminaries

We will discuss here some majorant functions and estimates that will be used in our proof. We denote by $\mathbb{N}$ the set of nonnegative integers, by D^{-1} and D the usual anti-differentiation and differentiation operators, respectively, and by ∂_t^{-1} the contour integral over the line segment from zero to t given $\partial_t^{-1}u(t, x) = \int_{[0,t]} u(s, x)ds$.

We will use the usual definition of a majorant; that is, given two formal power series $f(X) = \sum a_k X^k$ and $g(X) = \sum b_k X^k$, where $a_k \in \mathbb{C}$ and $b_k \geq 0$, we say that $f \ll g$ if, and only if, $|a_k| \leq b_k$ for all k.

We introduce the power series $\phi(X)$ given by

$$\phi(X) = K^{-1} \sum_{k\in\mathbb{N}} \frac{X^k}{(k+1)^2}, \quad \text{where } K = 4\pi^2/3. \tag{3.1}$$

This majorant was used in [12] and was based on Lax's majorant function. This was also the form of the majorant used in [8] and in [7], except for the constant term. It is easy to show that ϕ converges in the unit disk, and that $\phi^2 \ll \phi$ (see [7]).

To deal with Gevrey functions, we introduce a formal series used in [3]. For $T, \zeta > 0$, we define

$$\Phi := \Phi^{\omega,d}_{T,\zeta}(t, x) = \sum_{k\in\mathbb{N}} \frac{(\zeta x)^k}{k!}(k!)^{d-1} D^k \phi\Big(\frac{t}{T}\Big).$$

It can be shown that Φ also satisfies $\Phi^2 \ll \Phi$. Since we are dealing functions of Gevrey index two, we can set $d = 2$.

Definition 3.1. *Let $T, \zeta > 0$. A function $u(t, x)$ is said to belong to the space $\mathcal{G}^{\omega,2}_{T,\zeta}(O_T)$ if $u \in C^{\omega,\infty}(O_T)$, and there exists a constant $C > 0$ such that for all $\alpha \in \mathbb{N}$ and $x \in \Omega$,*

$$\partial_x^\alpha u(t, x) \ll C\zeta^\alpha \alpha! D^\alpha \phi(t/T).$$

The above definition can be written in terms of Φ as $u(t, x) \ll C\Phi(t, x)$ for any fixed $x \in \Omega$. If we define the norm of $\mathcal{G}^{\omega,2}_{T,\zeta}(O_T)$ to be the infimum of the Cs that satisfy the majorant relation, then it becomes a Banach algebra. Furthermore, for all $0 < T' < T$ and $\zeta > 0$, the subset relation $\mathcal{G}^{\omega,2}_{T,\zeta}(O_T) \subset \mathbb{G}^{\omega,2}(O_{T'})$ holds (see Proposition 1 of [3]). Lastly, for the space of vectors $\vec{\tau}(t, x) = (\tau_j(t, x))_{j=1}^N \in \bigoplus \mathcal{G}^{\omega,2}_{T,\zeta}(O_T)$, we define the norm as $\|\vec{\tau}\|_N := \max_{j=1,2,\ldots,N} \|\tau_j(x)\|$, where the norm $\|\tau_j(x)\|$ is the infimum norm in $\mathcal{G}^{\omega,2}_{T,\zeta}(O_T)$.

To prove our main result, we shall extend some results from [12] and [3] to the Gevrey function spaces defined above.

Proposition 3.1. *(cf. [12]) Let $f(X) = f(X_1, \ldots, X_N) = \sum_{|\alpha|\geq 2} a_\alpha X^\alpha$ be a convergent power series that vanishes of second order at $X = 0$. If $\vec{\tau}(t, x)$, $\vec{\sigma}(t, x) \in \bigoplus \mathcal{G}^{\omega,2}_{T,\zeta}(O_T)$ have sufficiently small norms, then*

$f(\vec{\tau}(t,x))$ and $f(\vec{\sigma}(t,x))$ are well-defined as elements of $\mathcal{G}^{\omega,2}_{T,\zeta}(O_T)$. Moreover, there exists a constant $A = A(f) > 0$, depending only on f and independent of $\vec{\tau}, \vec{\sigma}, T, \zeta$ and Ω, such that

$$\|F(\vec{\tau}(x))\| \leq A\|\vec{\tau}\|_N^2,$$
$$\|F(\vec{\tau}(x)) - F(\vec{\sigma}(x))\| \leq A\|\vec{\tau} - \vec{\sigma}\|_N(\|\vec{\tau}\|_N + \|\vec{\sigma}\|_N).$$

Moreover, we give estimates involving the operator $\partial_t^{-k}\partial_x^{\alpha}$ acting on elements of $\mathcal{G}^{\omega,2}_{T,\zeta}(O_T)$.

Proposition 3.2. *[3, 12] Let $k, \alpha \in \mathbb{N}$ with $-k + 2\alpha \leq 0$. The operator $\partial_t^{-k}\partial_x^{\alpha}$ is an endomorphism of the Banach space $\mathcal{G}^{\omega,2}_{T,\zeta}(O_T)$. Moreover, there exists $B > 0$ such that*

$$\|\partial_t^{-k}\partial_x^{\alpha}u\| \leq BT^k\zeta^{\alpha}\|u\|.$$

If $\varphi(x) \in \mathbb{G}^2(\Omega)$, then we can find constants $p(\varphi)$ and $q(\varphi)$ such that for all $\alpha \in \mathbb{N}$,

$$\sup_{x\in\Omega} |\partial_x^{\alpha}\varphi(x)| \leq p(\varphi)q(\varphi)(\alpha!)^2. \tag{3.2}$$

The following result states that $\mathbb{G}^2(\Omega)$ is closed under differentiation and we can compute for the constants $p(\partial_x^j\varphi)$ and $q(\partial_x^j\varphi)$.

Proposition 3.3. *If $\varphi(x) \in \mathbb{G}^2(\Omega)$ satisfies (3.2) with $p(\varphi) = q(\varphi) = \varepsilon$ and m is a positive integer, then for $j = 1, 2, \ldots, m$,*

$$p(\partial_x^j\varphi) = (2^m m!)^2\varepsilon^{j+1}, \quad q(\partial_x^j\varphi) = 4\varepsilon.$$

Finally, functions in $\mathbb{G}^2(\Omega)$ are also functions in $\mathcal{G}^{\omega,2}_{T,\zeta}(O_T)$ and we can compute for their norms.

Proposition 3.4. *[3, 12] If $\psi(x) \in \mathbb{G}^2(\Omega)$, then for all $T > 0$ and $\zeta \geq e^2 q(\psi)$, we have $\psi(x) \in \mathcal{G}^{\omega,2}_{T,\zeta}(O_T)$ and $\|\psi\| \leq Kp(\psi)$, where K is the one in (3.1).*

4. Proof of Theorem 2.1

We first prove Theorem 2.1. From the assumptions on (CP), we can rewrite the Cauchy problem as

$$\begin{cases} \partial_t^2 u = \lambda\partial_x u + f(\partial_x u), \\ u(0,x) = \varphi(x), \ \partial_t u(0,x) = \psi(x), \end{cases} \tag{CP*}$$

where $\lambda \in \mathbb{C}$ and $f(v)$ vanishes of second order at $v = 0$. Set $u^* = u - \varphi - t\psi$ and $\partial_t^2 u^* = w$. Thus, (CP*) is reduced to $\mathcal{L}w = w$, where

$$\mathcal{L}w := \lambda\partial_x(\partial_t^{-2}w + \varphi + t\psi) + f(\partial_x(\partial_t^{-2}w + \varphi + t\psi)).$$

We use the method of successive approximations to solve this equation. We define the approximate solutions $\{w_n\}$ as follows: $w_{-1} \equiv 0$ and for $n \geq 0$, $w_n = \mathcal{L}(w_{n-1})$. Furthermore, define the sequence $\{d_n\}$ by $d_n = w_n - w_{n-1}$. By construction, it is enough to prove the convergence of $\sum_{i=0}^{n} d_i$ in $\mathcal{G}^{\omega,2}_{T,\zeta}(O_T)$ to show that the sequence $\{w_n\}$ converges also in $\mathcal{G}^{\omega,2}_{T,\zeta}(O_T)$.

Now, let $\varepsilon \in (0,1)$, $C = \max\{A, |\lambda|\}$, $\zeta = (2e)^2\varepsilon$ and $T = \mu/\varepsilon^\sigma$, where $\sigma > 0$. We will show that there exists a solution $u(t,x) \in \mathbb{G}^{\omega,2}(O_T)$ and that $\sigma = 1/2$.

We start with the case $k = 0$. By Proposition 3.3, the linear term is estimated by

$$\|\lambda\partial_x(\varphi + t\psi)\| \le C(\|\partial_x\varphi\| + T\|\partial_x\psi\|) \le 2^6 CK\varepsilon^2(1+T).$$

Consequently, by Proposition 3.1, $\|f(\partial_x(\varphi + t\psi))\| \le 2^{12}CK^2\varepsilon^4(1+T)^2$. Hence, as $T = \mu/\varepsilon^\sigma$ and $\mu < 1$, we have

$$\begin{aligned}\|d_0\| &\le 2^{12}CK^2\varepsilon^2(1+T)(1+\varepsilon^2(1+T))\\ &= 2^{12}CK^2(\varepsilon^2 + \mu\varepsilon^{2-\sigma})(1+\varepsilon^2+\mu\varepsilon^{2-\sigma})\\ &\le 2^{12}CK^2(\varepsilon^2+\varepsilon^4+2\varepsilon^{4-\sigma}+\varepsilon^{2-\sigma}+\varepsilon^{4-2\sigma}).\end{aligned}$$

For the right-hand side estimate to be bounded for any $\varepsilon \in (0,1)$, we must have $\sigma \le 2$. Lastly, since $\varepsilon^{4-\sigma} < \varepsilon^{2-\sigma}$ for $\varepsilon \in (0,1)$, we conclude that $\|d_0\| \le 6 \cdot 2^{12}CK^2\varepsilon^{2-\sigma}$.

The next proposition provides similar estimates for the case $k \ge 1$. It will also be a sufficient condition to show the existence of a solution in $\mathbb{G}^{\omega,2}(O_T)$.

Proposition 4.1. *If we choose $\mu < 1$ small enough such that*

$$(2e)^2BC\mu^2(1+(2^8e)^2BCK^2\mu^2+2^8K) \le \frac{1}{2},$$

then the following holds for $n \ge 1$:

$$\|d_n\| \le 6\cdot 2^{12}CK^2\varepsilon^{(n+2)-(2n+1)\sigma}\Big(\frac{1}{2}\Big)^n.$$

Proof. We prove by induction. Note that d_1 satisfies the equation $d_1 = \lambda\partial_x(\partial_t^{-2}d_0) + H_1$, where

$$H_1 = f(\partial_x(\partial_t^{-2}w_0 + \varphi + t\psi)) - f(\partial_x(\varphi + t\psi)).$$

Using Proposition 3.2, we can estimate the linear term as follows:

$$\begin{aligned}\|\lambda\partial_x(\partial_t^{-2}d_0)\| \le& BC(T^2\zeta)\|d_0\|\\ \le& BC(\mu^2\varepsilon^{-2\sigma})((2e)^2\varepsilon)(6\cdot 2^{12}CK^2\varepsilon^{2-\sigma})\\ =&6\cdot(2^7e)^2BC^2K^2\mu^2\varepsilon^{3-3\sigma}.\end{aligned}$$

For the nonlinear term H_1, by Proposition 3.1 we have

$$\|H_1\| \le C\|\partial_x\partial_t^{-2}d_0\|(\|\partial_x\partial_t^{-2}w_0\| + 2\|\partial_x(\varphi + t\psi)\|).$$

By the previous cases, $\|\partial_x(\varphi + t\psi)\| \le 2^7K\varepsilon^{2-\sigma}$. Since $w_0 = d_0$, we obtain

$$\|d_1\| \le 6\cdot(2^7e)^2BC^2K^2\mu^2\varepsilon^{3-3\sigma} + 6^2\cdot(2^7e)^4B^2C^3K^4\mu^4\varepsilon^{6-6\sigma} + 6\cdot 2^8(2^7e)^2BC^2K^3\mu^2\varepsilon^{5-4\sigma}.$$

Thus, for the right-hand side estimate to be bounded for any $\varepsilon \in (0,1)$, we must have $\sigma \le 1$. Furthermore, since $\max\{\varepsilon^{6-6\sigma}, \varepsilon^{5-4\sigma}\} < \varepsilon^{3-3\sigma}$, we get

$$\begin{aligned}\|d_1\| &\le 6\cdot(2^7e)^2BC^2K^2\mu^2\varepsilon^{3-3\sigma}(1+6(2^7e)^2BCK^2\mu^2+2^8K)\\ &= 6\cdot 2^{12}CK^2\varepsilon^{3-3\sigma}((2e)^2BC\mu^2(1+6(2^7e)^2BCK^2\mu^2+2^8K))\\ &\le 6\cdot 2^{12}CK^2\varepsilon^{3-3\sigma}\Big(\frac{1}{2}\Big)\end{aligned}$$

from our choice of μ.

Now, suppose the claim holds for $k \le n$. We will show that the claim holds for $k = n+1$. Recall that $d_{n+1} = \lambda\partial_x\partial_t^{-2}d_n + H_{n+1}$, where

$$H_{n+1} = f(\partial_x(\partial_t^{-2}w_n + \varphi + t\psi)) - f(\partial_x(\partial_t^{-2}w_{n-1} + \varphi + t\psi)).$$

By the inductive hypothesis and Proposition 3.2, we have

$$\|\partial_x\partial_t^{-2}d_n\| \le 6\cdot(2^7e)^2BCK^2\mu^2\varepsilon^{(n+3)-(2n+3)\sigma}\Big(\frac{1}{2}\Big)^n.$$

For the nonlinear term H_{n+1}, we first find an estimate for $\partial_x\partial_t^{-2}w_j$. Since the sequence $\left\{\frac{n+2}{2n+1}\right\}$ is decreasing, the quantity $\|d_k\|$ will be bounded for every $\varepsilon \in (0,1)$ and $k \le n$ if $\sigma \le \frac{n+2}{2n+1}$. Moreover, this implies that for any $k = 0,\ldots,n$,

$$\|d_k\| \le 6\cdot 2^{12}CK^2\varepsilon^{(n+2)-(2n+1)\sigma}\Big(\frac{1}{2}\Big)^k.$$

Hence, we have the following estimate for $0 \le j \le n$:

$$\begin{aligned}\|\partial_x\partial_t^{-2}w_j\| &\le BT^2\zeta\|d_0 + d_1 + \cdots + d_j\|\\ &\le B(\mu^2\varepsilon^{-2\sigma})((2e)^2\varepsilon)\cdot 6\cdot 2^{12}CK^2\varepsilon^{(n+2)-(2n+1)\sigma}\Big[1+\frac{1}{2}+\ldots+\Big(\frac{1}{2}\Big)^j\Big]\\ &\le 12\cdot(2^7e)^2BCK^2\mu^2\varepsilon^{(n+3)-(2n+3)\sigma}.\end{aligned}$$

Thus, by Proposition 3.1 we have

$$\begin{aligned}\|H_{n+1}\| &\le C\|\partial_x\partial_t^{-2}d_n\|\Big(\|\partial_x\partial_t^{-2}w_n\| + \|\partial_x\partial_t^{-2}w_{n-1}\| + 2\|\partial_x(\varphi+t\psi)\|\Big)\\ &\le 6^2\cdot(2^{15}e)^2B^2C^3K^4\mu^4\varepsilon^{(n+3)-(2n+3)\sigma}\Big(\frac{1}{2}\Big)^n + 6\cdot(2^{11}e)^2BC^2K^3\mu^2\varepsilon^{(n+5)-(2n+4)\sigma}\Big(\frac{1}{2}\Big)^n.\end{aligned}$$

As in the previous cases, we must have $\sigma \le \frac{n+3}{2n+3}$. Since $\varepsilon^{(n+5)-(2n+4)\sigma} \le \varepsilon^{(n+3)-(2n+3)\sigma}$, we finally obtain

$$\begin{aligned}\|d_{n+1}\| &\le 6\cdot(2^7e)^2BC^2K^2\mu^2\varepsilon^{(n+3)-(2n+3)\sigma}\Big(\frac{1}{2}\Big)^n(1+6\cdot(2^8e)^2BCK^2\mu^2+2^8K)\\ &\le 6\cdot 2^{12}CK^2\varepsilon^{(n+3)-(2n+3)\sigma}((2e)^2BC\mu^2(1+6\cdot(2^8e)^2BCK^2\mu^2+2^8K))\Big(\frac{1}{2}\Big)^n\\ &\le 6\cdot 2^{12}CK^2\varepsilon^{(n+3)-(2n+3)\sigma}\Big(\frac{1}{2}\Big)^{n+1},\end{aligned}$$

which proves our claim. □

It is good to note that following the proof of Proposition 4.1, the sequence $(\|d_n\|)$ will converge if $\sigma \le \frac{n+2}{2n+1}$ for all $n \ge 0$, which will be satisfied if $\sigma = 1/2$. Thus, $T = \mu\varepsilon^{-1/2}$ and as a consequence, $T \to \infty$ as $\varepsilon \to 0$.

5. Proof of Theorem 2.2

For convenience of notation, we set $D^{0,1}u = (u, \partial_x u)$. We do the same preliminary steps as in the previous case; that is, by setting $v = u - \varphi - t\psi$ and $\partial_t^2 v = w$, (CP2) is reduced to the initial problem $\mathcal{L}_1 w = w$, where the operator $\mathcal{L}_1(w)$ is defined as

$$\mathcal{L}_1(w) := \lambda\partial_x(\partial_t^{-2}w + \varphi + t\psi) + g(D^{0,1}(\partial_t^{-2}w + \varphi + t\psi)).$$

The approximate solutions $\{w_n\}$, the sequence $\{d_n\}$ and the other parameters ε, C, ζ, T and σ are defined similarly as in the previous case. Moreover, following the same arguments as before, we can prove the following estimate:

Proposition 5.1. *If we choose $\mu < 1$ small enough such that $4BC\mu^2(1 + 6 \cdot 2^6 BCK^2\mu^2 + K) \leq 1/2$, then the following holds for $n \geq 0$:*

$$\|d_n\| \leq 6 \cdot 2^6 CK^2 \varepsilon^{(2^{n+1})-(3\cdot 2^{n+1}-4)\sigma}\left(\frac{1}{2}\right)^n. \tag{5.1}$$

Here, we can see that the sequence $\{\|d_n\|\}$ will converge if $\sigma \leq 1/3$. Hence, we conclude that $T = \mu\varepsilon^{-1/3}$.

6. Conclusions

In this paper we have studied two second-order nonlinear Cauchy problems with small data. In the first case where F does not depend on u, we see that the order of the lifespan to the solution to the Cauchy problem is 1/2 with respect to $1/\varepsilon$, which agrees to the order Gourdin and Mechab obtained in [3]. In the case where F depends on u, we also obtained an estimate of the order of the lifespan. In the second case, the lifespan tends to ∞ at a slower rate as $\varepsilon \to 0$. In both cases, the method of successive approximations was used in constructively obtaining estimates for the lifespan. We conclude that the order of the lifespan of the solutions to Cauchy problems with small Gevrey data depends on the data and on the terms on the right-hand side of the equation.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors have no conflicts of interest to declare.

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