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Research article

Analysis of traveling waves for nonlinear degenerate viscosity of chemotaxis model under general perturbations

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Abstract: In this paper, we generalized the results of the following chemotaxis model with the nonlinear degenerate viscosity

$$\begin{cases} u_t - \chi(uv)_x = D(u^m)_{xx}, \\ v_t - u_x = 0, \end{cases}$$

by introducing the following general initial perturbation

$$\int_{-\infty}^{+\infty} \kappa(Z_0|\tilde{Z})dx < \infty,$$

where κ is the relative entropy function defined in Eq (2.24). We further employed the relative entropy method by choosing the specific shift function. According to the estimates with the cutoff version, and overcoming the complexity caused by the porous media diffusion, the nonlinear orbital stability of traveling waves was established under small amplitude and general perturbations.

Keywords: chemotaxis; nonlinear diffusion; orbital stability; energy estimates cutoff; general perturbations
Mathematics Subject Classification: 35A01, 35B40, 35Q92, 92C17

1. Introduction

We are concerned with the analysis of traveling waves for the chemotaxis system with nonlinear diffusion with $u_+ > 0$. It follows from the general perturbations and small wave amplitude that we employ the energy estimates with cutoff version to prove the nonlinear orbital stability. Moreover, we need a different strategy by employing the relative entropy method to handle these general perturbations. We first define the PDE-ODE system of the chemotaxis model as follows:

$$\begin{cases} u_t = D(u^m)_{xx} - \chi\,(u(\ln c)_x)_x\,, \\ c_t = -uc + \beta c, \end{cases} \tag{1.1}$$

with $m > 0$ and initial data

$$(u, c)(0, x) = (u_0, c_0)(x) \to \begin{cases} (u^+, c^+) & \text{as} \quad x \to +\infty, \\ (u^-, c^-) & \text{as} \quad x \to -\infty. \end{cases} \tag{1.2}$$

System (1.1) represents the reinforced movement of cells (or bacterial) in porous media, where c, u, and $\beta > 0$ describe the concentration of chemical signals (e.g., nutrients), the population density of cells, and the growth rate respectively. Moreover, $D > 0$ represents the diffusion rate of cells and χ represents chemotactic coefficient. The chemotaxis is said to be attractive if $\chi > 0$ and repulsive if $\chi < 0$. The logarithmic sensitivity $\ln c$ was derived from Weber-Fechner law [13] and has been verified by experimental data [11].

When $m = 1$, the system (1.1) is identical to the chemotaxis model studied in [21] to represent the reinforced random walks. There are other interesting analytical works of this reinforced random walks. Othmer and Stevens [21] studied the model from random walk and presented the numerical simulations of the formation of spikes and blowup. Moreover, the analytical results were investigated in [20] to support some numerical results in [21]. Yang etc. [27, 28] was curious in the global existence and blowup of classical solutions on a bounded domain with no-flux boundary conditions. Li etc. [16] was interested in the global existence of smooth solutions to system (1.1). Zhang and Zhu [29] presented the weakness of solutions to (1.1) with the Robin boundary condition. Other global dynamics references were offered in [5, 14, 18, 25], including well-posedness for large time solutions in the whole space. Besides the spike solution and blowup solution, traveling wave is another biological pattern observed in chemotaxis [13]. The existence of traveling waves in (1.1) when $m = 1$ was first studied in [26]. The stability of such a traveling front in the case of $u_+ > 0$ was obtained in [17]. When $u_+ = 0$, the energy estimate has the singular term $1/U$, which is extremely difficult to overcome. This singular term of $1/U$ was presented in [10] by considering the weighted function of the singular term to establish the energy estimates. Recently, [15] considered the half-space case in (1.1) (when $m = 1$) under the nonzero flux boundary condition. The authors in [15] showed that the system still admits traveling wave profiles on the half-space by introducing a wave selection mechanism. For other related works on traveling waves of chemotaxis models, we refer the wide variety of readers to these references [9, 24].

When $m \neq 1$, system (1.1) is the chemotaxis model with diffusion in porous media. The problems of the chemotaxis model in porous media are both important in experiments and mathematical modelings. The experiments of bacterial chemotaxis in porous media were quantified in [19, 23], and [1, 8] introduced the porous medium diffusion in the chemotaxis model to prevent overcrowding. Tao and Winkler [22] established the existence of global solutions and boundedness for chemotaxis models of self-aggregation with any porous medium diffusion. However, the existence of compactly supported traveling waves of nonlinear diffusion was studied in [4]. The main issues of this paper are general perturbations and nonlinear diffusion, which are used to establish the energy estimates with cutoff and orbital stability. Due to the difficulty of logarithmic singularity, we further take the following Cole-Hopf transformation as in [10, 17]

$$v = -(\ln c)_x. \tag{1.3}$$

Hence, the system of (1.1) becomes

$$\begin{cases} u_t - \chi(uv)_x = D(u^m)_{xx}, \\ v_t - u_x = 0, \end{cases} \tag{1.4}$$

with the initial conditions

$$(u, v)(x, 0) = (u_0, v_0)(x) \to (u^\pm, v^\pm) \text{ as } x \to \pm\infty. \tag{1.5}$$

The stability analysis of shock waves involving the small antiderivative of perturbation $(u - \tilde{u}, v - \tilde{v})$ was studied in [17] in $H^2(\mathbb{R})$ space. Moreover, Choi etc. [2] employed the small amplitudes and removed both the mean-zero and small initial perturbation conditions. The technique used in the stability of traveling waves for the chemotaxis model in [2] was the relative entropy method. This method of relative entropy has the same role in L^2 space as the distance between (u, v) and $(\tilde{u}, \tilde{v})$, then we prove that the function of relative entropy is decreasing in time for any large perturbation. One needs to remark that the property of contraction is independent of the size of the perturbation. However, we need assumption that the traveling wave strength $|u^- - u^+|$ is small enough.

Based on the previous works, the main novelty presented in this paper is the stability of nonlinear diffusion case for parabolic-hyperbolic system (1.4) under general perturbations and small wave amplitude. Unlike the previous research in [6], which considered the following small initial perturbation

$$\int_{-\infty}^{+\infty} \begin{pmatrix} u(x) - \tilde{u}(x - x_0) \\ v(x) - \tilde{v}(x - x_0) \end{pmatrix} dx = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{for some } x_0 \in \mathbb{R},$$

this condition is also used to investigate the stability of viscous shocks as in [7, 12]. However, we generalize the results in [6], by removing the small initial perturbations above and instead use the following large initial perturbation

$$\int_{-\infty}^{+\infty} \kappa(Z_0|\tilde{Z})dx < \infty,$$

where κ is the relative entropy function which is defined in Eq (2.24). Moreover, the challenge of this porous media case is to employ the relative entropy method in the general system form of the viscous conservative laws of (1.4) involving nonlinear diffusion, which is presented in more detail in section two. Therefore, the purpose of this paper is to handle these barriers and study the contraction of a traveling wave for a system (1.4) under small amplitude and general perturbations.

2. Properties of traveling waves

We further provide smooth monotone traveling waves $\tilde{Z}(x - \varpi t) = \begin{pmatrix} \tilde{u}(x - \varpi t) \\ \tilde{v}(x - \varpi t) \end{pmatrix}$ of (1.4) connecting (u^-, v^-) and (u^+, v^+) in $\mathbb{R}^+ \times \mathbb{R}$,

$$\tilde{u}(-\infty) = u^- > 0, \quad \tilde{u}(+\infty) = u^+ > 0, \quad \tilde{v}(-\infty) = v^-, \quad \tilde{v}(+\infty) = v^+. \tag{2.1}$$

We then write $\lim_{x\to\pm\infty} g(x)$ as $g(\pm\infty)$ in short and assume that two states satisfy the following Rankine-Hugoniot and Lax entropy conditions:

$$\text{for some }\ \varpi\in\mathbb{R}\quad\text{such that}\quad \begin{cases}-\varpi(u^+-u^-)-\chi(u^+v^+-u^-v^-)=0,\\ \quad -\varpi(v^+-v^-)-(u^+-u^-)=0,\end{cases} \tag{2.2}$$
$$\text{and either}\quad u^->u^+\quad\text{and}\quad v^-<v^+\quad\text{or}\quad u^-<u^+\quad\text{and}\quad v^-<v^+\quad\text{holds.}$$

Here, the wave speed ϖ and v^+ are respectively determined by

$$\varpi:=\begin{cases}\frac{-\chi v^-+\sqrt{(\chi v^-)^2+\chi 4u^+}}{2}>0 & \text{if}\quad u^->u^+>0,\\ \frac{-\chi v^--\sqrt{(\chi v^-)^2+\chi 4u^+}}{2}<0 & \text{if}\quad 0<u^-<u^+,\end{cases} \tag{2.3}$$

and

$$v^+:=v^-+\frac{(u^--u^+)}{\varpi}. \tag{2.4}$$

Before the existence of traveling waves is established, we first change the variables in (1.4) from (t,x) to $(t,\zeta=x-\varpi t)$ with the wave speed ϖ defined in (2.7), then one has

$$\begin{cases}u_t-\varpi u_\zeta-\chi\,(uv)_\zeta=D(u^m)_{\zeta\zeta},\\ v_t-\varpi v_\zeta-u_\zeta=0.\end{cases} \tag{2.5}$$

Moreover, the traveling waves $\tilde{Z}=\begin{pmatrix}\tilde{u}\\ \tilde{v}\end{pmatrix}$ of (1.4) are as follows:

$$\begin{cases}-\varpi\tilde{u}_\zeta-\chi\,(\tilde{u}\tilde{v})_\zeta=D(\tilde{u}^m)_{\zeta\zeta},\\ -\varpi\tilde{v}_\zeta-\tilde{u}_\zeta=0.\end{cases} \tag{2.6}$$

Without losing the generality, we consider the traveling waves $(\tilde{u},\tilde{v})$ satisfying $\tilde{u}(0)=(u^-+u^+)/2$. Moreover, the existence of traveling waves $(\tilde{u},\tilde{v})$ are stated as follows.

Lemma 1. (1) *For any $u^\pm, v^\pm$ and $u^->u^+>0$ satisfying* (2.2)*, system* (1.4) *admits a smooth traveling wave $\begin{pmatrix}\tilde{u}\\ \tilde{v}\end{pmatrix}(x-\varpi t)$ connecting (u^-,v^-) and (u^+,v^+), with the wave speed*

$$\varpi=\frac{-\chi v^-+\sqrt{(\chi v^-)^2+\chi 4u^+}}{2}>0. \tag{2.7}$$

Moreover,

$$\tilde{u}'<0,\quad \tilde{v}'=-\frac{\tilde{u}'}{\varpi}>0,\quad \textit{and}\quad (\tilde{u}^m)'=\frac{\chi(\tilde{u}-u^-)(\tilde{u}-u^+)}{D\varpi}. \tag{2.8}$$

(2) *For any $(u^-,v^-)\in\mathbb{R}^+\times\mathbb{R}$, there exists positive constant γ_1 and C, such that for any $0<\gamma<\gamma_1$ and any $(u^+,v^+)\in\mathbb{R}^+\times\mathbb{R}$ satisfying* (2.1) *with $u^+=u^--\gamma$, the following properties hold.*

We consider the traveling waves $\begin{pmatrix}\tilde{u}\\ \tilde{v}\end{pmatrix}(x-\varpi t)$ *connecting* (u^-,v^-) *and* (u^+,v^+) *such that* $\tilde{u}(0) = (u^-+u^+)/2$.

Thus, one gets

$$-\frac{\gamma^2}{\varpi_-}e^{-\frac{\chi\gamma|\zeta|}{DC\varpi_-}} \leq \tilde{u}'(\zeta) \leq -\frac{\gamma^2}{4\varpi_-}e^{-\frac{C\chi\gamma|\zeta|}{D\varpi_-}}, \tag{2.9}$$

where

$$\varpi_- := \frac{-\chi v^- + \sqrt{(\chi v^-)^2 + \chi 4u^-}}{2}. \tag{2.10}$$

Moreover, we have

$$0 < \frac{\varpi_-}{2} \leq (\varpi_- - C\gamma) \leq \varpi < \varpi_-, \tag{2.11}$$

and

$$|(\tilde{u}^m)''(\zeta)| \leq \frac{4\chi\gamma}{D\varpi_-}|\tilde{u}'(\zeta)|. \tag{2.12}$$

Proof. We first show the proof of (1). It follows from the integration results of $(2.6)_1$, with respect to ζ, we get

$$\begin{aligned} D(\tilde{u}^m)' &= -\varpi(\tilde{u}-u^-) - \chi(\tilde{u}\tilde{v} - u^-v^-) \\ &= -\varpi(\tilde{u}-u^-) - \chi\tilde{u}(\tilde{v}-v^-) - \chi v^-(\tilde{u}-u^-). \end{aligned}$$

However, since $\tilde{v} - v^- = -\frac{\tilde{u}-u^-}{\varpi}$ from the integration results of $(2.6)_2$ with respect to ζ,

$$\begin{aligned} D(\tilde{u}^m)' &= -\varpi(\tilde{u}-u^-) + \chi\tilde{u}\left(\frac{\tilde{u}-u^-}{\varpi}\right) - \chi v^-(\tilde{u}-u^-) \\ &= \left(-\varpi + \frac{\chi\tilde{u}}{\varpi} - \chi v^-\right)(\tilde{u}-u^-). \end{aligned}$$

It follows from (2.7) that $\varpi^2 + \chi v^-\varpi = \chi u^+$, then we get

$$(\tilde{u}^m)' = \frac{\chi(\tilde{u}-u^+)(\tilde{u}-u^-)}{D\varpi}. \tag{2.13}$$

The above ODE has a smooth solution $\tilde{u}$ connecting u^- to u^+ and $\tilde{u}' < 0$.

To establish the proof of (2). It follows from (2.7) and $\gamma = u^- - u^+$ that one has

$$\varpi = \frac{-\chi v^- + \sqrt{(\chi v^-)^2 + \chi 4(u^- - \gamma)}}{2}.$$

By taking small enough γ, the proof of (2.11) is completed. To show (2.9), we first find an estimate of (2.13). Since $m(u^-/2)^{m-1} \leq m(\tilde{u})^{m-1} \leq m(u^-)^{m-1}$ and $(\tilde{u}^m)' = m\tilde{u}^{m-1}\tilde{u}'$, then we have

$$C^{-1}\frac{\chi(\tilde{u}-u^-)(\tilde{u}-u^+)}{D\varpi} \leq \tilde{u}' \leq C\frac{\chi(\tilde{u}-u^-)(\tilde{u}-u^+)}{D\varpi}, \tag{2.14}$$

for $C > 0$. Moreover, $\tilde{u}_\zeta = -\varpi\tilde{v}_\zeta < 0$ and $\tilde{u}(0) = (u^- + u^+)/2$ implies

$$\begin{aligned} \zeta \le 0 &\Rightarrow \tilde{u}(0) - u^+ = (u^- - u^+)/2 \le \tilde{u}(\zeta) - u^+ \le u^- - u^+, \\ \zeta \ge 0 &\Rightarrow u^- - \tilde{u}(0) = (u^- - u^+)/2 \le u^- - \tilde{u}(\zeta) \le u^- - u^+. \end{aligned} \tag{2.15}$$

Thus, employing (2.14), (2.15) and $u^- - u^+ = \gamma$, we have

$$\begin{aligned} \zeta \le 0 &\Rightarrow -C\frac{\chi\gamma}{2D\varpi}(u^- - \tilde{u}) \le (u^- - \tilde{u})' \le -C^{-1}\frac{\chi\gamma}{D\varpi}(u^- - \tilde{u}), \\ \zeta \ge 0 &\Rightarrow -C\frac{\chi\gamma}{D\varpi}(\tilde{u} - u^+) \le (\tilde{u} - u^+)' \le -C^{-1}\frac{\chi\gamma}{2D\varpi}(\tilde{u} - u^+). \end{aligned}$$

Employing the above inequalities and $\tilde{u}(0) = (u^- + u^+)/2$, we get

$$\begin{aligned} \zeta \le 0 &\Rightarrow \frac{\gamma}{2}e^{-\frac{\chi\gamma|\zeta|}{DC\varpi}} \le (u^- - \tilde{u})' \le \frac{\gamma}{2}e^{-\frac{C\chi\gamma|\zeta|}{2D\varpi}}, \\ \zeta \le 0 &\Rightarrow \frac{\gamma}{2}e^{-\frac{\chi\gamma\zeta}{DC\varpi}} \le (\tilde{u} - u^+)' \le \frac{\gamma}{2}e^{-\frac{C\chi\gamma\zeta}{2D\varpi}}. \end{aligned}$$

By applying the above inequalities together with (2.14) and (2.15), we have

$$-\frac{\gamma^2}{2\varpi}e^{-\frac{\chi\gamma|\zeta|}{2DC\varpi}} \le \tilde{u}'(\zeta) \le -\frac{\gamma^2}{4\varpi}e^{-\frac{C\chi\gamma|\zeta|}{D\varpi}}.$$

By using (2.11), then we have the desired estimates in (2.9). Moreover, to show (2.12), we differentiate $(\tilde{u}^m)'$ in (2.8) with respect to ζ, and one has

$$|(\tilde{u}^m)''(\zeta)| = \left|\frac{\chi}{D}\tilde{u}'\left(\frac{\tilde{u} - u^-}{\varpi} + \frac{\tilde{u} - u^+}{\varpi}\right)\right|.$$

It follows from (2.11) and (2.15) that one can provide

$$\begin{aligned} |(\tilde{u}^m)''(\zeta)| &\le \left|\frac{\chi}{D}\tilde{u}'\left(\frac{\tilde{u} - u^-}{\varpi} + \frac{\tilde{u} - u^+}{\varpi}\right)\right| \\ &\le \frac{2\chi\gamma}{D\varpi}|\tilde{u}'(\zeta)| \le \frac{4\chi\gamma}{D\varpi_-}|\tilde{u}'(\zeta)|. \end{aligned}$$

□

Definition 1 (Weighted function w). *For a given stationary solution $\tilde{Z}$ and a constant $\nu > 0$, the weighted function $w(\cdot)$ can be defined as follows:*

$$w := 1 + \frac{\nu}{\gamma}(u^- - \tilde{u}^m) \ \text{for} \ m > 0. \tag{2.16}$$

Since $(u^- - \tilde{u}^m) \le (u^- - \tilde{u})$, one has

$$w(+\infty) \le 1 + \nu, \quad w(-\infty) \le 1, \quad w' = -\frac{\nu}{\gamma}(\tilde{u}^m)' > 0. \tag{2.17}$$

Remark 1. *It follows from* (2.9)*,* (2.12) *and* (2.17) *that we can find the second derivative of weighted function w*

$$\left|w''\right| = \left|\frac{\nu}{\gamma}(\tilde{u}^m)''\right| \le \frac{4\nu\chi}{D\varpi}|\tilde{u}'(\zeta)| \le \frac{2\nu\chi\gamma^2}{D\varpi_-^2}e^{-\frac{C\chi\gamma|\zeta|}{D\varpi_-}}.$$

Moreover, for the first derivative of weighted function w, we have

$$\left|w'\right| = \left|\frac{\nu}{\gamma}(\tilde{u}^m)'\right| \le C\frac{\nu\gamma}{4\varpi_-}e^{-\frac{C\chi\gamma|\zeta|}{D\varpi_-}} \ \text{, where } C = m(u^-)^{m-1} > 0.$$

2.1. Relative entropy

We first rewrite the system (2.5) into the following general system of viscous conservative laws to employ the relative entropy method

$$\partial_t Z + \partial_\zeta[A(Z)] = \partial_\zeta[M(Z)\partial_\zeta \nabla\kappa(Z)], \tag{2.18}$$

where

$$\begin{aligned} Z &:= \begin{pmatrix} u \\ v \end{pmatrix}, \quad A(Z) := \begin{pmatrix} -\chi(uv) & -\varpi u \\ -u & -\varpi v \end{pmatrix}, \quad M(Z) := \begin{pmatrix} D\chi m u^m & 0 \\ 0 & 0 \end{pmatrix}, \\ \kappa(Z) &:= \frac{|v|^2}{2} + \Theta(u), \quad \Theta(u) := \frac{u}{\chi m}\log\left(\frac{\chi u^m}{m}\right) - \frac{u}{\chi}. \end{aligned} \tag{2.19}$$

Moreover, since

$$\nabla\kappa(Z) := (\partial_u\kappa(Z)\partial_v\kappa(Z)) = \begin{pmatrix} \frac{1}{\chi m}\log\left(\frac{\chi u^m}{m}\right) & v \end{pmatrix}, \tag{2.20}$$

then (2.5) is similar to (2.18). We notice that the entropy κ for system of (2.18) is strictly convex and the entropy function $G(Z)$ is the flux of entropy κ given as follows:

$$G(Z) := -\frac{vu}{m}\log\left(\frac{\chi u^m}{m}\right) - \varpi\kappa(Z), \tag{2.21}$$

such that $\partial_i G(Z) := \Sigma_{k=1}^2 \partial_k\kappa(Z)\partial_i A_k(Z), 1 \le i \le 2$. In general, for a given function r, we define $r(\cdot|\cdot)$ as its relative function of two variables by

$$r(k|l) := r(k) - r(l) - \nabla r(l)(k-l). \tag{2.22}$$

Since $Z_i := \begin{pmatrix} u_i \\ v_i \end{pmatrix}$, for $i = 1, 2$, then one has

$$\begin{aligned} A(Z_1|Z_2) &= A(Z_1) - A(Z_2) - \nabla A(Z_2)(Z_1 - Z_2) \\ &= \begin{pmatrix} -\chi(u_1-u_2)(v_1-v_2) \\ 0 \end{pmatrix}, \end{aligned} \tag{2.23}$$

and

$$\kappa(Z_1|Z_2) = \kappa(Z_1) - \kappa(Z_2) - \nabla\kappa(Z_2)(Z_1 - Z_2) = \frac{|v_1 - v_2|^2}{2} + \Theta(u_1|u_2), \tag{2.24}$$

where

$$\Theta(u_1|u_2) = \Theta(u_1) - \Theta(u_2) - \nabla\Theta(u_2)(u_1 - u_2). \tag{2.25}$$

It follows from $\Theta(u) := \frac{u}{\chi m}\log\left(\frac{\chi u^m}{m}\right) - \frac{u}{\chi}$, and we find that

$$\Theta(u_1|u_2) = \frac{u_1}{\chi m}\log\left(\frac{u_1}{u_2}\right)^m - \frac{1}{\chi}(u_1 - u_2).$$

Due to the law of logarithmic function, the above equation becomes

$$\Theta(u_1|u_2) = \frac{u_1}{\chi}\log\left(\frac{u_1}{u_2}\right) - \frac{1}{\chi}(u_1 - u_2). \tag{2.26}$$

The relative flux $G(\cdot;\cdot)$ for our relative entropy $\kappa(\cdot|\cdot)$ is defined as follows:

$$\begin{aligned} G(Z_1;Z_2) &:= G(Z_1) - G(Z_2) - \nabla\kappa(Z_2)(A(Z_1) - A(Z_2)) \\ &:= -\frac{\chi}{m}(v_1 - v_2)\Theta(u_1|u_2) - \frac{\chi}{m}v_2\Theta(u_1|u_2) - (u_1 - u_2)(v_1 - v_2) - \varpi\kappa(Z_1|Z_2). \end{aligned} \tag{2.27}$$

Global existence and uniqueness of solutions to (1.4) belonging to the space

$$\mathcal{M}_T := \left\{(u,v) \in L^\infty((0,T)\times\mathbb{R})^2 | u > 0, u^{-1} \in L^\infty((0,T)\times\mathbb{R}), u_x \in L^2((0,T)\times\mathbb{R})\right\}, \tag{2.28}$$

for each $T > 0$ are studied in [3].

Moreover, for any shift function $X : [0,\infty) \to \mathbb{R}$ and function $h : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$, we employ the following notation

$$h^{\pm X}(t,\zeta) := h(t, \zeta \pm X(t)). \tag{2.29}$$

We also introduce the following function space

$$\mathcal{K} := \left\{(u,v) \in (L^\infty(\mathbb{R}))^2 | u > 0, u^{-1} \in L^\infty(\mathbb{R}), \left(\log\left(\frac{u}{\tilde{u}}\right)^m\right)_\zeta \in L^2(\mathbb{R})\right\}. \tag{2.30}$$

Remark 2. *We suppose the solution of Z to* (1.4) *in $\mathcal{M}_T$. Since $(u_\zeta, \tilde{u}') \in L^2((0,T)\times\mathbb{R}) \times L^2(\mathbb{R})$ and $(u^{-1}, \tilde{u}) \in L^\infty((0,T)\times\mathbb{R}) \times L^\infty(\mathbb{R})$, one gets*

$$\left(\log\left(\frac{u}{\tilde{u}}\right)^m\right)_\zeta \in L^2((0,T)\times\mathbb{R}), \tag{2.31}$$

which implies $Z(t) \in \mathcal{K}$ for almost everywhere $t \in [0,T]$.

3. Main results

The following useful inequalities of the relative quantity $\Theta(\cdot|\cdot)$ are useful throughout this paper. The construction of shift function X is unique and exists by Picard's iteration.

Lemma 2. *For given constants $\delta \in (0, \frac{1}{2}]$ and $u^- > 0$, there exists positive constants $C_1(\delta), C_2(\delta)$ and $C_3(\delta)$, such that the following ones are true:*
(1) *For any $u_1 > 0$ and $u_2 > 0$ with $\frac{u^-}{2} < u_2 < u^-$,*

$$\frac{1}{C_1(\delta)}|u_1 - u_2|^2 \le \Theta(u_1|u_2) \le C_1(\delta)|u_1 - u_2|^2 \quad \textit{if} \quad \left|\frac{u_1}{u_2} - 1\right| \le \delta, \tag{3.1}$$

$$\frac{1}{C_2(\delta)}\left(1 + u_1\log^+\frac{u_1}{u_2}\right) \le \Theta(u_1|u_2) \le C_2(\delta)\left(1 + u_1\log^+\frac{u_1}{u_2}\right) \quad \textit{if} \quad \left|\frac{u_1}{u_2} - 1\right| \ge \delta, \tag{3.2}$$

$$\frac{1}{C_3(\delta)}|u_1 - u_2| \leq \Theta(u_1|u_2) \leq C_3(\delta)|u_1 - u_2|^2 \quad \textit{if} \quad \left|\frac{u_1}{u_2} - 1\right| \geq \delta, \tag{3.3}$$

where $\log^+(r)$ *is the positive part of* $log(r)$*.*
(2) *For any* $u_1, u_2, s > 0$ *satisfying* $s \leq u_2 \leq u_1$ *or* $u_1 \leq u_2 \leq s$*,*

$$\Theta(u_1|s) \geq \Theta(u_2|s). \tag{3.4}$$

Proof. We first establish the proof of (3.1) by the relative function such that

$$\Theta(u_1|u_2) = (u_1 - u_2)^2 \int_0^1 \int_0^1 \Theta''(u_2 + st(u_1 - u_2))tdsdt.$$

It follows from (2.19) that one has $\Theta''(u) = \frac{1}{\chi u}$. Hence,

$$\Theta''(u_2 + st(u_1 - u_2)) = \frac{1}{\chi(stu_1 + (1 - st)u_2)}.$$

Since $\left|\frac{u_1}{u_2} - 1\right| \leq \delta \leq \frac{1}{2}$ and $\frac{u^-}{2} < u_2 < u^-$, we have

$$\frac{1}{2} \leq \frac{u_1}{u_2} \leq \frac{3}{2} \Rightarrow \frac{u_2}{2} \leq u_1 \leq \frac{3u_2}{2} \Rightarrow \frac{u^-}{4} \leq u_1 \leq \frac{3u^-}{2}.$$

Therefore, for any $0 \leq s, t \leq 1$,

$$\frac{1}{\chi\left(st\frac{3u^-}{2} + (1 - st)u^-\right)} \leq \Theta''(u_2 + st(u_1 - u_2)) \leq \frac{1}{\chi\left(st\frac{u^-}{4} + (1 - st)\frac{u^-}{2}\right)}.$$

Hence

$$c_1(u_1 - u_2)^2 \leq \Theta(u_1|u_2) \leq c_2(u_1 - u_2)^2,$$

where the constant c_1, c_2 only depend on u^- as

$$c_1 = \int_0^1 \int_0^1 \frac{1}{\chi\left(st\frac{3u^-}{2} + (1 - st)u^-\right)}dsdt, \ c_2 = \int_0^1 \int_0^1 \frac{1}{\chi\left(st\frac{u^-}{4} + (1 - st)\frac{u^-}{2}\right)}dsdt.$$

To establish the proof of (3.2). It follows from (2.26) that we have

$$\Theta(u_1|u_2) = \frac{u_2}{\chi}\hat{\Theta}\left(\frac{u_1}{u_2}\right), \quad \hat{\Theta}(r) := r\log r - (r - 1) \ \text{ for } \ r > 0. \tag{3.5}$$

Hence, according to (3.5), one has nonnegative and smooth relative function $\hat{\Theta}$ on $(0, \infty)$. Moreover, $\hat{\Theta}'' = 1/r > 0$, which means strictly convex on $(0, \infty)$, and $\hat{\Theta}(r) = 0$ when $r = 1$, which is a critical point. We will first estimate $\hat{\Theta}(y)$ as follows:
For constant $\delta \in (0, 1/2]$ and $\hat{\Theta}' = \log r < 0$ (monotone decreasing) for all $r \in (0, 1 - \delta]$,

$$\lim_{s\to 0+} \hat{\Theta}(s) = 1 \geq \hat{\Theta}(r) \geq \hat{\Theta}(1 - \delta) > 0, \ \ \text{for all} \ \ r \in (0, 1 - \delta]. \tag{3.6}$$

Moreover, for any δ, since $\hat{\Theta}' = \log r > 0$ (monotone increasing) for all $r \in [1+\delta, \infty)$, one has

$$\hat{\Theta}(1+\delta) \geq \hat{\Theta}(r) > 0, \quad \text{for all} \quad r \in [1+\delta, \infty).$$

We notice that $\hat{\Theta}$ is nonnegative from (3.5), which gives for all $r \in [1+\delta, \infty)$,

$$\hat{\Theta}(1+\delta) \geq \hat{\Theta}(r) = r\log r + 1 - r \geq \frac{1}{C(\delta)}(r\log r + 1) > 0, \tag{3.7}$$

and also

$$C(\delta)(r\log r + 1) \geq \hat{\Theta}(1+\delta) \geq \hat{\Theta}(r) = r\log r + 1 - r > 0. \tag{3.8}$$

Combining (3.6)–(3.8) for a constant $C(\delta) > 0$, and for all $|r-1| \geq \delta$, one has

$$\frac{1}{C(\delta)}(1 + r\log^+ r) \leq \hat{\Theta}(r) \leq \lim_{r\to 0^+} \hat{\Theta}(r) = 1 \leq C(\delta)(1 + r\log^+ r), \tag{3.9}$$

where sign + of $\log^+ r$ indicates that $\hat{\Theta}(r)$ is nonnegative on $(0, \infty)$. Therefore, by combining (3.9) together with (3.5) and $\frac{u^-}{2} < u_2 < u^-$, the proof of (3.2) is completed.
Proof of (3.3). By the similar way with the proof of (3.2). For a constant $C(\delta) > 0$, one has

$$\frac{1}{C(\delta)}|r-1| \leq \hat{\Theta}(r) \leq C(\delta)|r-1|^2 \quad \text{for any } |r-1| \geq \delta.$$

Proof of (3.4). We have that $d \mapsto \Theta(d|r)$ is convex when $d > 0$ and zero at $d = r$. Moreover, $d \mapsto \Theta(d|r)$ is increasing, implying that

$$\begin{aligned}\Theta(u_1|s) - \Theta(u_2|s) &= \Theta(u_1) - \Theta(s) - \nabla(s)(u_1 - s) - (\Theta(u_2) - \Theta(s) - \nabla(s)(u_2 - s)) \\ &= \Theta(u_1) - \Theta(u_2) - \nabla(s)(u_1 - u_2) \geq 0.\end{aligned}$$

Hence, $\Theta(u_1|s) \geq \Theta(u_2|s)$. □

For any fixed constant $\gamma > 0$, we present the following continuous function Φ_γ

$$\Phi_\gamma = \begin{cases} \frac{1}{\gamma^2} \ \text{ if } \ y \leq -\gamma^2, \\ -\frac{1}{\gamma^4}y \ \text{ if } \ |y| \leq \gamma^2, \\ -\frac{1}{\gamma^2} \ \text{ if } \ y \geq -\gamma^2. \end{cases} \tag{3.10}$$

For almost everywhere $t \in [0, T]$, we define a shift function $X(t)$ as the solution of nonlinear ODE

$$\begin{cases} \dot{X}(t) = \Phi_\gamma(\mathcal{L}(Z^X))(2|\mathcal{I}^{bad}(Z^X)| + 1), \\ X(0) = 0, \end{cases} \tag{3.11}$$

where $Z \in \mathcal{M}_T$ and the functionals of $\mathcal{L}$ and $\mathcal{I}^{bad}$ are as in (3.42).

We assume the righthand side of (3.11) by $H(t, X)$, then there exist functions of $\beta_1, \beta_2 \in L^2(0, T)$ such that for $t \in [0, T]$, one has

$$\sup_{x\in\mathbb{R}} |H(t,x)| \leq \beta_1(t) \quad \text{and} \quad \sup_{x\in\mathbb{R}} |D_x H(t,x)| \leq \beta_2(t).$$

By employing $Z \in \mathcal{M}_T$ and the changing of variables $\zeta \to \zeta - X(t)$ as in Lemma 7, we can establish the existence of local solution, continuity and uniqueness by Picard's iteration for shift function X satisfying (3.11).

3.1. Existence and uniqueness of shift function X

Let $\gamma > 0$ and $Z \in \mathcal{M}_T$, then we can define $H : [0,T] \times \mathbb{R} \to \mathbb{R}$ as follows:

$$H(t,X) = \Phi_\gamma(\mathcal{L}(Z^X))(2|\mathcal{I}^{bad}(Z^X)| + 1), \tag{3.12}$$

where Φ_γ is defined in (3.10) and $\mathcal{L}, \mathcal{I}^{bad}$ are as in (3.42).

It follows from the existence in section two and weighted function in (2.14) and (2.15), then, $\Phi_\gamma, w, \tilde{u}$ and $1/\tilde{u}$ are bounded, and $\tilde{u}'', w', \tilde{u}'$ and $\tilde{v}'$ are bounded and integrable. Moreover, we choose the forth term of $\mathcal{I}^{bad}(Z)$ in (3.42) that

$$\int_{\mathbb{R}} \frac{D}{\chi m}\left(w\frac{(\tilde{u}^m)'}{\tilde{u}^m} - w'\right) u^m \log\left(\frac{u^x}{\tilde{u}}\right)^m \partial_\zeta \log\left(\frac{u^x}{\tilde{u}}\right)^m d\zeta. \tag{3.13}$$

Substituting (3.13) into (3.12) and using (3.10), one has

$$|H(t,x)| \leq \frac{1}{\gamma^2}\left(2\int_{\mathbb{R}} \frac{D}{\chi m}\left(w\frac{(\tilde{u}^m)'}{\tilde{u}^m} - w'\right) u^m \log\left(\frac{u^x}{\tilde{u}}\right)^m \partial_\zeta \log\left(\frac{u^x}{\tilde{u}}\right)^m d\zeta + 1\right). \tag{3.14}$$

Due to the law of logarithmic function, one has

$$\frac{2D}{\chi m}\log\left(\frac{u^x}{\tilde{u}}\right)^m = \frac{2D}{\chi}\log\left(\frac{u^x}{\tilde{u}}\right). \tag{3.15}$$

We further employ (3.15) into (3.14), and one has

$$\begin{aligned} |H(t,x)| &\leq C\sqrt{\int_{\mathbb{R}} u^x \left(\log\frac{u^x}{\tilde{u}}\right)^2} + C\sqrt{\int_{\mathbb{R}} w(u^x)^m \left|\partial_\zeta \log\left(\frac{u^x}{\tilde{u}}\right)^m\right|^2 d\zeta} + 1 \\ &\leq C\sqrt{\int_{\mathbb{R}} 1 + u^x\left(\log^+\frac{u^x}{\tilde{u}}\right)^2 \mathbf{1}_{\{|(u^x/\tilde{u})-1|>\delta\}}} + C\sqrt{\int_{\mathbb{R}} w(u^x)^m \left|\partial_\zeta \log\left(\frac{u^x}{\tilde{u}}\right)^m\right|^2 d\zeta} + 1 \\ &\leq C\left(\sqrt{Dw(u^x)^m \left|\partial_\zeta \log\left(\frac{u^x}{\tilde{u}}\right)^m\right|^2} + 1\right) \quad \text{for } t \in [0,T] \text{ and } x \in \mathbb{R}. \end{aligned} \tag{3.16}$$

Due to $\sup_{x\in\mathbb{R}} Dw(u^x)^m \left|\partial_\zeta \log\left(\frac{u^x}{\tilde{u}}\right)^m\right|^2 \leq C\left(Dwu^m \left|\partial_\zeta \log\left(\frac{u}{\tilde{u}}\right)^m\right|^2 + 1\right)$ for each $t \in [0,T]$ and $Dwu^m \left|\partial_\zeta \log\left(\frac{u}{\tilde{u}}\right)^m\right|^2 \in L^1(0,T)$, one has

$$\sup_{x\in\mathbb{R}} |H(t,x)| \leq \beta_1(t), \tag{3.17}$$

for some function $\beta_1 \in L^2(0,T)$.

By a similar way, for some function $\beta_2(t) \in L^2(0,T)$, we can also estimate

$$\sup_{x\in\mathbb{R}} |D_x H(t,x)| \leq \beta_2(t), \quad \text{for } t \in [0,T]. \tag{3.18}$$

Based on the results of (3.17) and (3.18), we can derive the following lemma (the so-called Cauchy-Lipschitz theorem).

Lemma 3. *Consider $p > 1$, $T > 0$ and the function $H : [0,T] \times \mathbb{R} \to \mathbb{R}$ holds the following estimates:*

$$\sup_{x\in\mathbb{R}} |H(t,x)| \le \beta_1(t), \quad and \quad \sup_{x,y\in\mathbb{R},x\neq y} \left|\frac{H(t,x)-H(t,y)}{x-y}\right| \le \beta_2(t) \ \ for \ \ t \in [0,T], \tag{3.19}$$

for some function $(\beta_1(t), \beta_2(t)) \in L^1(0,T) \times L^p(0,T)$. Thus, for any $x_0 \in \mathbb{R}$, one has the shift function $X : [0,T] \to \mathbb{R}$ which is unique continuous and satisfies

$$\begin{cases} \dot{X}(t) = H(t, X(t)) \ \ for\ a.e \ \ t \in [0,T], \\ X(0) = x_0. \end{cases} \tag{3.20}$$

Proof. Note that (3.20) is equivalent to

$$X(t) = x_0 + \int_0^t H(r, X(r))dr \quad \text{for } t \in [0,T], \tag{3.21}$$

then, the proof is from the following Picard's iteration

$$\begin{aligned} x_0(t) &= x_0, \\ x_{n+1}(t) &= x_0 + \int_0^t H(r, x_n(r))dr \quad \text{for } n \ge 0. \end{aligned} \tag{3.22}$$

Moreover, $\beta_1 \in L^1$, $x_n : [0,T] \to \mathbb{R}$ is continuous, and the following one is satisfied

$$\begin{aligned} \|x_n - x_0\|_{L^\infty(0,T)} &= \sup_{x_{n-1}\in\mathbb{R}} \left| \int_0^T H(r, x_{n-1}(r))\, ds \right| \\ &\le \int_0^T \sup_{x_{n-1}\in\mathbb{R}} |H(r, x_{n-1}(r))|\, ds \\ &\le \int_0^T \beta_1(r)dr = \|\beta_1\|_{L^1(0,T)} \quad \text{for each } n \ (\textbf{using the fact}, \sup_{x\in\mathbb{R}} |H(t,x)| \le \beta_1(t)). \end{aligned}$$

For $\beta_2 \in L^p$ with $p > 1$ by taking $t_* > 0$, we have $\|\beta_2\|_{L^p(0,T)} .(t_*)^{1-(1/p)} \le \frac{1}{2}$ and $t_* \le T$. Thus, for every $n \ge 1$, one has

$$\begin{aligned} \|x_{n+1} - x_n\|_{L^\infty(0,t_*)} &= \sup_{x_n, x_{n-1}\in\mathbb{R}, x_n\neq x_{n-1}} \left| \int_0^{t_*} [H(r, x_n(r)) - H(r, x_{n-1}(r))]\, dr \right| \\ &\le \int_0^{t_*} \sup_{x_n, x_{n-1}\in\mathbb{R}, x_n\neq x_{n-1}} |H(r, x_n(r)) - H(r, x_{n-1}(r))|\, dr. \end{aligned}$$

By using the fact that $\sup_{x,y\in\mathbb{R},x\neq y} |H(t,x) - H(t,y)| \le \sup_{x,y\in\mathbb{R},x\neq y} |x-y|\beta_2(t)$, one has

$$\begin{aligned} &\|x_{n+1} - x_n\|_{L^\infty(0,t_*)} \\ &\le \int_0^{t_*} \sup_{x_n, x_{n-1}\in\mathbb{R}, x_n\neq x_{n-1}} |x_n(r) - x_{n-1}(r)|\beta_2(r)dr \\ &\le \sup_{x_n, x_{n-1}\in\mathbb{R}, x_n\neq x_{n-1}} \left| \int_0^{t_*} [x_n(r) - x_{n-1}(r)]dr \right| \int_0^{t_*} \beta_2(r)dr = \|x_n - x_{n-1}\|_{L^\infty(0,t_*)} \int_0^{t_*} \beta_2(r)dr \\ &\le \|x_n - x_{n-1}\|_{L^\infty(0,t_*)} \int_0^T (\beta_2(r)^p)^{\frac{1}{p}} (t_*)^{1-\frac{1}{p}}\, dr = \|x_n - x_{n-1}\|_{L^\infty(0,t_*)} \|\beta_2\|_{L^p(0,T)} (t_*)^{1-\frac{1}{p}} \\ &\le \frac{1}{2} \|x_n - x_{n-1}\|_{L^\infty(0,t_*)} . \end{aligned}$$

From the above calculation, one gets

$$\|x_{n+1} - x_n\|_{L^\infty(0,t_*)} \leq \frac{1}{2} \|x_n - x_{n-1}\|_{L^\infty(0,t_*)}.$$

By repeating the above inequality and employing $t_* \leq T$, we have

$$\begin{aligned} \|x_{n+1} - x_n\|_{L^\infty(0,T)} &\leq \frac{1}{2} \|x_n - x_{n-1}\|_{L^\infty(0,T)} \leq \frac{1}{2^2} \|x_{n-1} - x_{n-2}\|_{L^\infty(0,t_*)} \\ &\leq \frac{1}{2^3} \|x_n - x_{n-1}\|_{L^\infty(0,T)} \ldots \leq \frac{1}{2^n} \|x_1 - x_0\|_{L^\infty(0,T)} \leq \frac{1}{2^n} \|\beta_1\|_{L^1(0,T)}. \end{aligned}$$

Thus, one has $\|x_{n+1} - x_n\|_{L^\infty(0,T)} \leq \frac{1}{2^n} \|\beta_1\|_{L^1(0,T)}$. Finally, we can prove the existence of limit $X : [0, t_*] \to \mathbb{R}$ of the sequence $x_n : [0, t_*] \to \mathbb{R}_{n=1}^\infty$ satisfying (3.21) for every $0 \leq t \leq t_*$, then, we are concerned with the uniqueness of the shift function. Let $q_1(t)$ and $q_2(t)$ be two solutions satisfying (3.20) and also $k(t) = q_1(t) - q_2(t)$. We have to show that $k(t) = 0$, which means that $q_1(t) = q_2(t)$. From (3.22) one has

$$\begin{aligned} \|k(t)\|_{L^\infty(0,t)} = \|q_1(t) - q_2(t)\|_{L^\infty(0,t)} &= \sup_{q_1,q_2\in\mathbb{R},q_1\neq q_2} \left| \int_0^t [H(r, q_1(r)) - H(r, q_2(r))]\, dr \right| \\ &\leq \int_0^t \sup_{q_1,q_2\in\mathbb{R},q_1\neq q_2} |q_1(r) - q_2(r)| \beta_2(r) dr \ \ \textbf{(using Cauchy-Lipschitz theorem)} \\ &\leq \int_0^t \sup_{q_1,q_2\in\mathbb{R},q_1\neq q_2} |q_1(r) - q_2(r)|\, dr \int_0^t \beta_2(r) dr = \int_0^t \sup_{k\in\mathbb{R}} |k(r) dr| \int_0^t \beta_2(r) dr \\ &\leq \|k\|_{L^\infty(0,t)} \int_0^T (\beta_2(r)^p)^{\frac{1}{p}} (t_*)^{1-\frac{1}{p}} dr = \|k\|_{L^\infty(0,t)} \|\beta_2\|_{L^p(0,T)} \\ &\leq \frac{1}{2} \|k\|_{L^\infty(0,t)}. \end{aligned}$$

Thus, $\|k\|_{L^\infty(0,t)} = 0$ is the only one solution satisfying the above inequality. □

We present a lemma of a uniform bound of $w'\kappa(Z|\tilde{Z})$, which is useful to prove Lemma 9 where this Lemma 9 has an important role to establish Theorem 1.

Lemma 4. *Let $\delta_0 \in (0, 1/2)$, $C > 0$, and for any $\gamma, \nu > 0$ with $\nu \in (\delta_0^{-1}\gamma, \delta_0)$, then the following one is satisfied:*

$$\int_{\mathbb{R}} w'\Theta(u|\tilde{u})\, d\zeta + \int_{\mathbb{R}} w' \frac{|v - \tilde{v}|^2}{2}\, d\zeta \leq C\frac{\gamma^2}{\nu}, \tag{3.23}$$

where $|\mathcal{L}(Z)| \leq \gamma^2$ and $Z \in \mathcal{K}$.

Proof. By using $w' = -\frac{\nu}{\gamma}(\tilde{u}^m)'$ and $\tilde{v}' = -\frac{\tilde{u}'}{\varpi}$, we rewrite $\mathcal{L}(Z)$ in (3.42) as

$$\mathcal{L}(Z) = -\int_{\mathbb{R}} w' \left(\frac{|v - \tilde{v}|^2}{2} + \Theta(u|\tilde{u}) \right) d\zeta - \frac{\gamma}{\nu m \tilde{u}^{m-1}} \int_{\mathbb{R}} w w' \left(\frac{u - \tilde{u}}{\tilde{u}} - \frac{v - \tilde{v}}{\varpi} \right) d\zeta, \tag{3.24}$$

then, one has

$$\begin{aligned}
&\int_{\mathbb{R}} w'\kappa(Z|\tilde{Z})\, d\zeta \\
&\le |\mathcal{L}(Z)| + C\frac{\gamma}{\nu}\int_{\mathbb{R}} ww' \left|\frac{u-\tilde{u}}{\tilde{u}} - \frac{v-\tilde{v}}{\varpi}\right| d\zeta \\
&\le \gamma^2 + C\frac{\gamma}{\nu}\int_{\{|(u/\tilde{u})-1|\le 1/2\}} w'|u-\tilde{u}|\, d\zeta + C\frac{\gamma}{\nu}\int_{\{|(u/\tilde{u})-1|>1/2\}} w'|u-\tilde{u}|\, d\zeta + C\frac{\gamma}{\nu}\int_{\mathbb{R}} w'|v-\tilde{v}|\, d\zeta.
\end{aligned}$$

Applying (3.1) and (3.3) to the above inequality, we get

$$\begin{aligned}
&\int_{\mathbb{R}} w'\kappa(Z|\tilde{Z})\, d\zeta \\
&\le \gamma^2 + C\frac{\gamma}{\nu}\sqrt{\int_{\{|(u/\tilde{u})-1|\le 1/2\}} w'|u-\tilde{u}|^2\, d\zeta}.\sqrt{\int_{\mathbb{R}} w'\, d\zeta} \\
&\quad + C\frac{\gamma}{\nu}\sqrt{\int_{\mathbb{R}} w'|v-\tilde{v}|^2\, d\zeta}.\sqrt{\int_{\mathbb{R}} w'\, d\zeta} + C\frac{\gamma}{\nu}\int_{\{|(u/\tilde{u})-1|>1/2\}} w'|u-\tilde{u}|\, d\zeta.
\end{aligned}$$

Hence

$$\begin{aligned}
&\int_{\mathbb{R}} w'\kappa(Z|\tilde{Z})\, d\zeta \\
&\le \gamma^2 + C\frac{\gamma}{\nu}\sqrt{\int_{\{|(u/\tilde{u})-1|\le 1/2\}} w'\Theta(u|\tilde{u})\, d\zeta} + C\frac{\gamma}{\sqrt{\nu}}\sqrt{\int_{\mathbb{R}} w'|v-\tilde{v}|^2\, d\zeta} \\
&\quad + C\delta_0\int_{\{|(u/\tilde{u})-1|>1/2\}} w'\Theta(u|\tilde{u})\, d\zeta \le C\frac{\gamma^2}{\nu} + \frac{1}{2}\int_{\mathbb{R}} w'\kappa(Z|\tilde{Z})\, d\zeta,
\end{aligned}$$

then, one gets

$$\int_{\mathbb{R}} w'\kappa(Z|\tilde{Z})\, d\zeta \le C\frac{\gamma^2}{\nu},$$

which implies that the proof is completed. □

We further present the lemma as follows to get the estimation of the term $|u-\tilde{u}|\mathbf{1}_{\{|(u/\tilde{u})-1|>\delta\}}$.

Lemma 5. *Let $\delta_0 \in (0, 1/2)$ be a sufficiently small constant and $C > 0$, then for any $\gamma, \nu > 0$ with $\delta_0^{-1}\gamma < \nu < \delta_0$, and for any $Z \in \mathcal{K}$ satisfying $|\mathcal{L}(Z)| \le \gamma^2$, the following estimates hold:*

$$|u(\zeta) - \tilde{u}(\zeta)| \le C\left(\frac{1}{\gamma} + |\zeta|\right)\mathcal{D}(Z), \tag{3.25}$$

whenever $\zeta \in \mathbb{R}$ satisfies

$$\left|\frac{u(\zeta)}{\tilde{u}(\zeta)} - 1\right| \ge \delta. \tag{3.26}$$

Proof. We set $\gamma := \frac{1}{\nu}\int_{-1/\gamma}^{1/\gamma} w'\, d\zeta$. By using $\frac{1}{\nu}\int_{\mathbb{R}} a'\, d\zeta = 1$ and $w' = \nu/\gamma|(\tilde{u}^m)'|$ together with (2.9), gives

$$\frac{D}{4C\chi}\left(1 - e^{-\frac{C\chi}{D\varpi_-}}\right) \leq \gamma \leq 1.$$

It follows from (3.23) that

$$\int_{-1/\gamma}^{1/\gamma} w'\Theta(u|\tilde{u})\, d\zeta \leq C\frac{\gamma^2}{\nu},$$

which gives

$$\int_{-1/\gamma}^{1/\gamma} \frac{w'}{\nu\gamma}\Theta(u|\tilde{u})\, d\zeta \leq C\left(\frac{\gamma}{\nu}\right)^2.$$

Since $\int_{-1/\gamma}^{1/\gamma} \frac{w'}{\nu\gamma}\, d\zeta = 1$, there exists a point $\zeta \in [-\frac{1}{\gamma}, \frac{1}{\gamma}]$ such that

$$\Theta(u(\zeta_0)|\tilde{u}(\zeta_0)) \leq \tilde{C}\left(\frac{\gamma}{\nu}\right)^2 \leq \tilde{C}(\delta_0)^2, \tag{3.27}$$

for some constant $\tilde{C}$.

We assume small enough δ_0 to get

$$\tilde{C}(\delta_0)^2 \leq C_2/2,$$

where constant C_2 is defined in (3.2).

It follows from the lower bound of (3.1) and (3.2) that

$$\begin{aligned}\Theta(u(\zeta_0)|\tilde{u}(\zeta_0)) &\geq min\left(C_1^{-1}|u(\zeta_0) - \tilde{u}(\zeta_0)|^2, C_2^{-1}\left(1 + u(\zeta_0)\log^+ \frac{u(\zeta_0)}{\tilde{u}(\zeta_0)}\right)\right)\\ &\geq min\left(C_1^{-1}|u(\zeta_0) - \tilde{u}(\zeta_0)|^2, C_2^{-1}\right),\end{aligned}$$

and together with (3.27) it gives

$$|u(\zeta_0) - \tilde{u}(\zeta_0)| \leq \sqrt{C_1\Theta(u(\zeta_0)|\tilde{u}(\zeta_0))} \leq \sqrt{C_1}\tilde{C}(\delta_0).$$

For small enough δ_0, we assume

$$\left|\frac{u(\zeta_0)}{\tilde{u}(\zeta_0)} - 1\right| \leq min\left\{\frac{\delta}{2}, \frac{(\sqrt{1+\delta} - 1)(\sqrt{1-(\delta/2)} + 1)}{2}\right\}. \tag{3.28}$$

For any $\zeta \in \mathbb{R}$ and ζ_0,

$$\begin{aligned}\left|\sqrt{\frac{u(\zeta)}{\tilde{u}(\zeta)}} - \sqrt{\frac{u(\zeta_0)}{\tilde{u}(\zeta_0)}}\right| &= \left|\int_{\zeta_0}^{\zeta} \frac{d}{d\zeta}\sqrt{\frac{u(\zeta)}{\tilde{u}(\zeta)}}\, d\zeta\right|\\ &= \left|\int_{\zeta_0}^{\zeta} \frac{1}{2}\sqrt{\frac{\tilde{u}(\zeta)}{u(\zeta)}}\frac{d}{d\zeta}\frac{u(\zeta)}{\tilde{u}(\zeta)}\, d\zeta\right|\\ &= \left|\int_{\zeta_0}^{\zeta} \frac{1}{2m}\sqrt{\frac{u(\zeta)}{\tilde{u}(\zeta)}}\frac{d}{d\zeta}\log\left(\frac{u(\zeta)}{\tilde{u}(\zeta)}\right)^m\, d\zeta\right|,\end{aligned}$$

which gives

$$\begin{aligned}
&\left|\sqrt{\frac{u(\zeta)}{\tilde{u}(\zeta)}}-\sqrt{\frac{u(\zeta_0)}{\tilde{u}(\zeta_0)}}\right| \\
&\leq \sqrt{\int_{\zeta_0}^{\zeta}\frac{1}{4m^2Dw(\zeta)u(\zeta)}\frac{u}{\tilde{u}}(\zeta)\,d\zeta}\sqrt{\int_{\zeta_0}^{\zeta}Dw(\zeta)u(\zeta)^m\left|\frac{d}{d\zeta}\log\left(\frac{u}{\tilde{u}}(\zeta)\right)^m\right|^2\,d\zeta} \\
&\leq \sqrt{\frac{C}{2u^-}\int_{\zeta_0}^{\zeta}1\,d\zeta}\sqrt{\mathcal{D}(Z)} \\
&\leq \sqrt{\frac{C}{2u^-}}\sqrt{|\zeta-\zeta_0|}\sqrt{\mathcal{D}(Z)}\leq\sqrt{\frac{C}{2u^-}}\sqrt{|\zeta|+\frac{1}{\gamma}}\sqrt{\mathcal{D}(Z)}.
\end{aligned} \tag{3.29}$$

We then assume that $L = L(\delta) > 0$ exists, then if $r > 0, r_0 > 0$ with

$$|r_0-1|\leq min\left\{\frac{\delta}{2},\frac{(\sqrt{1+\delta}-1)(\sqrt{1-(\delta/2)}+1)}{2}\right\}\ \text{ and }\ |r-1|\geq\delta,$$

one has

$$|r-1|\leq L|\sqrt{r}-\sqrt{r_0}|^2. \tag{3.30}$$

We consider that $\psi := |\sqrt{r}-\sqrt{r_0}|$. Since $|r-1|\geq\delta$, then there are two cases:
(a) For case $0 < r \leq 1-\delta$.
Since $r\leq 1-\delta < 1-(\delta/2)\leq r_0\leq 1+(\delta/2)$, one has

$$\frac{\delta}{2}\leq|(r-1)-(r_0-1)|=|r-r_0|\leq\psi|\sqrt{r}+\sqrt{r_0}|\leq 2\psi\sqrt{r_0}\leq 2\psi\sqrt{1+(\delta/2)}\leq 4\psi,$$

which gives $1 \leq (8\psi/\delta)$. Therefore,

$$|r-1| = 1-r\leq 1 = 1^2\leq\frac{64\psi^2}{\delta^2}. \tag{3.31}$$

(b) For case $r \geq 1+\delta$.
Since

$$|\sqrt{r_0}-1|=\frac{|r_0-1|}{\sqrt{r_0}+1}\leq\frac{|r_0-1|}{\sqrt{1-(\delta/2)}+1}\leq\frac{\sqrt{1+\delta}-1}{2}\leq\frac{\sqrt{r}-1}{2},$$

one has

$$\begin{aligned}
\psi=|\sqrt{r}-\sqrt{r_0}|=|(\sqrt{r}-1)-(\sqrt{r_0}-1)|\leq|\sqrt{r}-1|-|\sqrt{r_0}-1|\\
\geq\frac{\sqrt{r}-1}{2}.
\end{aligned} \tag{3.32}$$

Therefore, one yields

$$1+\delta\leq r\leq(2\psi+1)^2,$$

which gives

$$0 < \delta \le (4\psi^2 + 4\psi + 1) - 1 = 4\psi(\psi + 1).$$

Let $\psi_0 = \psi_0(\delta)$ be the positive constant satisfying $4\psi_0(\psi_0 + 1) = \delta$.

Since $4\psi_0(\psi_0 + 1) \le 4\psi(\psi + 1)$, one has $1 \le (\psi/\psi_0)$, then by (3.32), one has

$$\begin{aligned}|r - 1| = r - 1 = (\sqrt{r} - 1)((\sqrt{r} - 1) + 2) \le 4\psi(\psi + 1)\\ \le 4\psi\left(\psi + \frac{\psi}{\psi_0}\right) = 4\psi^2\left(1 + \frac{1}{\psi_0}\right).\end{aligned}$$

It follows from (3.31), (3.32) and $\psi := |\sqrt{r} - \sqrt{r_0}|$ that the claims in (3.30) are completed by taking $L := (64\psi^2/\delta^2) + 4(1 + (1/\psi_0))$. By considering $r := (u/\tilde{u})(\zeta)$ and $r_0 := (u/\tilde{u})(\zeta_0)$ and employing (3.29) and (3.30), one has

$$|u(\zeta) - \tilde{u}(\zeta)| = |\tilde{u}(\zeta)|L|\sqrt{r} - \sqrt{r_0}|^2 \le u^-|L|\sqrt{r} - \sqrt{r_0}|^2 \le C\left(\frac{1}{\gamma} + |\zeta|\right)\mathcal{D}(Z),$$

which means that the proof is established. □

Lemma 6. *Under Lemma 5, one has*

$$\int_{\mathbb{R}} w'\left(1 + \left[u\log^+\frac{u}{\tilde{u}}\right]^2\right)\mathbf{1}_{\{|(u/\tilde{u})-1|\ge\delta\}}\,d\zeta \le C\sqrt{\frac{\gamma}{\nu}}\mathcal{D}(Z), \tag{3.33}$$

$$\int_{\mathbb{R}} w'\left(1 + u\log^+\frac{u}{\tilde{u}}\right)\mathbf{1}_{\{|(u/\tilde{u})-1|\ge\delta\}}\,d\zeta \le C\sqrt{\frac{\gamma}{\nu}}\mathcal{D}(Z), \tag{3.34}$$

$$\int_{\mathbb{R}} w'|v - \tilde{v}|\left(1 + u\log^+\frac{u}{\tilde{u}}\right)\mathbf{1}_{\{|(u/\tilde{u})-1|\ge\delta\}}\,d\zeta \le C\sqrt{\frac{\gamma}{\nu}}\mathcal{D}(Z). \tag{3.35}$$

Proof. (a) Proof of (3.33). Since $|(u/\tilde{u}) - 1| \ge \delta$, then there are two cases,
For case u satisfying $(u/\tilde{u}) - 1 \le -\delta$, then one gets

$$\log^+\frac{u}{\tilde{u}} = 0,$$

and

$$|u - \tilde{u}| = \tilde{u} - u \ge \delta\tilde{u} \ge \left(\delta\frac{u^-}{2}\right) > 0, \quad \Theta(u|\tilde{u}) \ge C_2 > 0.$$

Since $C = C(\delta) > 0$, one has

$$\left(1 + u\left(\log^+\frac{u}{\tilde{u}}\right)^2\right)\mathbf{1}_{\{(u/\tilde{u})-1\le-\delta\}} \le C\sqrt{\Theta(u|\tilde{u})}|u - \tilde{u}|\mathbf{1}_{\{(u/\tilde{u})-1\le-\delta\}}.$$

For case u satisfying $(u/\tilde{u}) - 1 \ge \delta$ together with (3.2), one has

$$\Theta(u|\tilde{u}) \ge \left(1 + u\left(\log^+\frac{u}{\tilde{u}}\right)\right)\mathbf{1}_{\{(u/\tilde{u})-1\ge\delta\}} \ge \left(1 + u\left(\log^+(1 + \delta)\right)\right)\mathbf{1}_{\{(u/\tilde{u})-1\ge\delta\}}. \tag{3.36}$$

By using the inequality

$$\left(1+u\left(\log^+\frac{u}{\tilde u}\right)\right)\le 1+u\tau^2\left(\frac{u}{\tilde u}\right)^{1/6}\le 1+(\tau^2(2/u^-)^{1/6})u^{7/6},\quad \text{where}\ \ \tau:=\sup_{y\in[1+\delta,\infty)}\frac{\log y}{y^{1/12}}<\infty, \tag{3.37}$$

then for some constant $C=C(\delta)>0$, we get

$$\left(1+u\left(\log^+\frac{u}{\tilde u}\right)^2\right)\mathbf{1}_{\{(u/\tilde u)-1\ge\delta\}}\le\ \sqrt{\Theta(u|\tilde u)}|u-\tilde u|\mathbf{1}_{\{(u/\tilde u)-1\ge\delta\}}. \tag{3.38}$$

Moreover, if u is large from (3.36)–(3.38), then the lefthand side is bounded above by $C(1+Cu^{7/6})$ and righthand side is bounded below by $\frac{1}{C}(1+\frac{1}{C}u^{7/6})$. We further combine those two cases to get

$$\left(1+u\left(\log^+\frac{u}{\tilde u}\right)^2\right)\mathbf{1}_{\{|(u/\tilde u)-1|\ge\delta\}}\le\ \sqrt{\Theta(u|\tilde u)}|u-\tilde u|\mathbf{1}_{\{|(u/\tilde u)-1|\ge\delta\}}.$$

Therefore, one can derive

$$\begin{aligned}&\int_{\mathbb{R}}w'\left(1+u\left(\log^+\frac{u}{\tilde u}\right)^2\right)\mathbf{1}_{\{|(u/\tilde u)-1|\ge\delta\}}\\&\le\int_{\mathbb{R}}w'\sqrt{\Theta(u|\tilde u)}|u-\tilde u|\mathbf{1}_{\{|(u/\tilde u)-1|\ge\delta\}}\\&\le\int_{|\zeta|\le\frac{1}{\gamma}\sqrt{\frac{\nu}{\gamma}}}w'\sqrt{\Theta(u|\tilde u)}|u-\tilde u|\mathbf{1}_{\{|(u/\tilde u)-1|\ge\delta\}}+\int_{|\zeta|\ge\frac{1}{\gamma}\sqrt{\frac{\nu}{\gamma}}}w'\sqrt{\Theta(u|\tilde u)}|u-\tilde u|\mathbf{1}_{\{|(u/\tilde u)-1|\ge\delta\}}.\end{aligned}$$

From (3.25) and (3.23), $C=C(\delta)>0$ and the first term above is estimated as

$$\begin{aligned}&\int_{|\zeta|\le\frac{1}{\gamma}\sqrt{\frac{\nu}{\gamma}}}w'\sqrt{\Theta(u|\tilde u)}|u-\tilde u|\mathbf{1}_{\{|(u/\tilde u)-1|\ge\delta\}}\\&\le\left(\sup_{[-\sqrt{\nu/\gamma^3},\sqrt{\nu/\gamma^3}]}|u-\tilde u|\mathbf{1}_{\{|(u/\tilde u)-1|\ge\delta\}}\right)\int_{|\zeta|\le\frac{1}{\gamma}\sqrt{\frac{\nu}{\gamma}}}w'\sqrt{\Theta(u|\tilde u)}\\&\le\frac{C}{\gamma}\sqrt{\frac{\nu}{\gamma}\mathcal{D}(Z)}\int_{\mathbb{R}}a'\Theta(u|\tilde u)\,d\zeta\le C\sqrt{\frac{\gamma}{\nu}\mathcal{D}(Z)},\end{aligned}$$

and the second term

$$\begin{aligned}&\int_{|\zeta|\ge\frac{1}{\gamma}\sqrt{\frac{\nu}{\gamma}}}w'\sqrt{\Theta(u|\tilde u)}|u-\tilde u|\mathbf{1}_{\{|(u/\tilde u)-1|\ge\delta\}}\\&\le C\mathcal{D}(Z)\left(\int_{\mathbb{R}}w'\Theta(u|\tilde u)\,d\zeta\right)^{1/2}\left(\int_{|\zeta|\ge\frac{1}{\gamma}\sqrt{\frac{\nu}{\gamma}}}w'|\zeta|^2\,d\zeta\right)^{1/2}\\&\le C\mathcal{D}(Z)\sqrt{\frac{\gamma^2}{\nu}}\left(\int_{|\zeta|\ge\frac{1}{\gamma}\sqrt{\frac{\nu}{\gamma}}}w'|\zeta|^2\,d\zeta\right)^{1/2}.\end{aligned}$$

Note that

$$\int_{|\zeta|\geq\frac{1}{\gamma}\sqrt{\frac{\nu}{\gamma}}} w'|\zeta|^2\,d\zeta \leq C\gamma\nu\int_{|\zeta|\geq\frac{1}{\gamma}\sqrt{\frac{\nu}{\gamma}}} e^{-C\gamma|\zeta|}|\zeta|^2\,d\zeta$$
$$\leq C\frac{\nu}{\gamma^2}\int_{|\zeta|\geq\frac{1}{\gamma}\sqrt{\frac{\nu}{\gamma}}} e^{-C|\zeta|}|\zeta|^2\,d\zeta.$$

By taking small enough δ_0 such that for any $\gamma/\nu \leq \delta_0$, $|\zeta|^2 \leq e^{(C/2)|\zeta|}$ for $\zeta \geq \sqrt{\nu/\gamma}$ and

$$\int_{|\zeta|\geq\frac{1}{\gamma}\sqrt{\frac{\nu}{\gamma}}} e^{-C|\zeta|}|\zeta|^2\,d\zeta \leq \int_{|\zeta|\geq\sqrt{\frac{\nu}{\gamma}}} e^{-\frac{C}{2}|\zeta|}\,d\zeta = Ce^{-\frac{C}{2}\sqrt{\frac{\nu}{\gamma}}} \leq C\frac{\gamma}{\nu},$$

then the second term becomes

$$\int_{|\zeta|\geq\frac{1}{\gamma}\sqrt{\frac{\nu}{\gamma}}} w'\sqrt{\Theta(u|\tilde{u})}|u-\tilde{u}|\mathbf{1}_{\{|(u/\tilde{u})-1|\geq\delta\}} \leq C\sqrt{\frac{\gamma}{\nu}}\mathcal{D}(Z).$$

Thus, (3.33) is completed.
(b) Proof of (3.34). To establish (3.34), we apply the similar steps in (3.33) by using the following inequality

$$\log^+\frac{u}{\tilde{u}} \leq \frac{1}{\log(1+\delta)}\left(\log^+\frac{u}{\tilde{u}}\right)^2,$$

for $|(u/\tilde{u})-1| \geq \delta$, then (3.34) is completed.
(c) Proof of (3.35). It follows from (3.33) and by using the following inequality that

$$\log^+\frac{u}{\tilde{u}} \leq \frac{1}{\log(1+\delta)}\left(\log^+\frac{u}{\tilde{u}}\right)^2,$$

then for some constant $C > 0$, one can derive

$$\left(1+u\log^+\frac{u}{\tilde{u}}\right)\mathbf{1}_{\{(u/\tilde{u})-1\geq\delta\}} \leq \Theta(u|\tilde{u})^{1/4}|u-\tilde{u}|\mathbf{1}_{\{(u/\tilde{u})-1\geq\delta\}}.$$

For large u, the righthand side of the above inequality is bounded below by $\frac{1}{C}(1+\frac{1}{C}u^{5/4})$, then one has

$$\int_{\mathbb{R}} w'|u-\tilde{u}|\left(1+u\log^+\frac{u}{\tilde{u}}\right)\mathbf{1}_{\{(u/\tilde{u})-1\geq\delta\}}\,d\zeta$$
$$\leq \int_{|\zeta|\leq\frac{1}{\gamma}\sqrt{\frac{\nu}{\gamma}}} w'|u-\tilde{u}|\Theta(u|\tilde{u})^{1/4}|u-\tilde{u}|\mathbf{1}_{\{(u/\tilde{u})-1\geq\delta\}}\,d\zeta$$
$$+\int_{|\zeta|\geq\frac{1}{\gamma}\sqrt{\frac{\nu}{\gamma}}} w'|u-\tilde{u}|\Theta(u|\tilde{u})^{1/4}|u-\tilde{u}|\mathbf{1}_{\{(u/\tilde{u})-1\geq\delta\}}\,d\zeta.$$

Using the same steps as in (3.33),

$$\begin{aligned}
&\int_{|\zeta|\le\frac{1}{\gamma}\sqrt{\frac{\nu}{\gamma}}} w'|u-\tilde{u}|\Theta(u|\tilde{u})^{1/4}|u-\tilde{u}|\mathbf{1}_{\{(u/\tilde{u})-1\ge\delta\}}\,d\zeta\\
&\le C\frac{1}{\gamma}\sqrt{\frac{\nu}{\gamma}}\mathcal{D}(Z)\int_{\mathbb{R}} w'|u-\tilde{u}|\Theta(u|\tilde{u})^{1/2}\,d\zeta\\
&\le C\frac{1}{\gamma}\sqrt{\frac{\nu}{\gamma}}\mathcal{D}(Z)\left(\int_{\mathbb{R}} w'|u-\tilde{u}|^2\,d\zeta\right)^{1/2}\left(\int_{\mathbb{R}} w'\Theta(u|\tilde{u})\,d\zeta\right)^{1/2}\\
&\le C\sqrt{\frac{\nu}{\gamma}}\mathcal{D}(Z).
\end{aligned}$$

Since $|u-\tilde{u}|\Theta(u|\tilde{u})^{1/4}\le C\kappa(Z|\tilde{Z})^{3/4}$, the second term becomes

$$\begin{aligned}
&\int_{|\zeta|\ge\frac{1}{\gamma}\sqrt{\frac{\nu}{\gamma}}} w'|u-\tilde{u}|\Theta(u|\tilde{u})^{1/4}|u-\tilde{u}|\mathbf{1}_{\{(u/\tilde{u})-1\ge\delta\}}\,d\zeta\\
&\le C\mathcal{D}(Z)\left(\int_{\mathbb{R}} w'\kappa(Z|\tilde{Z})\,d\zeta\right)^{3/4}\left(\int_{|\zeta|\ge\frac{1}{\gamma}\sqrt{\frac{\nu}{\gamma}}} w'|\zeta|^4\,d\zeta\right)^{1/4}\\
&\le C\mathcal{D}(Z)\left(\frac{\gamma^2}{\nu}\right)^{3/4}\left(\frac{\nu}{\gamma^4}\right)^{1/4} = C\sqrt{\frac{\nu}{\gamma}}\mathcal{D}(Z).
\end{aligned}$$

which completes the proof of (3.35). □

Theorem 1 (Main results)**.** *Let $(u^-,v^-)\in\mathbb{R}^+\times\mathbb{R}$, $\delta_0\in(0,1/2)$ and $C>0$, then the following satisfies:*

Let $\gamma,\nu>0$ where $(\gamma,\nu)\in(0,u^-)\times(\delta_0^{-1}\gamma,\delta_0)$, and for any $(u^+,v^+)\in\mathbb{R}^+\times\mathbb{R}$ satisfying (2.4) *with $|u^- - u^+|=\gamma$, for some constants w^-,w^+ with $|w^+-w^-|=\nu$, $w:\mathbb{R}\to\mathbb{R}^+$ is a smooth monotone function and the limit of weighted function $\lim_{x\to\pm\infty}w(x)=1+w^\pm$, then one has*

Let $\tilde{Z}:=\begin{pmatrix}\tilde{u}\\ \tilde{v}\end{pmatrix}$ be traveling waves of (1.4) *with the boundary conditions defined in* (2.1) *and the wave speed ϖ defined in* (2.3)*. For fixed $T>0$, we consider that $Z(x,t):=\begin{pmatrix}u(x,t)\\ v(x,t)\end{pmatrix}$ is the solution of* (1.4) *in the space $\mathcal{M}_T$ with the initial values $Z_0(x):=\begin{pmatrix}u_0(x)\\ v_0(x)\end{pmatrix}$, satisfying*

$$\int_{-\infty}^{+\infty}\kappa(Z_0|\tilde{Z})dx<\infty.$$

Moreover, there exists shift function $X:[0,\infty)\to\mathbb{R}$ with $X\in W^{1,1}_{loc}$ and $X(0)=0$, such that

$$\begin{aligned}
&\int_{-\infty}^{+\infty} w(x-\varpi t)\kappa(Z(t,x-X(t))|\tilde{Z}(x-\varpi t))dx\\
&+D\delta_0\int_0^t\int_{-\infty}^{+\infty} w(x-\varpi\mu)u(\mu,x-X(\mu))^m\left|\partial_x\left(\log\frac{u(\mu,x-X(\mu))^m}{\tilde{u}(x-\varpi\mu)^m}\right)\right|^2 dxd\mu\\
&\le\int_{-\infty}^{+\infty} w(x)\kappa(Z_0(x)|\tilde{Z}(x))dx,
\end{aligned}\tag{3.39}$$

for the monotone function w defined in (2.16), *and*

$$
\begin{aligned}
&|\dot{X}(t) - \varpi| \leq \frac{1}{\gamma^2}\left(f(t) + C\int_{-\infty}^{+\infty} \kappa(Z_0|\tilde{Z})dx + 1\right) \quad \textit{for a.e.} \quad t \in [0, T], \\
&\textit{for some } f(t) > 0 \textit{ where } \|f\|_{L^1(0,T)} \leq C\frac{\nu}{\gamma}\int_{-\infty}^{+\infty} \kappa(Z_0|\tilde{Z})dx.
\end{aligned}
\tag{3.40}
$$

Remark 3. *From* (3.40)*, we can estimate it as follows:*

$$
|X(t)| \leq C\left(\int_{\mathbb{R}} \kappa(Z_0|\tilde{Z})dx + 1\right)(t + 1),
$$

for any $0 \leq t \leq T$ *and* $C > 0$ *is dependent of* u^-, v^-, γ*, and* ν*. Particularly, the* $C > 0$ *is independent of* T*.*

3.2. Proof of main results

In general, Theorem 1 can be proved through Lemmas 7 and 8, and also the construction of the shift function X, in which the uniqueness and existence of this shift function are proved through Picard's iteration, which is extended up to time T.

Lemma 7. *Let* $\tilde{Z} := \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix}$ *be the traveling waves in* (2.6) *and the weighted function* $w : \mathbb{R} \to \mathbb{R}^+$ *defined in* (2.16)*. For any solution* $Z := \begin{pmatrix} u \\ v \end{pmatrix} \in \mathcal{M}_T$ *of* (2.5)*, for some* $T > 0$ *and for any absolutely continuous shift* $X : [0, T] \to \mathbb{R}$*. For almost everywhere* $0 \leq t \leq T$*, one has*

$$
\frac{d}{dt}\int_{\mathbb{R}} w(\zeta)\kappa(Z^X(t,\zeta)|\tilde{Z}(\zeta))d\zeta = \dot{X}(t)\mathcal{L}(Z^X) + \mathcal{I}^{bad}(Z^X) - \mathcal{I}^{good}(Z^X),
\tag{3.41}
$$

where

$$
\begin{aligned}
\mathcal{L}(Z) &:= -\int_{\mathbb{R}} w'\kappa(Z|\tilde{Z})d\zeta + \int_{\mathbb{R}} w\partial_\zeta \nabla\kappa(\tilde{Z})(Z - \tilde{Z})d\zeta, \\
\mathcal{I}^{bad}(Z) &:= -\int_{\mathbb{R}} \frac{1}{m}\left[\chi w'\Theta(u|\tilde{u}) + \left(w' - w\frac{(\tilde{u}^m)'}{\tilde{u}^m}\right)(u - \tilde{u})\right](v - \tilde{v})d\zeta \\
&\quad - \int_{\mathbb{R}} \frac{\chi}{m}w'\tilde{v}\Theta(u|\tilde{u})d\zeta + \int_{\mathbb{R}} Dw\frac{(\tilde{u}^m)''}{\tilde{u}^m}\Theta(u|\tilde{n})d\zeta \\
&\quad + \int_{\mathbb{R}} \frac{D}{\chi m}\left(w\frac{(\tilde{u}^m)'}{\tilde{u}^m} - w'\right)u^m \log\left(\frac{u}{\tilde{u}}\right)^m \partial_\zeta \log\left(\frac{u}{\tilde{u}}\right)^m d\zeta, \\
\mathcal{I}^{good}(Z) &:= \varpi\int_{\mathbb{R}} w'\frac{|v - \tilde{v}|^2}{2}d\zeta + \varpi\int_{\mathbb{R}} w'\Theta(u|\tilde{u})d\zeta + \int_{\mathbb{R}} Dwu^m\left|\partial_\zeta \log\left(\frac{u}{\tilde{u}}\right)^m\right|^2 d\zeta.
\end{aligned}
\tag{3.42}
$$

Proof. We first change the variables $\zeta \to \zeta - X(t)$ such that

$$
\int_{\mathbb{R}} w(\zeta)\kappa(Z^X(t,\zeta)|\tilde{Z}(\zeta))d\zeta = \int_{\mathbb{R}} w^{-X}(\zeta)\kappa(Z(t,\zeta)|\tilde{Z}^{-X}(\zeta))d\zeta.
$$

By using the identity in (2.27), we get

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}} w^{-X}(\zeta)\kappa(Z(t,\zeta)|\tilde{Z}^{-X})d\zeta \\
&= \dot{X}\left(-\int_{\mathbb{R}} w'^{-X}\kappa(Z|\tilde{Z}^{-X})d\zeta + \int_{\mathbb{R}} w^{-X}\partial_\zeta\nabla\kappa(\tilde{Z}^{-X})(Z-\tilde{Z}^{-X})d\zeta\right) \\
&\quad + I_1 + I_2 + I_3,
\end{aligned}
$$

where

$$
\begin{aligned}
I_1 &= -\int_{\mathbb{R}} w^{-X}\partial_\zeta G(Z;\tilde{Z}^{-X})d\zeta, \\
I_2 &= -\int_{\mathbb{R}} w^{-X}\partial_\zeta\nabla\kappa(\tilde{Z}^{-X})A(Z|\tilde{Z}^{-X})d\zeta, \\
I_3 &= \int_{\mathbb{R}} w^{-X}(\nabla\kappa(Z)-\nabla\kappa(\tilde{Z}^{-X}))\partial_\zeta(M(Z)\partial_\zeta\nabla\kappa(Z))d\zeta \\
&\quad - \int_{\mathbb{R}} w^{-X}\nabla^{-X}\nabla^2\kappa(\tilde{Z}^{-X})(Z-\tilde{Z}^{-X})\partial_\zeta(M(\tilde{Z}^{-X})\partial_\zeta\nabla\kappa(\tilde{Z}^{-X})).
\end{aligned}
$$

By using (2.23), (2.26) and (2.27), we have

$$
\begin{aligned}
I_1 &= -\int_{\mathbb{R}} (w')^{-X}\frac{\chi}{m}(v-\tilde{v}^{-X})\Theta(u|\tilde{u}^{-X}) - \int_{\mathbb{R}} (w')^{-X}\frac{\chi}{m}\tilde{v}^{-X}\Theta(u|\tilde{u}^{-X}) \\
&\quad - \int_{\mathbb{R}} (w')^{-X}(u-\tilde{u}^{-X})(v-\tilde{v}^{-X})\partial_\zeta(M(\tilde{Z}^{-X})\partial_\zeta\nabla\kappa(\tilde{Z}^{-X}))d\zeta, \\
I_2 &= -\int_{\mathbb{R}} w^{-X}\frac{((\tilde{u}^{-X})^m)'}{(\tilde{u}^{-X})^m}(u-\tilde{u}^{-X})(v-\tilde{v}^{-X})d\zeta, \\
I_3 &= -\int_{\mathbb{R}} Dw^{-X}u^m\left|\partial_\zeta\log\left(\frac{u}{\tilde{u}^{-X}}\right)^m\right|^2 d\zeta + \int_{\mathbb{R}} Dw^{-X}\frac{((\tilde{u}^{-X})^m)''}{(\tilde{u}^{-X})^m}\Theta(u|\tilde{u}^{-X})d\zeta \\
&\quad + \int_{\mathbb{R}} \frac{D}{\chi m}\left(w^{-X}\frac{((\tilde{u}^{-X})^m)'}{(\tilde{u}^{-X})^m} - (w^{-X})'\right)u^m\log\left(\frac{u}{\tilde{u}^{-X}}\right)^m\partial_\zeta\log\left(\frac{u}{\tilde{u}^{-X}}\right)^m d\zeta.
\end{aligned}
$$

Combining all the results and changing again the variable $\zeta \to \zeta + X(t)$, the desired solution of Lemma 7 is established. □

Lemma 8. *Let* $\tilde{Z} := \begin{pmatrix}\tilde{u}\\ \tilde{v}\end{pmatrix}$ *be the traveling waves defined in* (2.6) *and the weighted function* $w : \mathbb{R} \to \mathbb{R}^+$ *stated in* (2.16)*. Let constant* $\delta > 0$*, such that for any* $Z = \begin{pmatrix}u\\ v\end{pmatrix} \in \mathcal{K}$*, one gets*

$$
\mathcal{I}^{bad}(Z) - \mathcal{I}^{good}(Z) = \mathcal{B}_\delta(Z) - \mathcal{G}_\delta(Z), \tag{3.43}
$$

where

$$
\begin{aligned}
\varphi(n) &:= \frac{1}{\varpi m}\left[\chi\Theta(u|\tilde{u}) + \left(1 + \frac{\gamma}{\nu}\frac{w}{\tilde{u}^m}\right)(u - \tilde{u})\right],\\
\mathcal{B}_\delta(Z) &:= -\int_{\mathbb{R}} \frac{w'}{m}\left[\chi\Theta(u|\tilde{u}) + \left(1 + \frac{\gamma}{\nu}\frac{w}{\tilde{u}^m}\right)(u - \tilde{u})\right](v - \tilde{v})\mathbf{1}_{\{|(u/\tilde{u})-1|>\delta\}}d\zeta\\
&\quad - \int_{\mathbb{R}} \frac{\chi}{m} w^{'}\tilde{v}\Theta(u|\tilde{u})d\zeta - \int_{\mathbb{R}} \frac{Dw'}{\chi m}\left(1 + \frac{\gamma}{\nu}\frac{w}{\tilde{u}^m}\right)u^m \log\left(\frac{u}{\tilde{u}}\right)^m \partial_\zeta \log\left(\frac{u}{\tilde{u}}\right)^m d\zeta\\
&\quad - \frac{\gamma}{\nu}\int_{\mathbb{R}} Dw^{''}\frac{w}{\tilde{u}^m}\Theta(u|\tilde{u})d\zeta + \frac{\varpi}{2}\int_{\mathbb{R}} w^{'}|\varphi(u)|^2\mathbf{1}_{\{|(u/\tilde{u})-1|\le\delta\}}d\zeta,\\
\mathcal{G}_\delta(Z) &:= \frac{\varpi}{2}\int_{\mathbb{R}} w^{'}|v - \tilde{v} + \varphi(u)|^2\mathbf{1}_{\{|(u/\tilde{u})-1|\le\delta\}}d\zeta + \varpi\int_{\mathbb{R}} w^{'}\frac{|v-\tilde{v}|^2}{2}\mathbf{1}_{\{|(u/\tilde{u})-1|>\delta\}}d\zeta\\
&\quad + \varpi\int_{\mathbb{R}} w'\Theta(u|\tilde{u})d\zeta + \int_{\mathbb{R}} Dwu^m\left|\partial_\zeta \log\left(\frac{u}{\tilde{u}}\right)^m\right|^2 d\zeta.
\end{aligned}
\tag{3.44}
$$

Proof. By using $w' = -\frac{\nu}{\gamma}(\tilde{u}^m)'$, it follows from (3.42) that we have

$$
\begin{aligned}
\mathcal{I}^{bad}(Z) &:= \underbrace{-\int_{\mathbb{R}} \frac{w'}{m}\left[\chi\Theta(u|\tilde{u}) + \left(1 + \frac{\gamma}{\nu}\frac{w}{\tilde{u}^m}\right)(u - \tilde{u})\right](v - \tilde{v})\mathbf{1}_{\{|(u/\tilde{u})-1|\le\delta\}}d\zeta}_{J_1}\\
&\quad - \int_{\mathbb{R}} \frac{w'}{m}\left[\chi\Theta(u|\tilde{u}) + \left(1 + \frac{\gamma}{\nu}\frac{w}{\tilde{u}^m}\right)(u - \tilde{u})\right](v - \tilde{v})\mathbf{1}_{\{|(u/\tilde{u})-1|>\delta\}}d\zeta\\
&\quad - \int_{\mathbb{R}} \frac{\chi}{m} w^{'}\tilde{v}\Theta(u|\tilde{u})d\zeta - \int_{\mathbb{R}} \frac{Dw'}{\chi m}\left(1 + \frac{\gamma}{\nu}\frac{w}{\tilde{u}^m}\right)u^m \log\left(\frac{u}{\tilde{u}}\right)^m \partial_\zeta \log\left(\frac{u}{\tilde{u}}\right)^m d\zeta\\
&\quad - \frac{\gamma}{\nu}\int_{\mathbb{R}} Dw^{''}\frac{w}{\tilde{u}^m}\Theta(u|\tilde{n})d\zeta,\\
-\mathcal{I}^{good}(Z) &:= \underbrace{-\varpi\int_{\mathbb{R}} w^{'}\frac{|v-\tilde{v}|^2}{2}\mathbf{1}_{\{|(u/\tilde{u})-1|\le\delta\}}d\zeta}_{J_2} - \varpi\int_{\mathbb{R}} w^{'}\frac{|v-\tilde{v}|^2}{2}\mathbf{1}_{\{|(u/\tilde{u})-1|>\delta\}}d\zeta\\
&\quad - \varpi\int_{\mathbb{R}} w'\Theta(u|\tilde{u})d\zeta - \int_{\mathbb{R}} Dwu^m\left|\partial_\zeta \log\left(\frac{u}{\tilde{u}}\right)^m\right|^2 d\zeta.
\end{aligned}
$$

By using $\alpha x^2 + \beta x = \alpha\left(x + \frac{\beta}{2\alpha}\right)^2 - \frac{\beta^2}{4\alpha}$ and $x := v - \tilde{v}$, we have

$$
J_1 + J_2 = \frac{\varpi}{2}\int_{\mathbb{R}} w^{'}|\varphi(u)|^2\mathbf{1}_{\{|(u/\tilde{u})-1|\le\delta\}}d\zeta - \frac{\varpi}{2}\int_{\mathbb{R}} w^{'}|v - \tilde{v} + \varphi(u)|^2\mathbf{1}_{\{|(u/\tilde{u})-1|\le\delta\}}d\zeta
$$

Thus, the desired solution is established. □

Lemma 9. *There exists* $\delta_0 \in (0, 1/2)$ *and* $\delta \in (0, 1/2)$ *such that if positive constant* γ *and* ν *satisfy* $\delta_0^{-1}\gamma < \nu < \delta_0$, *then for any traveling wave* $\tilde{Z} := \begin{pmatrix}\tilde{u}\\ \tilde{v}\end{pmatrix}$ *in* (2.6) *and for any* $Z \in \mathcal{K}$ *satisfying* $|\mathcal{L}(Z)| \le \gamma^2$, *one has*

$$
\mathcal{R}(Z) := -\frac{1}{\gamma^4}|\mathcal{L}(Z)|^2 + \mathcal{B}_\delta(Z) + \delta_0\frac{\gamma}{\nu}|\mathcal{B}_\delta(Z)| - \mathcal{G}_\delta(Z) - (1 - \delta)\mathcal{D}(Z) \le 0.
$$

where the functional $\mathcal{L}$ *is defined in* (3.42) *and* $\mathcal{B}_\delta$ *and* $\mathcal{G}_\delta$ *are defined in* (3.44).

Proof. We separate for all the terms in $\mathcal{R}(Z)$ and estimate them as shown
(a) For term of $|\mathcal{B}_\delta(Z)|$. From (3.44), we calculate for each part in $\mathcal{B}_\delta$ that

$$\mathcal{B}_\delta(Z) = \mathcal{B}_1 + \mathcal{B}_2 + \mathcal{B}_3 + \mathcal{B}_4,$$

where

$$\begin{aligned}
\mathcal{B}_1 &:= -\int_{\mathbb{R}} \frac{w'}{m}\left[\chi\Theta(u|\tilde{u}) + \left(1 + \frac{\gamma}{\nu}\frac{w}{\tilde{u}^m}\right)(u-\tilde{u})\right](v-\tilde{v})\mathbf{1}_{\{|(u/\tilde{u})-1|>\delta\}}d\zeta,\\
\mathcal{B}_2 &:= -\int_{\mathbb{R}} \frac{\chi}{m} w' \tilde{v}\Theta(u|\tilde{u})d\zeta - \frac{\gamma}{\nu}\int_{\mathbb{R}} Dw'' \frac{w}{\tilde{u}^m}\Theta(u|\tilde{u})d\zeta,\\
\mathcal{B}_3 &:= \frac{\varpi}{2}\int_{\mathbb{R}} w' |\varphi(u)|^2 \mathbf{1}_{\{|(u/\tilde{u})-1|\le\delta\}}d\zeta,\\
\mathcal{B}_4 &:= -\int_{\mathbb{R}} \frac{Dw'}{\chi m}\left(1 + \frac{\gamma}{\nu}\frac{w}{\tilde{u}^m}\right)u^m \log\left(\frac{u}{\tilde{u}}\right)^m \partial_\zeta \log\left(\frac{u}{\tilde{u}}\right)^m d\zeta.
\end{aligned}$$

We further estimate for all parts of $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3$ and $\mathcal{B}_4$,

$$\begin{aligned}
|\mathcal{B}_1| &\le \int_{\mathbb{R}} \frac{w'\chi}{m}\Theta(u|\tilde{u})(v-\tilde{v})\mathbf{1}_{\{|(u/\tilde{u})-1|>\delta\}} + C\int_{\mathbb{R}} \frac{w'}{m}(u-\tilde{u})(v-\tilde{v})\mathbf{1}_{\{|(u/\tilde{u})-1|>\delta\}}\\
&\le C\int_{\mathbb{R}} w'\Theta(u|\tilde{u})(v-\tilde{v})\mathbf{1}_{\{|(u/\tilde{u})-1|>\delta\}}\\
&\le C\int_{\mathbb{R}} w'\left(1 + u\log^+\frac{u}{\tilde{u}}\right)(v-\tilde{v})\mathbf{1}_{\{|(u/\tilde{u})-1|\ge\delta\}} \quad \text{(from (3.2))}\\
&\le C\sqrt{\frac{\gamma}{\nu}}\mathcal{D}(Z) \quad \text{(from (3.35))},\\
|\mathcal{B}_2| &\le C\int_{\mathbb{R}} \Theta(u|\tilde{u}) \le C\int_{\mathbb{R}} w'\kappa(Z|\tilde{Z}) \le C\frac{\gamma^2}{\nu},\\
|\mathcal{B}_3| &= \frac{\varpi}{2}w'\left|\frac{1}{\varpi m}\left[\chi\Theta(u|\tilde{u}) + \left(1 + \frac{\gamma}{\nu}\frac{w}{\tilde{u}^m}\right)(u-\tilde{u})\right]\right|^2 \mathbf{1}_{\{|(u/\tilde{u})-1|\le\delta\}}\\
&\le C\int_{\mathbb{R}} w'|\Theta(u|\tilde{u})\mathbf{1}_{\{|(u/\tilde{u})-1|\le\delta\}}|^2 + C\int_{\mathbb{R}} w'|(u-\tilde{u})\mathbf{1}_{\{|(u/\tilde{u})-1|\le\delta\}}|^2\\
&\le C\int_{\mathbb{R}} w'(|u-\tilde{u}|^2 + \Theta(u|\tilde{u})^2) \quad \text{(from (3.1))}\\
&\le C\int_{\mathbb{R}} w'\Theta(u|\tilde{u}) \le C\int_{\mathbb{R}} w'\kappa(Z|\tilde{Z}) \le C\frac{\gamma^2}{\nu} \quad \text{(from Lemma 4)},\\
|\mathcal{B}_4| &\le C\sqrt{\int_{\mathbb{R}} u\left(\log\frac{u}{\tilde{u}}\right)^2} + C\sqrt{\int_{\mathbb{R}} wu^m\left|\partial_\zeta \log\left(\frac{u}{\tilde{u}}\right)^m\right|^2 d\zeta}\\
&\le C\sqrt{\int_{\mathbb{R}} 1 + u\left(\log^+\frac{u}{\tilde{u}}\right)^2 \mathbf{1}_{\{|(u/\tilde{u})-1|\ge\delta\}}} + C\sqrt{\int_{\mathbb{R}} wu^m\left|\partial_\zeta \log\left(\frac{u}{\tilde{u}}\right)^m\right|^2 d\zeta}\\
&\le C\sqrt{C\sqrt{\frac{\gamma}{\nu}}\mathcal{D}(Z)} + C\sqrt{\mathcal{D}(Z)} \quad \text{(from (3.34))}\\
&\le 2C\sqrt{\mathcal{D}(Z)}.
\end{aligned}$$

Since, all the terms in $\mathcal{B}(Z)$ are negative, then

$$\mathcal{B}_\delta(Z) + \delta_0 \frac{\gamma}{\nu} |\mathcal{B}_\delta(Z)| \leq \delta_0 \frac{\gamma}{\nu} |\mathcal{B}_\delta(Z)|.$$

(b) For term of $|\mathcal{G}_\delta(Z)|$.
By using the similar way with the previous one

$$\mathcal{G}_\delta(Z) = \mathcal{G}_1 + \mathcal{G}_2 + \mathcal{G}_3 + \mathcal{G}_4,$$

where

$$\begin{aligned}
\mathcal{G}_1 &:= \frac{\varpi}{2} \int_{\mathbb{R}} w' |v - \tilde{v} + \varphi(u)|^2 \mathbf{1}_{\{|(u/\tilde{u})-1| \leq \delta\}} d\zeta, \\
\mathcal{G}_2 &:= \varpi \int_{\mathbb{R}} w' \frac{|v - \tilde{v}|^2}{2} \mathbf{1}_{\{|(u/\tilde{u})-1| > \delta\}} d\zeta, \\
\mathcal{G}_3 &:= \varpi \int_{\mathbb{R}} w' \Theta(u|\tilde{u}) d\zeta, \\
\mathcal{G}_4 &:= \int_{\mathbb{R}} D w u^m \left| \partial_\zeta \log \left(\frac{u}{\tilde{u}} \right)^m \right|^2 d\zeta := \mathcal{D}(Z),
\end{aligned}$$

which gives the following estimates

$$\begin{aligned}
\mathcal{G}_1 &\leq C \int_{\mathbb{R}} w' |v - \tilde{v}|^2 + C \int_{\mathbb{R}} w' |\varphi(u)|^2 \mathbf{1}_{\{|(u/\tilde{u})-1| \leq \delta\}} \\
&\leq C \int_{\mathbb{R}} w' |v - \tilde{v}|^2 + C \frac{\gamma^2}{\nu} \quad \text{(from the estimate of } \mathcal{B}_3\text{)} \\
&\leq C \int_{\mathbb{R}} w' \kappa(Z|\tilde{Z}) + C \frac{\gamma^2}{\nu} \leq 2C \frac{\gamma^2}{\nu}, \\
\mathcal{G}_2 &\leq C \int_{\mathbb{R}} w' (|v - \tilde{v}|^2) + C \int_{\mathbb{R}} w' \Theta(u|\tilde{u}) \mathbf{1}_{\{|(u/\tilde{u})-1| > \delta\}} \\
&\leq C \int_{\mathbb{R}} w' (|v - \tilde{v}|^2) + C \int_{\mathbb{R}} w' \left(1 + u \log^+ \frac{u}{\tilde{u}} \right) \\
&\leq C \int_{\mathbb{R}} w' (|v - \tilde{v}|^2 + \Theta(u|\tilde{u})) + C \int_{\mathbb{R}} w' \left(1 + u \log^+ \frac{u}{\tilde{u}} \right) \mathbf{1}_{\{|(u/\tilde{u})-1| > \delta\}} \\
&\leq C \frac{\gamma^2}{\nu} + C \sqrt{\frac{\gamma}{\nu}} \mathcal{D}(Z) \quad \text{(from (3.23) and (3.34))}, \\
\mathcal{G}_3 &\leq \varpi \int_{\mathbb{R}} w' \kappa(Z|\tilde{Z}) \leq C \frac{\gamma^2}{\nu}, \\
\mathcal{G}_4 &= \mathcal{D}(Z).
\end{aligned}$$

Finally, we combine all the estimates of (a) and (b) and one has

$$\begin{aligned}\mathcal{R}(Z) &= -\frac{1}{\gamma^4}|\mathcal{L}(Z)|^2 + \mathcal{B}_\delta(Z) + \delta_0\frac{\gamma}{\nu}|\mathcal{B}_\delta(Z)| - \mathcal{G}_\delta(Z) - (1-\delta)\mathcal{D}(Z)\\ &\leq \delta_0\frac{\gamma}{\nu}\left(2C\frac{\gamma^2}{\nu} + 2C\sqrt{\mathcal{D}(Z)} + C\sqrt{\frac{\gamma}{\nu}}\mathcal{D}(Z)\right)\\ &\quad -\left(4C\frac{\gamma^2}{\nu} + \mathcal{D}(Z) + C\sqrt{\frac{\gamma}{\nu}}\mathcal{D}(Z)\right) - (1-\delta)\mathcal{D}(Z)\\ &\leq \left(\delta_0\frac{\gamma}{\nu} - 1\right)\left(2C\frac{\gamma^2}{\nu} + 2C\sqrt{\mathcal{D}(Z)} + C\sqrt{\frac{\gamma}{\nu}}\mathcal{D}(Z)\right) - (2-\delta)\mathcal{D}(Z).\end{aligned}$$

Since $(\delta, \delta_0) \in (0, 1/2)$ and $\delta_0^{-1} < \nu < \delta_0$,

$$\left(\delta_0\frac{\gamma}{\nu} - 1\right) < 0,$$

which indicates that $\mathcal{R}(Z) \leq 0$, then Lemma 9 is completed. □

Proof of Theorem 1. We are concerned with Theorem 1 through Lemmas 7 and 8, (3.41) and (3.11). It is enough for the righthand side in (3.41) that for a.e. $t \in [0, T]$ we have

$$\Phi_\gamma(\mathcal{L}(Z^X))(2|\mathcal{I}^{bad}(Z^X)| + 1)\mathcal{L}(Z^X) + \mathcal{I}^{bad}(Z^X) - \mathcal{I}^{good}(Z^X) \leq 0.$$

For every $Z \in \mathcal{K}$ we define

$$\mathcal{F}(Z) = \Phi_\gamma(\mathcal{L}(Z))(2|\mathcal{I}^{bad}(Z)| + 1)\mathcal{L}(Z) + \mathcal{I}^{bad}(Z) - \mathcal{I}^{good}(Z) \leq 0.$$

From (3.10), one can provide two cases:
(i). For $|\mathcal{L}| \geq \gamma^2$,

$$\Phi_\gamma(\mathcal{L}(Z))(2|\mathcal{I}^{bad}(Z)| + 1)\mathcal{L}(Z) = -\frac{(2|\mathcal{I}^{bad}| + 1)}{\gamma^2}\mathcal{L} \leq -2|\mathcal{I}^{bad}|.$$

(ii). For $|\mathcal{L}| \leq \gamma^2$,

$$\Phi_\gamma(\mathcal{L}(Z))(2|\mathcal{I}^{bad}(Z)| + 1)\mathcal{L}(Z) = -\frac{(2|\mathcal{I}^{bad}| + 1)}{\gamma^4}\mathcal{L}^2 \leq -\frac{\mathcal{L}^2}{\gamma^4}.$$

Note that for all $Z \in \mathcal{K}$ satisfying $|\mathcal{L}| \geq \gamma^2$,

$$\mathcal{F}(Z) \leq -2|\mathcal{I}^{bad}(Z)| + \mathcal{I}^{bad}(Z) - \mathcal{I}^{good}(Z) = -|\mathcal{I}^{bad}(Z)| - \mathcal{I}^{good}(Z) \leq 0.$$

By using (3.43), for any $\delta > 0$ and any $Z \in \mathcal{K}$ satisfying $|\mathcal{L}| \leq \gamma^2$, one has

$$\mathcal{F}(Z) \leq -\frac{1}{\gamma^4}\mathcal{L}(Z)^2 + \mathcal{I}^{bad}(Z) - \mathcal{I}^{good}(Z) = -\frac{1}{\gamma^4}\mathcal{L}(Z)^2 + \mathcal{B}_\delta(Z) - \mathcal{G}_\delta(Z).$$

For any $Z \in \mathcal{K}$ satisfying $|\mathcal{L}| \le \gamma^2$, we apply Lemma 9 into the above inequality and one has

$$\mathcal{F}(Z) \le -\delta_0 \frac{\gamma}{\nu} |\mathcal{B}_\delta| - \delta_0 \mathcal{D}(Z) \le 0.$$

Therefore, by employing $\mathcal{F}(Z)$ for $|\mathcal{L}| \le \gamma^2$ and $|\mathcal{L}| \ge \gamma^2$ and $Z = Z^X$ and $\delta_0 < \frac{1}{2}$ into (3.41), for a.e. $0 \le t \le T$ one has

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}} w\kappa(Z^X|\tilde{Z}) d\zeta + \delta_0 \mathcal{D}(Z^X) &= \mathcal{F}(Z^X) + \delta_0 \mathcal{D}(Z^X) \\ &\le -|\mathcal{I}^{bad}(Z^X)| \mathbf{1}_{\{|\mathcal{L}(Z^X)| \ge \gamma^2\}} - \delta_0 \frac{\gamma}{\nu} |\mathcal{B}_\delta(Z^X)| \mathbf{1}_{\{|\mathcal{L}(Z^X)| \le \gamma^2\}} \le 0, \end{aligned} \tag{3.45}$$

and by the initial data $\int_{\mathbb{R}} \kappa(Z_0|\tilde{Z}) d\zeta < \infty$, then we have

$$\int_{\mathbb{R}} w\kappa(Z^X|\tilde{Z}) d\zeta + \delta_0 \int_0^t \mathcal{D}(Z^X) \le \int_{\mathbb{R}} w\kappa(Z_0|\tilde{Z}) d\zeta, \tag{3.46}$$

where $\mathcal{D}(Z^X)$ is defined by

$$\mathcal{D}(Z^X) := \int_{\mathbb{R}} Dw(u^X)^m \left| \partial_\zeta \log \left(\frac{u^X}{\tilde{u}} \right)^m \right|^2 d\zeta.$$

From $Z^X(t,\zeta) := Z(t, \zeta + X(t)) = Z(t, x - \varpi t + X(t))$, we redefine $X(t)$ by $\varpi t - X(t)$ to get the desired solution of (3.39), such that

$$Z^X(t,\zeta) := Z(t, x - \varpi t + \varpi t - X(t)) = Z(t, x - X(t)). \tag{3.47}$$

For the estimate $|\dot{X}|$, we first observe the shift function X in (3.10) and (3.11) such that

$$|\dot{X}| \le \frac{1}{\gamma^2} (2|\mathcal{I}^{bad}| + 1). \tag{3.48}$$

From (3.43) together with the definitions of $\mathcal{I}^{good}$ and $\mathcal{G}_\delta$, we have

$$\begin{aligned} |\mathcal{I}^{bad}(Z^X)| &= |\mathcal{I}^{bad}(Z^X)| \mathbf{1}_{\{|\mathcal{L}(Z^X)| \ge \gamma^2\}} + |\mathcal{I}^{bad}(Z^X)| \mathbf{1}_{\{|\mathcal{L}(Z^X)| \le \gamma^2\}} \\ &= |\mathcal{I}^{bad}(Z^X)| \mathbf{1}_{\{|\mathcal{L}(Z^X)| \ge \gamma^2\}} + |\mathcal{I}^{good}(Z^X) + \mathcal{B}_\delta(Z^X) - \mathcal{G}_\delta(Z^X)| \mathbf{1}_{\{|\mathcal{L}(Z^X)| \le \gamma^2\}} \\ &\le |\mathcal{I}^{bad}(Z^X)| \mathbf{1}_{\{|\mathcal{L}(Z^X)| \ge \gamma^2\}} + |\mathcal{B}_\delta(Z^X)| \mathbf{1}_{\{|\mathcal{L}(Z^X)| \le \gamma^2\}} \\ &\quad + C \int_{\mathbb{R}} |w'| (|v^X - \tilde{v}|^2 + \Theta(u^X|\tilde{u})^2 + |u^X - \tilde{u}|^2) \mathbf{1}_{\{|(u^X/\tilde{u}) - 1| \le \delta\}} d\zeta. \end{aligned}$$

Since (3.1) implies that

$$\Theta(u^X|\tilde{u}) \le C_1 |u^X - \tilde{u}|^2 \le C_1 (\delta u^-)^2, \quad \text{whenever } |(u^X/\tilde{u}) - 1| \le \delta,$$

by using (3.1) and Remark 1 where $w' \le C \frac{\nu\gamma}{4\varpi_-} e^{-\frac{C_X \gamma |\zeta|}{D\varpi_-}} \le C\delta_0 \le w$, one has

$$|\mathcal{I}^{bad}(Z^X)| \le |\mathcal{I}^{bad}(Z^X)| \mathbf{1}_{\{|\mathcal{L}(Z^X)| \ge \gamma^2\}} + |\mathcal{B}_\delta(Z^X)| \mathbf{1}_{\{|\mathcal{L}(Z^X)| \le \gamma^2\}} + C \int_{\mathbb{R}} w\kappa(Z^X|\tilde{Z}) d\zeta. \tag{3.49}$$

Moreover, we substitute (3.49) into (3.48) to get

$$|\dot{X}| \leq \frac{2}{\gamma^2}\left(|\mathcal{I}^{bad}(Z^X)|\mathbf{1}_{\{|\mathcal{L}(Z^X)|\geq\gamma^2\}} + |\mathcal{B}_\delta(Z^X)|\mathbf{1}_{\{|\mathcal{L}(Z^X)|\leq\gamma^2\}}\right) + \frac{C}{\gamma^2}\int_{\mathbb{R}} w\kappa(Z_0|\tilde{Z})d\zeta + \frac{1}{\gamma^2},$$

where

$$\int_0^T \left(|\mathcal{I}^{bad}(Z^X)|\mathbf{1}_{\{|\mathcal{L}(Z^X)|\geq\gamma^2\}} + |\mathcal{B}_\delta(Z^X)|\mathbf{1}_{\{|\mathcal{L}(Z^X)|\leq\gamma^2\}}\right) \leq \frac{2\nu}{\delta_0\gamma}\int_{\mathbb{R}} \kappa(Z_0|\tilde{Z})d\zeta.$$

We further apply (3.47) again to get (3.40). Finally, Eq (3.39) is established. □

4. Conclusions

This paper provided the orbital stability with the cutoff version, because of the effect of the following large initial perturbations

$$\int_{-\infty}^{+\infty} \kappa(Z_0|\tilde{Z})dx < \infty,$$

where κ is the relative entropy function, which is defined in Eq (2.24). The difference with the previous study is the following small initial perturbations

$$\int_{-\infty}^{+\infty} \begin{pmatrix} u(x) - \tilde{u}(x - x_0) \\ v(x) - \tilde{v}(x - x_0) \end{pmatrix} dx = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{for some } x_0 \in \mathbb{R}.$$

Moreover, the large initial perturbations can be handled by introducing the appropriate relative entropy method for nonlinear degenerate viscosity, where this relative entropy method has the same role in L^2 (as in the small initial perturbations) to provide the distance between (u, v) and $(\tilde{u}, \tilde{v})$.

Use of AI tools declaration

The author declares no use of Artificial Intelligence (AI) tools in the creation of this paper.

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Conflict of interest

The author declares no conflict of interest in this paper.

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