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Research article

Existence and uniqueness of positive solution of a nonlinear differential equation with higher order Erdélyi-Kober operators

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Abstract: In this paper, the initial value problem of a nonlinear differential equation with higher order Caputo type modification of the Erdélyi-Kober fractional derivatives was studied. Based on the transmutation method, the well-posedness of initial value problem of the higher order linear model was proved and an explicit solution was presented. Then some new Gronwall type inequalities involving Erdélyi-Kober fractional integral were established. By applying these results and some fixed point theorems, the existence and uniqueness of the positive solution of the nonlinear differential equation were proved. The method is applicable to the fractional differential equation with any order $\gamma \in (n-1, n]$.

Keywords: fractional differential equation; Caputo type fractional derivative; Erdélyi-Kober operators; Gronwall type integral inequality; positive solution
Mathematics Subject Classification: 26A33, 26D07, 34A08, 34A12

1. Introduction

In this paper, the initial value problem of the nonlinear differential equation

$$\begin{cases} t^{-\beta\gamma}{}_*D_\beta^{\alpha,\gamma}u(t) - \lambda u(t) = f(t, u(t)), \\ \lim_{t\to 0} t^{\beta(\alpha+n-k)}\Pi_{i=1}^k(\alpha + n - i + \frac{1}{\beta}t\frac{d}{dt})u(t) = u_k, \\ \qquad\qquad\qquad\qquad k = 0, 1, 2, \cdots, n-1 \end{cases} \tag{1.1}$$

is studied, where ${}_*D_\beta^{\alpha,\gamma}$ is given in Definition 2.3 and called a Caputo type modification of the Erdélyi-Kober fractional derivative (EKFD) with γ-th order, $\gamma \in (n-1, n]$, $n \in \mathbb{N}^+$, $\beta \in \mathbb{R}^+$ and $\alpha \in \mathbb{R}^+$.

Caputo type fractional derivative is an important subject with numerous applications to several fields outside mathematics, such as engineering, biology, economics, physics, etc., because it describes the property of memory and heredity of many materials. In general, every type of fractional derivative has its own merits and also the shortages. The Riemann-Liouville fractional derivative (RLFD) is a natural generalization of the integer order derivative and usually employed in mathematical texts and not so frequently in applications, and the Caputo fractional derivative (CFD) is often used to formulate mathematical models of some applied problems by means of fractional differential equations since its physical interpretation is clear. The EKFD as a generalization of RLFD and CFD is often applied both in the mathematical texts and applications. The so-called Caputo type modification of the EKFD was first introduced by Gorenflo, Luchko and Mainardi in [1] and applied to describe of the scale-invariant solutions of the diffusion-wave equation. In fact, fractional differential equations with Caputo type initial data are natural and useful in modeling the reality. Finding simple and effective methods to solve these kinds of models is fundamental and has become more and more active [2, 3]. Integral transform method is suitable for solving Cauchy problems of fractional differential equations with constant coefficients [4] in half space. For fractional differential equations with variable coefficients, Kiryakova and her collaborators established the explicit solution by use of the transmutation method [5, 6]. In [7], the series method works only for relatively special domain, although it allows for finding solutions of fractional differential equations with arbitrary order. For more analytical approaches and numerical algorithm, one can refer to [8, 9] and the references therein. As the deepening of research on calculus theory and its applications, the so-called Caputo type modification of the EKFD has attracted more and more attention [10–12]. In this paper, we develop some new related Gronwall integral inequality to enrich the analytical methods and overcome the difficulties that are caused by nonlinearity of equations and the low regularity of solutions. Recently, this idea has been generalized for studying the qualitative theory of fractional differential equations [13–17].

In order to establish the solution of problem (1), we focus on a weakly singular integral whose kernel involves Mittag-Leffler functions. Some new related Gronwall type integral inequalities are established, then based on these inequalities and fixed point theorems, we show the existence and uniqueness of the positive solution of nonlinear fractional differential equations with higher order Caputo type modification of the EKFDs. This method is not confined with the order of the equation.

The main contributions of this work are that:

(i) The well posedness of the initial value problem of linear fractional differential equation with higher order Caputo type modification of the EKFDs is established.

(ii) The established new Gronwall-type integral inequalities extends the results in [18].

(iii) The equivalence between the nonlinear differential equation with higher order Caputo type modification of the EKFDs and a nonlinear Volterra-type integral equation with a complicated singular kernel is established.

(iv) The existence and uniqueness of the positive solution of the nonlinear fractional differential equation with higher order Caputo type modification of the EKFDs are obtained.

This paper is organized as follows: In Section 2, the basic knowledge of some fractional calculus and Mittag-Leffler functions are recalled. In Section 3, we show the well posedness of the initial value problem of fractional differential equation. In Section 4, we analyze an integral whose kernel involves Mittag-Leffler functions and singular power functions, and then derive some new useful Gronwall-type inequalities. In Section 5, we study nonlinear fractional differential equations by use of Gronwall-type inequalities and fixed point theorems. In final, we establish the existence and uniqueness of the positive solution of the nonlinear fractional differential equation with higher order Caputo type modification of the EKFD operators.

2. Preliminary

In this section, we start by writing down definitions of the Erdélyi-Kober fractional integral (EKFI) given in [5, 11, 19], and then give some recalls of fractional calculus that will be used later.
Definition 2.1. *The EKFI of $f(t) \in \mathbb{C}_\mu$ is defined by*

$$I_\beta^{\alpha,\gamma} f(t) = \frac{t^{-\beta(\alpha+\gamma)}}{\Gamma(\gamma)} \int_0^t (t^\beta - \tau^\beta)^{\gamma-1} \tau^{\beta\alpha} f(\tau) d(\tau^\beta),$$

with arbitrary parameters $\mu \in \mathbb{R}$, $\gamma \in \mathbb{R}^+$, $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}^+$, where weighted space of continuous functions $\mathbb{C}_\mu^{(n)}$ is defined by

$$\mathbb{C}_\mu^{(n)} = \{f(t) = t^p \tilde{f}(t) : p > \mu, \tilde{f}(t) \in \mathbb{C}^{(n)}[0, \infty)\}, n \in \mathbb{N}.$$

The special case for $\gamma = 0$, the EKFI is defined as the identity operator. For $\gamma < 0$, the interpretation is derived by

$$I_\beta^{\alpha,\gamma} f(t) = (\alpha + \gamma + 1) I_\beta^{\alpha,\gamma+1} f(t) + \frac{1}{\beta} I_\beta^{\alpha,\gamma+1} (t \frac{d}{dt}) f(t).$$

For $\alpha = 0, \beta = 1$, the EKFI is reduced to the well-known Riemann-Liouville fractional integral with a power weight.
Definition 2.2. *The Riemann-Liouille type modification of the EKFD of order γ of a function $f(t) \in \mathbb{C}_\mu^{(n)}$, $\mu > -\beta(\alpha + 1)$ is defined by*

$$D_\beta^{\alpha,\gamma} f(t) = D_n I_\beta^{\alpha+\gamma,n-\gamma} f(t),$$

where $D_n = \Pi_{i=1}^n (\alpha + i + \frac{1}{\beta} t \frac{d}{dt})$, $\gamma \in (n-1, n]$, $n \in \mathbb{N}$.
Definition 2.3. *The Caputo type modification of the EKFD of order γ of a function $f(t) \in \mathbb{C}_\mu^{(n)}$, $\mu > -\beta(\alpha + 1)$ is defined by*

$${}_*D_\beta^{\alpha,\gamma} f(t) = I_\beta^{\alpha+\gamma,n-\gamma} D_n f(t),$$

where $\gamma \in (n-1, n]$, $n \in \mathbb{N}$ and $D_n = \Pi_{i=1}^n (\alpha + i + \frac{1}{\beta} t \frac{d}{dt})$.
Lemma 2.4. *For $f(t) \in C_\mu$, $\mu \geq -\beta(\alpha + 1)$, the right-hand sided EKFD ${}_*D_\beta^{\alpha,\gamma}$ and $D_\beta^{\alpha,\gamma}$ satisfy*

$$D_\beta^{\alpha,\gamma} I_\beta^{\alpha,\gamma} f(t) = {}_*D_\beta^{\alpha,\gamma} I_\beta^{\alpha,\gamma} f(t) = f(t).$$

Lemma 2.5. *For $f(t) \in C_\mu^{(n)}$, $\mu \geq -\beta(\alpha + \gamma + 1)$ and $\gamma \in (n-1, n]$, $n \in \mathbb{N}^+$, the right-hand sided EKFD ${}_*D_\beta^{\alpha,\gamma}$ and integral $I_\beta^{\alpha,\gamma}$ satisfy*

$$I_\beta^{\alpha,\gamma} {}_*D_\beta^{\alpha,\gamma} f(t) = f(t) - \sum_{k=0}^{n-1} p_k t^{-\beta(1+\alpha+k)},$$

where constants $p_k = \lim_{t\to 0} t^{\beta(1+\alpha+k)} \Pi_{i=k+1}^{n-1} (1 + \alpha + i + \frac{1}{\beta} t \frac{d}{dt}) f(t)$, $k = 0, 1, 2, \cdots, n-1$, and the right-hand sided EKFD $D_\beta^{\alpha,\gamma}$ and integral $I_\beta^{\alpha,\gamma}$ satisfy

$$I_\beta^{\alpha,\gamma} D_\beta^{\alpha,\gamma} f(t) = f(t) - \sum_{k=0}^{n-1} c_k t^{-\beta(1+\alpha+k)},$$

where constants $c_k = \frac{\Gamma(n-k)}{\Gamma(\alpha-k)} \lim_{t\to 0} t^{\beta(1+\alpha+k)} \prod_{i=k+1}^{n-1} (1+\alpha+i+\frac{1}{\beta}t\frac{d}{dt})(I_\beta^{\alpha+\gamma,n-\gamma} f)(t)$, $k = 0,1,2,\cdots,n-1$, and $\Gamma(\cdot)$ is a Gamma function.

The Mittag-Leffler function is defined by the convergent series

$$E_{\alpha,\beta}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha k+\beta)}, \quad \Re(\alpha) > 0, \Re(\beta) > 0, \tag{2.1}$$

where $\alpha \in \mathbb{C}$, $\beta \in \mathbb{C}$, $\Re(\cdot)$ denotes the real part of a complex number. The Mittag-Leffler function is an entire function that is a natural extension of the exponential, trigonometric and incomplete gamma functions. Particularly, it is easy to verify that $E_{\alpha,\beta}(t)$ is positive for $t > 0$, $\alpha \in \mathbb{R}^+$, $\beta \in \mathbb{R}^+$. The following asymptotic expansions of the Mittag-Leffler function are given in [20].

Lemma 2.6. *For* $0 < \alpha < 2$, $\beta \in \mathbb{R}$ *and* $\frac{\pi\alpha}{2} < \mu < \min\{\pi, \pi\alpha\}$, *we have*

$$E_{\alpha,\beta}(t) = \frac{1}{\alpha} t^{\frac{1-\beta}{\alpha}} exp(t^{\frac{1}{\alpha}}) - \sum_{r=1}^{N} \frac{1}{\Gamma(\beta - r\alpha)t^r} + O(\frac{1}{t^{N+1}}) \tag{2.2}$$

for large $|t|$ and $|\arg t| \le \mu$.

For $\alpha \ge 2, \beta \in \mathbb{R}$, we have

$$E_{\alpha,\beta}(t) = \frac{1}{\alpha} \sum_{n} t^{\frac{1}{n}} \exp[\exp(\frac{2n\pi i}{\alpha}) t^{\frac{1}{\alpha}}]^{1-\beta} - \sum_{r=1}^{N} \frac{1}{\Gamma(\beta - r\alpha)t^r} + O(\frac{1}{t^{N+1}}) \tag{2.3}$$

for large $|t|$, $|\arg t| \le \frac{\pi\alpha}{2}$ and where the first sum is taken over all integers n such that $|\arg t + 2n\pi| \le \frac{\pi\alpha}{2}$.

Fixed point theory is one of the most powerful and fruitful tools of various theoretical and applied fields, such as linear inequalities, variational inequalities, the approximation theory, integral and differential equations and inclusions, nonlinear analysis, the dynamic systems theory, mathematics of fractals, mathematical physics, economics and mathematical modeling. In particular, the fixed point theory is always considered a core subject of nonlinear analysis. The following results can be found in [21, 22].

Theorem 2.7. *Assume* S_0 *is a Banach space,* S_1 *is a closed, convex subset of* S_0, S_2 *is an open subset of* S_1, *and* $p \in S_2$. *Suppose that* $M : \overline{S_2} \to S_1$ *is a continuous, compact map, then either*
(a) F has a fixed point in $\overline{S_2}$; or
(b) there are $s \in \partial S_2$ (the boundary of S_2 in S_1) and $\nu \in (0,1)$ with $s = \nu M(s) + (1-\nu)p$.

Theorem 2.8. *Assume* S_0 *is a Hausdorff locally convex linear topological space,* S_1 *is a convex subset of* S_0, S_2 *is an open subset of* S_1, *and* $p \in S_2$. *Suppose that* $M : \overline{S_2} \to S_1$ *is a continuous, compact map, then either*
(a) F has a fixed point in $\overline{S_2}$; or
(b) there are $s \in \partial S_2$ (the boundary of S_2 in S_1) and $\nu \in (0,1)$ with $s = \nu M(s) + (1-\nu)p$.

3. Well posedness of the initial value problem of linear fractional differential equation

In this section, we consider the initial value problem of the linear fractional differential equation

$$\begin{cases} t^{-\beta\gamma} {}_*D_\beta^{\alpha,\gamma} u(t) - \lambda u(t) = f(t), \\ \lim_{t\to 0} t^{\beta(\alpha+n-k)} \prod_{i=1}^{k} (\alpha+n-i+\frac{1}{\beta}t\frac{d}{dt})u(t) = u_k, \\ \qquad k = 0,1,2,\cdots,n-1, \end{cases} \tag{3.1}$$

where u_k $(k = 0, 1, 2, \cdots, n-1)$ are positive numbers.

We refer to [23] for a study of the initial value problem involving Riemann-Liouille type modification of EKFD

$$\begin{cases} t^{-\beta\gamma} D_\beta^{\alpha,\gamma} u(t) - \lambda u(t) = g(t), \\ \lim_{t\to 0} \frac{\Gamma(n-k)}{\Gamma(\gamma-k)} t^{\beta(1+\alpha+k)} \Pi_{i=k+1}^{n-1} (1 + \alpha + i + \frac{1}{\beta} t \frac{d}{dt})(I_\beta^{\alpha+\gamma, n-\gamma} u)(t) = \bar{u}_k, \\ \qquad\qquad k = 0, 1, 2, \cdots, n-1, \end{cases} \tag{3.2}$$

where $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}^+$, $\gamma \in (n-1, n]$ and $n \in \mathbb{N}^+$. The explicit solution was established by the transmutation method.

Lemma 3.1. *Let $f \in C_{\beta\mu}$, $\mu \geq \max\{0, -\alpha - \gamma\} - 1$, then there exists a solution $u \in C_{\beta\mu}^{(n)}$ of problem (3.2) with the form*

$$\begin{aligned} u(t) = &\sum_{k=0}^{n-1} \bar{u}_k t^{\beta(\gamma-k-1)} E_{\gamma, \alpha+2\gamma-k}(\lambda t^{\beta\gamma}) \\ &+ t^{-\beta(\alpha+\gamma)} \int_0^t (t^\beta - \tau^\beta)^{\gamma-1} \tau^{\beta(\alpha+\gamma)} E_{\gamma,\gamma}(\lambda(t^\beta - \tau^\beta)^\gamma) g(\tau) d\tau^\beta. \end{aligned}$$

The parameter that appears in initial conditions of the initial value problem and is defined by the Riemann-Liouille type modification of the EKFDs whose physical interpretation is not clear. For this reason, the Caputo type modification of the EKFDs is introduced and the following problem is considered:

$$\begin{cases} t^{-\beta\gamma} {}_*D_\beta^{\alpha,\gamma} u(t) - \lambda u(t) = f(t), \\ \lim_{t\to 0} t^{\beta(1+\alpha+k)} \Pi_{i=k+1}^{n-1} (1 + \alpha + i + \frac{1}{\beta} t \frac{d}{dt}) u(t) = \tilde{u}_k, \\ \qquad\qquad k = 0, 1, 2, \cdots, n-1, \end{cases} \tag{3.3}$$

where $\alpha \in \mathbb{R}, \beta \in \mathbb{R}^+, \gamma \in (n-1, n]$ and $n \in \mathbb{N}^+$.

In order to use of the result given in Lemma 3.1, we recall the relation between ${}_*D_\beta^{\alpha,\gamma}$ and $D_\beta^{\alpha,\gamma}$, which was established in [12] by use of operational methods.

Lemma 3.2. *The Caputo type modification of the EKFD ${}_*D_\beta^{\alpha,\gamma}$ coincides with Riemann-Liouille type modification of the EKFD $D_\beta^{\alpha,\gamma}$ for a function $u \in C_\nu^n$, $\gamma \in (n-1, n]$, $n \in \mathbb{N}$, $\nu \geq -\beta(\alpha + 1)$, if and only if for conditions $\tilde{u}_k = \bar{u}_k, k = 0, 1, 2, \cdots, n-1$ are fulfilled.*

Since ${}_*D_\beta^{\alpha,\gamma}$ and $D_\beta^{\alpha,\gamma}$ are on the right-hand sided EKFDs of order γ, then applying Lemmas 2.4 and 2.5, we have

$${}_*D_\beta^{\alpha,\gamma} u(t) = D_\beta^{\alpha,\gamma} I_\beta^{\alpha,\gamma} {}_*D_\beta^{\alpha,\gamma} u(t) = D_\beta^{\alpha,\gamma} \Big(u(t) - \sum_{k=0}^{n-1} \tilde{u}_k t^{-\beta(1+\alpha+k)} \Big),$$

then, there exists

$$I_\beta^{\alpha,\gamma} {}_*D_\beta^{\alpha,\gamma} u(t) = I_\beta^{\alpha,\gamma} D_\beta^{\alpha,\gamma} \Big(u(t) - \sum_{k=0}^{n-1} \tilde{u}_k t^{-\beta(1+\alpha+k)} \Big),$$

which is equivalent to

$$\sum_{k=0}^{n-1} \tilde{u}_k t^{-\beta(1+\alpha+k)} = \sum_{k=0}^{n-1} \bar{u}_k t^{-\beta(1+\alpha+k)} + \sum_{k=0}^{n-1} \tilde{u}_k I_\beta^{\alpha,\gamma} D_\beta^{\alpha,\gamma} t^{-\beta(1+\alpha+k)},$$

in terms of Lemma 2.5. According to formulas (45) and (67) in [11], there exists

$$I_\beta^{\alpha,\gamma} D_\beta^{\alpha,\gamma} t^{-\beta(1+\alpha+k)} = t^{-\beta(1+\alpha+k)}.$$

Then, these yield $\bar{u}_k = 0$. Hence, the problem (3.3) is equivalent to the following problem

$$\begin{cases} t^{-\beta\gamma} D_\beta^{\alpha,\gamma} u(t) - \lambda u(t) = f(t) + t^{-\beta\gamma} \sum_{k=0}^{n-1} \tilde{u}_k D_\beta^{\alpha,\gamma}(t^{-\beta(1+\alpha+k)}), \\ \lim_{t\to 0} \frac{\Gamma(n-k)}{\Gamma(\gamma-k)} t^{\beta(1+\alpha+k)} \Pi_{i=k+1}^{n-1} (1+\alpha+i+\frac{1}{\beta} t \frac{d}{dt})(I_\beta^{\alpha+\gamma,n-\gamma} u)(t) = 0, \\ \qquad\qquad\qquad\qquad k = 0, 1, 2, \cdots, n-1, \end{cases}$$

where $\alpha \in \mathbb{R}, \beta \in \mathbb{R}^+$, $\gamma \in (n-1, n]$ and $n \in \mathbb{N}^+$. Based on the above analysis and Lemmas 3.1 and 3.2, we confirm the well-posedness of the initial value problem of inhomogeneous linear differential equation involving Caputo type modification of the EKFDs.

Theorem 3.3. *Let $f \in C_{\beta\mu}$, $\mu \geq \max\{0, -\alpha-\gamma\} - 1$, then there exists a solution $u \in C_{\beta\mu}^{(n)}$ of problem (3.3) with the form*

$$\begin{aligned} u(t) =& t^{-\beta(\alpha+\gamma)} \int_0^t (t^\beta - \tau^\beta)^{\gamma-1} \tau^{\beta(\alpha+\gamma)} E_{\gamma,\gamma}(\lambda(t^\beta - \tau^\beta)^\gamma)\Big(f(\tau) \\ &+ \tau^{-\beta\gamma} \sum_{k=0}^{n-1} \tilde{u}_k D_\beta^{\alpha,\gamma}(\tau^{-\beta(1+\alpha+k)})\Big) d\tau^\beta. \end{aligned}$$

According to formula (67) in [11], there exists

$$D_\beta^{\alpha,\gamma} t^q = \frac{\Gamma(\alpha+\gamma+1+\frac{q}{\beta})}{\Gamma(\alpha+1+\frac{q}{\beta})} t^q, q > -\beta(\alpha+1).$$

then, we obtain

$$D_\beta^{\alpha,\gamma}(t^{-\beta(n+\alpha-k)}) = \frac{\Gamma(\gamma+k+1-n)}{\Gamma(k+1-n)} t^{-\beta(n+\alpha-k)}.$$

Hence, set

$$I = \int_0^t (t^\beta - \tau^\beta)^{\gamma-1} \tau^{\beta\alpha} E_{\gamma,\gamma}(\lambda(t^\beta - \tau^\beta)^\gamma) D_\beta^{\alpha,\gamma}(\tau^{-\beta(n+\alpha-k)}) d\tau^\beta,$$

and we arrive at

$$\begin{aligned} I =& \frac{\Gamma(\gamma+k+1-n)}{\Gamma(k+1-n)} \int_0^t (t^\beta - \tau^\beta)^{\gamma-1} \tau^{\beta(k-n)} E_{\gamma,\gamma}(\lambda(t^\beta - \tau^\beta)^\gamma) d\tau^\beta \\ =& \frac{\Gamma(\gamma+k+1-n)}{\Gamma(k+1-n)} \sum_{j=0}^\infty \frac{\lambda^j}{\Gamma(\gamma j+\gamma)} \int_0^t (t^\beta - \tau^\beta)^{\gamma+\gamma j-1} \tau^{\beta(k-n)} d\tau^\beta \\ =& \frac{\Gamma(\gamma+k+1-n)}{\Gamma(k+1-n)} \sum_{j=0}^\infty \frac{(\lambda t^{\beta\gamma})^j t^{\beta(\gamma+k-n)}}{\Gamma(\gamma j+\gamma)} \int_0^1 (1-\tau)^{\gamma+\gamma j-1} \tau^{k-n} d\tau \\ =& \frac{\Gamma(\gamma+k+1-n)}{\Gamma(k+1-n)} t^{\beta(\gamma+k-n)} \sum_{j=0}^\infty \frac{(\lambda t^{\beta\gamma})^j \mathrm{B}(\gamma j+\gamma, k-n+1)}{\Gamma(\gamma j+\gamma)} \\ =& \Gamma(\gamma+k+1-n) t^{\beta(\gamma+k-n)} E_{\gamma,\gamma-n+k+1}(\lambda t^{\beta\gamma}), \end{aligned}$$

where $\mathrm{B}(\cdot,\cdot)$ denotes a Beta function.

Finally, applying the above results and Theorem 3.3 with $u_k = \tilde{u}_{n-1-k}$, we obtain Theorem 3.4.

Theorem 3.4. *Set* $f \in C_{\beta\mu}$, $\mu \geq \max\{0, -\alpha-\gamma\} - 1$, *then there exists an explicit solution* $u \in C_{\beta\mu}^{(n)}$ *of problem (3.1), which is given in the form*

$$
\begin{aligned}
u(t) = &\sum_{k=0}^{n-1} u_k \Gamma(\gamma + k + 1 - n) t^{\beta(k-n-\alpha)} E_{\gamma,\gamma-n+k+1}(\lambda t^{\beta\gamma}) \\
&+ t^{-\beta(\alpha+\gamma)} \int_0^t (t^\beta - \tau^\beta)^{\gamma-1} \tau^{\beta(\alpha+\gamma)} E_{\gamma,\gamma}(\lambda(t^\beta - \tau^\beta)^\gamma) f(\tau) d\tau^\beta.
\end{aligned}
$$

Remark 3.5. *If the continuous function* $f(t)$ *is nonnegative and constants* λ, $u_k (k = 0, 1, 2, \cdots, n-1)$ *are positive, then the explicit solution of problem (5) presented in Theorem 3.4 is positive.*

4. Gronwall type integral inequalities involving Erdélyi-Kober fractional integral

Gronwall type inequalities play an important role in solving solutions of numerous differential and integral equations. The classical form of this type of inequality [24] is described as follows.

Theorem 4.1. *Assume* $u(t)$, $v_i(t)$ $(i = 1, 2)$ *and* $g(t)$ *are continuous, nonnegative functions on* $[0, +\infty)$ *with* $p \geq 1$ *such that*

$$u(t) \leq v_1(t) + v_2(t) \Big(\int_0^t g(s) u^p(\tau) d\tau \Big)^{\frac{1}{p}},$$

then,

$$u(t) \leq v_1(t) + v_2(t) \frac{(\int_0^t w(\tau) g(\tau) v_1^p(\tau) d\tau)^{\frac{1}{p}}}{1 - (1 - w(t))^{\frac{1}{p}}},$$

where $w(t) = \exp(-\int_0^t g(\tau) v_2^p(\tau) d\tau)$.

In order to establish a new useful Gronwall inequalities for studying the problem (1.1), we recall Kalla's results with the form

$$\int_0^t K(\frac{\tau}{t}) \tau^\gamma f(\tau) d\tau, \quad t > \tau > 0$$

that one can refer to see ([25–29]). Some extended theories of this integral are developed and have been applied to different topics, like as Gronwall inequalities and other operational calculus, integral transforms, and classes of integral and differential equations.

Consider integral

$$\int_0^t K(\alpha, \beta, \gamma, \lambda, t, \tau) f(\tau) d\tau, \quad t > \tau > 0 \tag{4.1}$$

with parameters $\alpha \in \mathbb{R}, \beta \in \mathbb{R}^+$, $\gamma \in (n-1, n]$ and $n \in \mathbb{N}^+$, the kernel is expressed by

$$K(\alpha, \beta, \gamma, \lambda, t, \tau) = (t^\beta - \tau^\beta)^{\gamma-1} \tau^{\beta(\alpha+\gamma)+\beta-1} E_{\gamma,\gamma}(\lambda(t^\beta - \tau^\beta)^\gamma), \tag{4.2}$$

where $E_{\gamma,\gamma}(\cdot)$ denotes the Mittag-Leffler function, function $f(\cdot)$ is given. It is easy to find that integral (4.1) is a direct generalization of Kalla's and closely connected with EKFD formally. To establish some new related Gronwall inequalities based on (4.1)–(4.2), one should consider the more complicated kernel with singularity compared with the form [3, 14, 17, 18].

In the following, we show our results.

Theorem 4.2. *Let $\lambda \in \mathbb{R}^+$, $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}^+$, $\gamma \in (n-1, n]$ and $n \in \mathbb{N}^+$, assume $v_i(t)$ $(i = 1, 2)$, $g(t)$ and $u(t)$ are continuous, nonnegative functions on $(0, +\infty)$ with*

$$u(t) \le v_1(t) + v_2(t) \int_0^t K(\alpha,\beta,\gamma,\lambda,t,\tau) g(\tau) u(\tau) d\tau,$$

then,

$$u(t) \le V_1(t) + V_2(t) \frac{(\int_0^t w(\tau)(gV_1)^r(\tau) d\tau)^{\frac{1}{r}}}{1 - (1 - w(t))^{\frac{1}{r}}}, \tag{4.3}$$

where $w(t) = \exp(-\int_0^t (gV_2)^r(\tau) d\tau)$, $V_1(t) = v_1(t)$, $V_2(t) = Cv_2(t)h(t)t^{\frac{1-\beta}{p}+\frac{1}{q}}$, constant $C = C(\alpha,\beta,\gamma,p)$ only depends on α,β,γ,p, and

$$h(t) = \begin{cases} t^{\beta(\alpha+2\gamma+\frac{1}{p})-1}, \quad t \in (0,1), \\ t^{\beta(\alpha+\gamma+1+\frac{1}{p})-1} \exp(\lambda^{\frac{1}{\gamma}} t^{\beta}), \quad t \in [1, +\infty), \gamma \in (0, 2), \\ t^{\beta(\alpha+2\gamma+\frac{1}{p})-1} \sum_{k \le \frac{\gamma}{4}} t^{\frac{\beta\gamma}{k}} \exp(\lambda^{\frac{1-\gamma}{\gamma}} t^{\beta(1-\gamma)}), \quad t \in [1, +\infty), \gamma \in [2, +\infty) \end{cases}$$

for $(1-\beta)q + p > 0$, $p(\alpha + \gamma + 1 - \frac{1}{\beta}) + 1 > 0$, $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$, $p, q, r \in \mathbb{R}^+$.

Proof: By a direct computation, it is easy to verify that

$$\begin{aligned} &\int_0^t K^p(\alpha,\beta,\gamma,\lambda,t,\tau) d\tau^{\beta} \\ =&\int_0^t ((t^{\beta} - \tau^{\beta})^{\gamma-1} \tau^{\beta(\alpha+\gamma)+\beta-1} E_{\gamma,\gamma}(\lambda(t^{\beta} - \tau^{\beta})^{\gamma}))^p d\tau^{\beta} \\ =&\int_0^t (\tau^{\beta(\gamma-1)}(t^{\beta} - \tau^{\beta})^{\alpha+\gamma+1-\frac{1}{\beta}} E_{\gamma,\gamma}(\lambda\tau^{\beta\gamma}))^p d\tau^{\beta} \\ =&t^{p(\beta(\alpha+2\gamma+\frac{1}{p})-1)} \int_0^1 (\tau^{\gamma-1}(1-\tau)^{\alpha+\gamma+1-\frac{1}{\beta}} E_{\gamma,\gamma}(\lambda t^{\beta\gamma}\tau^{\gamma}))^p d\tau. \end{aligned} \tag{4.4}$$

In terms of the series expansion (2.2) and the asymptotic behavior of the entire function $E_{\alpha,\beta}(t)$ in Lemma 2.6, we have

$$\int_0^t K^p(\alpha,\beta,\gamma,\lambda,t,\tau) d\tau^{\beta} \le C_1 h^p(t) \tag{4.5}$$

for $C_1 = C_1(\alpha,\beta,\gamma,p)$, $p(\alpha + \gamma + 1 - \frac{1}{\beta}) + 1 > 0$ and any $t \in [0, +\infty)$.

Meanwhile, by a direct computation, we obtain

$$\int_0^t \tau^{\frac{q(1-\beta)}{p}} d\tau = \frac{p}{p + q(1-\beta)} t^{\frac{q(1-\beta)}{p}+1} \tag{4.6}$$

for $(1-\beta)q + p > 0$.

Then, the H$\ddot{o}$lder inequality and (4.5)–(4.6) yield

$$u(t) \le v_1(t) + v_2(t) \int_0^t K(\alpha,\beta,\gamma,\lambda,t,\tau) g(\tau) u(\tau) d\tau$$

$$
\begin{aligned}
&\leq v_1(t)+v_2(t)\Big(\int_0^t (gu)^r(\tau)d\tau\Big)^{\frac{1}{r}}\Big(\int_0^t K^p(\alpha,\beta,\gamma,\lambda,t,\tau)d\tau^{\beta}\Big)^{\frac{1}{p}}\Big(\int_0^t \tau^{\frac{q(1-\beta)}{p}}d\tau\Big)^{\frac{1}{q}}\\
&\leq V_1(t)+V_2(t)\Big(\int_0^t (gu)^r(\tau)d\tau\Big)^{\frac{1}{r}},
\end{aligned}
\tag{4.7}
$$

where $V_1(t)=v_1(t)$, $V_2(t)=Cv_2(t)h(t)t^{\frac{1-\beta}{p}+\frac{1}{q}}$ for $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}=1$, $p,q,r\in\mathbb{R}^+$.

Hence, in terms of (4.7), Theorem 4.1 yields (4.3).

Thus, we complete the proof.

In solving the fractional differential equations with nonlinear term $f(t,u(t))$, the Gronwall type integral inequality is a powerful tool [15, 18]. Here and in the following, we define an invertible operator Θ by

$$
\Theta(\delta)=\int_{\epsilon}^{\delta}\frac{dt}{\theta(t)},\quad \epsilon>0,\quad \delta>0 \tag{4.8}
$$

for some positive and nondecreasing function $\theta(t)$.

Theorem 4.3. *Let $\lambda\in\mathbb{R}^+$ $\alpha\in\mathbb{R}$, $\beta\in\mathbb{R}^+$, $\gamma\in(n-1,n]$ and $n\in\mathbb{N}^+$, assume functions $v_i(t)\in C[0,T]$, $i=1,2$ are nondecreasing and nonnegative, $g(t)\in C[0,T]$ is nonnegative, $f(t)\in C[0,T]$ is nondecreasing and nonnegative, and $u(t)\in C[0,T]$ is nonnegative, such that*

$$
u(t)\leq v_1(t)+v_2(t)\int_0^t K(\alpha,\beta,\gamma,\lambda,t,\tau)g(\tau)f(u(\tau))d\tau
$$

for arbitrary positive number T, then

$$
u(t)\leq\Big(\Theta^{-1}\Big(\Theta(2^{r-1}V_1^r(t))+2^{r-1}V_2^r(t)\int_0^t g^r(\tau)d\tau\Big)\Big)^{\frac{1}{r}},
$$

where $V_1(t)=v_1(t)$, $V_2(t)=Cv_2(t)h(t)t^{\frac{1-\beta}{p}+\frac{1}{q}}$, $C=C(\alpha,\beta,\gamma,p)$ is a constant which only depends on α,β,γ,p, and

$$
h(t)=\begin{cases} t^{\beta(\alpha+2\gamma+\frac{1}{p})-1}, \quad t\in(0,1),\\ \begin{cases} t^{\beta(\alpha+\gamma+1+\frac{1}{p})-1}\exp(\lambda^{\frac{1}{\gamma}}t^{\beta}), \quad t\in[1,+\infty),\gamma\in(0,2),\\ t^{\beta(\alpha+2\gamma+\frac{1}{p})-1}\sum_{k\leq\frac{\gamma}{4}}t^{\frac{\beta\gamma}{k}}\exp(\lambda^{\frac{1-\gamma}{\gamma}}t^{\beta(1-\gamma)}), \quad t\in[1,+\infty),\gamma\in[2,+\infty)\end{cases}\end{cases}
$$

for $(1-\beta)q+p>0$, $p(\alpha+\gamma+1-\frac{1}{\beta})+1>0$, $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}=1$, $p,q,r\in\mathbb{R}^+$.

Proof: Similar to the proof in proving (4.7) of Theorem 4.2, we have

$$
\begin{aligned}
u(t)\leq&V_1(t)+V_2(t)\Big(\int_0^t (g(\tau)f(u(\tau)))^r d\tau\Big)^{\frac{1}{r}}\\
\leq&V_1(T_0)+V_2(T_0)\Big(\int_0^t (g(\tau)f(u(\tau)))^r d\tau\Big)^{\frac{1}{r}}
\end{aligned}
$$

for some $T_0\in(0,T]$. Take $U=u^r$, then the last inequality becomes

$$
U(t)\leq 2^{r-1}\Big(V_1^r(T_0)+V_2^r(T_0)\int_0^t (g(\tau)f(u(\tau)))^r d\tau\Big). \tag{4.9}
$$

Next, we show

$$\begin{cases} d\Theta(V(t)) \leq 2^{r-1}V_2^r(T_0)g^r(t), \\ V(0) = 2^{r-1}V_1^r(T_0), \end{cases} \tag{4.10}$$

where Θ is given by (4.8) with $\epsilon = V(0)$, and

$$V(t) = 2^{r-1}\Big(V_1^r(T_0) + V_2^r(T_0)\int_0^t (g(t)f(u(t)))^r d\tau\Big).$$

In addition, if we let $\theta(t) = f^r(t^{\frac{1}{r}})$, then

$$\begin{aligned} V'(t) \leq& 2^{r-1}V_2^{pr_2}(T_0)g^r(t)\theta(U(t)) \\ \leq& 2^{r-1}V_2^r(T_0)g^r(t)\theta(V(t)) \end{aligned}$$

holds and (4.10) is derived.

In terms of initial value problem (4.10), we obtain

$$\Theta(V(t)) \leq \Theta(2^{r-1}V_1^r(T_0)) + 2^{r-1}V_2^r(T_0)\int_0^t g^r(\tau)d\tau. \tag{4.11}$$

Since Θ is inversible, then (4.11) is equivalent to the following equation

$$V(t) \leq \Theta^{-1}\Big(\Theta(2^{r-1}V_1^r(T_0)) + 2^{r-1}V_2^r(T_0)\int_0^t g^r(\tau)d\tau\Big).$$

Furthermore, applying (4.9), there exists

$$U(t) \leq \Theta^{-1}\Big(\Theta(2^{r-1}V_1^r(T_0)) + 2^{r-1}V_2^r(T_0)\int_0^t g^r(\tau)d\tau\Big).$$

For the arbitrariness of $t \in [0, T_0]$, we conclude

$$u(T_0) \leq \Big(\Theta^{-1}\Big(\Theta(2^{r-1}V_1^r(T_0)) + 2^{r-1}V_2^r(T_0)\int_0^{T_0} g^r(\tau)d\tau\Big)\Big)^{\frac{1}{r}}.$$

Furthermore, for the arbitrariness of $T_0 \in [0, T]$, we obtain

$$\Theta(2^{r-1}V_1^r(t)) + 2^{r-1}V_2^r(t)\int_0^t g^r(\tau)d\tau \in \mathrm{Dom}(\Theta^{-1}).$$

This completes the proof.

5. Existence and uniqueness of solution of nonlinear differential equation

In this section, we establish the existence and uniqueness of the positive solution of the initial value problem of the nonlinear differential equation with higher order Caputo type modification of the Erdélyi-Kober operators in terms of Theorems 4.2 and 4.3 with parameters γ-th order, $\gamma \in (n-1, n]$, $n \in \mathbb{N}^+, \beta \in \mathbb{R}^+$ and

$$\alpha \geq \max_{k\in(0,1,2,\cdots,n-1)}\{n-k-\gamma k-2\gamma-1, n-k-\frac{1}{\beta}(\gamma+\gamma k+1)\}.$$

Theorem 5.1. *Set C_0 as a positive number, f as a nonnegative and continuous function that satisfies $f(t,u)|_{u=0} = 0$, $|f(t,u) - f(t,v)| \leq C_0|u - v|$ and initial conditions $u_k > 0$ for all $k = 0, 1, 2, \cdots, n-1$, then there exists a unique positive solution $u(t) \in C(0,+\infty)$ of problem (1.1) and ${}_*D^{\alpha,\gamma}_{\beta}u(t) \in C(0,+\infty)$.*

Proof: Set $S_1 = \{u(t) \in C[0,T] : t^{\beta(\alpha+n)}u(t) \geq u_0\}$, $T \in (0,+\infty)$. Consider the map $M : S_1 \to S_1$ defined by

$$\begin{aligned}(Mu)(t) = &\sum_{k=0}^{n-1} u_k \Gamma(\gamma + k + 1 - n) t^{\beta(k-n-\alpha)} E_{\gamma,\gamma-n+k+1}(\lambda t^{\beta\gamma})\\ &+ t^{-\beta(\alpha+\gamma)} \int_0^t (t^\beta - \tau^\beta)^{\gamma-1} \tau^{\beta(\alpha+\gamma)} E_{\gamma,\gamma}(\lambda(t^\beta - \tau^\beta)^\gamma) f(\tau, u(\tau)) d\tau^\beta.\end{aligned}$$

Since f is nonnegative, then Remark 3.5 implies Mu is positive.

Based on Theorem 4.2, we obtain that any fixed point of map M is a solution of the Volterra-type integral equation

$$\begin{aligned}u(t) = &\sum_{k=0}^{n-1} u_k \Gamma(\gamma + k + 1 - n) t^{\beta(k-n-\alpha)} E_{\gamma,\gamma-n+k+1}(\lambda t^{\beta\gamma})\\ &+ t^{-\beta(\alpha+\gamma)} \int_0^t (t^\beta - \tau^\beta)^{\gamma-1} \tau^{\beta(\alpha+\gamma)} E_{\gamma,\gamma}(\lambda(t^\beta - \tau^\beta)^\gamma) f(\tau, u(\tau)) d\tau^\beta.\end{aligned}$$

We know that the fixed point of the Volterra-type integral equation is also a solution of problem (1.1). Set

$$S_2 = \{u(t) \in S_1 : u(t) < V_1(t) + \frac{C_0 V_2(t) (\int_0^t w(\tau) V_1^r(\tau) d\tau)^{\frac{1}{r}}}{1 - (1 - w(t))^{\frac{1}{r}}}, t \geq 0\},$$

where

$$\begin{cases} V_1(t) = & \sum_{k=0}^{n-1} u_k \Gamma(\gamma + k + 1 - n) t^{\beta(k-n-\alpha)} E_{\gamma,\gamma-n+k+1}(\lambda t^{\beta\gamma}),\\ V_2(t) = & h(t) t^{\frac{1-\beta}{p} + \frac{1}{q} - \beta(\alpha+\gamma)},\\ w(t) = & \exp(-C_0 \int_0^t V_2^r(\tau) d\tau), \end{cases}$$

and $h(t)$ is given in Theorem 4.2 for $(1-\beta)q + p > 0$, $p(\alpha + \gamma + 1 - \frac{1}{\beta}) + 1 > 0$, $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$, $p, q, r \in \mathbb{R}^+$. Take a similar procedure used in [18], and we derive that $M : \overline{S_2} \to S_1$ is continuous and compact.

If $u \in \overline{S_2}$ is any solution of equation

$$\begin{aligned}u(t) =&(1 - \nu) V_1(t) + \nu (Fu)(t)\\ =&V_1(t) + \nu t^{-\beta(\alpha+\gamma)} \int_0^t (t^\beta - \tau^\beta)^{\gamma-1} \tau^{\beta(\alpha+\gamma)} E_{\gamma,\gamma}(\lambda(t^\beta - \tau^\beta)^\gamma) f(\tau, u(\tau)) d\tau^\beta\end{aligned}$$

for $\nu \in (0,1)$, then it is easy to verify that

$$u(t) \leq V_1(t) + C_0 \nu t^{-\beta(\alpha+\gamma)} \int_0^t K(\alpha, \beta, \gamma, \lambda, t, \tau) u(t) d\tau.$$

Applying (10), we have

$$u(t) \leq V_1(t) + \frac{C_0 V_2(t) (\int_0^t w(\tau) V_1^r(\tau) d\tau)^{\frac{1}{r}}}{1 - (1 - w(t))^{\frac{1}{r}}}.$$

Hence, by use of Theorem 2.7, we claim that F has a fixed point in $\overline{S_2}$, then this fixed point is a positive solution of problem (1.1). Since the arbitrariness of $T \in (0, +\infty)$, then we claim that there exists a positive solution $u \in C(0, +\infty)$ of problem (1.1).

Furthermore, consider the continuity of function $f(\cdot, \cdot)$ and the expression

$$t^{-\beta\gamma}{}_*D_\beta^{\alpha,\gamma}u(t) = \lambda u(t) + f(t, u(t)),$$

and we obtain ${}_*D_\beta^{\alpha,\gamma}u(t) \in C(0, +\infty)$.

If there exists two solutions u and v in $\overline{U}$ to problem (15), then it follows

$$\begin{aligned}
&|u(t) - v(t)| \\
=&t^{-\beta(\alpha+\gamma)} \int_0^t (t^\beta - \tau^\beta)^{\gamma-1}\tau^{\beta(\alpha+\gamma)}E_{\gamma,\gamma}(\lambda(t^\beta - \tau^\beta)^\gamma)|f(\tau, u(\tau)) - f(\tau, v(\tau))|d\tau^\beta \\
\leq&C_0t^{-\beta(\alpha+\gamma)} \int_0^t (t^\beta - \tau^\beta)^{\gamma-1}\tau^{\beta(\alpha+\gamma)}E_{\gamma,\gamma}(\lambda(t^\beta - \tau^\beta)^\gamma)|u(\tau) - v(\tau)|d\tau^\beta,
\end{aligned}$$

then by use of Theorem 4.2 with $V_1(t) = 0$, we obtain $u = v$. This yields the uniqueness of the solution.

Thus, we complete the proof.

Theorem 5.2. *Set $w(t) \in C[0, T]$ and the nondecreasing function $g(t) \in C[0, T]$ as nonnegative, assume $f(t, u(t)) \leq w(t)g(u(t))$, which is nonnegative and continuous, and initial conditions $u_k > 0$ for all $k = 0, 1, 2, \cdots, n-1$, then there exists a positive solution $u(t) \in C(0, T]$ of problem (1.1) and ${}_*D_\beta^{\alpha,\gamma}u(t) \in C(0, T]$.*

Proof: For any positive number $T > 0$, we form a map $M : S_1 \to S_1$, which is defined by

$$\begin{aligned}
(Mu)(t) =& \sum_{k=0}^{n-1} u_k\Gamma(\gamma + k + 1 - n)t^{\beta(k-n-\alpha)}E_{\gamma,\gamma-n+k+1}(\lambda t^{\beta\gamma}) \\
&+ t^{-\beta(\alpha+\gamma)} \int_0^t (t^\beta - \tau^\beta)^{\gamma-1}\tau^{\beta(\alpha+\gamma)}E_{\gamma,\gamma}(\lambda(t^\beta - \tau^\beta)^\gamma)f(\tau, u(\tau))d\tau^\beta,
\end{aligned}$$

where $S_1 = \{u(t) \in C(0, T] : t^{\beta(\alpha+n)}u(t) \geq u_0, T > 0\}$. It is easy to verify that Mu is positive if f is nonnegative. Set

$$S_2 = \{u \in S_1 : u^r(t) < \Theta^{-1}\Big(\Theta(2^{r-1}V_1^r(t)) + 2^{r-1}V_2^r(t) \int_0^t w^r(\tau)d\tau\Big), 0 < t \leq T\},$$

where

$$\begin{cases}
V_1(t) = & \sum_{k=0}^{n-1} u_k\Gamma(\gamma + k + 1 - n)t^{\beta(k-n-\alpha)}E_{\gamma,\gamma-n+k+1}(\lambda t^{\beta\gamma}), \\
V_2(t) = & h(t)t^{\frac{1-\beta}{p}+\frac{1}{q}-\beta(\alpha+\gamma)},
\end{cases}$$

and $h(t)$ is defined in Theorem 4.3 for $(1-\beta)q+p > 0$, $p(\alpha+\gamma+1-\frac{1}{\beta})+1 > 0$, $\frac{1}{p}+\frac{1}{q}+\frac{1}{r} = 1$, $p, q, r \in \mathbb{R}^+$. By applying the usual techniques used in Theorem 5.1, we obtain that the map $M : \overline{S_2} \to S_1$ is continuous and compact. It follows of Theorem 3.4 that the fixed points of operator M are solutions of problem (1.1).

If $u \in \overline{S_2}$ is any solution of the Volterra-type integral equation

$$\begin{aligned} u(t) =&(1-\gamma)V_1(t) + \gamma(Fu)(t) \\ =&V_1(t) + \gamma t^{-\beta(\alpha+\gamma)} \int_0^t (t^\beta - \tau^\beta)^{\gamma-1} \tau^{\beta(\alpha+\gamma)} E_{\gamma,\gamma}(\lambda(t^\beta - \tau^\beta)^\gamma) f(\tau, u(\tau)) d\tau^\beta \end{aligned}$$

for $\gamma \in (0, 1)$, then it is easy to verify that

$$u(t) \leq V_1(t) + \gamma t^{-\beta(\alpha+\gamma)} \int_0^t K(\alpha, \beta, \gamma, \lambda, t, \tau) w(\tau) g(u)(\tau) d\tau.$$

Applying Theorem 4.3 that we obtain

$$u(t) < \Big(\Theta^{-1}\Big(\Theta\Big(2^{r-1}\Big(V_1(t)\Big)^r\Big) + 2^{r-1} V_2^r(t) \int_0^t w^r(s) ds\Big)\Big)^{\frac{1}{r}}.$$

Hence, Theorem 2.8 yields that F has a fixed point in $\overline{U}$.

Meanwhile, Eq (1.1) and the continuity of function $f(\cdot, \cdot)$ implies ${}_{*}D_\beta^{\alpha,\gamma} u(t) \in C(0, T]$.

Thus, we confirm the result of Theorem 5.2.

6. Conclusions

In this paper, the positive solution of a nonlinear differential equation with higher order Caputo type modification of the EKFD was studied. First, the well-posedness of the initial value problem of the higher order linear model was proved and an explicit positive solution was presented based on the transmutation method. Second, some new Gronwall type inequalities involving EKFI with singular kernels were established. At last, by applying the derived results and some fixed point theorems, the existence and uniqueness of the positive solution of this kinds of nonlinear differential equation were obtained. The method is applicable to such kinds of fractional differential equations with any order $\gamma \in (n-1, n]$, $n \in \mathbb{N}^+$.

Use of AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares to have no competing interests.

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