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Research article

Sufficient conditions for triviality of Ricci solitons

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Abstract: We found conditions on an n-dimensional Ricci soliton $(M, g, \mathbf{u}, \lambda)$ to be trivial. First, we showed that under an appropriate upper bound on the squared length of the covariant derivative of the potential field $\mathbf{u}$, the Ricci soliton $(M, g, \mathbf{u}, \lambda)$ reduces to a trivial soliton. We also showed that appropriate upper and lower bounds on the Ricci curvature $Ric(\mathbf{u}, \mathbf{u})$ of a compact Ricci soliton $(M, g, \mathbf{u}, \lambda)$ with potential field $\mathbf{u}$ geodesic vector field makes it a trivial soliton. We showed that if the Ricci operator S of the Ricci soliton $(M, g, \mathbf{u}, \lambda)$ is invariant under the potential field $\mathbf{u}$, then $(M, g, \mathbf{u}, \lambda)$ is trivial and the converse is also true. Finally, it was shown that if the potential field $\mathbf{u}$ of a connected Ricci soliton $(M, g, \mathbf{u}, \lambda)$ is a concurrent vector field, then the Ricci soliton is shrinking.

Keywords: Ricci soliton; potential field; geodesic vector field; concurrent vector field
Mathematics Subject Classification: 35Q51, 53B25, 53B50

1. Introduction

In first decade of the nineteenth century, Poincare made the following conjecture: "A compact simply connected three-manifold without boundary is diffeomorphic to the three sphere S^3."A more general conjecture than Poincare conjecture is Thurston's geometrization conjecture, which says that any closed three-manifold can be decomposed into pieces such that each piece has a locally homogeneous metric and are S^3, R^3, H^3, $S^2 \times R$, $H^2 \times R$, $SL(2, R)$, nil^3 and sol^3. With the aim of proving the geometrization conjecture, Hamilton [1] initiated a program in 1982 called Ricci flow that starts with a given Riemannian metric g_0 on a smooth n-dimensional manifold M and evolves it as a one-parameter family of metrics $g(s)$ satisfying

$$\partial_s g = -2Ric, \quad g(0) = g_0,$$

where Ric is the Ricci tensor of the evolving metric $g(s)$. The generalized fixed points of the Ricci flow are those manifolds that change by a diffeomorphism and a rescaling under the Ricci flow. More precisely, let M be an n-dimensional smooth manifold and $(M, g(s))$ be a solution of the Ricci flow such that $g(0) = g_0$. Let $f_s : M \to M$ be a one-parameter family of diffeomorphisms generated by the family of vector fields $X(s)$ and let $\rho(s)$ be a time-dependent scale factor, then a solution of the Ricci flow of the form $g(s) = \rho(s) f_s^*(g)$ is called Ricci soliton. Thus, a Ricci soliton is a generalized fixed point of the Ricci flow, viewed as a dynamical system on the space of Riemannian metrics modulo diffeomorphisms and scalings. Taking the derivative of the above equation with respect to s, substituting $s = 0$and assuming $\dot{\rho}(0) = -2\lambda$, $\rho(0) = 1$, $f_0 = id$, $X(0) = \mathbf{u}$, we get

$$\frac{1}{2}\pounds_{\mathbf{u}} g + \mathrm{Ric} = \lambda g, \tag{1.1}$$

where λ is a constant, $\pounds_{\mathbf{u}}$ is the Lie-derivative of g with respect to $\mathbf{u}$ and Ric is the Ricci tensor of (M, g). We shall denote a Ricci soliton by $(M, g, \mathbf{u}, \lambda)$. The topic Ricci soliton is important in geometry as well as global analysis, especially since it was deployed in settling the famous Poincarè conjecture. The vector field $\mathbf{u}$ appearing in Ricci soliton $(M, g, \mathbf{u}, \lambda)$ is called the potential field of the Ricci soliton. If the potential field $\mathbf{u}$ is Killing; that is, $\pounds_{\mathbf{u}} g = 0$, then the definition of Ricci soliton implies

$$\mathrm{Ric} = \lambda g;$$

that is, the Ricci soliton is an Einstein manifold. In this case the Ricci soliton $(M, g, \mathbf{u}, \lambda)$ is called a trivial Ricci soliton.

In [1–7], authors found different conditions under which a Ricci soliton $(M, g, \mathbf{u}, \lambda)$ is a trivial Ricci soliton. For compact gradient Ricci solitons in [5], the author derived several identities, and later in [6] these identities were used to prove that a compact gradient shrinking Ricci soliton, which is locally conformally flat, must be trivial. In [8], authors used the Ricci mean value δ of an n-dimensional compact gradient Ricci soliton $(M, g, \nabla f, \lambda)$ defined by

$$\delta = \frac{1}{nV} \int_M Ric(\nabla f, \nabla f),$$

where V is the volume of M and $Ric(\nabla f, \nabla f)$ is the Ricci curvature in the direction of ∇f, to prove for $n \geq 2$ that $\delta \geq 0$; the equality holds if, and only if, the Ricci soliton is trivial. Similarly, in [9] the author has considered Ricci soliton $(M, g, \mathbf{u}, \lambda)$ of positive Ricci curvature and has shown that if the potential field $\mathbf{u}$ is a Jacobi-type vector field, then the Ricci soliton is trivial.

Apart from finding conditions under which a Ricci soliton is trivial, there are several important aspects of the geometry of Ricci solitons. For instance, on space times such as spherically symmetric static space times Lorentzian plane-symmetric static space times and Kantowski Sachs space times, treated as Ricci solitons, the role of potential field on these respective space times is studied in [3, 10–13], respectively.

We denote by η the smooth one-form dual to $\mathbf{u}$ that is

$$\eta(X) = g(X, \mathbf{u}),$$

for smooth vector field X on M, then we obtain a skew-symmetric tensor field F defined on M by

$$\frac{1}{2} d\eta(X, Y) = g(F(X), Y)$$

for smooth vector fields X and Y on M.

One of the important questions on Ricci solitons is to find conditions under which a Ricci soliton $(M, g, \mathbf{u}, \lambda)$ is trivial. Note that the squared length of the covariant derivative of potential field $\mathbf{u}$ is given by

$$\|\nabla \mathbf{u}\|^2 = \sum_{i=1}^{n} g\left(\nabla_{u_i}\mathbf{u}, \nabla_{u_i}\mathbf{u}\right),$$

where $\{u_1, u_2, \ldots, u_n\}$ is a local orthonormal frame on M, $n = \dim M$. Also, we define

$$\|F\|^2 = \sum_{i=1}^{n} g\left(F\left(u_i\right), F\left(u_i\right)\right).$$

Our first result is the following:

Theorem 1. *If the covariant derivative of the potential field* $\mathbf{u}$ *of a connected Ricci soliton* $(M, g, \mathbf{u}, \lambda)$ *satisfies*

$$\|\nabla \mathbf{u}\|^2 \leq \|F\|^2,$$

then the Ricci soliton $(M, g, \mathbf{u}, \lambda)$ *is trivial.*

Recall that if potential field $\mathbf{u}$ is Killing makes $(M, g, \mathbf{u}, \lambda)$ a trivial Ricci soliton. In [14], authors introduced the notion of the geodesic vector field. Note that a Killing vector field of constant length is a geodesic vector field, and there are many examples of geodesic vector fields that are not Killing. Recall that a vector field ξ on a Riemannian (M, g) is said to be a geodesic vector field if

$$\nabla_{\xi}\xi = 0;$$

that is, the integral curves of ξ are geodesics. An interesting example is provided by the vector field ξ of a proper tran-Sasakian manifold $(M, g, \phi, \xi, \eta, \alpha, \beta)$, which is a geodesic vector field that is not killing [15, 16]. A similar example is provided by the vector field ξ of a Kenmotsu manifold (M, g, ϕ, ξ, η) [17].

In our next result on an n-dimensonal Ricci soliton $(M, g, \mathbf{u}, \lambda)$, we use the condition that the potential field $\mathbf{u}$ is a geodesic vector field to prove the following:

Theorem 2. *If the Ricci curvature* $Ric(\mathbf{u}, \mathbf{u})$ *of an n-dimensional compact Ricci soliton* $(M, g, \mathbf{u}, \lambda)$ *with scalar curvature* τ *satisfies*

$$\|F\|^2 \leq Ric(\mathbf{u}, \mathbf{u}) \leq \frac{\tau}{n}\|\mathbf{u}\|^2$$

and the potential field $\mathbf{u}$ *is a geodesic vector field, then the Ricci soliton* $(M, g, \mathbf{u}, \lambda)$ *is trivial.*

For an n-dimensional Ricci soliton $(M, g, \mathbf{u}, \lambda)$, we let $\{\varphi_t\}$ be the local flow of the potential field $\mathbf{u}$, then the Ricci operator S of $(M, g, \mathbf{u}, \lambda)$ is said to be invariant under $\mathbf{u}$ if

$$S \circ d\varphi_t = d\varphi_t \circ S$$

or, equivalently,

$$\pounds_{\mathbf{u}} S = 0.$$

If the Ricci operator S of the Ricci soliton $(M, g, \mathbf{u}, \lambda)$ is invariant under the potential field $\mathbf{u}$, then we have the following characterization of a trivial Ricci soliton.

Theorem 3. *If the Ricci operator S an n-dimensional compact Ricci soliton $(M, g, \mathbf{u}, \lambda)$ is invariant under the potential field* $\mathbf{u}$ *and satisfies*

$$(\nabla S)(U, \mathbf{u}) = (\nabla S)(\mathbf{u}, U)$$

for each vector field U on M, then $(M, g, \mathbf{u}, \lambda)$ is a trivial Ricci soliton and the converse also holds.

A Ricci soliton $(M, g, \mathbf{u}, \lambda)$ is said to be shrinking if the constant is $\lambda > 0$ [1]; it is an important question to find geometric conditions under which a Ricci soliton is shrinking. One of important classical vector fields is concurrent vector field ξ on a Riemannian manifold (M, g), which obeys

$$\nabla_U \xi = U$$

for any smooth vector field U on M. This means that the holonomy group of M leaves a point of M invariant [18, 19]. In our final result, we use the condition that the potential field $\mathbf{u}$ of the Ricci soliton $(M, g, \mathbf{u}, \lambda)$ is a concurrent vector field to prove the following:

Theorem 4. *Let $(M, g, \mathbf{u}, \lambda)$ be an n-dimensional connected Ricci soliton with* $\mathbf{u}$ *as a concurrent vector field, then $(M, g, \mathbf{u}, \lambda)$ is a shrinking Ricci soliton.*

2. Preliminaries

Suppose $(M, g, \mathbf{u}, \lambda)$ is an n-dimensional Ricci soliton, then by Eq 1.1 we have

$$\frac{1}{2}\pounds_{\mathbf{u}} g + \mathrm{Ric} = \lambda g.$$

We denote by S the Ricci operator of $(M, g, \mathbf{u}, \lambda)$ satisfying

$$\mathrm{Ric}(U, V) = g(S(U), V), \quad U, V \in \mathfrak{X}(M),$$

where $\mathfrak{X}(M)$ is the Lie-algebra of vector field on M. Using the following expressions

$$(\pounds_{\mathbf{u}} g)(U, V) = g(\nabla_U \mathbf{u}, V) + g(\nabla_V \mathbf{u}, U)$$

and

$$(d\eta)(U, V) = g(\nabla_U \mathbf{u}, V) - g(\nabla_V \mathbf{u}, U),$$

we derive

$$g(\nabla_U \mathbf{u}, V) = \frac{1}{2}(\pounds_{\mathbf{u}} g)(U, V) + \frac{1}{2} d\eta(U, V);$$

that is,

$$g(\nabla_U \mathbf{u}, V) = \lambda g(U, V) - \mathrm{Ric}(U, V) + g(F(U), V). \tag{2.1}$$

Equation 2.1 implies

$$\nabla_U \mathbf{u} = \lambda U - S(U) + F(U). \tag{2.2}$$

Recall that the scalar curvature τ of the Ricci soliton $(M, g, \mathbf{u}, \lambda)$ is given by

$$\tau = \mathrm{tr}.S = \sum_{i=1}^{n} \mathrm{Ric}(u_i, u_i),$$

where $\{u_1, ..., u_n\}$ is a local orthonormal frame on M, $n = \dim M$.

Lemma 2.1. *Let* $(M, g, \mathbf{u}, \lambda)$ *be an n-dimnesional Ricci soliton, then*

(i) $\mathrm{div}\mathbf{u} = n\lambda - \tau$*;*

(ii) $\|S - \frac{\tau}{n}I\|^2 = -n\left(\lambda - \frac{\tau}{n}\right)^2 + \|\nabla\mathbf{u}\|^2 - \|F\|^2$.

Proof. Using Eq 2.2, we have

$$\mathrm{div}\mathbf{u} = n\lambda - \mathrm{tr}S + 0 = n\lambda - \tau,$$

where we used $\mathrm{tr}.F = 0$. This proves (i). Also, Eq 2.2 implies

$$S(U) = \lambda U + F(U) - \nabla_U \mathbf{u};$$

that is,

$$S(U) - \frac{\tau}{n}U = \left(\lambda - \frac{\tau}{n}\right)U + F(U) - \nabla_U \mathbf{u}.$$

Thus, we have

$$\|S - \frac{\tau I}{n}\|^2 = n\left(\lambda - \frac{\tau}{n}\right)^2 + \|F\|^2 + \|\nabla\mathbf{u}\|^2 - 2\left(\lambda - \frac{\tau}{n}\right)\mathrm{div}\mathbf{u} - 2\sum_{i=1}^{n} g\left(\nabla_{u_i}\mathbf{u}, F(u_i)\right),$$

where $\{u_1, ..., u_n\}$ is a local orthonormal frame on M.

Now, using Eq 2.2 and (i) in Lemma 2.1, we obtain the result in (ii).

3. Proof of Theorem 1

Using the definition of Ricci soliton 1.1, we have

$$\begin{aligned}
\frac{1}{4}\left|\pounds_{\mathbf{u}}g\right|^2 &= \frac{1}{4}\sum_{i=1}^{n}\left(\left(\pounds_{\mathbf{u}}g\right)\left(u_i, u_j\right)\right)^2 \\
&= \sum_{i=1}^{n}\left(\lambda g\left(u_i, u_j\right) - \mathrm{Ric}\left(u_i, u_j\right)\right)^2 \\
&= \sum_{i=1}^{n}\left(\lambda g\left(u_i, u_j\right) - g\left(S(u_i), u_j\right)\right)^2 \\
&= n\lambda^2 + \|S\|^2 - 2\lambda\sum_{i=1}^{n} g\left(u_i, u_j\right) g\left(S(u_i), u_j\right) \\
&= n\lambda^2 + \|S\|^2 - 2\lambda\sum_{i=1}^{n} g\left(S(u_i), u_i\right) \\
&= n\lambda^2 + \|S\|^2 - 2\lambda\tau \\
&= \left(\|S\|^2 - \frac{\tau^2}{n}\right) + n\lambda^2 - 2\lambda\tau + \frac{\tau^2}{n}.
\end{aligned}$$

Thus,

$$\frac{1}{4}\left|\pounds_{\mathbf{u}}g\right|^2 = \left(\|S\|^2 - \frac{\tau^2}{n}\right) + n\left(\lambda - \frac{\tau}{n}\right)^2. \tag{3.1}$$

Also, note that

$$\begin{aligned} \|S-\frac{\tau}{n}I\|^2 &= \|S\|^2+\frac{\tau^2}{n}-2\frac{\tau}{n}\sum_{i=1}^{n}g\left(S\left(u_i\right),u_i\right) \\ &= \|S\|^2-\frac{\tau^2}{n}. \end{aligned} \tag{3.2}$$

Using 3.2 in (ii) of Lemma 2.1, we obtain

$$\|S\|^2-\frac{\tau^2}{n}=-n\left(\lambda-\frac{\tau}{n}\right)^2+\|\nabla\mathbf{u}\|^2-\|F\|^2.$$

Combining it with Eq 3.1, we conclude

$$\frac{1}{4}\left|\pounds_{\mathbf{u}}g\right|^2=\|\nabla\mathbf{u}\|^2-\|F\|^2.$$

Hence, using the condition in the statement, we conclude $\pounds_{\mathbf{u}}g=0$; that is, $(M,g,\mathbf{u},\lambda)$ is a trivial soliton.

Remark 3.1. *As Theorem 1 suggests, the Ricci soliton $(M,g,\mathbf{u},\lambda)$ with potential field $\mathbf{u}$ satisfying*

$$\|\nabla\mathbf{u}\|^2\leq\|F\|^2$$

is trivial. It is natural to expect to see through an example of a nontrivial Ricci soliton $(M,g,\mathbf{u},\lambda)$ that the potential field $\mathbf{u}$ does not satisfy the above condition. We consider the n-dimensional Euclidean space $(\mathbf{E}^n,g)$ with Euclidean metric g and the vector field $\mathbf{u}$ defined by

$$\mathbf{u}=\sum_{i=1}^{n}u^i\frac{\partial}{\partial u^i},$$

where $u^1,...,u^n$ are the Euclidean coordinates, then we have

$$\pounds_{\mathbf{u}}g=2g$$

and

$$\frac{1}{2}\pounds_{\mathbf{u}}g+Ric=g.$$

This shows that $(\mathbf{E}^n,g,\mathbf{u},1)$ is an n-dimensional nontrivial Ricci soliton. It follows that $\mathbf{u}$ is a closed field and, therefore, $F=0$. Moreover, we have

$$\|\nabla\mathbf{u}\|^2=n;$$

that is, we have

$$\|\nabla\mathbf{u}\|^2>\|F\|^2.$$

4. Proof of Theorem 2

Assume that the potential field $\mathbf{u}$ of the Ricci soliton $(M, g, \mathbf{u}, \lambda)$ is a geodesic vector field; that is,

$$\nabla_{\mathbf{u}}\mathbf{u} = 0. \tag{4.1}$$

By virtue of Eqs 2.2 and 4.4, we have

$$S(\mathbf{u}) = \lambda\mathbf{u} + F(\mathbf{u}); \tag{4.2}$$

that is,

$$\mathrm{Ric}\,(\mathbf{u}, \mathbf{u}) = \lambda\|\mathbf{u}\|^2, \tag{4.3}$$

where we used $g\,(F(\mathbf{u}), \mathbf{u}) = 0$, owing to skew-symmetry of F. Also, on using (i) in Lemma 2.1, we have that

$$\begin{aligned} \mathrm{div}\left(\frac{1}{2}\|\mathbf{u}\|^2\mathbf{u}\right) &= g\,(\nabla_{\mathbf{u}}\mathbf{u}, \mathbf{u}) + \frac{1}{2}\|\mathbf{u}\|^2\mathrm{div}(\mathbf{u}) \\ &= \frac{1}{2}\|\mathbf{u}\|^2\,(n\lambda - \tau)\,. \end{aligned}$$

Integrating the above equation and using Eq 4.3, we conclude

$$\int_M \left(\mathrm{Ric}(\mathbf{u}, \mathbf{u}) - \frac{\tau}{n}\|\mathbf{u}\|^2\right) = 0.$$

Using the condition in the statement, we conclude

$$\mathrm{Ric}(\mathbf{u}, \mathbf{u}) = \frac{\tau}{n}\|\mathbf{u}\|^2. \tag{4.4}$$

Comparing Eqs 4.3 and 4.4, we have

$$\tau = n\lambda; \tag{4.5}$$

using it in (ii) of Lemma 2.1, we have

$$\|S - \frac{\tau}{n}I\|^2 = \|\nabla\mathbf{u}\|^2 - \|F\|^2; \tag{4.6}$$

in (i) of Lemma 2.1, we have

$$\mathrm{div}\mathbf{u} = 0. \tag{4.7}$$

Finally, the integral formula of Yano [20] and Eq 4.7 implies

$$\int_M \left(\mathrm{Ric}(\mathbf{u}, \mathbf{u}) + \frac{1}{2}\,|\pounds_{\mathbf{u}}g|^2 - \|\nabla\mathbf{u}\|^2\right) = 0. \tag{4.8}$$

Using Eqs 4.6 and 4.8, we conclude

$$\int_M \left(\mathrm{Ric}(\mathbf{u}, \mathbf{u}) + \frac{1}{2}\,|\pounds_{\mathbf{u}}g|^2 - \|S - \frac{\tau}{n}I\|^2 - \|F\|^2\right) = 0. \tag{4.9}$$

Note that in view of Eq 4.9, Eq 3.1 takes the form

$$\frac{1}{4}\left|\pounds_{\mathbf{u}}g\right|^2=\left(S^2-\frac{\tau^2}{n}\right),$$

and using Eq 3.2, we conclude

$$\frac{1}{4}\left|\pounds_{\mathbf{u}}g\right|^2=\|S-\frac{\tau}{n}I\|^2.$$

Using the above equation in integral 4.9, we have

$$\int_M \|S-\frac{\tau}{n}I\|^2=\int_M\left(\|F\|^2-\operatorname{Ric}(\mathbf{u},\mathbf{u})\right).$$

Now, using the lower bound on $\operatorname{Ric}(\mathbf{u},\mathbf{u})$ in the statement, we conclude

$$\|S-\frac{\tau}{n}I\|=0;$$

that is, in view of Eq 4.5,

$$S=\lambda I.$$

Hence, $(M,g,\mathbf{u},\lambda)$ is trivial.

5. Proof of Theorem 3

Suppose $(M,g,\mathbf{u},\lambda)$ is an n-dimensional compact Ricci soliton such that the Ricci operator S satisfies

$$(\nabla S)(U,\mathbf{u})=(\nabla S)(\mathbf{u},U),\quad U\in\mathfrak{X}(M) \tag{5.1}$$

and is invariant under the potential field $\mathbf{u}$; that is,

$$\pounds_{\mathbf{u}}S=0. \tag{5.2}$$

Using Eq 5.2, we have

$$[\mathbf{u},SU]=S\,[\mathbf{u},U],\quad U\in\mathfrak{X}(M),$$

which in view of Eq 2.2 gives

$$(\nabla S)(\mathbf{u},U)=F(SU)-S(FU),\quad U\in\mathfrak{X}(M). \tag{5.3}$$

Now, define a function ψ on M by

$$\psi=\frac{1}{2}\|\mathbf{u}\|^2,$$

then using Eq 2.2 and symmetry of the Ricci operator and skew symmetry of the operator F, we find the gradient $\nabla\psi$ of ψ as

$$\nabla\psi=\lambda\mathbf{u}-S(\mathbf{u})-F(\mathbf{u}). \tag{5.4}$$

Note that by using Lemma 2.1, we have

$$\operatorname{div}(\lambda\mathbf{u})=\left(n\lambda^2-\lambda\tau\right). \tag{5.5}$$

Next, we compute the divergence of $S\,(\mathbf{u})$ while using Eqs 2.2, 5.1 and 5.3 through a local orthonormal frame $\{u_1, .., u_n\}$, and we have

$$\begin{aligned}
\operatorname{div}(S\mathbf{u}) &= \sum_{j=1}^{n} g\left(\nabla_{u_j} S\mathbf{u}, u_j\right) = \sum_{j=1}^{n} g\left((\nabla S)(u_j, \mathbf{u}) + S\left(\nabla_{u_j}\mathbf{u}\right), u_j\right) \\
&= \sum_{j=1}^{n} g\left((\nabla S)(\mathbf{u}, u_j) + S\left(\lambda u_j - S\left(u_j\right) + F(u_j)\right), u_j\right) \\
&= \sum_{j=1}^{n} g\left(F\left(S u_j\right) - S\left(F u_j\right), u_j\right) + \lambda\tau - \|S\|^2 + \sum_{j=1}^{n} g\left(F(u_j), S\left(u_j\right)\right).
\end{aligned}$$

Since S is symmetric and F is skew symmetric, it follows that

$$\sum_{j=1}^{n} g\left(F(u_j), S\left(u_j\right)\right) = 0. \tag{5.6}$$

Thus, we confirm

$$\operatorname{div}(S\mathbf{u}) = \lambda\tau - \|S\|^2. \tag{5.7}$$

Thus, using Eq 5.4, we have

$$\Delta\psi = \operatorname{div}(\lambda\mathbf{u} - S\,(\mathbf{u}) - F\,(\mathbf{u})),$$

which in view of Eqs 5.4 and 5.7, we have

$$\Delta\psi = n\lambda^2 - 2\lambda\tau + \|S\|^2 - \operatorname{div}(F\,(\mathbf{u}));$$

integrating the above equation, we reach

$$\int_M \left(n\lambda^2 - 2\lambda\tau + \|S\|^2\right) = 0. \tag{5.8}$$

We rearrange the above integral as

$$\int_M \frac{1}{n}(n\lambda - \tau)^2 + \int_M \left(\|S\|^2 - \frac{1}{n}\tau^2\right) = 0, \tag{5.9}$$

and as by Schwartz's inequality $\|S\|^2 \geq \frac{1}{n}\tau^2$, both integrands in the above equation are nonnegative and we can confirm

$$\int_M \frac{1}{n}(n\lambda - \tau)^2 = 0 \text{ and } \int_M \left(\|S\|^2 - \frac{1}{n}\tau^2\right) = 0. \tag{5.10}$$

Thus, we conclude that

$$\tau = n\lambda \tag{5.11}$$

and

$$\|S\|^2 = \frac{1}{n}\tau^2. \tag{5.12}$$

Now, the Eq 5.12 is equal in the Schwartz's inequality and it holds if, and only if,

$$S = \frac{\tau}{n} I, \tag{5.13}$$

and combining it with Eq 5.11, we confirm

$$Ric = \lambda g. \tag{5.14}$$

Hence, $(M, g, \mathbf{u}, \lambda)$ is trivial Ricci soliton.

Conversely, if $(M, g, \mathbf{u}, \lambda)$ is trivial Ricci soliton, then the potential field $\mathbf{u}$ is Killing and, therefore, the flow $\{\varphi_t\}$ consists of isometries. Hence, the Ricci operator $S = \lambda I$ is invariant under $\mathbf{u}$. Moreover, the condition

$$(\nabla S)(U, \mathbf{u}) = (\nabla S)(\mathbf{u}, U), \quad U \in \mathfrak{X}(M) \tag{5.15}$$

holds. This completes the proof.

6. Proof of Theorem 4

Suppose the potential field $\mathbf{u}$ is a concurrent vector field, then we have [18, 19]

$$\nabla_U \mathbf{u} = U \tag{6.1}$$

and, consequently, we have

$$R(U, V)\mathbf{u} = \nabla_U V - \nabla_V U - [U, V] = 0.$$

Using a local frame $\{u_1, ...u_n\}$, we have

$$Ric(U, \mathbf{u}) = \sum_{i=1}^{n} g(R(u_i, U)\mathbf{u}, u_i) = 0.$$

On the basis of it, we conclude

$$S(\mathbf{u}) = 0. \tag{6.2}$$

Now, using Eq 2.2, we have

$$\nabla_{\mathbf{u}} \mathbf{u} = \lambda \mathbf{u} - S(\mathbf{u}) + F(\mathbf{u}),$$

which in view of Eqs 6.1 and 6.2 yields

$$(1 - \lambda)\mathbf{u} = F(\mathbf{u}).$$

Taking the inner product with $\mathbf{u}$ in the above equation and paying attention to skew-symmetry of F, we reach

$$(1 - \lambda)\|\mathbf{u}\|^2 = 0.$$

Since $\mathbf{u} = 0$ together with Eq 6.1 gives a contradiction, the above equation on connected M implies $\lambda = 1$; that is, the Ricci soliton is shrinking.

7. Conclusions

In Theorems 1 and 2, we discussed conditions under which a Ricci soliton $(M, g, \mathbf{u}, \lambda)$ is trivial. In Theorem 2, we used that the potential field $\mathbf{u}$ is a geodesic vector field, and in Theorem 4, it was shown that if the potential field $\mathbf{u}$ is a concurrent vector field, then the Ricci soliton $(M, g, \mathbf{u}, \lambda)$ is shrinking. There is yet another important vector field defined on a Riemannian manifold called the incompressible vector field [21]; this notion was taken from fluid dynamics, where the velocity field of an incompressible fluid satisfies the equation of continuity. A smooth vector field ξ on a Riemannian manifold (M, g) is said to be incompressible if $\mathrm{div}\xi = 0$. In view of Theorems 2 and 4, it will be interesting to study the behavior of the Ricci soliton $(M, g, \mathbf{u}, \lambda)$ under the condition that the potential field $\mathbf{u}$ is incompressible. It is worth finding a condition under which a Ricci soliton $(M, g, \mathbf{u}, \lambda)$ with the potential field $\mathbf{u}$ is incompressible and trivial.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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