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Research article

The Burgers-KdV limit in one-dimensional plasma with viscous dissipation: A study of dispersion and dissipation effects

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Abstract: The Burgers-KdV equation as a highly nonlinear model, is commonly used in weakly nonlinear analysis to describe small but finite amplitude ion-acoustic waves. In this study, we demonstrate that by considering viscous dissipation, we can derive the Burgers-KdV limit from a one-dimensional plasma system by using the Gardner-Morikawa transformation. This transformation allows us to obtain both homogeneous and inhomogeneous Burgers-KdV equations, which incorporate dissipative and dispersive terms, for the ionic acoustic system. To analyze the remaining system, we employ the energy method in Sobolev spaces to estimate its behavior. As a result, we are able to capture the Burgers-KdV dynamics over a time interval of order $O(\varepsilon^{-1})$, where ε represents a small parameter.

Keywords: Burgers-KdV; plasma; viscous dissipation; Gardner-Morikawa transformation
Mathematics Subject Classification: 35Q53, 35Q35

1. Introduction

Recently, highly nonlinear models have attracted the focus of many scientists due to their ability to provide more meaningful insights into physical phenomena with memory effects [1]. The Burgers-Korteweg-de Vries (Burgers-KdV) equation has gained attention for the purpose of modeling various natural phenomena, such as the propagation of undular bores in shallow water, the flow of liquids containing gas bubbles and the propagation of waves in elastic tubes filled with viscous fluids [2–5]. This equation has attracted physicists, engineers and applied mathematicians from different disciplines who are interested in studying these phenomena. In the field of weak nonlinearity, the Burgers-KdV equation (or KdV equation) is commonly used to describe dispersion waves of finite but small magnitude, and it is not limited to waves in bubble streams. On the other hand, the nonlinear Schrödinger equation (or Ginzburg-Landau equation) is a widely used nonlinear wave equation (or nonlinear evolution equation) for cases involving strong dispersion.

Yatabe et al. [6] employed a multi-scale method to derive two KdV-Burgers equations that incorporate a drag force correction term. They verified that the time evolution of wave dissipation, caused by the drag force, differs from that caused by acoustic radiation. From a mathematical perspective, there exists a close relationship between the Burgers equation ($s = 0$) [7] and the KdV equation ($\beta = 0$) [8]. The standard Burgers-KdV equation is given by

$$u_t + \alpha u u_x + \beta u_{xx} + s u_{xxx} = 0,$$

where the real constants α, β and s satisfy that $\alpha\beta s \neq 0$. The nonexistence of a spectral solution for this equation presents a challenge in studying its integrability. Currently, there is no effective analytical method available to solve this type of equation.

Indeed, there have been numerous studies on the Burgers-KdV equation, addressing various aspects such as the existence, uniqueness, well-posedness, stability and solution properties [9–15]. Many physical processes can be perturbed by external factors, and the nature of these perturbations can vary across different problems. In recent years, investigating the limit problem with viscous dissipation has received significant attention.

Significant progress has been made in understanding the Burgers-KdV equation. Luc and Francis [16] showed that the Burgers-KdV equations are globally well-posed. They established the low regularity of solutions through the use of an algebraic inequality and an a priori estimate. Dlotko [17] proved the local and global solvability in $H^2(\mathbb{R})$ of the Cauchy problem for the generalized KdV-Burgers equation by using the parabolic regularization technique. Wang et al. [18] obtained an approximate solution to the KdV-Burgers equation with boundary conditions by employing the Adomian decomposition method. Feng and Knobel [19] obtained traveling wave solutions from a KdV-Burgers-type equation with higher-order nonlinearities. Zhao and colleagues [20–22] introduced some localized wave solutions of the high-dimensional integrable systems for the nonlinear mathematical physics. These are just a few examples of the extensive research findings in this area.

Recently, there has been significant interest in the asymptotic connection between ionic dynamical systems and hydrodynamic models. The Euler-Poisson system has been used to derive various nonlinear dispersive equations through the use of reduced perturbation methods, including the KdV equation [23], Kadomtsev-Petviashvili equation [24], Zakharov-Kuznetsov equation [25, 26], Burgers equation [27] and Schrödinger equation [28]. However, directly applying this method to nonlinear systems involving both dissipative and dispersive effects can be challenging. Based on the literature mentioned, this paper focuses on the question of whether the Burgers-KdV system can converge to a similar solution. The reduced perturbation method is not directly applicable when both dispersion and dissipation are present.

When regardless of the magnetic field, we consider the one-dimensional plasma with viscous dissipation [29, 30], the two-fluid system describing ionic sound waves is reduced to

$$
\begin{cases}
\partial_{t'} n_i + \partial_{x'}(n_i v_i) = 0, & (1.1a)\\
m_i n_i(\partial_{t'} v_i + v_i \partial_{x'} v_i) = -\partial_{x'} p_i - n_i e \partial_{x'}\phi + \mu \partial^2_{x'} v_i, & (1.1b)\\
0 = e\partial_{x'}\phi - \dfrac{KT_e}{n_e}\partial_{x'} n_e, & (1.1c)\\
\partial^2_{x'}\phi = 4\pi e(n_e - n_i), & (1.1d)
\end{cases}
$$

where ϕ is the perturbed potential, μ is interpreted as the equivalent viscosity coefficient and n_i and v_i represent the density and velocity, respectively.

Standardize the physical quantities in (1.1) as follows:

$$
x = x'/L,\ \ t = t' \cdot c_s/L,\ \ v = v_i/c_s,\ \ \Phi = e\phi/KT_e,\ \ n = n_i/n_0,\ \ n_e = n_e/n_0,
$$

where L is the characteristic scale of fluctuation, $c_s = \sqrt{\frac{KT_e}{m_i}}$ denotes the ion-acoustic velocity, K is the Boltzmann constant and n_0 is the undisturbed density.

When ignoring the influence of the ion pressure term, the dimensionless equations can be simplified to

$$
\begin{cases}
\partial_t n + \partial_x(nv) = 0, & (1.2a)\\
\partial_t v + v\partial_x v + \partial_x \Phi - \dfrac{\nu}{Lc_s}\dfrac{1}{n}\partial^2_x v = 0, & (1.2b)\\
\partial_x \Phi = \dfrac{1}{n_e}\partial_x n_e, & (1.2c)\\
\dfrac{\lambda_D^2}{L^2}\partial^2_x \Phi = n_e - n, & (1.2d)
\end{cases}
$$

where $\nu = \frac{\mu}{m_i n_0}$ represents the equivalent kinematic viscosity coefficient and $\lambda_D = \sqrt{KT_e/4\pi n_0 e^2}$ is the Debye length.

Assuming a finite small quantity of density perturbation, let $\Delta n = n_i - n_0$; we have that $n - 1 = \frac{\Delta n}{n_0} \ll 1$; taking $\frac{\Delta n}{n_0} = \varepsilon \ll 1$, let $\lambda_D/L^2 = \alpha\varepsilon$ and $\nu/Lc_s = \beta\varepsilon$; this is what the weak dispersion and viscosity require. The quantity ε is a quantity that describes the strength of the nonlinearity. When both the weak dispersion and the weak dissipative effect are equivalent to ε, we can obtain the Burgers-KdV equation via the perturbation method. The system (1.1a) can be rewritten as follows:

$$
\begin{cases}
\partial_t n + \partial_x(nv) = 0, & (1.3a)\\
\partial_t v + v\partial_x v + \partial_x \Phi - \dfrac{\beta\varepsilon}{n}\partial^2_x v = 0, & (1.3b)\\
n_e - n = \alpha\varepsilon\partial^2_x \Phi, & (1.3c)\\
\partial_x \Phi = \dfrac{1}{n_e}\partial_x n_e. & (1.3d)
\end{cases}
$$

1.1. Derivation of Burgers-KdV equation

By applying the following Gardner-Morikawa transformation [29] to (1.3),

$$
x \to x - \lambda_0 t,\ \ t \to \varepsilon t, \tag{1.4}
$$

we obtain the following parameterized system:

$$\begin{cases} \varepsilon\partial_t n - \lambda_0\partial_x n + \partial_x(nv) = 0, & (1.5a)\\ \varepsilon\partial_t v - \lambda_0\partial_x v + v\partial_x v + \partial_x\Phi - \dfrac{\beta\varepsilon}{n}\partial_x^2 v = 0, & (1.5b)\\ n_e - n = \alpha\varepsilon\partial_x^2\Phi, & (1.5c)\\ \partial_x\Phi = \dfrac{1}{n_e}\partial_x n_e, & (1.5d)\end{cases}$$

where the small quantity ε is also the amplitude of initial disturbance and λ_0 is the velocity parameter.

Assume that the variables have the following expansions:

$$\begin{cases} n = 1 + \varepsilon n^{(1)} + \varepsilon^2 n^{(2)} + \varepsilon^3 n^{(3)} + \varepsilon^4 n^{(4)} + \cdots, & (1.6a)\\ n_e = 1 + \varepsilon n_e^{(1)} + \varepsilon^2 n_e^{(2)} + \varepsilon^3 n_e^{(3)} + \varepsilon^4 n_e^{(4)} + \cdots, & (1.6b)\\ v = \varepsilon v^{(1)} + \varepsilon^2 v^{(2)} + \varepsilon^3 v^{(3)} + \varepsilon^4 v^{(4)} + \cdots; & (1.6c)\end{cases}$$

we then incorporate (1.6) into system (1.5), terminating each expansion at different orders of magnitude ε.

At the order of $O(\varepsilon)$, we have

$$\begin{cases} -\lambda_0\partial_x n^{(1)} + \partial_x v^{(1)} = 0, & (1.7a)\\ -\lambda_0\partial_x v^{(1)} + \dfrac{1}{n_e}\partial_x n_e^{(1)} = 0, & (1.7b)\\ n_e^{(1)} - n^{(1)} = 0; & (1.7c)\end{cases}$$

a simpler calculation gives

$$\begin{cases} v^{(1)} = \lambda_0 n^{(1)} = \lambda_0 n_e^{(1)}, & (1.8a)\\ \lambda_0^2 n_e = 1. & (1.8b)\end{cases}$$

At the order of $O(\varepsilon^2)$, we have

$$\begin{cases} \partial_t n^{(1)} - \lambda_0\partial_x n^{(2)} + \partial_x(v^{(2)} + n^{(1)}v^{(1)}) = 0, & (1.9a)\\ \partial_t v^{(1)} - \lambda_0\partial_x v^{(2)} + v^{(1)}\partial_x v^{(1)} + \dfrac{1}{n_e}\partial_x n_e^{(2)} - \dfrac{\beta}{n}\partial_x^2 v^{(1)} = 0, & (1.9b)\\ n_e^{(2)} - n^{(1)} = \dfrac{\alpha}{n_e}\partial_x^2 n_e^{(1)}. & (1.9c)\end{cases}$$

Taking ∂_x of (1.9c), multiplying (1.9a) by λ_0 and then putting them into (1.9b), we obtain the homogeneous Burgers-KdV equation for $n_e^{(1)}$:

$$\partial_t n_e^{(1)} + \frac{3\lambda_0}{2} n_e^{(1)}\partial_x n_e^{(1)} + \frac{\alpha}{2}\lambda_0^3\partial_x^3 n_e^{(1)} - \frac{\beta}{2n}\partial_x^2 n_e^{(1)} = 0, \tag{1.10}$$

where we applied (1.8). Note that system (1.10) and (1.9) are self contained for $(n^{(1)}, v^{(1)}, \phi^{(1)})$.

From (1.9), we can express $(n^{(2)}, v^{(2)})$ in the following form:

$$\begin{cases} v^{(2)} = \lambda_0 n_e^{(2)} + f(1), & (1.11a)\\ n^{(2)} = n_e^{(2)} - \alpha\lambda_0^2\partial_x^2 n_e^{(1)}, & (1.11b)\end{cases}$$

where we have used (1.8b) and the function $f(1)$ only depends on $n_e^{(1)}$ and its derivatives.

At the order of $O(\varepsilon^3)$, we have

$$\begin{cases} \partial_t n^{(2)} - \lambda_0 \partial_x n^{(3)} + \partial_x (v^{(3)} + n^{(1)} v^{(2)} + n^{(2)} v^{(1)}) = 0, & (1.12a) \\ \partial_t v^{(2)} - \lambda_0 \partial_x v^{(3)} + v^{(1)} \partial_x v^{(2)} + v^{(2)} \partial_x v^{(1)} + \dfrac{1}{n_e} \partial_x n_e^{(3)} - \dfrac{\beta}{n} \partial_x^2 v^{(2)} = 0, & (1.12b) \\ n_e^{(3)} - n^{(3)} = -\dfrac{\alpha}{n_e^2} (\partial_x n_e^{(1)})^2 + \dfrac{\alpha}{n_e} \partial_x^2 n_e^{(2)}. & (1.12c) \end{cases}$$

Taking ∂_x of (1.12c), multiplying (1.12a) by λ_0 and then putting them into (1.12b), we obtain the inhomogeneous Burgers-KdV equation for $n_e^{(2)}$:

$$\partial_t n_e^{(2)} + \frac{3\lambda_0}{2} \partial_x (n_e^{(1)} n_e^{(2)}) + \frac{\alpha}{2} \lambda_0^3 \partial_x^3 n_e^{(2)} - \frac{\beta}{2n} \partial_x^2 n_e^{(2)} = G(1), \tag{1.13}$$

where we have applied (1.11) and $G(1)$ depends only on $n_e^{(1)}$ and its derivatives.

At the order of $O(\varepsilon^{k+1})$, we can get the linearized inhomogeneous Burgers-KdV equation for $n_e^{(k)}$:

$$\partial_t n_e^{(k)} + \frac{3\lambda_0}{2} \partial_x (n_e^{(1)} n_e^{(k)}) + \frac{\alpha}{2} \lambda_0^3 \partial_x^3 n_e^{(k)} - \frac{\beta}{2n} \partial_x^2 n_e^{(k)} = G(k-1), \quad k \geq 3, \tag{1.14}$$

where $G(k-1)$ only depends on $n_e^{(1)}, \ldots, n_e^{(k-1)}$, which can all be determined from the preceding k-1 steps. The system (1.14) is also self-contained and only depends on $n_e^{(k)}$.

Equations (1.10), (1.13) and (1.14) contain both third derivative terms and second derivative terms which arise from the interplay of dispersion and dissipation in the system. Finding a general Gardner-Morikawa transformation that keeps the equation unchanged is not possible in this case. This implies that the general reduced perturbation method fails to preserve the original form of the equation. Therefore we assumed that the magnitudes of the dispersion and dissipation terms are at the same level.

Remark 1.1. *If the magnitudes of dispersion and dissipation terms are not on the same order, the approximate evolution equation may be reduced to either the KdV equation ($\lambda_D^2/L^2 \gg \nu/L_{c_s}$) or the Burgers equation ($\lambda_D^2/L^2 \ll \nu/L_{c_s}$) respectively. These reduced equations capture the dominant behavior of the system when either dispersion or dissipation is significantly stronger than the other.*

Remark 1.2. *Through qualitative analysis, (1.10), (1.13) and (1.14) can be likened to a nonlinear oscillator equation with a damping term, which can be solved in the form of an oscillating shock wave by applying qualitative analysis for sufficiently large damping. On the contrary, when the damping is small enough, the soliton-mode solution with slow attenuation of amplitude can be obtained.*

Theorem 1.1. *Let $s_1 \geq 2$ be a sufficiently large integer. Then for any given initial data $n_0^{(1)} \in H^{s_1}(\mathbb{R})$, there exists $\tau_* > 0$ such that the initial value problem (1.8) and (1.10) has the unique solution*

$$(n^{(1)}, v^{(1)}, n_e^{(1)}) \in L^\infty(-\tau_*, \tau_*; H^{s_1}(\mathbb{R}))$$

with initial data $(n_{e0}^{(1)}, \lambda_0 n_{e0}^{(1)}, n_{e0}^{(1)})$. Moreover, we can extend the solution to any time interval $[-\tau, \tau]$ by the conservation of the Burgers-KdV equation [30, 31].

Theorem 1.2. *Let $k \geq 2$ and $s_k \leq s_1 - 3(k-1)$ be sufficiently large integers. Then, for any $\tau > 0$ and any given initial data $(n_0^{(k)}, v_0^{(k)}, n_{e0}^{(k)}) \in H^{s_k}(\mathbb{R})$, the initial value problem of (1.14) with initial data $(n_0^{(k)}, v_0^{(k)}, n_{e0}^{(k)})$ has the unique solution*

$$(n^{(k)}, v^{(k)}, n_e^{(k)}) \in L^\infty(-\tau, \tau; H^{s_k}(\mathbb{R})).$$

1.2. Remainder system

Next, we will give a strict proof to show that $n_e^{(1)}$ converges to a solution of the Burgers-KdV equation as $\varepsilon \to 0$. Suppose that $(n^{(k)}, v^{(k)}, n_e^{(k)})$, $1 \le k \le 4$ are sufficiently smooth. Let (n, v, n_e) be a solution of (1.5) and have the following expansions:

$$\begin{cases} n = 1 + \varepsilon n^{(1)} + \varepsilon^2 n^{(2)} + \varepsilon^3 n^{(3)} + \varepsilon^4 n^{(4)} + \varepsilon^3 N, & (1.15a) \\ n_e = 1 + \varepsilon n_e^{(1)} + \varepsilon^2 n_e^{(2)} + \varepsilon^3 n_e^{(3)} + \varepsilon^4 n_e^{(4)} + \varepsilon^3 N_e, & (1.15b) \\ v = \varepsilon v^{(1)} + \varepsilon^2 v^{(2)} + \varepsilon^3 v^{(3)} + \varepsilon^4 v^{(4)} + \varepsilon^3 V, & (1.15c) \end{cases}$$

where $(n^{(k)}, v^{(k)}, n_e^{(k)})$ for $1 \le k \le 4$ satisfies (1.8), (1.10) and (1.13) respectively, and (N, V, N_e) is the remainder.

Just for the sake of calculation, we denote

$$\tilde{n} = n^{(1)} + \varepsilon n^{(2)} + \varepsilon^2 n^{(3)} + \varepsilon^3 n^{(4)},$$
$$\tilde{n}_e = n_e^{(1)} + \varepsilon n_e^{(2)} + \varepsilon^2 n_e^{(3)} + \varepsilon^3 n_e^{(4)},$$
$$\tilde{v} = v^{(1)} + \varepsilon v^{(2)} + \varepsilon^2 v^{(3)} + \varepsilon^3 v^{(4)}.$$

Putting (1.15) into the scaled system (1.5), a simper calculation gives the remainder system, as follows

$$\begin{cases} \partial_t N - \dfrac{\lambda_0 - v}{\varepsilon}\partial_x N + \dfrac{n}{\varepsilon}\partial_x V + \partial_x \tilde{n} V + \partial_x \tilde{v} N + \varepsilon \mathcal{R}_1 = 0, & (1.16a) \\ \partial_t V - \dfrac{\lambda_0 - v}{\varepsilon}\partial_x V + \dfrac{1}{n_e \varepsilon}\partial_x N_e + \partial_x \tilde{v} V - \dfrac{\beta}{n}\partial_x^2 V + \varepsilon \mathcal{R}_2 = 0, & (1.16b) \\ N_e - N = \dfrac{\alpha\varepsilon}{n_e}\partial_x^2 N_e - \dfrac{2\alpha}{n_e^2}\varepsilon^2 \partial_x \tilde{n}_e \partial_x N_e + \varepsilon \mathcal{R}_3, & (1.16c) \end{cases}$$

where

$$\begin{cases} \mathcal{R}_1 = \partial_x n^{(4)} + \partial_x(n^{(2)}v^{(4)} + n^{(3)}v^{(3)} + n^{(4)}v^{(3)}) + \varepsilon\partial_x(n^{(3)}v^{(4)} + n^{(4)}v^{(3)}) + \varepsilon^2\partial_x(n^{(4)}v^{(4)}), \\ \mathcal{R}_2 = -\partial_t v^{(4)} + (v^{(1)}\partial_x v^{(4)} + v^{(2)}\partial_x v^{(3)} + v^{(3)}\partial_x v^{(2)} + v^{(4)}\partial_x v^{(1)}) + \varepsilon(v^{(2)}\partial_x v^{(4)} + v^{(3)}\partial_x v^{(3)} + v^{(3)}\partial_x v^{(2)}) \\ \qquad + \varepsilon^2\partial_x(v^{(3)}v^{(4)}) - \frac{\beta}{n}\partial_x v^{(3)} - \frac{\beta}{n}\varepsilon\partial_x v^{(4)}, \\ \mathcal{R}_3 = n^{(4)} - n_e^{(4)} - \frac{2\alpha}{n_e^2}(\partial_x n_e^{(1)}\partial_x n_e^{(2)}) - \frac{\alpha}{n_e^2}\varepsilon(2\partial_x n_e^{(3)}\partial_x n_e^{(1)} + \partial_x n_e^{(2)}\partial_x n_e^{(2)}) \\ \qquad - \frac{2\alpha}{n_e^2}\varepsilon^2(\partial_x n_e^{(4)}\partial_x n_e^{(1)} + \partial_x n_e^{(3)}\partial_x n_e^{(2)}) + \frac{\alpha}{n_e}(\partial_x^2 n_e^{(3)} + \varepsilon\partial_x^2 n_e^{(4)}). \end{cases} \quad (1.17)$$

Next, we give some basic estimates for the remainder term $\mathcal{R}_3$.

Lemma 1.1. *For the $s = 1, 2, \ldots$ integers, there exists some constant $C = C(\|n_e^{(i)}\|_{H^\delta})$ and $C = C(\|n_e^{(i)}\|_{H^\delta}, \sqrt{\varepsilon}\|N_e\|_{H^s})$ such that $\|\mathcal{R}_1, \mathcal{R}_2\|_{H^s} \le C(\|n_e^{(i)}\|_{H^\delta})$ and*

$$\|\mathcal{R}_3\|_{H^s} \le C(\|n_e^{(i)}\|_{H^\delta}, \varepsilon\|N_e\|_{H^s})\|N_e\|_{H^s}, \quad s = 1, 2, \ldots, \quad (1.18)$$

$$\|\partial_T \mathcal{R}_3\|_{H^s} \le C(\|{n_e}^{(i)}\|_{H^\delta}, \varepsilon\|N_e\|_{H^s})\|\partial_T N_e\|_{H^s}, \quad s = 1, 2, \ldots, \quad (1.19)$$

where $\delta = \max\{2, s-1\}$; due to the fact that H^s is an algebra, the proof of Lemma 1.1 is obvious; also, the constant C is nondecreasing.

1.3. Main results

The main results are given as follows:

Theorem 1.3. *Let $s_i \geq 2$ in Theorems 1.1 and 1.2 be sufficiently large and $(n^{(1)}, v^{(1)}, n_e^{(1)}) \in H^{s_1}$ be a solution constructed in Theorem 1.1 for the Burgers-KdV equation with initial data $(n_0^{(1)}, v_0^{(1)}, n_{e0}^{(1)})$ in H^{s_1} satisfying (1.8). Let $(n^{(i)}, v^{(i)}, n_e^{(i)}) \in H^{s_i}(i = 2, 3, 4)$ be a solution of (1.14), as constructed in Theorem 1.2 with initial data $(n_0^{(i)}, v_0^{(i)}, n_{e0}^{(i)})$ in H^{s_i}. Let $(N_0^\varepsilon, V_0^\varepsilon, N_{e0}^\varepsilon) \in H^s$ and assume the following:*

$$\begin{cases} n_0 = 1 + \varepsilon n_0^{(1)} + \varepsilon^2 n_0^{(2)} + \varepsilon^3 n_0^{(3)} + \varepsilon^4 n_0^{(4)} + \varepsilon^3 N_0, \\ n_{e0} = 1 + \varepsilon n_{e0}^{(1)} + \varepsilon^2 n_{e0}^{(2)} + \varepsilon^3 n_{e0}^{(3)} + \varepsilon^4 n_{e0}^{(4)} + \varepsilon^3 N_{e0}, \\ v_0 = \varepsilon v_0^{(1)} + \varepsilon^2 v_0^{(2)} + \varepsilon^3 v_0^{(3)} + \varepsilon^4 v_0^{(4)} + \varepsilon^3 V_0. \end{cases} \tag{1.20}$$

Then, for any $\tau > 0$, there exists $\varepsilon_0 > 0$ such that, if $0 < \varepsilon < \varepsilon_0$, the solution of the system (1.5) with initial data (n_0, v_0, n_{e0}) can be expressed as follows:

$$\begin{cases} n = 1 + \varepsilon n^{(1)} + \varepsilon^2 n^{(2)} + \varepsilon^3 n^{(3)} + \varepsilon^4 n^{(4)} + \varepsilon^3 N, \\ n_e = 1 + \varepsilon n_e^{(1)} + \varepsilon^2 n_e^{(2)} + \varepsilon^3 n_e^{(3)} + \varepsilon^4 n_e^{(4)} + \varepsilon^3 N_e, \\ v = \varepsilon v^{(1)} + \varepsilon^2 v^{(2)} + \varepsilon^3 v^{(3)} + \varepsilon^4 v^{(4)} + \varepsilon^3 V, \end{cases} \tag{1.21}$$

such that, for all $0 < \varepsilon < \varepsilon_0$,

$$\begin{aligned} &\sup_{[0,\tau]}\{\|(N, V, N_e)\|_{H^2}^2 + \varepsilon\|(\partial_x^3 V, \partial_x^3 N_e)\|_{L^2}^2 + \varepsilon^2\|\partial_x^4 N_e\|_{L^2}^2\} \\ &\leq C_\tau(1 + \|(N_0, V_0, N_{e0})\|_{H^2}^2 + \varepsilon\|(\partial_x^3 V_0, \partial_x^3 N_{e0})\|_{L^2}^2 + \varepsilon^2\|\partial_x^4 N_{e0}\|_{L^2}^2). \end{aligned} \tag{1.22}$$

From (1.22), we obtain the uniform H^2-norm of the remainder (N, V, N_e) on ε, and it satisfies

$$\sup_{[0,\varepsilon^{-1}\tau]} \left\| \begin{pmatrix} (n-1)/\varepsilon \\ (n_e - 1)/\varepsilon \\ v/\varepsilon \end{pmatrix} - BKdV \right\|_{H^2} \leq C\varepsilon \tag{1.23}$$

where C is independent of ε and BKdV denotes the Burgers-KdV equation.

In order to give the proof of Theorem1.3, we define the following weighted norm:

$$|||(V, N_e)|||_\varepsilon^2 = \|V\|_{H^2}^2 + \|N_e\|_{H^2}^2 + \varepsilon\|\partial_x^3 V\|^2 + \varepsilon\|\partial_x^3 N_e\|^2 + +\varepsilon^2\|\partial_x^4 N_e\|^2. \tag{1.24}$$

2. The uniform estimate

In this section, we will give the uniform estimates of the system (1.16) and show that (1.16) has smooth solutions for significantly small $\tau_\varepsilon > 0$, as dependent on $\varepsilon > 0$. Let $\tilde{C}$ be a constant that is independent of ε. By the classical theorem, we know that there exists $\tau_\varepsilon > 0$ such that on $[0, \tau_\varepsilon]$,

$$\|N\|_{H^2}^2, |||(V, N_e)(t)|||_\varepsilon^2 \leq \tilde{C}. \tag{2.1}$$

Note that n is bounded, i.e., $1/2 < n < 3/2$, and $|v| < 1/2$ for $\varepsilon < \varepsilon_1$. There exists some constant $C_1 = C_1(\varepsilon\tilde{C})$ for any $\alpha, \beta \geq 0$ such that

$$|\partial_{n_e^{(i)}}^\alpha \partial_{N_e}^\beta \mathcal{R}_3| \leq C_1 = C_1(\varepsilon\tilde{C}), \tag{2.2}$$

where C_1 is chosen to be nondecreasing in its argument.

Next, we will give some lemmas to show the relation between N and N_e.

Lemma 2.1. *Let* (N, V, N_e) *be a solution to (1.16) and* $\alpha \geq 0$ *be an integer. There exist some constants* $0 < \varepsilon_1 < 1$ *and* $C_1 = C_1(\varepsilon\tilde{C})$ *such that for every* $0 < \varepsilon < \varepsilon_1$,

$$C_1^{-1}\|\partial_x^\alpha N\|^2 \leq \|\partial_x^\alpha N_e\|^2 + \varepsilon\|\partial_x^{\alpha+1}N_e\|^2 + \varepsilon^2\|\partial_x^{\alpha+2}N_e\|^2 \leq C_1\|\partial_x^\alpha N\|^2. \tag{2.3}$$

Proof. When $\alpha = 0$, taking the inner product of (1.16c) with N_e, we have

$$\|N_e\|^2 + \varepsilon\frac{\alpha}{n_e}\|\partial_x N_e\|^2 = \int NN_e - \varepsilon\alpha\int\partial_x\Big(\frac{1}{n_e}\Big)\partial_x N_e N_e - \varepsilon^2\frac{2\alpha}{n_e^2}\int\partial_x\tilde{n}_e\partial_x N_e N_e + \varepsilon\int\mathcal{R}_3 N_e, \tag{2.4}$$

where α is a constant and $\frac{1}{2} < n_e < \frac{3}{2}$; therefore, there exists a constant C such that

$$\varepsilon\frac{\alpha}{n_e}\|\partial_x N_e\|^2 \geq C\varepsilon\|\partial_x N_e\|^2, \tag{2.5}$$

so the left of (2.5) is larger than $\|N_e\|^2 + \varepsilon\|\partial_x N_e\|^2$.

Due to the fact that

$$\Big|\partial_x\Big(\frac{1}{n_e}\Big)\Big| \leq C(\varepsilon|\partial_x\tilde{n}_e| + \varepsilon^3|\partial_x N_e|), \tag{2.6}$$

combined with Young's inequality, we get

$$\Big|-\varepsilon\alpha\int\partial_x\Big(\frac{1}{n_e}\Big)\partial_x N_e N_e\Big| \leq \frac{1}{8}\|N_e\|^2 + C\varepsilon^2\|\partial_x N_e\|^2 + C\varepsilon^4\tilde{C}\|\partial_x N_e\|^2. \tag{2.7}$$

Due to the fact that $\tilde{n}_e$ is bounded in L^∞, by the Hölder inequality, we have

$$\Big|-\varepsilon^2\frac{2\alpha}{n_e^2}\int\partial_x\tilde{n}_e\partial_x N_e N_e\Big| \leq \frac{1}{8}\|N_e\|^2 + C\varepsilon^2\|\partial_x N_e\|^2. \tag{2.8}$$

From Lemma 1.1, it holds that $\varepsilon < \varepsilon_1$, which is sufficiently small such that

$$\varepsilon\int\mathcal{R}_3 N_e \leq \frac{1}{4}\|N_e\|^2. \tag{2.9}$$

Therefore, by the Hölder inequality, we have

$$\|N_e\|^2 + \varepsilon\|\partial_x N_e\|^2 \leq C\|N\|^2. \tag{2.10}$$

Taking the inner product of (1.16c) with $\varepsilon\partial_x^2 N_e$, we have

$$\varepsilon\|\partial_x N_e\|^2 + \varepsilon^2\frac{\alpha}{n_e}\|\partial_x^2 N_e\|^2 = \varepsilon^3\frac{2\alpha}{n_e^2}\int\partial_x\tilde{n}_e\partial_x N_e\partial_x^2 N_e - \varepsilon\int N\partial_x^2 N_e - \varepsilon^2\int\mathcal{R}_3\partial_x^2 N_e; \tag{2.11}$$

by the Hölder inequality, the result is as follows

$$\varepsilon\|\partial_x N_e\|^2 + \varepsilon^2\|\partial_x^2 N_e\|^2 \leq C\|N\|^2. \tag{2.12}$$

Then, applying the L^2-norm of (1.16c), we obtain some C such that

$$\|N\|^2 \leq \|N_e\|^2 + \varepsilon^2\|\partial_x^2 N_e\|^2 + \varepsilon^4\|\partial_x N_e\|^2. \tag{2.13}$$

Putting (2.10), (2.12) and (2.13) together, we obtain (2.3) for $\alpha = 0$.

For the higher-order cases, we can differentiate (1.16c) with respect to ∂^α, take the inner product with $\partial_x^\alpha N_e$ and $\varepsilon\partial_x^{\alpha+2}N_e$ and then perform the same procedure as for the case $\alpha = 0$; we can complete this lemma.

Lemma 2.2. *Let (N, V, N_e) be a solution to (1.16) and $\alpha \geq 0$ be an integer. There exist some constants C and $C_1 = C_1(\varepsilon\tilde{C})$ such that*

$$\varepsilon\|\partial_t N\|^2 \leq C(\|N_e\|_{H^1}^2 + \|V\|_{H^1}^2 + \varepsilon\|\partial_x^2 N_e\|^2 + \varepsilon^2\|\partial_x^3 N_e\|^2) + C\varepsilon \tag{2.14}$$

and

$$\|\varepsilon\partial_{tx} N\|^2 \leq C_1(\|N_e\|_{H^2}^2 + \|V\|_{H^2}^2 + \varepsilon\|\partial_x^3 N_e\|^2 + \varepsilon^2\|\partial_x^4 N_e\|^2) + C\varepsilon. \tag{2.15}$$

Proof. Due to the fact that $1/2 < n < 3/2$ and $|v| < 1/2$, taking the L^2-norm of (1.16a) gives

$$\|\varepsilon\partial_t N\|^2 \leq C(\|\partial_x N\|^2 + \|\partial_x V\|^2) + C\varepsilon^2(\varepsilon^2 + \|N\|^2 + \|V\|^2). \tag{2.16}$$

By using Lemma 2.1 with $\alpha = 1$, we have (2.14).

Then, taking ∂_x of (1.16a), we obtain

$$\|\varepsilon\partial_{tx} N\|^2 \leq C(\|V\|_{H^2}^2 + \|N\|_{H^2}^2) + C\varepsilon^6 \int |\partial_x V|^2 |\partial_x N|^2 + C\varepsilon^4. \tag{2.17}$$

By the Sobolev embedding inequality, we have

$$C\varepsilon^6\|\partial_x V\|_{L^\infty}^2 \|\partial_x N\|^2 \leq C\varepsilon^6\|V\|_{H^2}^2 + \|N\|_{H^1}^2 \leq C(\varepsilon\tilde{C})\|V\|_{H^2}^2; \tag{2.18}$$

then by Lemma 2.1, we complete this lemma.

Lemma 2.3. *Let (N, V, N_e) be a solution to (1.16) and $\alpha \geq 0$ be an integer. There exist some constants $C_1 = C_1(\varepsilon\tilde{C})$ and $\varepsilon_1 > 0$ such that for any $0 < \varepsilon < \varepsilon_1$,*

$$\varepsilon\|\partial_t\partial_x^{\alpha+1} N_e\|^2 + \|\partial_t\partial_x^\alpha N_e\|^2 \leq C\|\partial_t\partial_x^\alpha N\|^2 + C_1. \tag{2.19}$$

Proof. For the case of $\alpha = 0$, taking ∂_t of (1.16c) and then taking the inner product with $\partial_t N_e$, we have

$$\begin{aligned}
&\|\partial_t N_e\|^2 + \varepsilon\frac{\alpha}{n_e}\int |\partial_{tx} N_e|^2 \\
&= \int \partial_t N \partial_t N_e + \alpha\varepsilon \int \partial_t\Big(\frac{1}{n_e}\Big)\partial_x^2 N_e \partial_t N_e - \alpha\varepsilon \int \partial_x\Big(\frac{1}{n_e}\Big)\partial_t\partial_x N_e \partial_t N_e \\
&\quad - 2\alpha\varepsilon^2 \int \partial_t\Big(\frac{1}{n_e^2}\Big)\partial_x \tilde{n}_e \partial_x N_e \partial_t N_e - \int \Big(\frac{2\alpha}{n_e^2}\varepsilon^2 \partial_t\partial_x \tilde{n}_e \partial_x N_e\Big)\partial_t N_e \\
&\quad - \int \Big(\frac{2\alpha}{n_e^2}\varepsilon^2 \partial_x \tilde{n}_e \partial_t\partial_x N_e\Big)\partial_t N_e + \int \varepsilon\partial_t \mathcal{R}_3 \partial_t N_e =: \sum_{i=1}^{7} A_i,
\end{aligned} \tag{2.20}$$

where α is a constant and $1/2 < n_e < 3/2$, so there exists a constant C such that $\varepsilon\frac{\alpha}{n_e}\int |\partial_{tx} N_e|^2 \geq C\varepsilon\int |\partial_{tx} N_e|^2$; therefore, the left of (2.20) is greater than or equal to $C(\|\partial_t N_e\|^2 + \varepsilon\int |\partial_{tx} N_e|^2)$. Now, we estimate the right of (2.20). For A_1, for any small $\gamma > 0$, by Young's inequality, we have

$$A_1 = \int \partial_t N \partial_t N_e \leq \gamma\|\partial_t N_e\|^2 + C_\gamma\|\partial_t N\|^2. \tag{2.21}$$

Due to the fact that

$$\left|\partial_t\left(\frac{1}{n_e}\right)\right|, \left|\partial_t\left(\frac{1}{n_e^2}\right)\right| \le C(\varepsilon|\partial_t\tilde{n}_e| + \varepsilon^3|\partial_t N_e|),$$
$$\left|\partial_x\left(\frac{1}{n_e}\right)\right| \le C(\varepsilon|\partial_x\tilde{n}_e| + \varepsilon^3|\partial_x N_e|), \tag{2.22}$$

by the Hölder inequality and Sobolev embedding inequality, we obtain

$$A_2 \le C\varepsilon^2 \int (|\partial_t\tilde{n}_e| + \varepsilon^2|\partial_t N_e|)\partial_x^2 N_e\partial_t N_e \le C_1(\|\partial_t N_e\|^2 + \varepsilon^2\|\partial_x^2 N_e\|^2) + C_1,$$
$$A_3 \le C\varepsilon^2 \int (|\partial_x\tilde{n}_e| + \varepsilon^2|\partial_x N_e|)\partial_t\partial_x N_e\partial_t N_e \le C_1(\|\partial_t N_e\|^2 + \varepsilon\|\partial_{xt} N_e\|^2) + C_1, \tag{2.23}$$

and

$$A_4 \le C\varepsilon^3 \int (|\partial_t\tilde{n}_e| + \varepsilon^2|\partial_t N_e|)\partial_x\tilde{n}_e\partial_x N_e\partial_t N_e \le C_1(\|\partial_t N_e\|^2 + \varepsilon\|\partial_x N_e\|^2) + C_1. \tag{2.24}$$

Similar to A_2, by the Hölder inequality, we obtain

$$A_{5,6,7} \le C_1(\|\partial_t N_e\|^2 + \varepsilon\|\partial_{xt} N_e\|^2) + C_1. \tag{2.25}$$

Therefore, we have

$$\|\partial_t N_e\|^2 + \varepsilon\|\partial_{xt} N_e\|^2 \le C\|\partial_t N\|^2 + C_1. \tag{2.26}$$

When $\alpha = 1$, we take ∂_{xt} of (1.16c); then, taking the inner product with $\varepsilon\partial_{xt}N_e$, we have

$$\begin{aligned}
&\varepsilon\|\partial_{xt}N_e\|^2 + \frac{\alpha\varepsilon^2}{n_e}\int |\partial_t\partial_x^2 N_e|^2 \\
&= \varepsilon\int \partial_{xt}N\partial_{xt}N_e + \varepsilon\int \partial_{xt}\left(\frac{\alpha\varepsilon}{n_e}\right)\partial_x^2 N_e\partial_{xt}N_e - \varepsilon\int \partial_x\left(\frac{\alpha\varepsilon}{n_e}\right)\partial_t\partial_x^2 N_e\partial_{xt}N_e \\
&\quad - \varepsilon\int \partial_{xt}\left(\frac{2\alpha}{n_e^2}\right)\varepsilon^2\partial_x\tilde{n}_e\partial_x N_e\partial_{xt}N_e - \varepsilon\int \left(\frac{2\alpha}{n_e^2}\varepsilon^2\partial_{xt}\partial_x\tilde{n}_e\partial_x N_e\right)\partial_{xt}N_e \\
&\quad - \varepsilon\int \partial_{xt}\left(\frac{2\alpha}{n_e^2}\varepsilon^2\partial_x\tilde{n}_e\partial_{xt}\partial_x N_e\right)\partial_{xt}N_e + \varepsilon\int \varepsilon\partial_{xt}\mathcal{R}_3\partial_{xt}N_e =: \sum_{i=1}^{7} B_i
\end{aligned}$$

Due to the fact that

$$\left|\partial_{xt}\left(\frac{1}{n_e}\right)\right|, \left|\partial_{xt}\left(\frac{1}{n_e^2}\right)\right| \le C(\varepsilon|\partial_{xt}\tilde{n}_e| + \varepsilon^3|\partial_{xt}N_e|), \tag{2.27}$$

for arbitrary $\gamma > 0$, by Young's inequality, we obtain

$$B_1 \le \gamma\varepsilon\|\partial_{xt}N_e\|^2 + C_\gamma\|\partial_{xt}N\|^2. \tag{2.28}$$

Using (2.1), by the Hölder inequality and Sobolev embedding inequality, we have

$$B_2 \le C\varepsilon^3(|\partial_{xt}\tilde{n}_e| + \varepsilon^2|\partial_{xt}N_e|)\partial_x^2 N_e\partial_{xt}N_e \le C_1(\varepsilon\|\partial_{xt}N_e\|^2 + \varepsilon^2\|\partial_t\partial_x^2 N_e\|^2) + C_1 \tag{2.29}$$

and

$$B_3 \le C\varepsilon^3 \int (|\partial_x\tilde{n}_e| + \varepsilon^2|\partial_x N_e|)\partial_t\partial_x^2 N_e\partial_{xt}N_e \le C_1(\varepsilon\|\partial_{xt}N_e\|^2 + \varepsilon^2\|\partial_t\partial_x^2 N_e\|^2) + C_1. \tag{2.30}$$

Similar to the process for B_2, by the Sobolev embedding inequality, we have

$$B_{4,5,6,7} \le C_1(\varepsilon\|\partial_{xt}N_e\|^2 + \varepsilon^2\|\partial_t\partial_x^2 N_e\|^2) + C_1; \tag{2.31}$$

therefore, we have

$$\varepsilon\|\partial_t N_e\|^2 + \varepsilon^2\|\partial_{xt}N_e\|^2 \le C\|\partial_{xt}N\|^2 + C_1. \tag{2.32}$$

We can give by the similar process for the case of $\alpha \ge 2$.

2.1. *Zeroth, first and second order estimates*

Proposition 2.1. *Let (N, V, N_e) be a solution to (1.16) and $\kappa = 0, 1, 2$; then,*

$$\frac{1}{2}\frac{d}{dt}\|\partial_x^\kappa V\|^2 + \frac{1}{2}\frac{d}{dt}\Big(\int \frac{1}{nn_e}|\partial_x^\kappa N_e|^2 + \int \frac{\alpha\varepsilon}{nn_e^2}|\partial_x^{\kappa+1}N_e|^2\Big) \le C_1(1 + \varepsilon^2|||(V, N_e)|||_\varepsilon^2)(1 + |||(V, N_e)|||_\varepsilon^2). \tag{2.33}$$

Proof. We take ∂_x^κ of (1.16b) and the inner product of $\partial_x^\kappa V$. Integrating by parts yields

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\partial_x^\kappa V\|^2 - \frac{\lambda_0}{\varepsilon}\int \partial_x^{\kappa+1}V\partial_x^\kappa V + \int \partial_x^\kappa((\tilde{v} + \varepsilon^2 V)\partial_x V)\partial_x^\kappa V + \int \partial_x^\kappa(\partial_x\tilde{v}V)\partial_x^\kappa V\\ &- \int \partial_x^\kappa\Big(\frac{\beta}{n}\partial_x^2 V\Big)\partial_x^\kappa V + \int \partial_x^\kappa(\varepsilon\mathcal{R}_2)\partial_x^\kappa V = \int \partial_x^{\kappa-1}\Big(\frac{1}{n_e}\partial_x N_e\Big)\frac{\partial_x^{\kappa+1}V}{\varepsilon}.\end{aligned} \tag{2.34}$$

The second term vanishes by integration by parts. The third term can be divided into two parts as follows

$$\int \partial_x^\kappa((\tilde{v} + \varepsilon^2 V)\partial_x V)\partial_x^\kappa V = \int \partial_x^\kappa(\tilde{v}\partial_x V)\partial_x^\kappa V + \int \partial_x^\kappa(\varepsilon^2 V\partial_x V)\partial_x^\kappa V;$$

for the first part, integration by parts gives

$$\int \partial_x^\kappa(\tilde{v}\partial_x V)\partial_x^\kappa V = -\frac{1}{2}\int \partial_x\tilde{v}\partial_x^\kappa V\partial_x^\kappa V + \sum_{0\le\gamma\le\kappa-1} C_\kappa^\gamma \int \partial_x^{\kappa-\gamma}\tilde{v}\partial_x^{\gamma+1}V\partial_x^\kappa V \le C\|V\|_{H^2}^2,$$

where, when $\kappa = 0$, there is no such "summation" term. Regarding the second part, after integration by parts, for $0 \le \kappa \le 2$, we have

$$\varepsilon^2\int \partial_x^\kappa(V\partial_x V)\partial_x^\kappa V = -\frac{\varepsilon^2}{2}\int \partial_x V\partial_x^\kappa V\partial_x^\kappa V + \sum_{0\le\gamma\le\kappa-1} C_\kappa^\gamma\varepsilon^2 \int \partial_x^{\kappa-\gamma}V\partial_x^{\gamma+1}V\partial_x^\kappa V \le C\varepsilon^2\|\partial_x V\|_{H^1}\|V\|_{H^\kappa}^2.$$

For the fourth therm, similar to the first term,

$$\int \partial_x^\kappa(\partial_x\tilde{v}V)\partial_x^\kappa V = \int(\partial_x\tilde{v}\partial_x^\kappa V)\partial_x^\kappa V + \sum_{0\le\gamma\le\kappa-1} C_\kappa^\gamma \int \partial_x^{\kappa+1-\gamma}(\tilde{v})\partial_x^\gamma V\partial_x^\kappa V \le C\|V\|_{H^2}^2.$$

For the fifth therm, similar to the first term, integration by parts gives

$$\begin{aligned}-\int \partial_x^\kappa\Big(\frac{\beta}{n}\partial_x^2 V\Big)\partial_x^\kappa V &= \int \partial_x\Big(\frac{\beta}{n}\Big)\partial_x^{\kappa+1}V\partial_x^\kappa V + \int \frac{\beta}{n}\partial_x^{\kappa+1}V\partial_x^{\kappa+1}V - \sum_{0\le\gamma\le\kappa-1} C_\kappa^\gamma \int \partial_x^{\kappa-\gamma}\Big(\frac{\beta}{n}\Big)\partial_x^{\gamma+2}V\partial_x^\kappa V\\ &\le C(\varepsilon\|\partial_x\tilde{n} + \varepsilon^3\partial_x N\|_{L^\infty})(\|\partial_x^{\kappa+1}V\|^2 + \|\partial_x^\kappa V\|^2) + C\|\partial_x^{\kappa+1}V\|^2.\end{aligned}$$

By Lemma (1.1), we have

$$\int \partial_x^\kappa(\varepsilon\mathcal{R}_2)\partial_x^\kappa V \leq C\varepsilon\|V\|_{H^\kappa}^2.$$

Now, we estimate the right side of (2.34) for $0 < \kappa \leq 2$; taking ∂_x^κ of (1.16a), we have

$$\begin{aligned}\frac{\partial_x^{\kappa+1}V}{\varepsilon} &= \frac{1}{n}\Big(\frac{\lambda_0 - \nu}{\varepsilon}\partial_x^{\kappa+1}N - \partial_t\partial_x^\kappa N - \sum_{0\leq\gamma\leq\kappa-1} C_\kappa^\gamma \partial_x^{\kappa-\gamma}(\tilde{v} + \varepsilon^2 V)\partial_x^{\gamma+1}N - \Big(\frac{n}{\varepsilon}\Big)\partial_x^{\kappa+1}V \\ &\quad - \sum_{0\leq\gamma\leq\kappa-1} C_\kappa^\gamma \partial_x^{\kappa-\gamma}\Big(\frac{n}{\varepsilon}\Big)\partial_x^{\gamma+1}V - \partial_x^\kappa(\partial_x\tilde{n}V) - \partial_x^\kappa(\partial_x\tilde{v}N) - \varepsilon\partial_x^\kappa\mathcal{R}_1\Big) =: \sum_{i=1}^{8} D_i\end{aligned}$$

Incorporating this into the right side of (2.34), we have

$$\sum_{i=1}^{8} I_i = \sum_{i=1}^{8}\int \partial_x^{\kappa-1}\Big(\frac{1}{n_e}\partial_x N_e\Big)D_i. \tag{2.35}$$

We first estimate the term for $3 \leq i \leq 8$; for the term I_3, by the Hölder inequality and Sobolev embedding inequality, we have

$$\begin{aligned}I_3 &= -\sum_{0\leq\gamma\leq\kappa-1} C_\kappa^\gamma \int \partial_x^{\kappa-1}\Big(\frac{1}{n_e}\partial_x N_e\Big)\frac{1}{n}\partial_x^{\kappa-\gamma}\tilde{v}\partial_x^{\gamma+1}N - \sum_{0\leq\gamma\leq\kappa-1} C_\kappa^\gamma\varepsilon^2 \int \partial_x^{\kappa-1}\Big(\frac{1}{n_e}\partial_x N_e\Big)\frac{1}{n}\partial_x^{\kappa-\gamma}V\partial_x^{\gamma+1}N \\ &\leq C(\|N\|_{H^2}^2 + \|N_e\|_{H^2}^2) + C_1(1 + \varepsilon^2\|N\|_{H^2}^2 + \varepsilon^2\|V\|_{H^2}^2)\varepsilon\|N_e\|_{H^3}.\end{aligned}$$

Similar to I_3, simple calculation gives

$$\begin{aligned}I_4 &= \int \partial_x^{\kappa-1}\Big(\frac{1}{n_e}\partial_x N_e\Big)\frac{1}{n}(\tilde{n} + \varepsilon^2 N)\partial_x^{\kappa+1}V \\ &\leq C(\|V\|_{H^2}^2 + \|N_e\|_{H^2}^2) + C_1(1 + \varepsilon^2\|N\|_{H^2}^2 + \varepsilon^2\|V\|_{H^2}^2)\varepsilon\|N_e\|_{H^3},\end{aligned}$$

and

$$\begin{aligned}I_5 &= -\sum_{0\leq\gamma\leq\kappa-1} C_\kappa^\gamma \int \partial_x^{\kappa-1}\Big(\frac{1}{n_e}\partial_x N_e\Big)\frac{1}{n}(\partial_x^{\kappa-\gamma}n + \varepsilon^2\partial_x^{\kappa-\gamma}N_e)\partial_x^{\gamma+1}V \\ &\leq C(\|V\|_{H^2}^2 + \|N_e\|_{H^2}^2) + C\varepsilon^2\|\partial_x^\kappa N_e\|_{L^\infty}\|N_e\|_{H^2}^2\|V\|_{H^2}^2.\end{aligned}$$

By Lemma (1.1), with a similar calculation to that for I_3, we obtain

$$I_6 + I_7 + I_8 \leq C(\|V\|_{H^2}^2 + \|N\|_{H^2}^2 + \|N_e\|_{H^2}^2).$$

Lemma 2.4. *Let (N, V, N_e) be a solution to (1.16); we have*

$$I_1 \leq C_1(1 + \varepsilon^2|||(V, N_e)|||_\varepsilon^2)(1 + |||(V, N_e)|||_\varepsilon^2), \tag{2.36}$$

where I_1 is defined in (2.35).

Proof. For the first term I_1, by using a simpler calculation, we can decompose it into two parts:

$$I_1 = \int \partial_x^{\kappa-1}\Big(\frac{1}{n_e}\partial_x N_e\Big)\frac{1}{n}\frac{\lambda_0 - v}{\varepsilon}\partial_x^{\kappa+1}N + \sum_{0<\gamma\le 2} C_\kappa^\gamma \int \partial_x^{\kappa-1}\Big(\frac{1}{n_e}\partial_x N_e\Big)\frac{1}{n}\partial_x^{\kappa-\gamma}\Big(\frac{\lambda_0 - v}{\varepsilon}\Big)\partial_x^{\gamma+1}N$$
$$=: I_{11} + I_{12}.$$

Taking $\partial_x^{\kappa+1}$ of (1.16c), we have

$$\partial_x^{\kappa+1}N = \partial_x^{\kappa+1}N_e - \partial_x^{\kappa+1}\Big(\frac{\alpha\varepsilon}{n_e}\partial_x^2 N_e\Big) + \partial_x^{\kappa+1}\Big(\frac{2\alpha}{n_e^2}\varepsilon^2\partial_x\tilde{n}_e\partial_x N_e\Big) - \varepsilon\partial_x^{\kappa+1}\mathcal{R}_3;$$

inserting it into I_{11} gives

$$I_{11} = \int \partial_x^{\kappa-1}\Big(\frac{1}{n_e}\partial_x N_e\Big)\frac{1}{n}\frac{\lambda_0 - v}{\varepsilon}\Big(\partial_x^{\kappa+1}N_e - \partial_x^{\kappa+1}\Big(\frac{\alpha\varepsilon}{n_e}\partial_x^2 N_e\Big) + \partial_x^{\kappa+1}\Big(\frac{2\alpha}{n_e^2}\varepsilon^2\partial_x\tilde{n}_e\partial_x N_e\Big) - \varepsilon\partial_x^{\kappa+1}\mathcal{R}_3\Big)$$
$$=: \sum_{i=1}^{4} I_{11i}.$$

Regarding estimate of I_{111}, by commutator estimation and Sobolev embedding, we obtain

$$I_{111} = \int \partial_x^{\kappa-1}\Big(\frac{1}{n_e}\partial_x N_e\Big)\frac{1}{n}\frac{\lambda_0 - v}{\varepsilon}\partial_x^{\kappa+1}N_e \le C_1\varepsilon(1+\varepsilon^2|||(N,V,N_e)|||_\varepsilon^2)|||(N,V,N_e)|||_\varepsilon^2.$$

Regarding estimate of I_{12}, by performing integration by parts twice, commutator estimation and Sobolev embedding, we have

$$I_{112} = \int \partial_x^{\kappa}\Big(\frac{1}{n_e}\partial_x N_e\Big)\frac{1}{n}\frac{\lambda_0 - v}{\varepsilon}\partial_x^{\kappa}\Big(\frac{\alpha\varepsilon}{n_e}\partial_x^2 N_e\Big) + \int \partial_x^{\kappa-1}\Big(\frac{1}{n_e}\partial_x N_e\Big)\partial_x\Big(\frac{1}{n}\frac{\lambda_0 - v}{\varepsilon}\Big)\partial_x^{\kappa}\Big(\frac{\alpha\varepsilon}{n_e}\partial_x^2 N_e\Big)$$
$$\le C_1\varepsilon(1+\varepsilon^2|||(N,V,N_e)|||_\varepsilon^2)(\varepsilon\|N_e\|_{H^{\kappa+2}}^2).$$

The estimate of I_{13} is similar to that of I_{12}:

$$I_{113} \le C_1(1+\varepsilon^2|||(N,V,N_e)|||_\varepsilon^2)(\varepsilon\|N_e\|_{H^{\kappa+1}}^2).$$

By Lemma 1.1, we have

$$I_{114} \le C\|\partial_x^\kappa N_e\|^2.$$

With a similar estimate to I_{11}, for $0 < \gamma \le 2$, we obtain

$$I_{12} \le C_1(1+\varepsilon^2|||(N,V,N_e)|||_\varepsilon^2)(\varepsilon\|N_e\|_{H^{\kappa+1}}^2).$$

Combining the estimate of I_{11} and I_{12}, we can get (2.36).

Lemma 2.5. *Let (N, V, N_e) be a solution to (1.16); then, the following inequality holds*

$$I_2 \le -\frac{1}{2}\frac{d}{dt}\int \frac{1}{nn_e}|\partial_x^\kappa N_e|^2 - \frac{1}{2}\frac{d}{dt}\int \frac{\alpha\varepsilon}{nn_e^2}|\partial_x^{\kappa+1}N_e|^2 + C_1(1+\varepsilon^2|||(V,N_e)|||_\varepsilon^2)(1+|||(V,N_e)|||_\varepsilon^2), \tag{2.37}$$

where I_2 is defined in (2.35).

Proof. Taking $\partial_t\partial_x^\kappa$ of (1.16c), we have

$$\partial_t\partial_x^\kappa N = \partial_t\partial_x^\kappa N_e - \partial_t\partial_x^\kappa\Big(\frac{\alpha\varepsilon}{n_e}\partial_x^2 N_e\Big) + \partial_t\partial_x^\kappa\Big(\frac{2\alpha}{n_e^2}\varepsilon^2\partial_x\tilde{n}_e\partial_x N_e\Big) - \varepsilon\partial_t\partial_x^\kappa\mathcal{R}_3;$$

inserting it into I_2 gives

$$I_2 = -\int \partial_x^{\kappa-1}\Big(\frac{1}{n_e}\partial_x N_e\Big)\frac{1}{n}\Big(\partial_t\partial_x^\kappa N_e - \partial_t\partial_x^\kappa\Big(\frac{\alpha\varepsilon}{n_e}\partial_x^2 N_e\Big) + \partial_t\partial_x^\kappa\Big(\frac{2\alpha}{n_e^2}\varepsilon^2\partial_x\tilde{n}_e\partial_x N_e\Big) - \varepsilon\partial_t\partial_x^\kappa\mathcal{R}_3\Big) =: \sum_{i=1}^4 I_{2i}.$$

Regarding the estimate of I_{21}, through integration by parts, we obtain

$$I_{21} = -\frac{1}{2}\frac{d}{dt}\int \frac{1}{nn_e}|\partial_x^\kappa N_e|^2 + \frac{1}{2}\int \partial_t\Big(\frac{1}{nn_e}\Big)|\partial_x^\kappa N_e|^2 - C_{\kappa-1}^\gamma\int \partial_x^{\kappa-1-\gamma}\Big(\frac{1}{nn_e}\Big)\partial_x^{\gamma+1}N_e\frac{1}{n}\partial_t\partial_x^\kappa N_e,$$

where the second term on the right-hand side is bounded as follows

$$\frac{1}{2}\int \partial_t\Big(\frac{1}{nn_e}\Big)|\partial_x^\kappa N_e|^2 \le C_1(1+\varepsilon^2|||(V,N_e)|||_\varepsilon^2)(\varepsilon\|N_e\|_{H^\kappa}^2),$$

where we have used the fact that

$$\Big|\partial_t\Big(\frac{1}{nn_e}\Big)\Big| \le C(\varepsilon(|\partial_t\tilde{n}| + |\partial_t\tilde{n}_e|) + \varepsilon^3(|\partial_t N| + |\partial_t N_e|));$$

for the third term, by the Hölder inequality and Sobolev embedding inequality, we have

$$\Big| - C_{\kappa-1}^\gamma\int \partial_x^{\kappa-1-\gamma}\Big(\frac{1}{nn_e}\Big)\partial_x^{\gamma+1}N_e\frac{1}{n}\partial_t\partial_x^\kappa N_e\Big| \le C\varepsilon(\varepsilon\|\partial_t\partial_x^\kappa N_e\|^2 + \varepsilon\|\partial_x^{\kappa+1}N_e\|^2)(\varepsilon^2\|N_e\|_{H^1} + \varepsilon^2\|N\|_{H^1});$$

therefore, we have

$$I_{21} = -\frac{1}{2}\frac{d}{dt}\int \frac{1}{nn_e}|\partial_x^\kappa N_e|^2 + C_1(1+\varepsilon^2|||(V,N_e)|||_\varepsilon^2)(\varepsilon\|N_e\|_{H^\kappa}^2). \tag{2.38}$$

For the estimate of I_{22}, we have the following decomposition:

$$\begin{aligned} I_{22} = &\int \partial_x^{\kappa-1}\Big(\frac{1}{n_e}\partial_x N_e\Big)\frac{1}{n}\frac{\alpha\varepsilon}{n_e}\partial_t\partial_x^{\kappa+2}N_e + \int \partial_x^{\kappa-1}\Big(\frac{1}{n_e}\partial_x N_e\Big)\frac{1}{n}\partial_t\Big(\frac{\alpha\varepsilon}{n_e}\Big)\partial_x^{\kappa+2}N_e \\ &+ \sum_{0<\gamma\le\kappa-1} C_\kappa^\gamma\int \partial_x^{\kappa-1}\Big(\frac{1}{n_e}\partial_x N_e\Big)\frac{1}{n}\partial_t\partial_x^{\kappa-\gamma}\Big(\frac{\alpha\varepsilon}{n_e}\Big)\partial_x^{\gamma+2}N_e \\ &+ \sum_{0<\gamma\le\kappa-1} C_\kappa^\gamma\int \partial_x^{\kappa-1}\Big(\frac{1}{n_e}\partial_x N_e\Big)\frac{1}{n}\partial_x^{\kappa-\gamma}\Big(\frac{\alpha\varepsilon}{n_e}\Big)\partial_t\partial_x^{\gamma+2}N_e =: \sum_{i=1}^4 I_{22i}. \end{aligned}$$

For the term I_{221}, through integration by parts, and by using the Hölder inequality and Sobolev embedding inequality, we have

$$\begin{aligned} I_{221} = &-\frac{1}{2}\frac{d}{dt}\int \frac{\alpha\varepsilon}{nn_e^2}|\partial_x^{\kappa+1}N_e|^2 + \frac{1}{2}\int \partial_t\Big(\frac{\alpha\varepsilon}{nn_e^2}\Big)|\partial_x^{\kappa+1}N_e|^2 \\ &- C_\kappa^\gamma\int \partial_x^{\kappa-\gamma}\Big(\frac{1}{n_e}\Big)\partial_x^{\kappa+1}N_e\frac{\alpha\varepsilon}{nn_e}\partial_t\partial_x^{\kappa+1}N_e - \int \partial_x^{\kappa-1}\Big(\frac{1}{n_e}\partial_x N_e\Big)\partial_x\Big(\frac{1}{n}\frac{\alpha\varepsilon}{n_e}\Big)\partial_t\partial_x^{\kappa+1}N_e \\ \le &-\frac{1}{2}\frac{d}{dt}\int \frac{\alpha\varepsilon}{nn_e^2}|\partial_x^{\kappa+1}N_e|^2 + C_1(1+\varepsilon^2|||(V,N_e)|||_\varepsilon^2)(\varepsilon\|N_e\|_{H^\kappa}^2), \end{aligned}$$

where we used the fact that

$$\left|\partial_t\left(\frac{\alpha\varepsilon}{nn_e^2}\right)\right| \le C\varepsilon^2(1+\varepsilon^2(|\partial_t N|+|\partial_t N_e|)),$$
$$\left|\partial_x\left(\frac{\alpha\varepsilon}{nn_e}\right)\right| \le C\varepsilon^2(1+\varepsilon^2(|\partial_x N|+|\partial_x N_e|)).$$

For the term I_{222}, by the Hölder inequality and Sobolev embedding inequality, we have

$$I_{222} \le C\varepsilon^2(\|\partial_x^\kappa N_e\|^2+\|\partial_x^{\kappa+2}N_e\|^2)+C\varepsilon^4\|\partial_x^\kappa N_e\|_{L^\infty}(\|\partial_t N_e\|^2+\|\partial_x^{\kappa+2}N_e\|^2).$$

For the term I_{223}, by the Hölder inequality and Sobolev embedding inequality, we have

$$I_{223} \le C_1(1+\varepsilon^2|||(V,N_e)|||_\varepsilon^2)(\varepsilon\|N_e\|_{H^\kappa}^2).$$

Therefore, we have the estimate

$$I_{22} \le -\frac{1}{2}\frac{d}{dt}\int\frac{\alpha\varepsilon}{nn_e^2}|\partial_x^{\kappa+1}N_e|^2+C_1(1+\varepsilon^2|||(V,N_e)|||_\varepsilon^2)(\varepsilon\|N_e\|_{H^\kappa}^2). \tag{2.39}$$

For the estimate of I_{23}, by the Hölder inequality and Sobolev embedding inequality, we have

$$I_{23} \le C\varepsilon^2(\|\partial_x^\kappa N_e\|^2+\varepsilon\|\partial_x^{\kappa+1}N_e\|^2)+C\varepsilon^5\|\partial_x^\kappa N_e\|_{L^\infty}\|\partial_t N_e\|^2+C(\varepsilon^2\|\partial_x^\kappa N_e\|^2+\|\varepsilon\partial_t\partial_x^{\kappa+1}N_e\|^2). \tag{2.40}$$

Regarding the estimate of I_{24}, by Lemma 1.1, we have

$$I_{24} \le C\varepsilon\|\partial_x^\kappa N_e\|^2. \tag{2.41}$$

Combining (2.38), (2.39), (2.40) and (2.41), we complete this lemma.

2.2. Third order estimates

Proposition 2.2. *Let (N,V,N_e) be a solution to (1.16); then,*

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\varepsilon\|\partial_x^3V\|^2)+\frac{1}{2}\frac{d}{dt}\left(\int\frac{\varepsilon}{nn_e}|\partial_x^3N_e|^2+\int\frac{\alpha\varepsilon^2}{n_e^2n}|\partial_x^4N_e|^2\right)\\&\le C_1(1+\varepsilon^2|||(V,N_e)|||_\varepsilon^2)(1+|||(V,N_e)|||_\varepsilon^2).\end{aligned} \tag{2.42}$$

Proof. Taking ∂_x^3 of (1.16b) and then taking the inner product with $\varepsilon\partial_x^3V$, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\varepsilon\|\partial_x^3V\|^2)-\int\frac{\lambda_0}{\varepsilon}\partial_x^4V\varepsilon\partial_x^3V+\int\frac{1}{\varepsilon}\partial_x^3(v\partial_xV)\varepsilon\partial_x^3V+\int\partial_x^3(\partial_x\tilde{v}V)\varepsilon\partial_x^3V-\int\partial_x^3\left(\frac{\beta}{n}\partial_x^2V\right)\varepsilon\partial_x^3V\\&\quad+\int\varepsilon\partial_x^3\mathcal{R}_2\varepsilon\partial_x^3V=\int\partial_x^2\left(\frac{1}{n_e}\partial_xN_e\right)\partial_x^4V.\end{aligned} \tag{2.43}$$

Estimate the left-hand side of (2.43). The second term vanishes by integration by parts. For the third term, by the Hölder inequality and Sobolev embedding inequality, we get

$$\begin{aligned}\int\frac{1}{\varepsilon}\partial_x^3(v\partial_xV)\varepsilon\partial_x^3V&=\frac{1}{2}\int\varepsilon\partial_x(\tilde{v}+\varepsilon^2V)|\partial_x^3V|^2+\int\varepsilon\partial_x^2(\tilde{v}+\varepsilon^2V)\partial_x^2V\partial_x^3V+\int\varepsilon\partial_x^3(\tilde{v}+\varepsilon^2V)\partial_xV\partial_x^3V\\&\le C(1+\varepsilon^2\|\partial_xV\|_{L^\infty}+\varepsilon^2\|\partial_x^2V\|_{L^\infty}+\varepsilon^2\|\partial_xV\|_{L^\infty})(\varepsilon\|\partial_x^2V\|^2+\varepsilon\|\partial_x^3V\|^2)\\&\le C_1(1+\varepsilon^2|||(V,N_e)|||_\varepsilon^2)(1+|||(V,N_e)|||_\varepsilon^2).\end{aligned}$$

Regarding the estimate for the fourth term, similar to the third term, we have

$$\int \partial_x^3(\partial_x \tilde{v} V)\varepsilon\partial_x^3 V \leq C\varepsilon(1 + \|V\|_{H^3}^2).$$

Regarding the estimate for the fifth term, we have the following decomposition:

$$\begin{aligned}\int \partial_x^3\Big(\frac{\beta}{n}\partial_x^2 V\Big)\varepsilon\partial_x^3 V &= \beta\varepsilon\int \frac{1}{n}\partial_x^5 V\partial_x^3 V + \beta\varepsilon\int \partial_x\Big(\frac{1}{n}\Big)\partial_x^4 V\partial_x^3 V + \beta\varepsilon\int \partial_x^2\Big(\frac{1}{n}\Big)\partial_x^3 V\partial_x^3 V\\ &\quad + \beta\varepsilon\int \partial_x^3\Big(\frac{1}{n}\Big)\partial_x^2 V\partial_x^3 V =: E_1 + E_2 + E_3 + E_4.\end{aligned}$$

Integration by parts twice yields

$$\begin{aligned}E_1 &= \frac{\beta}{2}\varepsilon\int \partial_x^2\Big(\frac{1}{n}\Big)|\partial_x^3 V|^2 - \beta\varepsilon\int \frac{1}{n}|\partial_x^4 V|^2\\ &\leq C\varepsilon\|\partial_x^4 V\|^2 + C\varepsilon^2\|\partial_x^3 V\|^2 + C\varepsilon^3\|\partial_x^3 V\|_{L^\infty}(\|\partial_x^2 N\|^2 + \varepsilon\|\partial_x^3 V\|^2).\end{aligned}$$

Through by parts, and by using the Sobolev embedding inequality, we have

$$E_2 + E_3 = \frac{\beta}{2}\varepsilon\int \partial_x^2\Big(\frac{1}{n}\Big)\partial_x^3 V\partial_x^3 V \leq C\varepsilon^2(1 + \varepsilon(\|\partial_x^2 N\|^2 + \varepsilon\|\partial_x^3 V\|^2))\|\partial_x^3 V\|^2.$$

Similarly, through integration by parts, and by using the Sobolev embedding inequality, we have

$$\begin{aligned}E_4 &= -\beta\varepsilon\int \partial_x^2\Big(\frac{1}{n}\Big)|\partial_x^3 V|^2 - \beta\varepsilon\int \partial_x^2\Big(\frac{1}{n}\Big)\partial_x^2 V\partial_x^4 V\\ &\leq C\varepsilon^2(1 + \varepsilon(\|\partial_x^2 N\|^2 + \varepsilon\|\partial_x^3 V\|^2))\|\partial_x^3 V\|^2 + C\varepsilon^2\|\partial_x^4 V\|^2 + C\varepsilon^2\|\partial_x^2 V\|_{L^\infty}(\|\partial_x^2 N\|^2 + \varepsilon^2\|\partial_x^4 V\|^2).\end{aligned}$$

For the last term, by Lemma 1.1, we have

$$\int \varepsilon\partial_x^3\mathcal{R}_2\varepsilon\partial_x^3 V \leq C\varepsilon^2(1 + \|V\|_{H^3}^2).$$

Estimate the right-hand side of (2.43). Taking ∂_x^3 of (1.16a) and inserting it into the right-hand side of (2.43) gives

$$\begin{aligned}\sum_{i=1}^{6}\int \partial_x^2\Big(\frac{1}{n_e}\partial_x N_e\Big)\frac{1}{n}\Big(&\partial_x^3((\lambda_0 - \nu)\partial_x N) - \varepsilon\partial_t\partial_x^3 N - \sum_{\gamma=1}^{2} C_3^\gamma\partial_x^{3-\gamma} n\partial_x^{\gamma+1} V - \varepsilon\partial_x^3(\partial_x\tilde{n}V)\\ &- \varepsilon\partial_x^3(\tilde{v}N) - \varepsilon^2\partial_x^3\mathcal{R}_1\Big) =: \sum_{i=1}^{6}\tilde{I}_i.\end{aligned}$$

We first give the estimate of $\tilde{I}_i$ for $3 \leq i \leq 6$. For $\tilde{I}_3$, we have the following decomposition:

$$\begin{aligned}\tilde{I}_3 &= -\sum_{\gamma=1}^{3} C_3^\gamma\int \partial_x^2\Big(\frac{1}{n_e}\partial_x N_e\Big)\frac{1}{n}\partial_x^\gamma\varepsilon\tilde{n}\partial_x^{4-\gamma} V - \sum_{\gamma=1}^{3} C_3^\gamma\int \partial_x^2\Big(\frac{1}{n_e}\partial_x N_e\Big)\frac{1}{n}\partial_x^\gamma\varepsilon^3 N\partial_x^{4-\gamma} V\\ &=: \tilde{I}_{31} + \tilde{I}_{32}.\end{aligned}$$

Regarding the estimate for $\tilde{I}_{31}$, by the Hölder inequality and Sobolev embedding, we have

$$\begin{aligned}\tilde{I}_{31} =& -\sum_{\gamma=1}^{3} C_3^\gamma \varepsilon \int \Big(\frac{1}{n_e}\partial_x^3 N_e\Big)\frac{1}{n}\partial_x^\gamma \tilde{n}\partial_x^{4-\gamma} V - \sum_{\gamma=1}^{3} C_3^\gamma \varepsilon \int \partial_x\Big(\frac{1}{n_e}\Big)\frac{1}{n}\partial_x^2 N_e \partial_x^\gamma \tilde{n}\partial_x^{4-\gamma} V\\ &- \sum_{\gamma=1}^{3} C_3^\gamma \varepsilon \int \partial_x^2\Big(\frac{1}{n_e}\Big)\frac{1}{n}\partial_x N_e \partial_x^\gamma \tilde{n}\partial_x^{4-\gamma} V\\ &\le C(\varepsilon^2\|\partial_x^3 N_e\|^2 + \varepsilon^2\|V\|_{H^3}^2 + \varepsilon^2\|\partial_x N_e\|^2)(\|\partial_x N_e\|_{L^\infty} + \|\partial_x^2 N_e\|_{L^\infty}).\end{aligned}$$

Regarding the estimate for $\tilde{I}_{32}$, through integration by parts, and by using the Hölder inequality and Sobolev embedding, we have

$$\tilde{I}_{32} \le C_1(1 + |||(V, N_e)|||_\varepsilon^2)|||(V, N_e)|||_\varepsilon^2.$$

Via a similar calculation process, we have

$$\tilde{I}_{4,5} \le C(1 + \varepsilon^3\|\partial_x^2 N_e\|_{L^\infty})(\varepsilon\|N_e\|_{H^3}^2 + \varepsilon\|V\|_{H^3}^2).$$

By Lemma 1.1, we have

$$\tilde{I}_6 \le C\varepsilon(\varepsilon^2\|N_e\|_{H^3}^2).$$

Lemma 2.6. *Let (N, V, N_e) be a solution to (1.16); we have*

$$\tilde{I}_1 \le C_1(1 + \varepsilon^2|||(V, N_e)|||_\varepsilon^2)(1 + |||(V, N_e)|||_\varepsilon^2). \tag{2.44}$$

Proof. We have the following decomposition:

$$\begin{aligned}\tilde{I}_1 =& \int \partial_x^2\Big(\frac{1}{n_e}\partial_x N_e\Big)\frac{\lambda_0 - v}{n}\partial_x^4 N - \int \partial_x^2\Big(\frac{1}{n_e}\partial_x N_e\Big)\frac{1}{n}\partial_x(\varepsilon\tilde{v} + \varepsilon^3 V)\partial_x^3 N\\ &- \int \partial_x^2\Big(\frac{1}{n_e}\partial_x N_e\Big)\frac{1}{n}[\partial_x^3(\varepsilon\tilde{v} + \varepsilon^3 V)\partial_x N + \partial_x^2(\varepsilon\tilde{v} + \varepsilon^3 V)\partial_x^2 N]\\ =&: \tilde{I}_{11} + \tilde{I}_{12} + \tilde{I}_{13}\end{aligned}$$

Regarding the estimate for $\tilde{I}_{11}$, taking ∂_x^4 of (1.16c) and inserting it into $\tilde{I}_{11}$, we have

$$\begin{aligned}\tilde{I}_{11} =& \int \partial_x^2\Big(\frac{1}{n_e}\partial_x N_e\Big)\frac{\lambda_0 - v}{n}\Big(\partial_x^4 N_e - \partial_x^4\Big(\frac{\alpha\varepsilon}{n_e}\partial_x^2 N_e\Big) + \partial_x^4\Big(\frac{2\alpha}{n_e^2}\varepsilon^2\partial_x\tilde{n}_e\partial_x N_e\Big) - \varepsilon\partial_x^4\mathcal{R}_3\Big)\\ =&: \sum_{i=1}^{4}\tilde{I}_{11i}\end{aligned}$$

Regarding the estimate for $\tilde{I}_{111}$, by the Hölder inequality and Sobolev embedding, we have

$$\tilde{I}_{111} \le C_1(1 + \varepsilon^2|||(V, N_e)|||_\varepsilon^2)(1 + |||(V, N_e)|||_\varepsilon^2),$$

where we used the fact that

$$\left|\partial_x\left(\frac{\lambda_0 - v}{n}\right)\right| \leq C\varepsilon(1 + \varepsilon^2|\partial_x N| + \varepsilon^2|\partial_x V| + \varepsilon^2|\partial_x N_e|).$$

Regarding the estimate for $\tilde{I}_{112}$ and $\tilde{I}_{113}$, through integration by parts, using the Hölder inequality and Sobolev embedding, we have

$$\tilde{I}_{112,3} \leq C_1(1 + \varepsilon^2 |||(V, N_e)|||_\varepsilon^2)(1 + |||(V, N_e)|||_\varepsilon^2).$$

By Lemma 1.1, we have

$$\tilde{I}_{114} \leq C\varepsilon(\varepsilon\|\partial_x^3 N_e\|^2 + \varepsilon\|\partial_x^2 N_e\|^2 + \varepsilon\|\partial_x N_e\|^2).$$

Therefore, we have

$$\tilde{I}_{11} \leq C_1(1 + \varepsilon^2 |||(V, N_e)|||_\varepsilon^2)(1 + |||(V, N_e)|||_\varepsilon^2).$$

Regarding the estimate for $\tilde{I}_{12}$, through integration by parts, and by using the the Hölder inequality and Sobolev embedding, we have

$$\begin{aligned}\tilde{I}_{12} = &\int \partial_x^3\left(\frac{1}{n_e}\partial_x N_e\right)\frac{1}{n}\partial_x(\varepsilon\tilde{v} + \varepsilon^3 V)\partial_x^2 N + \int \partial_x^2\left(\frac{1}{n_e}\partial_x N_e\right)\partial_x\left(\frac{1}{n}\right)\partial_x(\varepsilon\tilde{v} + \varepsilon^3 V)\partial_x^2 N \\ &+ \int \partial_x^2\left(\frac{1}{n_e}\partial_x N_e\right)\frac{1}{n}\partial_x^2(\varepsilon\tilde{v} + \varepsilon^3 V)\partial_x^2 N \\ \leq\ & C_1(1 + \varepsilon^2 |||(V, N_e)|||_\varepsilon^2)(1 + |||(V, N_e)|||_\varepsilon^2).\end{aligned}$$

Regarding the estimate for $\tilde{I}_{13}$, through integration by parts, and by using the Hölder inequality and Sobolev embedding, we have

$$\tilde{I}_{13} \leq C_1(1 + \varepsilon^2 |||(V, N_e)|||_\varepsilon^2)(1 + |||(V, N_e)|||_\varepsilon^2).$$

Lemma 2.7. *Let (N, V, N_e) be a solution to (1.16); we have*

$$\tilde{I}_2 \leq -\frac{1}{2}\frac{d}{dt}\int \frac{\varepsilon}{nn_e}|\partial_x^3 N_e|^2 - \frac{1}{2}\frac{d}{dt}\int \frac{\alpha\varepsilon^2}{n_e^2 n}|\partial_x^4 N_e|^2 + +C_1(1 + \varepsilon^2 |||(V, N_e)|||_\varepsilon^2)(1 + |||(V, N_e)|||_\varepsilon^2). \tag{2.45}$$

Proof. Taking $\partial_t\partial_x^3$ of (1.16c), and then inserting the result in $\tilde{I}_2$, we have

$$\tilde{I}_2 = -\int \partial_x^2\left(\frac{1}{n_e}\partial_x N_e\right)\frac{\varepsilon}{n}\left(\partial_t\partial_x^3 N_e - \partial_t\partial_x^3\left(\frac{\alpha\varepsilon}{n_e}\partial_x^2 N_e\right) + \partial_t\partial_x^3\left(\frac{2\alpha}{n_e^2}\varepsilon^2\partial_x\tilde{n}_e\partial_x N_e\right) - \varepsilon\partial_t\partial_x^3\mathcal{R}_3\right) =: \sum_{i=1}^4 \tilde{I}_{2i}.$$

Regarding the estimate for $\tilde{I}_{21}$, through integration by parts, we have

$$\begin{aligned}\tilde{I}_{21} = &-\frac{1}{2}\frac{d}{dt}\int \frac{\varepsilon}{nn_e}|\partial_x^3 N_e|^2 + \frac{1}{2}\int \partial_t\left(\frac{\varepsilon}{nn_e}\right)|\partial_x^3 N_e|^2 - \int \partial_x\left(\frac{1}{n_e}\right)\partial_x^2 N_e\frac{\varepsilon}{n}\partial_t\partial_x^3 N_e \\ &- \int \partial_x^2\left(\frac{1}{n_e}\right)\partial_x N_e\frac{\varepsilon}{n}\partial_t\partial_x^3 N_e =: -\frac{1}{2}\frac{d}{dt}\int \frac{\varepsilon}{nn_e}|\partial_x^3 N_e|^2 + \sum_{i=1}^3 \tilde{I}_{21i}.\end{aligned}$$

For $\tilde{I}_{211}$, by using the Hölder inequality and Sobolev embedding, we have

$$\tilde{I}_{211} \le C\varepsilon(1+\varepsilon^2\|\partial_t N\|_{L^\infty}+\varepsilon^2\|\partial_t N_e\|_{L^\infty})(\varepsilon\|\partial_x^3 N_e\|^2).$$

For $\tilde{I}_{212}$, through integration by parts, by using the Hölder inequality and Sobolev embedding, we have

$$\begin{aligned}\tilde{I}_{212} &= \int \partial_x^2\Big(\frac{1}{n_e}\Big)\partial_x^2 N_e \frac{\varepsilon}{n}\partial_t\partial_x^2 N_e + \int \partial_x\Big(\frac{1}{n_e}\Big)\partial_x^3 N_e \frac{\varepsilon}{n}\partial_t\partial_x^2 N_e + \int \partial_x\Big(\frac{1}{n_e}\Big)\partial_x^2 N_e \partial_x(\frac{\varepsilon}{n})\partial_t\partial_x^2 N_e \\ &\le C\varepsilon(1+\varepsilon^3\|\partial_x^2 N\|_{L^\infty}+\varepsilon\|\partial_x N\|_{L^\infty}+\varepsilon\|\partial_x N_e\|_{L^\infty})(\|\partial_x^2 N_e\|^2+\varepsilon\|\partial_x^3 N_e\|^2+\varepsilon\|\partial_t\partial_x^2 N_e\|^2).\end{aligned}$$

For $\tilde{I}_{213}$, it is similar to $\tilde{I}_{212}$:

$$\begin{aligned}\tilde{I}_{213} &= \int \partial_x^3\Big(\frac{1}{n_e}\Big)\partial_x N_e \frac{\varepsilon}{n}\partial_t\partial_x^2 N_e + \int \partial_x^2\Big(\frac{1}{n_e}\Big)\partial_x^2 N_e \frac{\varepsilon}{n}\partial_t\partial_x^2 N_e + \int \partial_x^2\Big(\frac{1}{n_e}\Big)\partial_x N_e \partial_x\Big(\frac{\varepsilon}{n}\Big)\partial_t\partial_x^2 N_e \\ &\le C\varepsilon(1+\varepsilon^2\|\partial_x^3 N_e\|_{L^\infty}+\varepsilon^2\|\partial_x^2 N_e\|_{L^\infty}+\varepsilon\|\partial_x N\|_{L^\infty})(\varepsilon\|\partial_t\partial_x^2 N_e\|^2+\|\partial_x N_e\|^2+\|\partial_x^2 N_e\|^2).\end{aligned}$$

Therefore, we have

$$\tilde{I}_{21} \le -\frac{1}{2}\frac{d}{dt}\int \frac{\varepsilon}{nn_e}|\partial_x^3 N_e|^2 + C_1(1+\varepsilon^2|||(V,N_e)|||_\varepsilon^2)(1+|||(V,N_e)|||_\varepsilon^2).$$

Regarding the estimate for $\tilde{I}_{22}$, we have the following decomposition:

$$\begin{aligned}\tilde{I}_{22} &= -\int \frac{\varepsilon}{n_e n}\partial_x^4 N_e \partial_t\partial_x^2\Big(\frac{\alpha\varepsilon}{n_e}\partial_x^2 N_e\Big) - \int \partial_x\Big(\frac{\varepsilon}{n_e n}\Big)\partial_x^3 N_e \partial_t\partial_x^2\Big(\frac{\alpha\varepsilon}{n_e}\partial_x^2 N_e\Big) \\ &\quad + \int \partial_x\Big(\frac{1}{n_e}\Big)\partial_x^2 N_e \frac{\varepsilon}{n}\partial_t\partial_x^3\Big(\frac{\alpha\varepsilon}{n_e}\partial_x^2 N_e\Big) + \int \partial_x^2\Big(\frac{1}{n_e}\Big)\partial_x N_e \frac{\varepsilon}{n}\partial_t\partial_x^3\Big(\frac{\alpha\varepsilon}{n_e}\partial_x^2 N_e\Big) =: \sum_{i=1}^{4}\tilde{I}_{22i}.\end{aligned}$$

For the term $\tilde{I}_{221}$, through integration by parts,we have

$$\begin{aligned}\tilde{I}_{221} &= -\frac{1}{2}\frac{d}{dt}\int \frac{\alpha\varepsilon^2}{n_e^2 n}|\partial_x^4 N_e|^2 + \frac{1}{2}\int \partial_t\Big(\frac{\alpha\varepsilon^2}{n_e^2 n}\Big)|\partial_x^4 N_e|^2 - \int \frac{\varepsilon}{n_e n}\partial_x^4 N_e \partial_t\Big(\frac{\alpha\varepsilon}{n_e}\Big)\partial_x^4 N_e \\ &\quad - \int \frac{\varepsilon}{n_e n}\partial_x^4 N_e \partial_t\partial_x\Big(\frac{\alpha\varepsilon}{n_e}\Big)\partial_x^3 N_e - \int \frac{\varepsilon}{n_e n}\partial_x^4 N_e \partial_x\Big(\frac{\alpha\varepsilon}{n_e}\Big)\partial_t\partial_x^3 N_e - \int \frac{\varepsilon}{n_e n}\partial_x^4 N_e \partial_t\partial_x^2\Big(\frac{\alpha\varepsilon}{n_e}\Big)\partial_x^2 N_e \\ &\quad - \int \frac{\varepsilon}{n_e n}\partial_x^4 N_e \partial_x^2\Big(\frac{\alpha\varepsilon}{n_e}\Big)\partial_t\partial_x^2 N_e \\ &=: -\frac{1}{2}\frac{d}{dt}\int \frac{\alpha\varepsilon^2}{n_e^2 n}|\partial_x^4 N_e|^2 + \sum_{i=1}^{6}\tilde{I}_{221i}.\end{aligned}$$

By using the Hölder inequality and Sobolev embedding, we have

$$\tilde{I}_{2211,2,3} \le C\varepsilon(1+\varepsilon^2\|\partial_t N_e\|^2+\varepsilon^2\|\partial_t N\|^2+\varepsilon^2\|\partial_t\partial_x N_e\|_{L^\infty}+\varepsilon^3\|\partial_x N_e\|_{L^\infty})(\varepsilon^2\|\partial_x^4 N_e\|^2+\varepsilon\|\partial_x^3 N_e\|^2).$$

Regarding the term $\tilde{I}_{224}$, due to the term $\partial_t\partial_x^3 N_e$, even if we raise the Sobolev order or the expansion order, it cannot be controlled in terms of $|||(V,N_e)|||_\varepsilon$, the essential reason is related to Lemmas 2.2 and

2.3. But, by observing its features, we can use (1.16c) to complete this estimate. Simple calculation gives

$$\tilde{I}_{2214} = -\int \frac{\varepsilon}{n}\partial_x^4 N_e \partial_x\Big(\frac{1}{n_e}\Big)\partial_t\partial_x\Big(\frac{\alpha\varepsilon}{n_e}\partial_x^2 N_e - N_e\Big) + \int \frac{\varepsilon}{n}\partial_x^4 N_e \partial_x\Big(\frac{1}{n_e}\Big)\partial_t\partial_x N_e$$
$$+ \int \frac{\varepsilon}{n}\partial_x^4 N_e \partial_x\Big(\frac{1}{n_e}\Big)\Big(\partial_x\partial_t\Big(\frac{\alpha\varepsilon}{n_e}\Big)\partial_x^2 N_e + \partial_t\Big(\frac{\alpha\varepsilon}{n_e}\Big)\partial_x^3 N_e + \partial_x\Big(\frac{\alpha\varepsilon}{n_e}\Big)\partial_t\partial_x^2 N_e\Big);$$

by using (1.16c), we have

$$\frac{\alpha\varepsilon}{n_e}\partial_x^2 N_e - N_e = \frac{2\alpha}{n_e^2}\varepsilon^2\partial_x\tilde{n}_e\partial_x N_e - \varepsilon\mathcal{R}_3 - N.$$

Inserting it into $\tilde{I}_{2214}$, and using the Hölder inequality and Sobolev embedding, we have

$$\begin{aligned}\tilde{I}_{2214} &= -\int \frac{\varepsilon}{n}\partial_x^4 N_e \partial_x\Big(\frac{1}{n_e}\Big)\partial_t\partial_x\Big(\frac{2\alpha}{n_e^2}\varepsilon^2\partial_x\tilde{n}_e\partial_x N_e - \varepsilon\mathcal{R}_3 - N\Big)\\ &\quad + \int \frac{\varepsilon}{n}\partial_x^4 N_e \partial_x\Big(\frac{1}{n_e}\Big)\Big(\partial_x\partial_t\Big(\frac{\alpha\varepsilon}{n_e}\Big)\partial_x^2 N_e + \partial_t\Big(\frac{\alpha\varepsilon}{n_e}\Big)\partial_x^3 N_e + \partial_x\Big(\frac{\alpha\varepsilon}{n_e}\Big)\partial_t\partial_x^2 N_e\Big)\\ &\quad + \int \frac{\varepsilon}{n}\partial_x^4 N_e \partial_x\Big(\frac{1}{n_e}\Big)\partial_t\partial_x N_e\\ &\le C_1(1+\varepsilon^2 |||(V,N_e)|||_\varepsilon^2)(1+|||(V,N_e)|||_\varepsilon^2).\end{aligned}$$

Regarding the terms $\tilde{I}_{2215}$ and $\tilde{I}_{2216}$, by using the Hölder inequality, we obtain

$$\tilde{I}_{2215} + \tilde{I}_{2216} \le C(1+\varepsilon^2\|\partial_x^2 N_e\|_{L^\infty})(\varepsilon^2\|\partial_x^4 N_e\|^2 + \varepsilon\|\partial_x^2 N_e\|^2 + \varepsilon\|\partial_t\partial_x^2 N_e\|^2).$$

Therefore, we have

$$\tilde{I}_{221} \le -\frac{1}{2}\frac{d}{dt}\int \frac{\alpha\varepsilon^2}{n_e^2 n}|\partial_x^4 N_e|^2 + C_1(1+\varepsilon^2 |||(V,N_e)|||_\varepsilon^2)(1+|||(V,N_e)|||_\varepsilon^2).$$

Regarding the estimate of $\tilde{I}_{222}$, by using the Hölder inequality and Sobolev embedding, we have

$$\tilde{I}_{222} \le C_1(1+\varepsilon^2 |||(V,N_e)|||_\varepsilon^2)(1+|||(V,N_e)|||_\varepsilon^2) - \alpha\varepsilon^2\int \partial_x\Big(\frac{1}{n_e n}\Big)\frac{1}{n_e}\partial_x^3 N_e\partial_t\partial_x^4 N_e + \mathcal{F}_1,$$

where we note that

$$\mathcal{F}_1 =: -\alpha\varepsilon^2\int \partial_x\Big(\frac{1}{n_e n}\Big)\frac{1}{n_e}\partial_x^3 N_e\partial_t\partial_x^4 N_e.$$

The difficulty is that the term $\partial_t\partial_x^4 N_e$ cannot be controlled in terms of $|||(V,N_e)|||_\varepsilon$; we can use (1.16c) to complete this estimate. Simple calculation gives

$$\begin{aligned}\mathcal{F}_1 &= = -\varepsilon\int \partial_x\Big(\frac{1}{n_e n}\Big)\partial_x^3 N_e\partial_t\partial_x^2\Big(\frac{\alpha\varepsilon}{n_e}\partial_x^2 N_e - N_e\Big) - \varepsilon\int \partial_x\Big(\frac{1}{n_e n}\Big)\partial_x^3 N_e\partial_t\partial_x^2 N_e\\ &\quad + \varepsilon\int \partial_x\Big(\frac{1}{n_e n}\Big)\partial_x^3 N_e\Big(\partial_t\partial_x^2\Big(\frac{\alpha\varepsilon}{n_e}\Big)\partial_x^2 N_e + \partial_x^2\Big(\frac{\alpha\varepsilon}{n_e}\Big)\partial_t\partial_x^2 N_e + \partial_t\partial_x\Big(\frac{\alpha\varepsilon}{n_e}\Big)\partial_x^3 N_e\\ &\quad + \partial_x\Big(\frac{\alpha\varepsilon}{n_e}\Big)\partial_t\partial_x^3 N_e + \partial_t\Big(\frac{\alpha\varepsilon}{n_e}\Big)\partial_x^4 N_e\Big);\end{aligned}$$

by using (1.16c), through integration by parts, by using the Hölder inequality and Sobolev embedding, we have

$$\begin{aligned}\mathcal{F}_1 &= -\varepsilon \int \partial_x\Big(\frac{1}{n_e n}\Big)\partial_x^3 N_e \partial_t \partial_x^2\Big(\frac{2\alpha}{n_e^2}\varepsilon^2 \partial_x \tilde{n}_e \partial_x N_e - \varepsilon \mathcal{R}_3 - N\Big) - \varepsilon \int \partial_x\Big(\frac{1}{n_e n}\Big)\partial_x^3 N_e \partial_t \partial_x^2 N_e \\ &\quad + \varepsilon \int \partial_x\Big(\frac{1}{n_e n}\Big)\partial_x^3 N_e \Big(\partial_t \partial_x^2\Big(\frac{\alpha\varepsilon}{n_e}\Big)\partial_x^2 N_e + \partial_x^2\Big(\frac{\alpha\varepsilon}{n_e}\Big)\partial_t \partial_x^2 N_e + \partial_t \partial_x\Big(\frac{\alpha\varepsilon}{n_e}\Big)\partial_x^3 N_e \\ &\quad + \partial_x\Big(\frac{\alpha\varepsilon}{n_e}\Big)\partial_t \partial_x^3 N_e + \partial_t\Big(\frac{\alpha\varepsilon}{n_e}\Big)\partial_x^4 N_e\Big) \\ &\le C_1(1+\varepsilon^2 |||(V,N_e)|||_\varepsilon^2)(1+|||(V,N_e)|||_\varepsilon^2).\end{aligned}$$

Regarding the estimate of $\tilde{I}_{223}$, through integration by parts, and by using the Hölder inequality and Sobolev embedding, we have

$$\begin{aligned}\tilde{I}_{223} &= -\alpha\varepsilon^2 \int \partial_x^2\Big(\frac{1}{n_e}\Big)\partial_x^2 N_e \frac{1}{n}\partial_t \partial_x^2\Big(\frac{1}{n_e}\partial_x^2 N_e\Big) - \alpha\varepsilon^2 \int \partial_x\Big(\frac{1}{n_e}\Big)\partial_x^3 N_e \frac{1}{n}\partial_t \partial_x^2\Big(\frac{1}{n_e}\partial_x^2 N_e\Big) \\ &\quad - \alpha\varepsilon^2 \int \partial_x^2\Big(\frac{1}{n_e}\Big)\partial_x^2 N_e \partial_x\Big(\frac{1}{n}\Big)\partial_t \partial_x^2\Big(\frac{1}{n_e}\partial_x^2 N_e\Big) \\ &\le C_1(1+\varepsilon^2 |||(V,N_e)|||_\varepsilon^2)(1+|||(V,N_e)|||_\varepsilon^2) + \mathcal{F}_2,\end{aligned}$$

where we note that

$$\mathcal{F}_2 =: -\alpha\varepsilon^2 \int \partial_x\Big(\frac{1}{n_e}\Big)\partial_x^3 N_e \frac{1}{n}\frac{1}{n_e}\partial_t \partial_x^4 N_e;$$

by using (1.16c), we have

$$\begin{aligned}\mathcal{F}_2 &= -\varepsilon \int \partial_x\Big(\frac{1}{n_e}\Big)\partial_x^3 N_e \frac{1}{n}\frac{\alpha\varepsilon}{n_e}\partial_t \partial_x^2 N_e \Big(\frac{2\alpha}{n_e^2}\varepsilon^2 \partial_x \tilde{n}_e \partial_x N_e - \varepsilon \mathcal{R}_3 - N\Big) - \varepsilon \int \partial_x\Big(\frac{1}{n_e}\Big)\partial_x^3 N_e \frac{1}{n}\partial_t \partial_x^2 N_e \\ &\quad + \varepsilon \int \partial_x\Big(\frac{1}{n_e}\Big)\partial_x^3 N_e \frac{1}{n}\Big(\partial_t \partial_x^2\Big(\frac{\alpha\varepsilon}{n_e}\Big)\partial_x^2 N_e + \partial_x^2\Big(\frac{\alpha\varepsilon}{n_e}\Big)\partial_t \partial_x^2 N_e + \partial_t \partial_x\Big(\frac{\alpha\varepsilon}{n_e}\Big)\partial_x^3 N_e \\ &\quad + \partial_t \partial_x\Big(\frac{\alpha\varepsilon}{n_e}\Big)\partial_t \partial_x^3 N_e + \partial_t\Big(\frac{\alpha\varepsilon}{n_e}\Big)\partial_x^4 N_e\Big) \\ &\le C_1(1+\varepsilon^2 |||(V,N_e)|||_\varepsilon^2)(1+|||(V,N_e)|||_\varepsilon^2);\end{aligned}$$

therefore, we have

$$\tilde{I}_{223} \le C_1(1+\varepsilon^2 |||(V,N_e)|||_\varepsilon^2)(1+|||(V,N_e)|||_\varepsilon^2).$$

Regarding the estimate of $\tilde{I}_{224}$, through integration by parts, and by using the Hölder inequality and Sobolev embedding, we have

$$\tilde{I}_{224} \le C_1(1+\varepsilon^2 |||(V,N_e)|||_\varepsilon^2)(1+|||(V,N_e)|||_\varepsilon^2) + \mathcal{F}_3,$$

where we note that

$$\mathcal{F}_3 =: \alpha\varepsilon^2 \int \partial_x^4\Big(\frac{1}{n_e}\Big)\partial_x N_e \frac{1}{n}\frac{1}{n_e}\partial_t \partial_x^3 N_e;$$

by using (1.16c) we have

$$\begin{aligned}\mathcal{F}_3 &= \varepsilon \int \partial_x^4\Big(\frac{1}{n_e}\Big)\partial_x N_e \frac{1}{n}\partial_t \partial_x\Big(\frac{2\alpha}{n_e^2}\varepsilon^2 \partial_x \tilde{n}_e \partial_x N_e - \varepsilon \mathcal{R}_3 - N\Big) + \varepsilon \int \partial_x^4\Big(\frac{1}{n_e}\Big)\partial_x N_e \frac{1}{n}\partial_t \partial_x^2 N_e \\ &\le C_1(1+\varepsilon^2 |||(V,N_e)|||_\varepsilon^2)(1+|||(V,N_e)|||_\varepsilon^2).\end{aligned}$$

Therefore, we have

$$\tilde{I}_{22} \leq -\frac{1}{2}\frac{d}{dt}\int \frac{\alpha\varepsilon^2}{n_e^2 n}|\partial_x^4 N_e|^2 + C_1(1+\varepsilon^2|||(V,N_e)|||_\varepsilon^2)(1+|||(V,N_e)|||_\varepsilon^2).$$

Regarding the estimate for $\tilde{I}_{23}$, similar to $\tilde{I}_{22}$, we have

$$\tilde{I}_{23} \leq C_1(1+\varepsilon^2|||(V,N_e)|||_\varepsilon^2)(1+|||(V,N_e)|||_\varepsilon^2).$$

By Lemma 1.1, we have

$$\tilde{I}_{24} \leq C\varepsilon(\varepsilon\|\partial_x^3 N_e\|^2 + \varepsilon^2\|\partial_x^2 N_e\|^2 + \varepsilon^2\|\partial_x N_e\|^2).$$

Combining these estimates, we complete this lemma.

3. Conclusions

Proof of Theorem 1.3. Combining Propositions 2.1 and 2.2, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|V\|_{H^2}^2 + \varepsilon\|\partial_x^3 V\|^2) + \frac{1}{2}\frac{d}{dt}\Big(\Big(\int \frac{1}{nn_e}|N_e|^2 + \int \frac{\alpha\varepsilon}{nn_e^2}|\partial_x N_e|^2\Big) + \Big(\int \frac{1}{nn_e}|\partial_x N_e|^2 + \int \frac{\alpha\varepsilon}{nn_e^2}|\partial_x^2 N_e|^2\Big) \\
&\quad + \Big(\int \frac{1}{nn_e}|\partial_x^2 N_e|^2 + \int \frac{\alpha\varepsilon}{nn_e^2}|\partial_x^3 N_e|^2\Big) + \Big(\int \frac{\varepsilon}{nn_e}|\partial_x^3 N_e|^2 + \int \frac{\alpha\varepsilon^2}{n_e^2 n}|\partial_x^4 N_e|^2\Big)\Big) \\
&\leq C_1(1+\varepsilon^2|||(V,N_e)|||_\varepsilon^2)(1+|||(V,N_e)|||_\varepsilon^2).
\end{aligned} \tag{3.1}$$

Integrating the inequality over $(0,t)$ yields

$$\begin{aligned}
|||(V,N_e)(t)|||_\varepsilon^2 &\leq C|||(V,N_e)(0)|||_\varepsilon^2 + \int_0^t C_1(1+|||(V,N_e)|||_\varepsilon^2)(1+|||(V,N_e)|||_\varepsilon^2)ds \\
&\leq C|||(V,N_e)(0)|||_\varepsilon^2 + \int_0^t C_1(1+\varepsilon\tilde{C})(1+|||(V,N_e)|||_\varepsilon^2)ds,
\end{aligned}$$

where C is an absolute constant.

Since C_1 is nondecreasing and depends on $|||(V,N_e)|||_\varepsilon^2$ through $\varepsilon|||(V,N_e)|||_\varepsilon^2$, let $C_1^* = c(1)$ and $C_2 > C sup_{\varepsilon<1}|||(V,N_e)(0)|||_\varepsilon^2$. For any arbitrarily given $\tau > 0$, we choose $\tilde{C}$ such that $\tilde{C} > e^{4C_1'}\tau(1+C_2)(1+C_1')$. Then there exists $\varepsilon_0 > 0$ such that $\varepsilon\tilde{C} \leq 1$ for all $\varepsilon < \varepsilon_0$; we have

$$\sup_{0\leq t\leq\tau} |||(V,N_e)(t)|||_\varepsilon^2 < \tilde{C}/2.$$

By Lemma 2.1, we have

$$\sup_{0\leq t\leq\tau} |||(N)(t)|||_{H^2}^2 \leq \tilde{C}/2.$$

By the Grönwall inequality, we complete the proof of Theorem 1.3.

Use of AI tools declaration

The authors declare that they have not used artificial intelligence tools in the creation of this article.

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Conflict of interest

All authors declare no conflict of interest that may influence the publication of this paper.

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