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Research article

Local well-posedness and blow-up criterion to a nonlinear shallow water wave equation

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Abstract: The initial data problem to a nonlinear shallow water wave equation in nonhomogeneous Besov space is discussed. Using the decomposition of Littlewood-Paley and the properties of nonhomogeneous Besov space, we establish the well-posedness of short time solutions for the equation in the Besov space. A blow-up criterion of solutions is also obtained.

Keywords: shallow water wave equation; local well-posedness; blow-up criterion; Besov space
Mathematics Subject Classification: 35Q35, 35Q51

1. Introduction

Consider the shallow water wave equation

$$W_t - W_{txx} + k_0 W_x + mWW_x = 3\alpha W_x W_{xx} + \alpha W W_{xxx}, \tag{1.1}$$

where constants $m > 0$, $\alpha > 0$ and $k_0 \in (-\infty, +\infty)$, $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$. The hydrodynamical models in Constantin and Lannes [1] includes Eq (1.1) as a special equation.

For Eq (1.1), we write out its Cauchy problem

$$\begin{cases} W_t - W_{txx} + k_0 W_x + mWW_x = 3\alpha W_x W_{xx} + \alpha W W_{xxx}, \\ W(0, x) = W_0(x), \end{cases} \tag{1.2}$$

which possesses the equivalent form

$$\begin{cases} W_t + \alpha W W_x = P(D)\left(-k_0 W + \dfrac{\alpha - m}{2} W^2\right), \\ W(0, x) = W_0(x), \end{cases} \tag{1.3}$$

where $P(D) = \partial_x \left(1 - \partial_x^2\right)^{-1}$. If $m = \alpha = \frac{3}{2}$ and $k_0 = -1$, Eq (1.1) is turned into the Fornberg-Whitham (FW) equation [2, 3]

$$W_t - W_{txx} - W_x + \frac{3}{2} W W_x = \frac{9}{2} W_x W_{xx} + \frac{3}{2} W W_{xxx}. \tag{1.4}$$

In 1967, Whitham [2] utilized the variational method to investigate water waves and wrote Eq (1.4) in the nonlocal form. Whitham and Fornberg [3] obtain that Eq (1.4) possesses the peaked solitary wave solution $W(x,t) = \frac{8}{9} e^{-\frac{1}{2}\left|x - \frac{4}{3}t\right|}$. Haziot [4] provides a concrete conditions imposing on the initial value to ensure occurrence of the wave breaking for Eq (1.4). Holmes [5] utilizes the Galerkin type approximation arguments to investigate that Eq (1.4) possesses short time solution in $H^s(\mathbb{R})$ associated with Sobolev index $s > \frac{3}{2}$ in the periodic case, and obtains that its solution is Hölder continuous in weak topology. Wu and Zhang [6] consider blow-up conditions to guarantee that the wave breaking of Eq (1.4) happens. Hörmann [7] finds discontinuous traveling waves of weak solutions to Eq (1.4). Based on the structure and conservation law of Eq (1.4), Yang [8] gives a sufficient condition to confirm the appearance of wave breaking for Eq (1.4).

If $m = 4, k_0 = 0$ and $\alpha = 1$, Eq (1.1) becomes the Degasperis-Procesi (DP) equation [9]

$$W_t - W_{txx} + 4 W W_x = 3 W_x W_{xx} + W W_{xxx}. \tag{1.5}$$

Mustafa [10] finds that smooth solutions of Eq (1.5) have infinite propagation speed. The DP model is integrable and possesses bi-Hamiltonian structure [11, 12]. Lundmark and Szmigielski [13] employ the inverse scattering technique to compute n-peakon solutions to Eq (1.5). The periodic and solitary wave solutions of the DP model are dicussed in Vakhnenko and Parkes [14]. Escher et al. [15] investigate that Eq (1.5) has global weak solutions under the sign condition (also see [16]). Liu and Yin [17] discuss the existence of global solutions under certain assumptions and analyze the formation of singularities for Eq (1.5) on the line (also see [18–25]). The numerical investigations about the DP equation and the relating partial differential equations are in detail carried out in [26–32].

The motivation of our job comes from the works in [33, 34]. Gui and Liu [33] investigated the local well-posedness of solutions for the DP equation in the Besov space, while Holmes and Thompson [34] utilized the induction methods to prove well-posedness of short time solutions for the Fornberg-Whitham model in Besov space. As the shallow water wave model Eq (1.1) includes the Degasperis-Procesi and Fornberg-Whitham equations, we study the well-posedness of the short time solutions for Eq (1.1) in the nonhomogeneous Besov space. Our conclusions contain the results of the well-posedness in the nonhomogeneous Besov space presented in [33, 34], namely, we extend parts of the conclusions in [33, 34].

The structure of this work is arranged as follows. Several lemmas are presented in section two. Local well-posedness of Eq (1.1) in the nonhomogeneous Besov space is verified in section three and a blow-up criterion result of problem (1.3) is given in section four.

2. Several lemmas

Several conclusions invloving the Littlewood Paley decomposition, the nonhomogeneous Besov spaces and their properties are stated in this part.

Let $\left(1-\partial_x^2\right)^{-1} f = p * f$ with $p(x)=\frac{1}{2}e^{-|x|}$, where $*$ represents the convolution. For a Banach space X, $\|\cdot\|_X$ stands for the norm of Banach space X. Notation $C(I;X)$ represents the continuous functions from I to X where I is an interval in $\mathbb{R}_+$. $\mathcal{F}f(\xi)=\widehat{f}(\xi)$ denotes the Fourier transform of $f(x)$.

Lemma 2.1 (Littlewood-Paley decomposition)**.** *[35] Suppose that* $\mathcal{B} := \{\xi \in \mathbb{R} : |\xi| \le \frac{4}{3}\}$ *and* $C := \{\xi \in \mathbb{R} : \frac{3}{4} \le |\xi| \le \frac{8}{3}\}$. *There exists two functions* $\chi \in C_c^\infty(\mathcal{B})$ *with* $|\chi| \le 1$, *and* $\varphi \in C_c^\infty(C)$ *with* $|\varphi| \le 1$ *satisfying the identity*

$$\chi(\xi) + \sum_{j\ge 0} \varphi\left(2^{-j}\xi\right) = 1,\ \forall \xi \in \mathbb{R}.$$

For any $u \in \mathcal{S}'(\mathbb{R})$, *the Littlewood-Paley dyadic blocks* Δ_j *satisfy*

$$\begin{aligned}
&\Delta_q u \le 0,\ \textit{if}\ q \le -2,\\
&\Delta_{-1} u \le \chi(D)u = \mathcal{F}^{-1}(\chi\mathcal{F}u),\\
&\Delta_q u \le \varphi\left(2^{-q}D\right)u = \mathcal{F}^{-1}\left(\varphi\left(2^{-q}\right)\mathcal{F}u\right),\ \ \textit{if}\ \ q \ge 0.
\end{aligned}$$

From Lemma 2.1, we denote the inhomogeneous cut-off operator S_q in the form

$$S_q u = \sum_{p=-1}^{q-1} \Delta_p u = \chi\left(2^{-q}D\right)u = \mathcal{F}^{-1}\left(\chi\left(2^{-q}\xi\right)\mathcal{F}u\right),$$

where q is an arbitrary nonnegative natural number. Thus, the following identities

$$\begin{aligned}
\Delta_p\Delta_q u &\equiv 0,\ \text{if}\ |p-q| \ge 2,\\
\Delta_q\left(S_{p-1}u\Delta_p v\right) &\equiv 0,\ \text{if}\quad |p-q| \ge 5
\end{aligned}$$

hold for any $u, v \in \mathcal{S}'(\mathbb{R})$. In addition, for any $1 \le p \le \infty$, we have the inequality

$$\left\|\Delta_q u\right\|_{L^p(\mathbb{R})},\ \left\|S_q u\right\|_{L^p(\mathbb{R})} \le c\|u\|_{L^p(\mathbb{R})},$$

in which constant $c > 0$ does not depend on p and q.

Definition 2.1 (Nonhomogeneous Besov space)**.** *[35] Let* $s \in \mathbb{R}$, $(p,r) \in [1,\infty]^2$ *and* $f \in \mathcal{S}'(\mathbb{R})$. *The nonhomogeneous Besov space* $B^s_{p,r}$ *is defined by*

$$B^s_{p,r} = B^s_{p,r}(\mathbb{R}) = \left\{f \in \mathcal{S}'(\mathbb{R}) : \|f\|_{B^s_{p,r}(\mathbb{R})} < \infty\right\},$$

in which

$$\|u\|_{B^s_{p,r}} \doteq \begin{cases} \left(\sum_{q\ge -1}\left(2^{sq}\left\|\Delta_q u\right\|_{L^p}\right)^r\right)^{1/r}, & \textit{if}\quad 1 \le r < \infty,\\ \sup_{q\ge -1} 2^{sq}\left\|\Delta_q u\right\|_{L^p}, & \textit{if}\quad r = \infty. \end{cases}$$

In particular, $B^\infty_{p,r} = \bigcap_{s\in\mathbb{R}} B^s_{p,r}$.

Lemma 2.2. *[35] Assume $s \in \mathbb{R}$. Let p, r, p_j and r_j $(j = 1, 2)$ belong to interval $[0, \infty]$. Then the following properties hold:*

(1) $B^s_{p,r}$ is a Banach space which is continuously embedded in $\mathcal{S}'(\mathbb{R})$.

(2) C^∞_c is dense in $B^s_{p,r} \Longleftrightarrow p, r \in [1, \infty)$.

(3) $B^s_{p_1,r_1} \hookrightarrow B^{s-\left(\frac{1}{p_1}-\frac{1}{p_2}\right)}_{p_2,r_2}$, if $p_1 \le p_2$ and $r_1 \le r_2$. $B^{s_2}_{p,r_2} \hookrightarrow B^{s_1}_{p,r_1}$ is locally compact if $s_1 < s_2$.

(4) Algebraic properties: $\forall s > 0, B^s_{p,r} \cap L^\infty$ is a Banach algebra. $B^s_{p,r}$ is a Banach algebra $\Longleftrightarrow$ $B^s_{p,r} \hookrightarrow L^\infty \Longleftrightarrow s > \frac{1}{p}$ or $s \ge \frac{1}{p}$ and $r = 1$.

In particular, $B^{1/2}_{2,1}$ is continuously embedded in $B^{1/2}_{2,\infty} \cap L^\infty$ and $B^{1/2}_{2,\infty} \cap L^\infty$ is a Banach algebra.

(5) 1-D Moser-type estimates:

(i) If $s > 0$, then

$$\|fg\|_{B^s_{p,r}} \le C\left(\|f\|_{B^s_{p,r}}\|g\|_{L^\infty} + \|f\|_{L^\infty}\|g\|_{B^s_{p,r}}\right).$$

(ii) If $\forall s_1 \le \frac{1}{p} < s_2$ and $s_1 + s_2 > 0$, then

$$\|fg\|_{B^{s_1}_{p,r}} \le C\|f\|_{B^{s_1}_{p,r}}\|g\|_{B^{s_2}_{p,r}}.$$

(6) Interpolation:

$$\|f\|_{B^{\theta s_1+(1-\theta))_2}_{p,r}} \le \|f\|^{\theta}_{B^{s_1}_{p,r}}\|g\|^{1-\theta}_{B^{s_2}_{p,r}}, \ \forall\, f \in B^{s_1}_{p,r} \cap B^{s_2}_{p,r}, \ \forall\, \theta \in [0, 1].$$

(7) Real interpolation: $\forall\, \theta \in (0, 1)$, $s_1 > s_2$, $s = \theta s_1 + (1-\theta)s_2$, there exists a constant C such that

$$\|W\|_{B^s_{p,1}} \le \frac{C(\theta)}{s_1 - s_2}\|W\|^{\theta}_{B^{s_1}_{p,\infty}}\|W\|^{1-\theta}_{B^{s_2}_{p,\infty}}, \ for\ W \in B^{s_1}_{p,\infty}.$$

In particular, for any $\theta \in (0, 1)$, then

$$\|W\|_{B^{1/2}_{2,1}} \le \|W\|_{B^{\frac{3}{2}-\theta}_{2,1}} \le C(\theta)\|W\|^{\theta}_{B^{1/2}_{2,\infty}}\|W\|^{1-\theta}_{B^{3/2}_{2,\infty}}.$$

(8) Provided that $\{W_n\}_{n\in\mathbb{N}}$ is bounded in $B^s_{p,r}$ and $W_n \to W$ in $\mathcal{S}'(\mathbb{R})$, then $W \in B^s_{p,r}$ and

$$\|W\|_{B^s_{p,r}} \le \liminf_{n\to\infty} \|W_n\|_{B^s_{p,r}}.$$

(9) Assume that $l \in \mathbb{R}$ and f is a S^l − mutiplier. Then the operator f(D) is continuous from $B^s_{p,r}$ to $B^{s-l}_{p,r}$.

(10) The map $B^{-\frac{1}{p}}_{p,r} \times B^{\frac{1}{p}}_{p,\infty}$ to $B^{-\frac{1}{p}}_{p,\infty}$ is continuous.

Lemma 2.3. *[35] Let (p, r) belong to the domain $[1, \infty]^2$. Suppose that $g_0 \in B^s_{p,r}(\mathbb{R})$, $G \in L^1\left([0, T]; B^s_{p,r}(\mathbb{R})\right)$ and $\partial_x u \in L^1\left([0, T]; B^{s-1}_{p,r}(\mathbb{R})\right)$. Suppose that $s > 1 + \frac{1}{p}$ or $s = 1 + \frac{1}{p}, r = 1$. Let $g \in L^\infty([0, T]; B^s_{p,r}(\mathbb{R})) \cap C([0, T]; \mathcal{S}'(\mathbb{R}))$ be a solution of the initial value problem*

$$\partial_t g + u\partial_x g = G, \ g(0, x) = g_0.$$

Then

$$\|g(t)\|_{B^s_{p,r}} \le e^{CZ(t)}\left(\|g_0\|_{B^s_{p,r}} + \int_0^t e^{-CZ(\tau)}\|G(\tau)\|_{B^s_{p,r}}d\tau\right),$$

where $Z(t) = \int_0^t \|\partial_x u(\tau)\|_{B^{s-1}_{p,r}}\, \mathrm{d}\tau$ and positive constant C relies on p, r and s.

3. Local well-posedness in the Besov space

In this part, we prove the well-posedness of the short time solution for Eq (1.1) in the Besov space. For $s \in \mathbb{R}$, $T > 0$, and $p \in [1, \infty]$, we define

$$E^s_{p,r}(T) = C\left([0,T]; B^s_{p,r}\right) \cap C^1\left([0,T]; B^{s-1}_{p,r}\right), \text{ if } r < \infty,$$
$$E^s_{p,\infty}(T) = L^\infty\left([0,T]; B^s_{p,\infty}\right) \cap \text{Lip}\left([0,T]; B^{s-1}_{p,\infty}\right), \text{ if } r = \infty.$$

Now we state our local well-posedness result for Eq (1.1).

Theorem 3.1. *Let $(p, r) \in [1, \infty]^2$, $s > \frac{3}{2}$ (or $s = \frac{p+1}{p}, r = 1, p \in [1, \infty)$) and $W_0 \in B^s_{p,r}$. Then, there exists a time $T > 0$ such that problem (1.3) has a unique solution $W \in B^s_{p,r}(T)$, which continuously depends on initial value W_0.*

Proof. We prove Theorem 3.1 by the following steps.

First step: constructing approximate solutions.

For $t > 0$ and $x \in \mathbb{R}$, let $(W_{(n)}(x, t))_{n\in\mathbb{N}}$ denote a sequence of smooth functions satisfying

$$\begin{cases} \partial_t W_{(n)} - \alpha W_{(n)}\partial_x W_{(n)} = P(D)\left(-k_0 W_{(n)} + \dfrac{\alpha - m}{2}(W_{(n)})^2\right), \\ W_{(n+1)}(0, x) = S_{(n+1)} W_0(x). \end{cases} \tag{3.1}$$

Making use of $W_0 \in B^s_{p,r}$ derives that $S_{(n+1)} W_0 \in B^\infty_{p,r}$ and $\left\|S_{(n+1)} W_0\right\|_{B^s_{p,r}} \leq C \|W_0\|_{B^s_{p,r}}$. By induction, using Lemma 2.3, for every $n \geq 1$, problem (1.3) has a unique solution $W_{(n)}$ in $C([0, T]; B^\infty_{p,r})$. We obtain that $W_{(n)}$ belongs to $E^s_{p,r}(T)$.

Second step: $W_{(n)} \in E^s_{p,r}(T)$.

From Lemmas 2.1–2.3, we notice that operator $P(D)$ is a $S^0 - \textit{multiplier}$ for each positive integer n. Then, we have the estimate

$$\begin{aligned} \left\|W_{(n+1)}(t)\right\|_{B^s_{p,r}} \leq e^{U_{(n)}(t)}(&\|W_0\|_{B^s_{p,r}} \\ &+ \int_0^t e^{-U_{(n)}(\tau)} \|P(D)(G(t))\|_{B^s_{p,r}} d\tau), \end{aligned} \tag{3.2}$$

in which $U_{(n)}(t) = \int_0^t \left\|W_{(n)}\right\|_{B^s_{p,r}} d\tau$ and $G(t) = -k_0 W_{(n)} + \frac{\alpha - m}{2} (W_{(n)})^2$.

Choose $0 < T < \frac{1}{2C\|W\|_{B^s_{p,r}}}$, $\forall t \in [0, T]$ and assume that

$$\left\|W_{(n)}(t)\right\|_{B^s_{p,r}} \leq \frac{\|W_0\|_{B^s_{p,r}}}{1 - 2C \|W_0\|^s_{B^s_{p,r}} t}. \tag{3.3}$$

Utilizing Lemma 2.3 and inequality (3.2) yields

$$\begin{aligned}
&\left\|W_{(n+1)}(t)\right\|_{B^s_{p,r}}\\
\leq &Ce^{CU_{(n)}(t)}\left\|W_0\right\|_{B^s_{p,r}}+C\int_0^t e^{CU_{(n)}(t)-CU_{(n)}(t')}\left\|W_{(n)}(t')\right\|^2_{B^s_{p,r}}dt'\\
\leq &C\exp\left\{-\frac{1}{2}\int_0^t\frac{d\left(1-2C^2\tau\left\|W_0\right\|_{B^s_{p,r}}\right)}{1-2C^2\tau\left\|W_0\right\|_{B^s_{p,r}}}\right\}\left\|W_0\right\|_{B^s_{p,r}}\\
&+C\int_0^t\exp\left\{-\frac{1}{2}\int_{t'}^t\frac{d\left(1-2C^2\tau\left\|W_0\right\|_{B^s_{p,r}}\right)}{1-2C^2\tau\left\|W_0\right\|_{B^s_{p,r}}}\right\}\frac{C^2\left\|W_0\right\|^2_{B^s_{p,r}}}{\left(1-2C^2t'\left\|W_0\right\|_{B^s_{p,r}}\right)^2}dt'\\
\leq &\left(\frac{1}{1-2C^2t\left\|W_0\right\|_{B^s_{p,r}}}\right)^{\frac{1}{2}}\left(C\left\|W_0\right\|_{B^s_{p,r}}+\int_0^t\left(\frac{C^3\left\|W_0\right\|^2_{B^s_{p,r}}}{\left(1-2C^2t'\left\|W_0\right\|_{B^s_{p,r}}\right)^{1+\frac{1}{2}}}\right)dt'\right)\\
\leq &\frac{\left\|W_0\right\|_{B^s_{p,r}}}{1-2C\left\|W_0\right\|^s_{B^s_{p,r}}t}.
\end{aligned}$$

From the above inequality, we know that $W_{(n)}(x,t)$ is uniformly bounded in $L^\infty\left([0,T];B^s_{p,r}\right)$. Employing the property of operator $P(D)$ and (3.3) yields

$$\begin{aligned}
\left\|P(D)(G(t))\right\|_{B^s_{p,r}}&=\left\|P(D)\left(-k_0W_n-\frac{\alpha-m}{2}(W_n)^2\right)\right\|_{B^s_{p,r}}\\
&\leq C\left\|-k_0W_n-\frac{\alpha-m}{2}(W_n)^2\right\|_{B^s_{p,r}}\\
&\leq C\left(\left\|W_n\right\|_{B^s_{p,r}}+\left\|(W_n)^2\right\|_{B^s_{p,r}}\right)\\
&\leq\frac{C\left\|W_0\right\|^2_{B^s_{p,r}}}{\left(1-2C^2\left\|W_0\right\|_{B^s_{p,r}}t\right)^2}+\frac{C\left\|W_0\right\|_{B^s_{p,r}}}{1-2C^2\left\|W_0\right\|_{B^s_{p,r}}t}.
\end{aligned}$$

Consequently, $W_{(n)}\in E^s_{p,r}(T)$.

Third step: convergence.

Now we verify that $(W_{(n)})_{n\in\mathbb{N}}$ is a Cauchy sequence in the space $C([0,T];B^s_{p,r})$. For any positive integers q and n, we deduce that

$$\begin{aligned}
\left(\partial_t+\alpha W_{(n+q)}\partial_x\right)\left(W_{(n+q+1)}-W_{(n+1)}\right)&=P(D)(H(x,t))\\
&+\alpha\left(W_{(n)}-W_{(n+q)}\right)\partial_xW_{(n+1)},
\end{aligned}$$

where

$$H(x,t)=-k_0\left(W_{(n+q)}-W_{(n)}\right)+\frac{\alpha-m}{2}\left(W_{(n+p)}-W_{(n)}\right)\left(W_{(n+q)}+W_{(n)}\right).$$

Utilizing Lemma 2.3 yields

$$
\begin{aligned}
&\left\|(W_{(n+q+1)}-W_{(n+1)})(t)\right\|_{B^s_{p,r}}\\
\leq& C\alpha e^{U_{(n+q)}(t)}\Big(\left\|(W_0)_{(n+q+1)}-(W_0)_{(n+1)}\right\|_{B^s_{p,r}}\\
&+\int_0^t e^{-CU_{(n+q)}(\tau)}\|(W_{(n)}-W_{(n+q)})\partial_x(W_0)_{(n+1)}\\
&+P(D)H(x,t)\|_{B^s_{p,r}}d\tau\Big).
\end{aligned}
\tag{3.4}
$$

Using the Banach algebra of $B^{s-1}_{p,r}$ and the uniform boundedness of $W_{(n)}$, we acquire

$$
\left\|\left(W_{(n)}-W_{(n+q)}\right)\partial_x W_{(n+1)}\right\|_{B^{s-1}_{p,r}}\leq C\left\|W_{(n)}-W_{(n+q)}\right\|_{B^{s-1}_{p,r}}
\tag{3.5}
$$

and

$$
\|P(D)H(x,t)\|_{B^s_{p,r}}\leq C\left\|W_{(n)}-W_{(n+q)}\right\|_{B^{s-1}_{p,r}}.
\tag{3.6}
$$

It is derived that

$$
\begin{aligned}
&\left\|(W_0)_{(n+q+1)}-(W_0)_{(n+1)}\right\|_{B^s_{p,r}}=\left\|\sum_{j=n+1}^{n+q}\Delta_j(W_0)\right\|_{B^s_{p,r}}\\
=&\left(\sum_{k\geq-1}2^{k(s-1r)}\left\|\Delta_k\left(\sum_{q=n+1}^{n+q}\Delta_q(W_0)\right)\right\|^r_{L^p}\right)^{\frac{1}{r}}\\
\leq& C\left(\sum_{k=n}^{n+q+1}2^{-kr}2^{ksr}\|\Delta_k(W_0)\|^r_{L^p}\right)^{\frac{1}{r}}\\
\leq& C2^{-n}\|W_0\|_{B^{s-1}_{p,r}}.
\end{aligned}
\tag{3.7}
$$

By induction, utilizing (3.4)–(3.7) gives rise to

$$
\left\|W_{(n+q+1)}-W_{(n+1)}(t)\right\|_{B^s_{p,r}}\leq C_T\left(2^{-n}+\int_0^t\left\|W_{(n+q)}-W_{(n)}\right\|_{B^{s-1}_{p,r}}d\tau\right).
$$

Since $\left\|W_{(q)}\right\|_{B^s_{p,r}}$ possesses the uniformly bounds in $B^s_{p,r}(T)$, we derive that there exists a new constant C'_T such that

$$
\left\|W_{(n+q+1)}-W_{(n+1)}\right\|_{B^s_{p,r}}\leq C'_T2^{-n}.
\tag{3.8}
$$

Utilizing (3.8), we conclude that $\left(W_{(n)}\right)_{n\in\mathbb{N}}$ is a Cauchy sequence in $C([0,T];B^s_{p,r})$. Note that $C([0,T];B^s_{p,r})$ is a Banach space. Then there is a $W\in C([0,T];B^s_{p,r})$ such that the sequence $W_{(n)}(x,t)$ converges to W in $C([0,T];B^s_{p,r})$.

Fourth step: existence of solution.

Now we check that $W\in B^s_{p,r}(T)$ solves Eq (1.3). Using the uniform boundedness of $W_{(n)}$ in $L^\infty\left(0,T;B^s_{p,r}\right)$ and Lemma 2.1, we obtain that $W\in E^s_{p,r}(T)$ and W solves problem (1.3).

Fifth step: uniqueness.

Assume $(p,r)\in[1,\infty]^2$, index $s>\frac{3}{2}$ (or $s=1+\frac{1}{p}$, $r=1$, $p\in[1,\infty)$). Suppose that two solutions W and $\tilde{W}\in L^\infty([0,T];B^s_{p,r})$ satisfy problem (1.3) corresponding to the initial values $W_0,\tilde{W}_0\in B^s_{p,r}$, respectively. Applying Lemmas 2.1–2.3, and letting $v=\tilde{W}-W$, we have

$$\begin{aligned}\|v\|_{B^s_{r,r}}\le{}& Ce^{C\int_0^t\|\tilde{W}\|_{B^s_{p,r}}dt'}\Bigg(\|v_0\|_{B^s_{p,r}}+C\int_0^t e^{-C\int_0^{t'}\|\tilde{W}\|_{B^s_{p,r}}}dt'\\&\times\|v\|_{B^s_{p,r}}\left[\|1-k_0W\|_{B^s_{p,r}}+\frac{|\alpha-m|}{2}\left(\|W\|_{B^s_{p,r}}+\|\tilde{W}\|_{B^s_{p,r}}\right)\right]dt'\Bigg),\end{aligned}$$

where $C=C(p,r,s)$ depends on $\|W_0\|_{B^s_{p,r}}$ and $\left\|\tilde{W}_0\right\|_{B^s_{p,r}}$. Making use of the Gronwall inequality yields

$$\|v(t)\|_{B^s_{p,r}}\le C\,\|v_0\|_{B^{s-1}_{p,r}}\,e^{C\int_0^t\left(\|\tilde{W}\|_{B^s_{p,r}}+\|W\|_{B^s_{p,r}}\right)\mathrm{d}t'},$$

from which we obtain $W=\tilde{W}$ if $v_0=0$. The proof of uniqueness is finished.

The sixth step: continuous dependence.

Let $W_0\in B^s_{p,r}$ and let $\{W_{0,h}\}_{h=0}^\infty$ be a sequence in $B^s_{p,r}$ such that $W_{0,h}$ converges to W_0 in $B^s_{p,r}$. Assume W and W_h are two solutions of problem (1.3) corresponding to initial values W_0 and $W_{0,h}$, respectively. We will prove

$$\lim_{h\to\infty}\|W_h-W\|_{C\left([0,T];B^s_{p,r}\right)}=0.$$

For each sufficiently small $\varepsilon>0$, there exists $N>0$, if $h\ge N$, we shall prove

$$\|W_h-W\|_{C\left([0,T];B^s_{p,r}\right)}<\varepsilon. \tag{3.9}$$

For notational convenience, we utilize W^δ and W^δ_h to denote the two solutions of problem (1.3) corresponding to initial data $J_\delta W_0$ and $J_\delta W_{0,h}$, respectively, where J_δ is a modifier operator. Using triangular inequalities, if we can prove the following three inequalities, then inequality (3.9) holds.

$$\left\|W_h-W^\delta_h\right\|_{C\left([0,T];B^s_{p,r}\right)}<\varepsilon/3, \tag{3.10}$$

$$\left\|W^\delta_h-W^\delta\right\|_{C\left([0,T];B^s_{p,r}\right)}<\varepsilon/3, \tag{3.11}$$

$$\left\|W^\delta-W\right\|_{C\left([0,T];B^s_{p,r}\right)}<\varepsilon/3. \tag{3.12}$$

Estimating $W^\delta_h-W^\delta$.

Now we prove

$$\left\|W^\delta_h-W^\delta\right\|_{C\left([0,T];B^s_{p,r}\right)}<\varepsilon/3.$$

We find that W and W_h are in $C([0,T];B^s_{p,r})$. Assume $v=W_h-W$ and $u=W_h+W$ yields

$$\partial_tv+u\partial_xv=-v\partial_xu+\Lambda^{-1}(\partial_xv).$$

Applying the energy estimate yields

$$\|v\|_{B^s_{p,r}}\le\|v_0\|_{B^s_{p,r}}=\left\|W^\delta_h(0)-W^\delta(0)\right\|_{B^s_{p,r}}\le\|W_h(0)-W(0)\|_{B^s_{p,r}}.$$

Taking $h > N$ large enough, we have

$$\left\|W_h^\delta - W^\delta\right\|_{B^s_{p,r}} < \varepsilon/3.$$

The proof of (3.10) and (3.12) is similar to that of (3.11). From (3.10)–(3.12), we complete the proof of continuous dependence. Up to now, we finish the proof of Theorem 3.1. □

4. Blow-up of solutions

In this part, we derive the blow-up criterion of the solutions to problem (1.3). We state the lemma.

Lemma 4.1. *Assume that $(p, r) \in [1, \infty]^2$, index $s > 1$. Suppose that $W \in L^\infty\left([0, T]; B^s_{p,r} \cap Lip\right)$ is the solution to problem (1.3) and $W_0 \in B^s_{p,r} \cap \mathrm{Lip}$. For every $t \in [0, T)$, then*

$$\|W(t)\|_{B^s_{p,r}} \le \|W_0\|_{B^s_{p,r}} e^{C\int_0^t\left(\|W(\tau)\|_{Lip}+1\right)d\tau} \tag{4.1}$$

and

$$\|W(t)\|_{Lip} + 1 \le \left(\|W_0\|_{Lip} + 1\right) e^{C\int_0^t \|\partial_x W(\tau)\|_{L^\infty(\mathbb{R})}d\tau}, \tag{4.2}$$

where $C = C(p, r)$.

Proof. Using Lemma 2.3 and Eq (1.3), we have

$$e^{-C\int_0^t\|\partial_x W(\tau)\|_{L^\infty}d\tau}\|W(t)\|_{B^s_{p,r}} \le \|W_0\|_{B^s_{p,r}} + C\int_0^t e^{-C\int_0^\tau\|\partial_x W(\tau')\|_{L^\infty}d\tau'}\|P(D)(G_0(x,\tau))\|_{B^s_{p,r}}d\tau,$$

where $G_0(x, \tau) = -k_0 W - \frac{\alpha-m}{2}W^2$. We obtain

$$\|P(D)(G_0(x,t))\|_{B^s_{p,r}} \le C\left\|-k_0 W - \tfrac{\alpha-m}{2}W^2\right\|_{B^{s-1}_{p,r}} \le C\left(\|W\|_{Lip} + 1\right)\|W\|_{B^s_{p,r}}.$$

Hence,

$$e^{-C\int_0^t\|\partial_x W(\tau)\|_{L^\infty}d\tau}\|W(t)\|_{B^s_{p,r}} \le \|W_0\|_{B^s_{p,r}} + C\int_0^t e^{-C\int_0^\tau\|\partial_x W(\tau')\|_{L^\infty}d\tau'}\left(\|W\|_{Lip} + 1\right)\|W\|_{B^s_{p,r}}d\tau. \tag{4.3}$$

Using (4.3) and the Gronwall inequality yields (4.1). Following the procedure in the proof of (4.1), we derive (4.2). □

Theorem 4.1. *Let W_0 be defined as in Theorem 3.1. Then*

$$T^*_{W_0} < \infty \Rightarrow \int_0^{T^*_{W_0}} \|\partial_x W(\tau)\|_{L^\infty} d\tau = \infty.$$

Proof. Assume that $W \in \bigcap_{0<T<T^*_{W_0}} E^s_{p,r}(T)$ satisfies $\int_0^{T^*_{W_0}} \|\partial_x W(\tau)\|_{L^\infty} d\tau < \infty$. From (4.2), we derive that $\int_0^{T^*_{W_0}}\left(\|W(\tau)\|_{Lip} + 1\right)d\tau$ is finite. Using (4.1) derives

$$\|W(t)\|_{B^s_{p,r}} \le M_{T^*_{W_0}} \le e^{\int_0^{T^*_{W_0}}\left(\|W(\tau)\|_{Lip}+1\right)d\tau},$$

where $t \in [0, T^*_{W_0})$. Let ε be a positive constant such that $\varepsilon < \frac{1}{2C^2 M_{T^*_{W_0}}}$, where C is a constant in Theorem 3.1. For initial value $W\left(T^*_{W_0} - \frac{\varepsilon}{2}\right)$ and solution $\tilde{W}(t) = W\left(t + T^*_{W_0} - \frac{\varepsilon}{2}\right)$ with $t \in \left[0, \frac{\varepsilon}{2}\right)$, we conclude that the solution $\tilde{W}(t)$ satisfies problem (1.3). Therefore, $\tilde{W}$ expands the solution W beyond $T^*_{W_0}$. We finish the proof by contradiction. □

5. Conclusions

In this work, utilizing the decomposition method of the Littlewood-Paley and the properties of nonhomogeneous Besov space, we have established the well-posedness of short time solutions for the shallow water wave equation (1.1) in the nonhomogeneous Besov space. A blow-up criterion of solutions is obtained. Using the Hirota bilinear method, the unified method or other methods to investigate the optical soliton solution, lump wave solution, periodic wave solution, kink and breather wave solutions like those in [27–31] would be our goal for future works.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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