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Research article

Prescribed-time control for spacecraft formation flying with uncertainties and disturbances

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Abstract: The prescribed-time spacecraft formation flying problem with uncertainties and unknown disturbances is investigated. First, based on Lie group SE(3), the coupled 6-degrees-of-freedom kinematics and dynamics for spacecraft with uncertainties and unknown disturbances are introduced. Second, with the aid of some key properties of a class of parametric Lyapunov equations, novel prescribed-time control laws are designed. It is proved that the proposed control laws can drive the relative motion between the leader spacecraft and follower spacecraft to zero in any prescribed time and are bounded. Finally, numerical simulations verify the effectiveness of the proposed control scheme.

Keywords: spacecraft formation flying; uncertainties; unknown disturbance; prescribed-time control; time-varying feedback
Mathematics Subject Classification: 34H05, 37N35, 93C10

1. Introduction

With several benefits, such as system robustness, flexibility and reconfigurability [5, 7, 12], spacecraft formation flying is attracting more and more attention. Among them, how to achieve high-precision control of spacecraft formation flying is an important topic due to the actual mission requirements, such as proximity operations [10]. The high precision control of spacecraft mainly depends on two main aspects: dynamics models and control algorithms. For dynamics models, a 6-degrees-of-freedom (6-DOF) relative motion model was proposed by using the dual-quaternion representation in [15]; a 6-DOF Euler-Lagrange form of the relative-motion model, where the rotational motion was described by modified Rodrigues parameters (MRPs), was studied in [14]. However, the attitude motion described by the dual quaternion and the MRPs method has some drawbacks, such as, the dual quaternion may cause unwinding problems [1], and the attitude motions described by MRPs in [1] are non-global and non-unique. As a set of positions and attitudes of a rigid body in 3-D Euclidean space, Lie group SE(3) can represent the spacecraft's motion in a unique and non-singularity way [1, 7]. Based on Lie group SE(3), a 6-DOF coupling relative-motion model was studied in [7, 15] and a decentralized consensus control problem of SFF was studied in [11].

Finite-/fixed-time control has received a lot of attention because of its higher tracking precision, faster convergence rate and greater robustness to disturbances [2, 8, 9, 17, 19]. This method has also been applied to spacecraft formation [18]. For finite-time control, by using terminal sliding mode control method, a finite-time control law was designed for spacecraft formation in [16]. For fixed-time control, with the aid of a fixed-time disturbance observer, a fixed-time sliding mode control law for spacecraft proximity operations with parameter uncertainties and disturbances was presented in [18]. However, the actual convergence times of the controllers mentioned above (not the upper bounds) are all dependent on the initial states, which may not meet the needs in practical engineering.

Recently, a time-varying high-gain based finite-time control method has regained the interest of researchers [13, 22–24]. A significant advantage of such method is that its convergence time can be independent of the initial states, which is called prescribed-time control. With the aid of a scaling of the state by a function of time that grows unbounded towards the terminal time, a controller that stabilizes the system in a prescribed finite time was designed in [13]. By using some key properties of a class of parametric Lyapunov equations (PLEs) and scalarization, the finite-time and prescribed-time feedback of linear systems, a class of nonlinear systems, and high-order nonholonomic systems were designed in [21, 23–26].

In this paper, the prescribed-time spacecraft formation flying problem with uncertainties and unknown disturbances is investigated. First, based on Lie group SE(3), the coupled 6-DOF kinematics and dynamics for spacecraft under uncertainties and unknown disturbances are introduced, where the relative configurations are expressed by exponential coordinates of SE(3). Second, with the aid of some key properties of a class of PLEs, novel prescribed-time control laws are designed. It is proved that the proposed control laws can drive the relative motion between the leader spacecraft and follower spacecraft to zero in any prescribed time and are bounded. Finally, numerical simulations verify the effectiveness of the proposed control scheme.

2. Models of spacecraft and preliminaries

In this section, we will introduce the kinematics and dynamics of the leader and the follower spacecrafts. Similar to [1, 7], we assume that all spacecrafts are rigid bodies in a gravitational field in the Earth's orbital environment.

2.1. Models of spacecraft

Similar to [1, 7], let the rotation matrix $R^0 \in \mathrm{SO}(3)$, $b^0 \in \mathbf{R}^3$, $\nu^0 \in \mathbf{R}^3$ and $\Omega^0 \in \mathbf{R}^3$ represent, respectively, the attitude, position, translational and angular velocities of the leader, and the configuration and velocities vector of the leader on SE(3) be represented as

$$g^0 = \begin{bmatrix} R^0 & b^0 \\ 0 & 1 \end{bmatrix} \in \mathrm{SE}(3), \quad \xi^0 = \begin{bmatrix} \Omega^0 \\ \nu^0 \end{bmatrix}.$$

Then the kinematics of the leader can be rewritten as

$$\dot{g}^0 = g^0(\xi^0)^\vee, \quad (\xi^0)^\vee = \begin{bmatrix} (\Omega^0)^\times & v^0 \\ 0 & 0 \end{bmatrix}, \tag{2.1}$$

where $(\cdot)^\times$ is the cross-product operator defined by

$$v^\times = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}^\times = \begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix}.$$

Define

$$\mathrm{Ad}_{g^0} = \begin{bmatrix} R^0 & 0_{3\times3} \\ b^{0\times}R^0 & R^0 \end{bmatrix}, \quad \mathrm{ad}_{\xi^0} = \begin{bmatrix} \omega^{0\times} & 0_{3\times3} \\ v^{0\times} & \omega^{0\times} \end{bmatrix},$$

and $\mathrm{ad}^*_{\xi^0} = \left(\mathrm{ad}_{\xi^0}\right)^{\mathrm{T}}$. The dynamics equations of the rotational and translational motions for a leader spacecraft can be expressed as

$$\Xi^0\dot{\xi}^0 = \mathrm{ad}^*_{\xi^0}\Xi^0\xi^0 + \varphi^0_{\mathrm{g}}, \tag{2.2}$$

where $\varphi^0_{\mathrm{g}} = \mathrm{diag}([M^0_{\mathrm{g}}, F^0_{\mathrm{g}}])$, $\Xi^0 = \mathrm{diag}([J^0, m^0 I_{3\times3}])$, m^0 and J^0 are the mass and moment of inertia matrix of the virtual leader, respectively, $M^0_{\mathrm{g}} \in \mathbf{R}^3$ and $F^0_{\mathrm{g}} \in \mathbf{R}^3$ are gravity gradient moment and gravity force, respectively.

The kinematics for the kth follower spacecraft have the same form as those for the leader, and are given by [1, 7]

$$\dot{g}^k = g^k(\xi^k)^\vee. \tag{2.3}$$

The dynamics of the follower can be expressed in the compact form [1,7] $\Xi^k\dot{\xi}^k = \mathrm{ad}^*_{\xi^k}\Xi^k\xi^k + \varphi^k_{\mathrm{g}} + \varphi^k_{\mathrm{c}} + \varphi^k_{\mathrm{d}}$, where $\varphi^k_{\mathrm{c}} = \left[\tau^{\mathrm{T}}_{\mathrm{c}}, f^{\mathrm{T}}_{\mathrm{c}}\right]^{\mathrm{T}}$, $\varphi^k_{\mathrm{d}} = \left[\tau^{k\mathrm{T}}_{\mathrm{d}}, f^{k\mathrm{T}}_{\mathrm{d}}\right]^{\mathrm{T}}$, $\varphi^k_{\mathrm{g}} = [M^{k\mathrm{T}}_{\mathrm{g}}, F^{k\mathrm{T}}_{\mathrm{g}} + mR^3 a^{k\mathrm{T}}_{J_2}]^{\mathrm{T}}$, in which $\varphi^k_{\mathrm{g}} \in \mathbf{R}^6$ are known gravity inputs, $\varphi^k_{\mathrm{c}} \in \mathbf{R}^6$ are control inputs, $\varphi^k_{\mathrm{d}} \in \mathbf{R}^6$ are external disturbances, $M^k_{\mathrm{g}} \in \mathbf{R}^3$ and $F^k_{\mathrm{g}} \in \mathbf{R}^3$ are gravity gradient moments and gravity forces, respectively, $f^k_{\mathrm{c}} \in \mathbf{R}^3$ and $\tau^k_{\mathrm{c}} \in \mathbf{R}^3$ are control forces and moments, $f^k_{\mathrm{d}} \in \mathbf{R}^3$ and $\tau^k_{\mathrm{d}} \in \mathbf{R}^3$ are unknown forces and moments on the follower spacecraft.

Let the configuration of the formation be given by $(h^1_f, h^2_f, \ldots, h^n_f) \in \mathrm{SE}(3)$, where h^k_f denotes the fixed relative configuration of the kth spacecraft to the virtual leader. Given the leader trajectory generated by (2.1) and (2.2), the desired states of the kth spacecraft are [1, 7] $g^{0k} = g^0(h^k_f)$ and $\xi^{0k} = \mathrm{Ad}_{(h^k_f)^{-1}}\xi^0$. The relative configuration between the follower and the leader spacecraft is

$$h^k = (g^0)^{-1}g^k. \tag{2.4}$$

This exponential coordinate vector for the configuration tracking error for the leader spacecraft is expressed as $\tilde{\eta} = [\tilde{\Theta}^{\mathrm{T}}, \tilde{\beta}^{\mathrm{T}}]^{\mathrm{T}}, (\tilde{\eta})^\vee = \log((h^k_f)^{-1}h^k) = \log((g^{0k})^{-1}g^k)$, where log: $\mathrm{SE}(3) \to \mathfrak{se}(3)$ is the logarithm map, $\tilde{\eta}$ is the exponential coordinate vector, describing the relative configuration between the desired configuration and the actual configuration of the kth spacecraft in the formation, while $\tilde{\Theta} \in \mathbf{R}^3$ and $\tilde{\beta} \in \mathbf{R}^3$ are the attitude and position tracking error in the exponential coordinate.

Taking the time-derivative of (2.4) and substituting (2.1) and (2.3), the relative velocities between the kth follower and the leader spacecraft are gaven by $\tilde{\xi}^k = \xi^k - \mathrm{Ad}_{(h^k)^{-1}}\xi^0$. The kinematics in the exponential coordinates can be expressed as $\dot{\tilde{\eta}}^k = G(\tilde{\eta}^k)\tilde{\xi}^k$, where the expression of $G(\tilde{\eta}^k)$ can be referred to (20) in [20].

According to [1, 7, 20], the coupled spacecraft nonlinear systems can be given as

$$\left\{\begin{aligned} \dot{\tilde{\eta}}^k &= G(\tilde{\eta}^k)\tilde{\xi}^k, \\ \Xi^k\dot{\tilde{\xi}}^k &= \mathrm{ad}^*_{\xi^k}\Xi^k\xi^k + \varphi^k_{\mathrm{g}} + \varphi^k_{\mathrm{d}} + \varphi^k_{\mathrm{c}} + \Xi^k(\mathrm{ad}_{\xi^k}\mathrm{Ad}_{(h^k)^{-1}}\xi^0 - \mathrm{Ad}_{(h^k)^{-1}}\dot{\xi}^0). \end{aligned}\right. \tag{2.5}$$

2.2. Preliminaries

In this subsection, we give some preliminaries. In light of (2.5), simple calculation yields

$$\begin{aligned} \ddot{\tilde{\eta}}^k =&\dot{G}(\tilde{\eta}^k)\tilde{\xi}^k + G(\tilde{\eta}^k)\dot{\tilde{\xi}}^k \\ =&G(\tilde{\eta}^k)\Xi^{k-1}(\mathrm{ad}^*_{\xi^k}\Xi^k\xi^k + \varphi^k_{\mathrm{g}} + \varphi^k_{\mathrm{d}} + \varphi^k_{\mathrm{c}}) + \dot{G}(\tilde{\eta}^k)\tilde{\xi}^k + G(\tilde{\eta}^k)(\mathrm{ad}_{\xi^k}\mathrm{Ad}_{(h^k)^{-1}}\xi^0 - \mathrm{Ad}_{(h^k)^{-1}}\dot{\xi}^0). \end{aligned} \tag{2.6}$$

Choose the controller as

$$\varphi^k_{\mathrm{c}} = -\varphi_{\mathrm{g}} - \Xi^k(\mathrm{ad}_{\xi^k}\mathrm{Ad}_{(h^k)^{-1}}\xi^0 - \mathrm{Ad}_{(h^k)^{-1}}\dot{\xi}^0) - \mathrm{ad}^*_{\xi^k}\Xi^k\xi^k + \Xi^k G^{-1}(\tilde{\eta}^k)(u - \dot{G}(\tilde{\eta}^k)\tilde{\xi}^k), \tag{2.7}$$

where u is an auxiliary controller to be designed. Denote

$$\begin{aligned} \varphi &= [\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5, \varphi_6]^{\mathrm{T}} \triangleq G(\tilde{\eta}^k)\Xi^{k-1}\varphi_{\mathrm{d}}, \\ u &= [u_1, u_2, u_3, u_4, u_5, u_6]^{\mathrm{T}}, \\ \tilde{\eta}^k &= [\tilde{\eta}^k_1, \tilde{\eta}^k_2, \tilde{\eta}^k_3, \tilde{\eta}^k_4, \tilde{\eta}^k_5, \tilde{\eta}^k_6]^{\mathrm{T}}, \\ \dot{\tilde{\eta}}^k &= [\dot{\tilde{\eta}}^k_1, \dot{\tilde{\eta}}^k_2, \dot{\tilde{\eta}}^k_3, \dot{\tilde{\eta}}^k_4, \dot{\tilde{\eta}}^k_5, \dot{\tilde{\eta}}^k_6]^{\mathrm{T}}, \\ x_i &= [\tilde{\eta}^k_i, \dot{\tilde{\eta}}^k_i]^{\mathrm{T}}, \quad i = 1, 2, \ldots, 6. \end{aligned}$$

In view of (2.7), system (2.6) can be re-expressed as

$$\dot{x}_i = Ax_i + b(u_i + \varphi_i), \tag{2.8}$$

where

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{2.9}$$

With the above preparations, we can give the following lemma:

Lemma 1. *[25] Suppose that Assumption 1 is satisfied, $L_2 = L_2(\gamma)$ is defined as (5.2) in Appendix and $\gamma_0 > 0$ is a constant. Then, for any $\gamma \geq \gamma_0 > 0$, and any $x_i \in \mathbf{R}^2$,*

$$(L_2\phi_i)^{\mathrm{T}}(L_2\phi_i) \leq d^2(L_2x_i)^{\mathrm{T}}(L_2x_i),$$

where

$$d^2 = d^2(\gamma_0) = \max\left\{c^2_{11} + \frac{2c^2_{21}}{\gamma^2_0}, 2c^2_{22}\right\}.$$

Definition 1. *Let $T > 0$ be a prescribed time. If the continuous function $\gamma(t) : [0, T) \to \mathbf{R}_{>0}$ satisfies $\lim_{t\uparrow T}\gamma(t) = \infty$, then it is called a T-finite-time escaping (T-FTE) function.*

3. Main results

In this section, we give a prescribed-time control scheme for each follower spacecraft, and the follower can arrive at its desired trajectory by maintaining a constant relative configuration with respect to the leader spacecraft. For clarification, we omit the superscript $()^k$ in the following. Consider the following PLE [23, 24]:

$$A^{\mathrm{T}}P + PA - Pbb^{\mathrm{T}}P = -\gamma P, \tag{3.1}$$

where $\gamma > 0$ is a (time-varying) scalar to be designed. The PLE has many interesting properties which are collected in Lemma 2 in Appendix. We will consider three cases, and the PLE will be used in the first two cases.

3.1. Case 1: Both mismatched and matched uncertainties

Rewrite system (2.8) as

$$\dot{x}_i = Ax_i + b(u_i + \varphi_i) + \phi_i, \tag{3.2}$$

where ϕ_i is the unmodeled dynamics. The following assumption is imposed on system (3.2).

Assumption 1. *There exist some positive known constants c_{i1}, c_{i2}, unknown constant δ, and continuous known functions $\psi_i = \psi_i(t, x)$, for $i = 1, 2, \ldots, 6$, such that*

$$|\phi_i| \leq c_{i1} \left|\tilde{\eta}_i^k\right| + c_{i2} \left|\dot{\tilde{\eta}}_i^k\right|,$$

and

$$|\varphi_i| \leq \psi_i \delta. \tag{3.3}$$

In [1], it is assumed that the unknown external disturbances $\varphi_{\mathrm{d}} = [\varphi_{\mathrm{d}1}, \varphi_{\mathrm{d}2}, \ldots, \varphi_{\mathrm{d}6}]^{\mathrm{T}}$ for the kth spacecraft is bounded by some known positive constants F_i, namely,

$$|\varphi_{\mathrm{d}i}| \leq F_i, \quad i = 1, 2, \ldots, 6.$$

In this paper, according to Assumption 1, we know that the constant δ in (3.3) can be unknown, namely, F_i can be unknown, which we believe is more reasonable.

Theorem 1. *Let Assumption 1 be satisfied, T be a prescribed time, $\lambda > 0$ be a constant, and γ_0 be a constant satifying*

$$\gamma_0 \geq \begin{cases} \frac{\beta}{1 - \mathrm{e}^{-\alpha\beta T}}, & d \neq 0, \\ \frac{1}{\alpha T}, & d = 0, \end{cases} \tag{3.4}$$

with $s \in (0, 1)$ and

$$\alpha = \frac{1 - s}{2 + \delta_{\mathrm{c}}}, \quad \beta = \frac{8d\hat{\lambda}}{1 - s}. \tag{3.5}$$

Consider the controller

$$\varphi_{\mathrm{c}}(t) = -\varphi_{\mathrm{g}} - \Xi^k(\mathrm{ad}_{\xi^k}\mathrm{Ad}_{(h^k)^{-1}}\xi^0 - \mathrm{Ad}_{(h^k)^{-1}}\dot{\xi}^0) - \mathrm{ad}^*_{\xi^k}\Xi^k\xi^k - \Xi^k G^{-1}(\tilde{\eta}^k)(u(t) + \dot{G}(\tilde{\eta}^k)\tilde{\xi}^k), \tag{3.6}$$

with

$$u(t) = [u_1(t), u_2(t), u_3(t), u_4(t), u_5(t), u_6(t)]^{\mathrm{T}},$$

$$
\begin{aligned}
u_i(t) &= -(\frac{1}{2} + 2\lambda\psi_i^2)b^{\mathrm{T}}P(\gamma)x_i, \\
\gamma(t) &= \begin{cases} \frac{\mathrm{e}^{\alpha\beta T}-1}{\mathrm{e}^{\alpha\beta T}-\mathrm{e}^{\alpha\beta t}}\gamma_0, & d \neq 0, \\ \frac{T}{T-t}\gamma_0, & d = 0. \end{cases}
\end{aligned} \tag{3.7}
$$

Then the state of the closed-loop system consisting of (3.2) and (3.6) converges to zero at the prescribed time T, and the control is bounded.

Proof. If $d = 0$, by using the L'Hospital rules, we have

$$
\begin{aligned}
\lim_{d\to 0}\gamma(t) = \lim_{\beta\to 0}\gamma(t) &= \lim_{\beta\to 0}\frac{\mathrm{e}^{\alpha\beta T}-1}{\mathrm{e}^{\alpha\beta T}-\mathrm{e}^{\alpha\beta t}}\gamma_0 \\
&= \lim_{\beta\to 0}\frac{\mathrm{e}^{\alpha\beta T}-1}{(\mathrm{e}^{\alpha\beta(T-t)}-1)\mathrm{e}^{\alpha\beta t}}\gamma_0 \\
&= \lim_{\beta\to 0}\frac{\mathrm{e}^{\alpha\beta T}-1}{\alpha\beta(T-t)\mathrm{e}^{\alpha\beta t}}\gamma_0 \\
&= \lim_{\beta\to 0}\frac{\mathrm{e}^{\alpha\beta(T-t)}-\mathrm{e}^{-\alpha\beta t}}{\alpha\beta(T-t)}\gamma_0 \\
&= \lim_{\beta\to 0}\frac{(T-t)\mathrm{e}^{\alpha\beta(T-t)}+t\mathrm{e}^{-\alpha\beta t}}{(T-t)}\gamma_0 \\
&= \frac{T}{T-t}\gamma_0,
\end{aligned}
$$

and

$$
\lim_{d\to 0}\frac{\beta}{1-\mathrm{e}^{-\alpha\beta T}} = \lim_{\beta\to 0}\frac{\beta}{1-\mathrm{e}^{-\alpha\beta T}} = \lim_{\beta\to 0}\frac{\beta\mathrm{e}^{\alpha\beta T}}{\mathrm{e}^{\alpha\beta T}-1} = \lim_{\beta\to 0}\frac{\mathrm{e}^{\alpha\beta T}}{\alpha T} = \frac{1}{\alpha T}.
$$

If $d \neq 0$, similar to [25], we will prove that there exists a $\gamma_* > 0$ such that (3.4) is satisfied for all $\gamma_0 \geq \gamma_*$. Denote

$$
\sigma(\gamma) = \frac{\beta(\gamma)}{1-\mathrm{e}^{-\alpha\beta(\gamma)T}}, \quad \gamma \in (0,\infty).
$$

Notice that $\lim_{\gamma_0\uparrow\infty} d^2(\gamma_0) \triangleq d_\infty^2 < \infty$, which implies that

$$
\lim_{\gamma_0\to\infty}\frac{\beta(\gamma_0)}{1-\mathrm{e}^{-\alpha\beta(\gamma_0)T}} = \frac{8d_\infty\hat{\lambda}}{(1-s)(1-\mathrm{e}^{-\alpha\beta(\gamma_0)T})} < \infty.
$$

Clearly, we have $\mathrm{d}\beta/\mathrm{d}\gamma \leq 0$. Then it can be obtained that

$$
\frac{\mathrm{d}\sigma(\gamma)}{\mathrm{d}\gamma} = \frac{\partial\sigma(\gamma)}{\partial\beta}\frac{\mathrm{d}\beta}{\mathrm{d}\gamma} = \frac{\mathrm{e}^{\alpha\beta T}(\mathrm{e}^{\alpha\beta T}-(T\alpha\beta+1))}{(\mathrm{e}^{\alpha\beta T}-1)^2}\frac{\mathrm{d}\beta}{\mathrm{d}\gamma} \leq 0.
$$

Therefore, there exists a $\gamma_* > 0$ such that (3.4) is satisfied for all $\gamma_0 \geq \gamma_*$. Particularly, γ_* can be chosen as the unique positive root (if it exists) of the following equation

$$
\gamma_* = \frac{\beta(\gamma_*)}{1-\mathrm{e}^{-\alpha\beta(\gamma_*)T}}.
$$

The closed-loop system consisting of (3.2), (3.6) and (3.7) can be written as

$$\dot{x}_i = Ax_i + b(u_i + \varphi_i) + \phi_i, \quad i = 1, 2, \ldots, 6. \tag{3.8}$$

Choose the Lyapunov-like function

$$V_i(t, x_i) = 2\gamma x_i^{\mathrm{T}} P(\gamma) x_i,$$

whose time-derivative along the closed-loop system (3.8) can be written as

$$\begin{aligned}
\dot{V}_i(t, x_i) =& 2\dot{\gamma} x_i^{\mathrm{T}} P x_i + 2\gamma \dot{x}_i^{\mathrm{T}} P x_i + 2\gamma\dot{\gamma} x_i^{\mathrm{T}} \frac{\mathrm{d}P}{\mathrm{d}\gamma} x_i + 2\gamma x_i^{\mathrm{T}} P \dot{x}_i \\
=& 2\dot{\gamma} x_i^{\mathrm{T}} P x_i + 2\gamma x_i^{\mathrm{T}} (A^{\mathrm{T}} P + PA - Pbb^{\mathrm{T}} P) x_i + 2\gamma\dot{\gamma} x_i^{\mathrm{T}} \frac{\mathrm{d}P}{\mathrm{d}\gamma} x_i \\
& - 4\lambda\psi_i^2 \gamma x_i^{\mathrm{T}} Pbb^{\mathrm{T}} P x_i + 4\gamma x_i^{\mathrm{T}} Pb\varphi_i + 4\gamma x_i^{\mathrm{T}} P\phi_i.
\end{aligned}$$

According to the Young's inequality with $k_0 > 0$ and $\lambda > 0$, we have

$$x_i^{\mathrm{T}} Pb\varphi_i \leq \lambda\psi_i^2 x_i^{\mathrm{T}} Pbb^{\mathrm{T}} P x_i + \frac{\delta^2}{4\lambda}, \quad x_i^{\mathrm{T}} P\phi_i \leq k_0 x_i^{\mathrm{T}} P x_i + \frac{\phi_i^{\mathrm{T}} P \phi_i}{k_0}.$$

By using Lemma 1 and (5.4) in Lemma 2 in Appendix, it follows that

$$\begin{aligned}
\phi_i^{\mathrm{T}} P \phi_i &= \phi_i^{\mathrm{T}} \gamma L_n P_n L_n \phi_i \\
&\leq \gamma \hat{\lambda} (L_n \phi_i)^{\mathrm{T}} (L_n \phi_i) \\
&\leq \gamma d^2 \hat{\lambda} (L_n x_i)^{\mathrm{T}} (L_n x_i) \\
&= \gamma d^2 \hat{\lambda}^2 (L_n x_i)^{\mathrm{T}} \hat{\lambda}^{-1} (L_n x_i) \\
&\leq \gamma d^2 \hat{\lambda}^2 (L_n x_i)^{\mathrm{T}} P_n (L_n x_i) \\
&= d^2 \hat{\lambda}^2 x_i^{\mathrm{T}} P x_i.
\end{aligned}$$

With this, $\dot{V}_i(t, x_i)$ can be continued as

$$\begin{aligned}
\dot{V}_i(t, x_i) \leq& 2\dot{\gamma} x_i^{\mathrm{T}} P x_i + 2\gamma x_i^{\mathrm{T}} (A^{\mathrm{T}} P + PA - Pbb^{\mathrm{T}} P) x_i + 2\gamma\dot{\gamma} x_i^{\mathrm{T}} \frac{\mathrm{d}P}{\mathrm{d}\gamma} x_i - 4\lambda\psi_i^2 \gamma x_i^{\mathrm{T}} Pbb^{\mathrm{T}} P x_i \\
& + 4\gamma \left(\lambda\psi_i^2 x_i^{\mathrm{T}} Pbb^{\mathrm{T}} P x_i + \frac{\delta^2}{4\lambda} \right) + 4\gamma \left(k_0 x_i^{\mathrm{T}} P x_i + \frac{\phi_i^{\mathrm{T}} P \phi_i}{k_0} \right) \\
\leq& 2\dot{\gamma} x_i^{\mathrm{T}} P x_i - 2\gamma^2 x_i^{\mathrm{T}} P x_i + 2\gamma\dot{\gamma} x_i^{\mathrm{T}} \frac{\mathrm{d}P}{\mathrm{d}\gamma} x_i + \frac{\gamma\delta^2}{\lambda} + 4\gamma k_0 x_i^{\mathrm{T}} P x_i + \frac{4\gamma d^2 \hat{\lambda}^2}{k_0} x_i^{\mathrm{T}} P x_i \\
\leq& 2\dot{\gamma} x_i^{\mathrm{T}} P x_i - \gamma^2 x_i^{\mathrm{T}} P x_i + 2\gamma\dot{\gamma} \frac{\delta_{\mathrm{c}}}{n\gamma} x_i^{\mathrm{T}} P x_i + \frac{\gamma\delta^2}{\lambda} + 4\gamma k_0 x_i^{\mathrm{T}} P x_i + \frac{4\gamma d^2 \hat{\lambda}^2}{k_0} x_i^{\mathrm{T}} P x_i \\
=& \left(2\dot{\gamma} - 2\gamma^2 + 2\dot{\gamma} \frac{\delta_{\mathrm{c}}}{n} + 8d\hat{\lambda}\gamma \right) x_i^{\mathrm{T}} P x_i + \frac{\gamma\delta^2}{\lambda} \\
\triangleq& \pi(\gamma) x_i^{\mathrm{T}} P x_i - s\gamma V_i(t, x_i) + \frac{\gamma\delta^2}{\lambda},
\end{aligned}$$

where we have taken $k_0 = d\hat{\lambda}$. It follows from (3.4) and (3.7) that

$$\pi(\gamma) = \frac{2(n+\delta_c)}{n}\left(\dot{\gamma} - \frac{n(1-s)}{(n+\delta_c)}\gamma^2 + \frac{4nd\hat{\lambda}}{(n+\delta_c)}\gamma\right) = 0.$$

Therefore, by using the comparison lemma in [4], $V_i(t, x_i)$ satisfies, for $t \in [0, T)$,

$$\begin{aligned} V_i(t,x_i) &\leq \exp\left(-s\int_0^t \gamma(\tau)\mathrm{d}\tau\right)V_i(0,x_i(0)) + \frac{\delta^2}{\lambda}\int_0^t \exp\left(-s\int_\tau^t \gamma(s)\mathrm{d}s\right)\gamma(\tau)\mathrm{d}\tau \\ &\leq \left(1-\frac{t}{T}\right)^{s\gamma_0 T} V_i(0,x(0)) + \frac{\delta^2}{\lambda}\int_0^t \exp\left(-s\int_\tau^t \gamma(s)\mathrm{d}s\right)\gamma(\tau)\mathrm{d}\tau \\ &= \left(1-\frac{t}{T}\right)^{s\gamma_0 T} V_i(0,x_i(0)) + \frac{\delta^2}{\lambda s}\left(1 - \frac{(T-t)^{s\gamma_0 T}}{T^{s\gamma_0 T}}\right). \end{aligned} \tag{3.9}$$

In view of

$$V_i(t,x_i) = 2\gamma x_i^{\mathrm{T}} P(\gamma) x_i \geq 2\lambda_{\min}(P(\gamma_0))\gamma(t)\left\|x_i(t)\right\|^2,$$

it follows from (3.9) that

$$\|x_i(t)\|^2 \leq \frac{1}{2\lambda_{\min}(P(\gamma))\gamma(t)}\left(1-\frac{t}{T}\right)^{s\gamma_0 T} V_i(0,x_i(0)) + \frac{1}{2\lambda_{\min}(P(\gamma))\gamma(t)}\frac{\delta^2}{\lambda s}\left(1-\frac{(T-t)^{s\gamma_0 T}}{T^{s\gamma_0 T}}\right),$$

namely, $\lim_{t\uparrow T}\|x_i(t)\| = 0$.

Choose the Lyapunov-like function

$$V(t,x) = \sum_{i=1}^{6} V_i(t,x_i).$$

According to (3.9), it is not difficult to show that

$$\dot{V}(t,x) \leq -s\gamma V(t,x) + \frac{\gamma\delta^2}{\lambda}, \quad t \in [0,T).$$

By using the comparison lemma in [4], $V(t, x)$ satisfies

$$V(t,x) \leq \left(1-\frac{t}{T}\right)^{s\gamma_0 T} V(0,x(0)) + \frac{\delta^2}{\lambda s}\left(1-\frac{(T-t)^{s\gamma_0 T}}{T^{s\gamma_0 T}}\right),$$

namely, $\lim_{t\uparrow T}\|x(t)\| = 0$. Next we prove that the controller (3.6) is bounded. Clearly, we just need to prove that $b^{\mathrm{T}}Px_i$ is bounded for $t \in [0, T)$. According to (3.9) and (5.5) in Lemma 2 in Appendix, we obtain, for $t \in [0, T)$,

$$\begin{aligned} \left\|b^{\mathrm{T}}Px_i\right\| &= x_i^{\mathrm{T}}Pbb^{\mathrm{T}}Px_i \\ &\leq x_i^{\mathrm{T}}P^{\frac{1}{2}}\mathrm{tr}(P^{\frac{1}{2}}bb^{\mathrm{T}}P^{\frac{1}{2}})P^{\frac{1}{2}}x_i \\ &= 2\gamma x_i^{\mathrm{T}}Px_i \\ &= V_i(t,x_i) \\ &\leq \left(1-\frac{t}{T}\right)^{s\gamma_0 T} V_i(0,x_i(0)) + \frac{\delta^2}{\lambda s}\left(1-\frac{(T-t)^{s\gamma_0 T}}{T^{s\gamma_0 T}}\right). \end{aligned}$$

The proof is finished. □

Remark 1. *It can be observed from (3.7) that* $\lim_{t\uparrow T}\gamma(t)=\infty$*, which may lead to some numerical problems in the simulation. According to [22], we can replace it by, for any* $t\in[0,T)$*,*

$$\gamma(t)=\begin{cases}\frac{e^{\alpha\beta T}-1}{e^{\alpha\beta(T+\varepsilon)}-e^{\alpha\beta t}}\gamma_0, & d\neq 0,\\ \frac{T}{T+\varepsilon-t}\gamma_0, & d=0,\end{cases}$$

with ε *being a small positive constant.*

3.2. Case 2: Matched uncertainties by adaptive control

Consider the system (2.8) in the form of

$$\dot{x}_i=Ax_i+b(u_i+\theta_i\psi_i), \tag{3.10}$$

where $x_i=[x_{i1},x_{i2}]^{\mathrm{T}}$, $\psi_i=\psi_i(t,x_i)$ is a known function and is bounded if t and x_i are bounded. The following assumption is imposed on system (3.10):

Assumption 2. *The known nonlinear smooth function* $\psi_i(t,x_i)$ *satisfies*

$$\lim_{\|x_i\|\to 0}\frac{\psi_i(t,x_i)}{\|x_i\|}<\infty.$$

Theorem 2. *Let Assumption 2 be satisfied,* T *be a prescribed time,* $\lambda>0$ *be a constant and* γ_0 *be a constant satisfying*

$$\gamma_0\geq\frac{2+\delta_c}{T}.$$

Consider the controller

$$u_i=-b^{\mathrm{T}}P(\gamma)x_i+v_i, \tag{3.11}$$

$$v_i=-\hat{\theta}_i\psi_i, \tag{3.12}$$

$$\dot{\hat{\theta}}_i=-2\gamma x_i^{\mathrm{T}}Pb\psi_i, \tag{3.13}$$

$$\gamma=\frac{T}{T-t}\gamma_0. \tag{3.14}$$

Then the state of the closed-loop system consisting of (3.10) and (3.11)–(3.14) converges to zero at the prescribed time T*, and the control is bounded.*

Proof. Choose the Lyapunov-like function

$$V_i=V_i(t,x_i,\tilde{\theta}_i)=2\gamma x_i^{\mathrm{T}}P(\gamma)x_i+\tilde{\theta}_i^2,$$

where $\tilde{\theta}_i=\theta_i-\hat{\theta}_i$. The time-derivative of V_i along (3.10) and (3.11)–(3.14) can be written as

$$\begin{aligned}\dot{V}_i=&2\dot{\gamma}x_i^{\mathrm{T}}Px_i+2\gamma\dot{x}_i^{\mathrm{T}}Px_i+2\gamma\dot{\gamma}x_i^{\mathrm{T}}\frac{\mathrm{d}P}{\mathrm{d}\gamma}x_i+2\gamma x_i^{\mathrm{T}}P\dot{x}_i+2\tilde{\theta}_i\dot{\tilde{\theta}}_i\\ =&2\dot{\gamma}x_i^{\mathrm{T}}Px_i+2\gamma x_i^{\mathrm{T}}(A^{\mathrm{T}}P+PA-Pbb^{\mathrm{T}}P)x_i\\ &+2\gamma\dot{\gamma}x_i^{\mathrm{T}}\frac{\mathrm{d}P}{\mathrm{d}\gamma}x_i+4\gamma x_i^{\mathrm{T}}Pbv_i+4\gamma x_i^{\mathrm{T}}Pb\theta_i\psi_i+2\tilde{\theta}_i\dot{\tilde{\theta}}_i\end{aligned}$$

$$
\begin{aligned}
&\le 2\dot{\gamma}x_i^{\mathrm{T}}Px_i - 2\gamma^2 x_i^{\mathrm{T}}Px_i + 2\gamma\dot{\gamma}\frac{\delta_{\mathrm{c}}}{n\gamma}x_i^{\mathrm{T}}Px_i + 4\gamma x_i^{\mathrm{T}}Pb(v_i + \theta_i\psi_i) + 2\tilde{\theta}_i\dot{\hat{\theta}}_i \\
&= 2\dot{\gamma}x_i^{\mathrm{T}}Px_i - \gamma^2 x_i^{\mathrm{T}}Px_i + 2\gamma\dot{\gamma}\frac{\delta_{\mathrm{c}}}{n\gamma}x_i^{\mathrm{T}}Px_i + 2\tilde{\theta}_i(2\gamma x_i^{\mathrm{T}}Pb\psi + \dot{\hat{\theta}}_i) \\
&= \left(2\dot{\gamma} - 2\gamma^2 + 2\gamma\dot{\gamma}\frac{\delta_{\mathrm{c}}}{n\gamma}\right)x_i^{\mathrm{T}}Px_i \\
&= 0,
\end{aligned}
$$

namely,

$$
V_i(t, x_i(t), \tilde{\theta}_i(t)) \le V_i(0, x_i(0), \tilde{\theta}_i(0)).
$$

In view of

$$
V_i(t, x_i, \tilde{\theta}_i) = 2\gamma x_i^{\mathrm{T}}P(\gamma)x_i + \tilde{\theta}_i^2 \ge 2\lambda_{\min}(L_nP_nL_n)\gamma^2(t)\,\|x_i(t)\|^2\,,
$$

it follows from (3.9) that

$$
\|x_i(t)\|^2 \le \frac{V_i(0, x_i(0), \tilde{\theta}_i(0))}{2\lambda_{\min}(L_nP_nL_n)\gamma^2(t)},
$$

namely, $\lim_{t\uparrow T}\|x_i(t)\| = 0$. According to (3.9) and (5.5) in Lemma 2 in Appendix, we obtain, for $t \in [0, T)$,

$$
\begin{aligned}
\left\|b^{\mathrm{T}}Px_i\right\|^2 &= x_i^{\mathrm{T}}Pbb^{\mathrm{T}}Px_i \\
&\le x_i^{\mathrm{T}}P^{\frac{1}{2}}\mathrm{tr}(P^{\frac{1}{2}}bb^{\mathrm{T}}P^{\frac{1}{2}})P^{\frac{1}{2}}x_i \\
&= 2\gamma x_i^{\mathrm{T}}Px_i \\
&\le V_i(t, x_i(t), \tilde{\theta}_i(t)) \\
&\le V_i(0, x_i(0), \tilde{\theta}_i(0)).
\end{aligned}
$$

And

$$
\begin{aligned}
\lim_{t\to T}\left\|\gamma b^{\mathrm{T}}Px_i\right\|^2\psi_i^2(t, x_i) &\le \lim_{t\to T}V_i(0, x_i(0), \tilde{\theta}_i(0))\gamma^2\|x_i\|^2\frac{\psi_i^2(t, x_i)}{\|x_i\|^2} \\
&\le V_i(0, x_i(0), \tilde{\theta}_i(0))\lim_{\|x_i\|\to 0}\frac{V_i(0, x_i(0), \tilde{\theta}_i(0))}{2\lambda_{\min}(L_nP_nL_n)}\frac{\psi_i^2(t, x_i)}{\|x_i\|} \\
&< \infty.
\end{aligned}
$$

The proof is finished. □

3.3. Case 3: Adaptive control with general T-FTE function

Rewrite system (2.5) as

$$
\begin{cases}
\dot{\tilde{\eta}}^k &= G(\tilde{\eta}^k)\tilde{\xi}^k, \\
\Xi^k\dot{\tilde{\xi}}^k &= \mathrm{ad}^*_{\xi^k}\Xi^k\xi^k + \varphi_{\mathrm{g}}^k + \varphi_{\mathrm{d}}^k + \varphi_{\mathrm{c}}^k + \Xi^k(\mathrm{ad}_{\xi^k}\mathrm{Ad}_{(h^k)^{-1}}\xi^0 - \mathrm{Ad}_{(h^k)^{-1}}\dot{\xi}^0).
\end{cases}
\tag{3.15}
$$

For clarification, we omit the superscript $()^k$ in the following:

Assumption 3. *There exists a known function $\psi(x)$ such that*

$$\psi(0) = 0, \quad \varphi_{\mathrm{d}} = \theta\psi,$$

where θ is an unknown parameter.

Theorem 3. *Let Assumption 3 be satisfied, $T > 0$ be a prescribed time and $\gamma(t) : [0, T) \to \mathbf{R}_{>0}$ be a T-FTE function such that*

$$\lim_{t\to T} \gamma^2 \exp\left(-\int_0^t \gamma(s)\mathrm{d}s\right) = 0, \tag{3.16}$$

$$\dot{\gamma} = K\gamma^2, \tag{3.17}$$

with $K \neq 1$, $K \neq 1/2$ and $K > 0$ being a constant. Consider the controller

$$\begin{aligned} \varphi_{\mathrm{c}}^k =& -\Xi^{-1}G^{\mathrm{T}}(\tilde{\eta})\tilde{\eta} - \Xi^k(\mathrm{ad}_{\xi^k}\mathrm{Ad}_{(h^k)^{-1}}\xi^0 - \mathrm{Ad}_{(h^k)^{-1}}\dot{\xi}^0) \\ & - \mathrm{ad}^*_{\xi^k}\Xi^k\xi^k - \varphi_{\mathrm{g}}^k - \hat{\theta}\psi - \gamma(\tilde{\xi} + \gamma\tilde{\eta}) - \dot{\gamma}\tilde{\eta} - \gamma G(\tilde{\eta})\tilde{\xi}, \end{aligned} \tag{3.18}$$

$$\dot{\hat{\theta}}_i = -(\tilde{\xi} + \gamma\tilde{\eta})^{\mathrm{T}}\psi. \tag{3.19}$$

Then the state of the closed-loop system consisting of (3.15), (3.18) and (3.19) converges to zero at the prescribed time T, and the control is bounded.

Proof. Choose the Lyapunov function

$$V_1 = \frac{1}{2}\tilde{\eta}^{\mathrm{T}}\tilde{\eta},$$

whose time-derivative along system (3.15) can be written as

$$\dot{V}_1 = \dot{\tilde{\eta}}^{\mathrm{T}}\tilde{\eta} = \tilde{\eta}^{\mathrm{T}}(G(\tilde{\eta})\tilde{\xi}_{\mathrm{r}} - G(\tilde{\eta})\tilde{\xi}_{\mathrm{r}} + G(\tilde{\eta})\tilde{\xi}) = -\gamma(t)\tilde{\eta}^{\mathrm{T}}\tilde{\eta} + \tilde{\eta}^{\mathrm{T}}G(\tilde{\eta})\tilde{\xi}_{\mathrm{e}},$$

where we have used $G(\tilde{\eta})\tilde{\eta} = \tilde{\eta}$ [1, 7], $\tilde{\xi}_{\mathrm{e}} = \tilde{\xi} - \tilde{\xi}_{\mathrm{r}}$, and the virtual controller is given as

$$\tilde{\xi}_{\mathrm{r}} = -\gamma\tilde{\eta}. \tag{3.20}$$

Therefore, we can obtain

$$\dot{\tilde{\xi}}_{\mathrm{r}} = -\dot{\gamma}\tilde{\eta} - \gamma\dot{\tilde{\eta}} = -\dot{\gamma}\tilde{\eta} - \gamma G(\tilde{\eta})\tilde{\xi}.$$

Choose the new Lyapunov function

$$V_2 = V_1 + \frac{1}{2}\tilde{\xi}_{\mathrm{e}}^{\mathrm{T}}\Xi\tilde{\xi}_{\mathrm{e}} + \frac{1}{2}\tilde{\theta}^2,$$

whose time-derivative along (3.15), (3.18) and (3.19) can be written as

$$\begin{aligned} \dot{V}_2 =& \dot{V}_1 + \tilde{\xi}_{\mathrm{e}}^{\mathrm{T}}\Xi\dot{\tilde{\xi}}_{\mathrm{e}} + \tilde{\theta}\dot{\hat{\theta}}_i \\ =& -\gamma(t)\tilde{\eta}^{\mathrm{T}}\tilde{\eta} + \tilde{\eta}^{\mathrm{T}}G(\tilde{\eta})\tilde{\xi}_{\mathrm{e}} + \tilde{\theta}\dot{\hat{\theta}}_i + \tilde{\xi}_{\mathrm{e}}^{\mathrm{T}}(v + \theta\psi) \\ =& -\gamma\tilde{\eta}^{\mathrm{T}}\tilde{\eta} - \gamma\tilde{\xi}_{\mathrm{e}}^{\mathrm{T}}\Xi\tilde{\xi}_{\mathrm{e}} + \tilde{\theta}(\tilde{\xi}_{\mathrm{e}}^{\mathrm{T}}\psi + \dot{\hat{\theta}}_i) \\ =& -\gamma\tilde{\eta}^{\mathrm{T}}\tilde{\eta} - \gamma\tilde{\xi}_{\mathrm{e}}^{\mathrm{T}}\Xi\tilde{\xi}_{\mathrm{e}}. \end{aligned}$$

By using Theorem 1 in [3], we can get $\lim_{t\uparrow T}\|\tilde{\eta}\|=0$ and $\lim_{t\uparrow T}\left\|\tilde{\xi}_{\mathrm{e}}\right\|=0$. On the one hand, we have

$$\dot{\tilde{\eta}}=G(\tilde{\eta})\tilde{\xi}=-\gamma\tilde{\eta}+G(\tilde{\eta})\tilde{\xi}_{\mathrm{e}}\triangleq-\gamma\tilde{\eta}+\sigma_1, \tag{3.21}$$

which can be solved as

$$\tilde{\eta}=\exp\left(-\int_0^t\gamma(s)\mathrm{d}s\right)\tilde{\eta}(0)+\exp\left(-\int_0^t\gamma(s)\mathrm{d}s\right)\int_0^t\sigma_1(\tau)\mathrm{e}^{\int_0^\tau\gamma(s)\mathrm{d}s}\mathrm{d}\tau.$$

Then by using the L'Hospital rules, (3.16) and (3.17), we have

$$\begin{aligned}\lim_{t\to T}\gamma\tilde{\eta}&=\lim_{t\to T}\gamma\exp\left(-\int_0^t\gamma(s)\mathrm{d}s\right)\tilde{\eta}(0)+\lim_{t\to T}\gamma\exp\left(-\int_0^t\gamma(s)\mathrm{d}s\right)\int_0^t\sigma_1(\tau)\mathrm{e}^{\int_0^\tau\gamma(s)\mathrm{d}s}\mathrm{d}\tau\\&=\lim_{t\to T}\frac{\int_0^t\sigma_1(\tau)\mathrm{e}^{\int_0^\tau\gamma(s)\mathrm{d}s}\mathrm{d}\tau}{\gamma^{-1}\exp(\int_0^t\gamma(s)\mathrm{d}s)}\\&=\lim_{t\to T}\frac{\sigma_1(t)\mathrm{e}^{\int_0^t\gamma(s)\mathrm{d}s}}{-2\gamma^{-2}\dot{\gamma}\exp(\int_0^t\gamma(s)\mathrm{d}s)+\exp(\int_0^t\gamma(s)\mathrm{d}s)}\\&=\lim_{t\to T}\frac{\sigma_1(t)}{1-2K}.\end{aligned}$$

Since $\lim_{t\uparrow T}\|\tilde{\eta}\|+\lim_{t\uparrow T}\left\|\tilde{\xi}_{\mathrm{e}}\right\|=0$, we have $\sigma_1(t)=0$, which implies $\lim_{t\uparrow T}\gamma\tilde{\eta}=0$. Then from (3.20) and (3.21), we can get that $\lim_{t\uparrow T}\dot{\tilde{\eta}}=0$ and $\lim_{t\uparrow T}\tilde{\xi}_{\mathrm{r}}=0$. In view of $\tilde{\xi}_{\mathrm{e}}=\tilde{\xi}-\tilde{\xi}_{\mathrm{r}}$, it can be obtained that

$$\lim_{t\to T}\left|\tilde{\xi}\right|\leq\lim_{t\to T}\left|\tilde{\xi}_{\mathrm{e}}\right|+\lim_{t\to T}\left|\tilde{\xi}_{\mathrm{r}}\right|=0.$$

On the other hand, according to (3.15) and (3.18), we can get

$$\begin{aligned}\dot{\tilde{\xi}}_{\mathrm{e}}&=\dot{\tilde{\xi}}-\dot{\tilde{\xi}}_{\mathrm{r}}\\&=-\gamma\tilde{\xi}_{\mathrm{e}}-\Xi^{-1}G^{\mathrm{T}}(\tilde{\eta})\tilde{\eta}-\tilde{\theta}\psi\\&\triangleq-\gamma\tilde{\xi}_{\mathrm{e}}+\sigma_2,\end{aligned}$$

which can be solved as

$$\tilde{\xi}_{\mathrm{e}}=\exp\left(-\int_0^t\gamma(s)\mathrm{d}s\right)\tilde{\xi}_{\mathrm{e}}(0)+\exp\left(-\int_0^t\gamma(s)\mathrm{d}s\right)\int_0^t\sigma_2(\tau)\mathrm{e}^{\int_0^\tau\gamma(s)\mathrm{d}s}\mathrm{d}\tau.$$

Therefore, it follows from the L'Hospital rules, (3.16) and (3.17) that

$$\begin{aligned}\lim_{t\to T}\gamma\tilde{\xi}_{\mathrm{e}}&=\lim_{t\to T}\gamma\exp\left(-\int_0^t\gamma(s)\mathrm{d}s\right)\left(\tilde{\xi}_{\mathrm{e}}(0)+\int_0^t\sigma_2(\tau)\mathrm{e}^{\int_0^\tau\gamma(s)\mathrm{d}s}\mathrm{d}\tau\right)\\&=\lim_{t\to T}\gamma\exp\left(-\int_0^t\gamma(s)\mathrm{d}s\right)\int_0^t\sigma_2(\tau)\mathrm{e}^{\int_0^\tau\gamma(s)\mathrm{d}s}\mathrm{d}\tau\\&=\lim_{t\to T}\frac{\int_0^t\sigma_2(\tau)\mathrm{e}^{\int_0^\tau\gamma(s)\mathrm{d}s}\mathrm{d}\tau}{\gamma^{-1}\exp(\int_0^t\gamma(s)\mathrm{d}s)}\end{aligned}$$

$$
\begin{aligned}
&=\lim_{t\to T}\frac{\sigma_2(t)e^{\int_0^t\gamma(s)ds}}{-\gamma^{-2}\dot\gamma\exp(\int_0^t\gamma(s)ds)+\exp(\int_0^t\gamma(s)ds)}\\
&=\lim_{t\to T}\frac{\sigma_2(t)}{1-K}\\
&=\frac{1}{1-K}\lim_{t\to T}(-\Xi^{-1}G^{\mathrm{T}}(\tilde\eta)\tilde\eta+\tilde\theta\psi)\\
&=0,
\end{aligned}\tag{3.22}
$$

where we have used $\psi(0)=0$. Then, we have

$$\lim_{t\to T}\gamma\sigma_1=\lim_{t\to T}\gamma G(\tilde\eta)\tilde\xi_{\mathrm{e}}=0.$$

By using the L'Hospital rules, (3.16) and (3.17), we can get

$$
\begin{aligned}
\lim_{t\to T}\gamma^2\tilde\eta&=\lim_{t\to T}\gamma^2\exp\left(-\int_0^t\gamma(s)ds\right)\tilde\eta(0)+\lim_{t\to T}\gamma^2\exp\left(-\int_0^t\gamma(s)ds\right)\int_0^t\sigma_1(\tau)e^{\int_0^\tau\gamma(s)ds}d\tau\\
&=\lim_{t\to T}\frac{\int_0^t\sigma_1(\tau)e^{\int_0^\tau\gamma(s)ds}d\tau}{\gamma^{-2}\exp(\int_0^t\gamma(s)ds)}\\
&=\lim_{t\to T}\frac{\sigma_1(t)e^{\int_0^t\gamma(s)ds}}{-2\gamma^{-3}\dot\gamma\exp(\int_0^t\gamma(s)ds)+\gamma^{-1}\exp(\int_0^t\gamma(s)ds)}\\
&=\lim_{t\to T}\frac{\sigma_1(t)}{-2\gamma^{-3}\dot\gamma+\gamma^{-1}}\\
&=\lim_{t\to T}\frac{\sigma_1(t)}{-2K\gamma^{-1}+\gamma^{-1}}\\
&=\lim_{t\to T}\frac{\gamma\sigma_1(t)}{1-2K}\\
&=0.
\end{aligned}\tag{3.23}
$$

Finally, we prove that the controller (3.18) is bounded. Clearly, we just need to prove that $\gamma\tilde\xi$, $\dot\gamma\tilde\eta$ and $\gamma G(\tilde\eta)\tilde\xi$ are bounded as t tends to T. According to (3.16), (3.17), (3.22), (3.23) and $\lim_{\tilde\eta\uparrow 0}G(\tilde\eta)=I_6$ [1, 7], we can get

$$
\begin{aligned}
&\lim_{t\to T}\gamma\left\|\tilde\xi\right\|\le\lim_{t\to T}\gamma\left\|\tilde\xi_{\mathrm{e}}\right\|+\lim_{t\to T}\gamma\left\|\tilde\xi_{\mathrm{r}}\right\|=\lim_{t\to T}\gamma\left\|\tilde\xi_{\mathrm{e}}\right\|+\lim_{t\to T}\gamma^2\left\|G^{-1}(\tilde\eta)\tilde\eta\right\|=0.\\
&\lim_{t\to T}\|\dot\gamma\tilde\eta\|=K\lim_{t\to T}\left\|\gamma^2\tilde\eta\right\|=0,\\
&\lim_{t\to T}\left\|\gamma G(\tilde\eta)\tilde\xi\right\|=\lim_{t\to T}\left\|\gamma\tilde\xi\right\|=0.
\end{aligned}
$$

The proof is finished. □

In Theorem 3, it is not difficult to satisfy Conditions (3.16) and (3.17). For example, similar to (3.14), we can take $\gamma(t)=\eta/(T-t)$. Clearly, when $K=1/\eta$ and $\eta>2$, (3.16) and (3.17) are satisfied.

4. Numerical simulations

In this section, a numerical simulation is given to verify the proposed approaches. The mass and the moment of inertia matrix of each spacecraft is $m = 8\text{kg}$ and $J = 0.1\text{diag}\{22, 20, 23\}\,\text{kg}\cdot\text{m}^2$, respectively. According to (5.3), we can get $\delta_c = 6.8284$ and $\hat{\lambda} = 2.618$. Take the prescribed time as $T = 100\text{s}$. For simplicity, similar to [11], the initial configurations and velocities of the leader spacecraft, and initial relative configurations and velocities of the follower spacecraft with respect to the leader are given by

$$g_L = \begin{bmatrix} 0.7956 & -0.2435 & 0.5547 & 3650.2\times10^3 \\ 0.6053 & 0.2839 & -0.7436 & -2526.9\times10^3 \\ 0.0236 & 0.9274 & 0.3733 & -5651.2\times10^3 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$g_{F/L} = \begin{bmatrix} -0.5754 & 0.2821 & 0.7677 & 3640.8\times10^3 \\ 0.8012 & 0.3830 & 0.4598 & -2829.2\times10^3 \\ -0.1643 & 0.8796 & -0.4464 & -5648.6\times10^3 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

and

$$V_L = \begin{bmatrix} 0 & 0 & 0.0150 & 9.7572\times10^3 & 0 & 0 \end{bmatrix}^T,$$
$$V_{F/L} = \begin{bmatrix} 0 & 0 & 0.0075 & 9.9737\times10^3 & 0 & 0 \end{bmatrix}^T,$$

where the displacements are meters, the velocities are in meters per second and angular velocities are radians per second.

Let the desired relative configuration between the leader spacecraft and the follower spacecraft be defined as

$$g_{dL/F} = \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Similar to [6], the unknown external disturbances on the follower spacecrafts are given by

$$\tau_d = [2, -2, -1.5]^T \cos(2\pi nt)\times 10^{-7}\text{Nm},$$
$$f_d = [1.92, -1.906, -1.517]^T \sin(2\pi nt)\times 10^{-5}\text{N},$$

where $n = \pi/6$. Here, we consider two cases. Case I: consider $\phi = 0$, namely $d = 0$. We take $s = 0.1$ and $\gamma_0 = 0.10501$. Clearly, (3.4) is satisfied. Case II: consider

$$\phi_1 = 5\times10^{-3}\begin{bmatrix} x_1 \\ \dot{x}_1 \end{bmatrix},\quad \phi_2 = 5\times10^{-3}\begin{bmatrix} x_2 \\ \dot{x}_2 \end{bmatrix},$$
$$\phi_3 = 5\times10^{-3}\begin{bmatrix} x_3 \\ \dot{x}_3 \end{bmatrix},\quad \phi_4 = 1\times10^{-4}\begin{bmatrix} 134x_4 \\ 50\dot{x}_3 \end{bmatrix},$$

$$\phi_5 = 1 \times 10^{-4} \begin{bmatrix} 134x_5 \\ 50\dot{x}_5 \end{bmatrix}, \quad \phi_6 = 1 \times 10^{-4} \begin{bmatrix} 134x_6 \\ 50\dot{x}_6 \end{bmatrix}.$$

We take $s = 0.1$. Then by (3.5) and (3.4), it follows that $\alpha = (1-s)0.11327 = 0.101943$, $\beta = 0.3972648/(1-s) = 0.44140533$ and

$$\frac{\beta}{1-\mathrm{e}^{-\alpha\beta T}} = \frac{0.44140533}{1-\mathrm{e}^{-0.11327\times 0.3972648\times T}} = 0.4464.$$

For $i = 1, 2, \ldots, 6$, we choose different initial values γ_0, denoted as γ_{01}, γ_{02}, γ_{03}, γ_{04}, γ_{05}, and γ_{06}. In order to satisfy (3.4), we can take $\gamma_{01} = 40/(\mathrm{e}^{\alpha\beta T}-1) = 0.4496$, $\gamma_{02} = 40/(\mathrm{e}^{\alpha\beta T}-1) = 0.4496$, $\gamma_{03} = 40/(\mathrm{e}^{\alpha\beta T}-1) = 0.4496$, $\gamma_{04} = 72/(\mathrm{e}^{\alpha\beta T}-1) = 0.8093$, $\gamma_{05} = 72/(\mathrm{e}^{\alpha\beta T}-1) = 0.8093$, and $\gamma_{06} = 72/(\mathrm{e}^{\alpha\beta T}-1) = 0.8093$. The tracking errors of the attitude and position for the follower spacecraft are plotted in Figures 1 and 4, while the tracking errors of the angular velocity and velocity are plotted in Figures 2 and 5. In addition, the control inputs are plotted in Figures 3 and 6. It can be observed from Figures 1, 2, 4 and 5 that the tracking errors converge to zero in the prescribed time $T = 100$ s, and the control inputs are bounded.

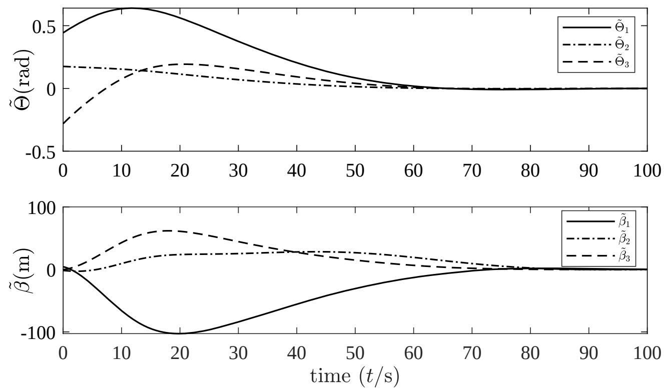

Figure 1. The tracking errors of the attitude and position for the follower spacecraft in Case I.

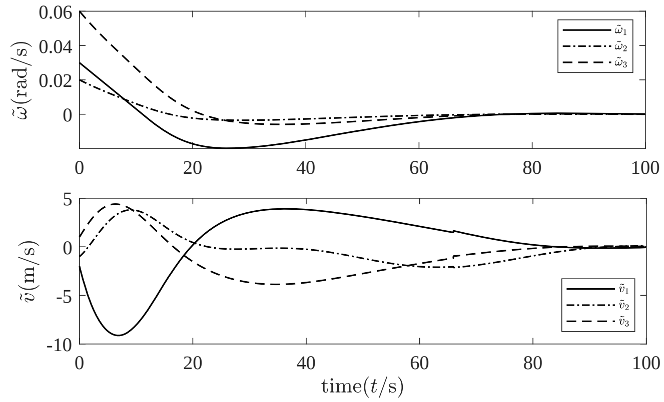

Figure 2. The tracking errors of the angular velocity and velocity for the follower spacecraft in Case I.

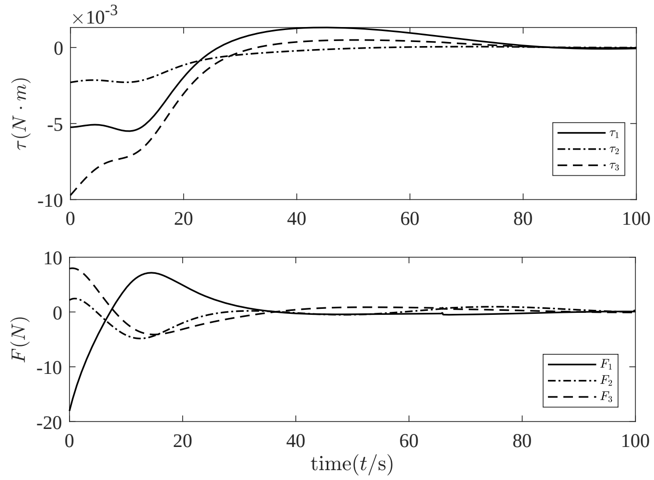

Figure 3. The control inputs for the follower spacecraft in Case I.

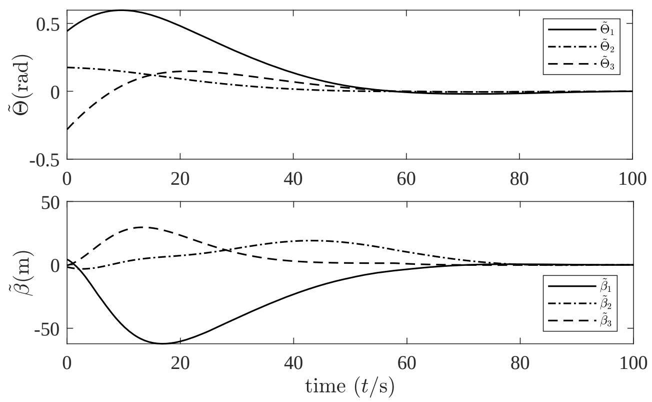

Figure 4. The tracking errors of the attitude and position for the follower spacecraft in Case II.

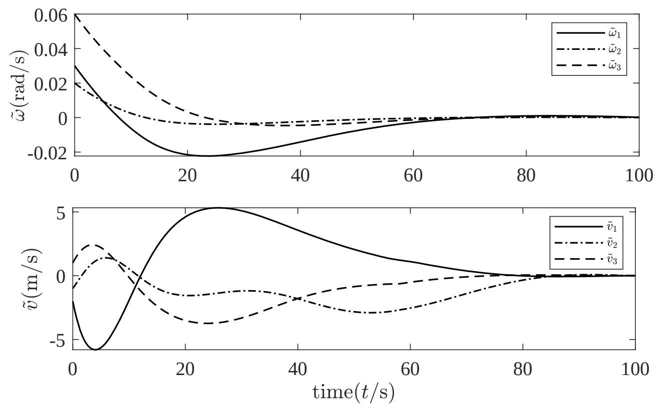

Figure 5. The tracking errors of the angular velocity and velocity for the follower spacecraft in Case II.

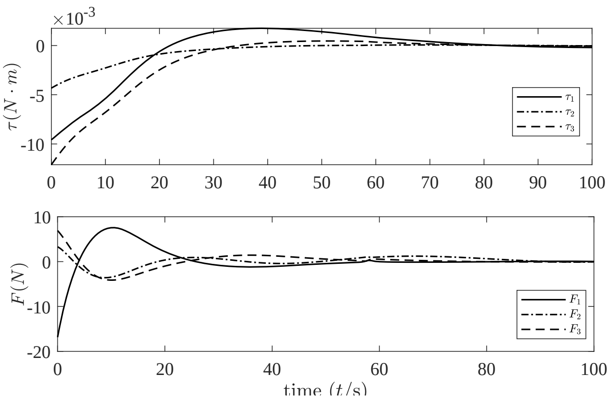

Figure 6. The control inputs for the follower spacecraft in Case II.

5. Conclusions

The prescribed-time spacecraft formation flying problem under uncertainties and unknown disturbances has been investigated. Firstly, based on Lie group SE(3), the coupled 6-degrees-of-freedom kinematics and dynamics for spacecraft under uncertainties and unknown disturbances were modeled. Secondly, with the aid of some key properties of a class of parametric Lyapunov equations, novel prescribed-time control laws were designed. It was proved that the proposed control laws can drive the relative motion between the leader spacecraft and follower spacecraft to zero in any prescribed time and are bounded. Finally, numerical simulations have demonstrated the effectiveness of the proposed control scheme. By simulation we observer that, if the convergence time is set to be small, then the magnitude of the control will be large, leading to actuator saturation, which will be studied in our future work.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare no conflicts of interest.

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Appendix

In this Appendix, we will give some properties of PLE (3.1).

Lemma 2. *[25] Let (A, b) be given by (2.9), $\gamma > 0$, $P(\gamma)$ be the unique positive definite solution to the PLE (3.1).*

(1) There holds

$$\frac{1}{n\gamma}P(\gamma) \leq \frac{\mathrm{d}P(\gamma)}{\mathrm{d}\gamma} \leq \frac{\delta_{\mathrm{c}}}{n\gamma}P(\gamma), \tag{5.1}$$

where

$$P(\gamma) = \gamma L_n P_n L_n,$$

with $P_n = P(1)$ and

$$L_n = L_n(\gamma) = \mathrm{diag}\left\{\gamma^{n-1}, \gamma^{n-2}, \ldots, 1\right\}, \tag{5.2}$$

$$\delta_{\mathrm{c}} = n(1 + \lambda_{\max}(E_n + P_n E_n P_n^{-1})), \tag{5.3}$$

in which $E_n = \mathrm{diag}\{n-1, n-2, \ldots, 1, 0\}$.

(2) There holds

$$\hat{\lambda} = \lambda_{\max}(P_n) = \lambda_{\min}^{-1}(P_n). \tag{5.4}$$

(3) The following equation holds

$$\mathrm{tr}(b^{\mathrm{T}} P b) = n\gamma. \tag{5.5}$$

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