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Research article

A new approach of overlapping generation model via fixed point technique

Abdelkader Belhenniche[1,2], **Monica-Felicia Bota**[3,*] **and Liliana Guran**[4]

[1] Research Center for Systems and Technologies (SYSTEC), ARISE, Faculty of Engineering, University of Porto, Rua Dr. Roberto Frias s/n, office i102, 4200-465 Porto, Portugal

[2] Laboratoire d'Etudes Pratiques en Sciences de Gestion et Sciences Commerciales, Ecole Supérieure de Commerce, 42003 Kolea, Tipaza, Algérie

[3] Department of Mathematics, Babeş-Bolyai University, M. Kogălniceanu Street, No. 1, 400084, Cluj-Napoca, Romania

[4] Department of Hospitality Services, Babeş-Bolyai University, Horea Street, No. 7, 400174, Cluj-Napoca, Romania

* **Correspondence:** Email: monica.bota@ubbcluj.ro.

Abstract: This article explored a specific variant of the overlapping generation model using a nonlinear Fredholm integral equation. We considered assumptions related to the Ćirić operator, offering a new perspective compared to existing research. To solve the equation, we employed the Galerkin method, which approximates it as a finite system of equations. By combining these approaches, we conducted a comprehensive analysis of the model, providing insights into its dynamics and potential applications.

Keywords: overlapping generation model; nonlinear Fredholm equation; Ćirić contractive maps; Galerkin method; fixed point theory
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1. Introduction and preliminaries

A strong and essential paradigm in the study of economics called the overlapping generation (OLG) model has important implications for forecasting future wages. By taking into account the entire economy, this model offers a holistic viewpoint that makes it possible to identify and address underlying problems that have an impact on economic dynamics and outcomes.

The OLG model is based on the idea that there are two different generations: The old generation and the new generation. These generational cohorts differ in their economic roles and behaviors, which define them. The awareness that people continuously enter and exit the economy throughout time, resulting in an overlapping of generations, is a crucial component of the OLG model.

The elder generation is often unable to actively contribute to the workforce due to factors such as retirement or diminished labor market involvement. They instead rely on consuming resources and things. The younger generation, on the other hand, is in the active working phase of their lives, where they play an important part in both production and savings.

To fully realize the OLG model's potential, it is critical to account for and study the interplay between the two generations. This entails examining consumption patterns, output levels, and savings accumulation across various age groups. Economists can acquire insights into the dynamics of resource allocation, economic growth and wealth distribution within a society by doing so.

The use of the OLG model in prior investigations adds to its significance. The understanding and development of this economic idea has been enriched by the contributions of notable researchers, including [1–3]. Their studies have set the way for a more in-depth examination of the intricacies and ramifications of overlapping generations, revealing important insights into the economic landscape.

In [4], I. Fredholm presented an uprising result in the theory of nonlinear integral equations, and it was defined by the fixed limits of the sum up of the data function and integration of the kernel function multiply by parameter.

The OLG model considers intergenerational dynamics as well as the interaction of various age cohorts. Nonlinear Fredholm integral equations offer a mathematical framework for analyzing and modeling complex nonlinear connections in the setting of overlapping generations. Researchers can investigate many aspects of the model, such as consumption patterns, savings behavior, resource allocation, and the influence of policy actions, by generating and solving these integral equations. The application of nonlinear Fredholm integral equations allows for a better understanding of the dynamics and consequences of the overlapping generation model, which contributes to advances in economic theory and policy analysis.

Furthermore, it is noteworthy that nonlinear equations possess the capacity to model a wide range of practical physical systems, underscoring the significance of addressing nonlinear equation solutions. Recent years have seen substantial advancements in this field, making it highly advisable for the author to reference pertinent literature pertaining to recent developments in solving nonlinear equations, (refer to [5–7]).

The Galerkin method introduced in [8] is a powerful numerical technique used for solving integral and differential equations like Fredholm equations. It is widely employed in scientific and engineering problems to approximate solutions efficiently. In this section, we introduce the Galerkin method with linearization, a variation of the standard Galerkin method designed to handle nonlinearities in equations. We outline the steps of the linearization method and demonstrate its effectiveness through numerical examples.

The Galerkin method with linearization provides an effective tool for solving integral and differential equations involving nonlinear terms. By approximating nonlinearities using linear combinations of basis functions, this method simplifies complex problems and produces accurate solutions. Its application to the OLG model showcases the potential of the linearization technique in tackling real-world problems. Numerical examples verify the superior performance of the Galerkin method with linearization, highlighting its importance in numerical analysis and scientific computing (see [9, 10]).

This paper provides a powerful display of the significance of the fixed point theorem to demonstrate how the iterative procedures solve an infinite-horizon problem under the weakest assumptions examined as yet.

In his work [11], Banach establishes that in a Banach space $(X, \|\cdot\|)$, if F is an operator that maps X to itself and satisfies the condition

$$\|Fx - Fy\| \leq \gamma \|x - y\|, \tag{1.1}$$

where $\gamma \in (0, 1)$ and $x, y \in X$, then F possesses a unique fixed point x^*. Furthermore, the sequence $\{x_n\}$ generated by the iterative process $x_{n+1} = Fx_n$ converges to x^* for a certain $x^* \in X$. Since then this concept has been extended by many authors and has obtained more strong fixed point results.

In his work [12], Ćirić introduced an extended version of contractive mappings as follows: a self-map $F : X \to X$ within a Banach space $(X, \|\cdot\|)$ is said to be a Ćirić mapping, if for a specific $\gamma \in (0, 1)$, and for any x and y within X, such that:

$$\|Fx - Fy\| \leq \gamma \max\left\{\|x - y\|, \|x - Fx\|, \|y - Fy\|, \frac{1}{2}\left(\|x - Fy\| + \|y - Fx\|\right)\right\}. \tag{1.2}$$

It is worth noting that there exists two other versions of Ćirić contractive maps, as documented in references [13, 14].

This article holds significance in providing a straightforward functional representation of the equation for the equilibrium fixed point. Specifically, intertemporal prices are the solution to a fixed point problem expressed as $p = F_\sigma p$.

The central innovation of this article resides in showcasing an approach for expressing the challenge of equilibrium fixed points through a broadly nonlinear integral equation. This specific integral equation concerns an unidentified intertemporal price function and requires solving under conditions that are comparatively less stringent than those previously considered. The approach employed in this study leverages the Ćirić operator to achieve these results.

2. Problem formulation

In the following discussion, we examine a model that incorporates overlapping generations with a finite time horizon, as described in [2]. The concept of time is continuous and denoted by $t \in [0, \infty)$.

From a demographic perspective, $\mathcal{G}(t) = (t - l, t]$ represents the set of all generations alive at time $t \geq 0$, where $l > 0$ denotes the lifespan of an individual.

Furthermore, $\mathcal{A}(\upsilon) := \{t \geq 0 \max[0, \upsilon] \leq t < \upsilon + l\}$ represents the interval of time during which generation $\upsilon \in (-l, +\infty)$ is alive, i.e., $[\upsilon, \upsilon + l)$.

Considering the concept of endowments, we have the following:

- $y(t, \upsilon) \geq 0$ represents the amount of endowment for generation υ at time t.
- If $y(t, \upsilon) = 0$, then $Y(t) = \int_{\mathcal{G}(t)} y(t, \upsilon) d\upsilon > 0$, which denotes the aggregate endowment at time $t \geq 0$.
- The ratio $\phi(t, \upsilon) = \frac{y(t,\upsilon)}{Y(t)} \geq 0$ provides the density of the endowment for generation υ.

Each individual expresses preferences for dated consumption goods through the integral function

$$\int_{\mathcal{A}(\upsilon)} e^{-\rho(t-\upsilon)} u[c(t, \upsilon)] dt, \quad u(c) = \frac{c^{1-\sigma} - 1}{1 - \sigma}, \quad \rho \geq 0, \quad \sigma > 0. \tag{2.1}$$

The intertemporal wealth of an individual, given consumption prices p at time $t = 0$, can be expressed as

$$\mathcal{W}(p, \upsilon) = \int_0^{\infty} p(s)y(s, \upsilon)ds + p(0)a(0, \upsilon). \tag{2.2}$$

The optimization problem implies that each individual selects their consumption to maximize the utility in Eq (2.1) while considering their intertemporal constraint:

$$\lambda_\sigma := \left[\int_{\mathcal{A}(\upsilon)} e^{-\frac{\rho}{\sigma}(s-\upsilon)} p(s)^{\frac{\sigma-1}{\sigma}} ds\right] \mathcal{W}(p, \upsilon)^{-\sigma}. \tag{2.3}$$

This problem leads to the consumption function

$$c_\sigma(p, t, \upsilon) := \alpha_\sigma(p, t, \upsilon)\frac{\mathcal{W}(p, \upsilon)}{p(t)}, \tag{2.4}$$

where $\alpha_\sigma(p, t, \upsilon)$ indicates the expenditure shares:

$$\alpha_\sigma(p, t, \upsilon) = \frac{e^{-\frac{\rho}{\sigma}t} p(s)^{\frac{\sigma-1}{\sigma}}}{\int_{\mathcal{A}(\upsilon)} e^{-\frac{\rho}{\sigma}s} p(s)^{\frac{\sigma-1}{\sigma}} ds}. \tag{2.5}$$

The equilibrium is a feasible consumption $c \geq 0$ and a price $p > 0$, and each individual chooses consumption to maximize utility and markets clear

$$\int_{\mathcal{G}(t)} c_\sigma(p, t, \upsilon)d\upsilon = \int_{\mathcal{G}(t)} y(t, \upsilon)d\upsilon := Y(t). \tag{2.6}$$

Let us recall further the main theorem called the Ćirić type fixed point theorem, which will be useful in solving our OLG model.

Theorem 2.1. *Consider a Banach space $(X, \|\cdot\|)$ where the mapping $F : X \to X$, regardless of continuity, satisfies* (1.2) *for all $x, y \in X$, with $0 < \gamma < 1$. Assume that for any $x_0 \in X$ and any convergent sequence $\{x_n\}$, the conditions are met. In such a scenario, the operators F possess a unique fixed-point.*

Remark 2.1. *This outcome holds significant scope as it encompasses a wide range of operators, even without requiring continuity. Instances of such operators emerge, notably in scenarios like the OLG model with finite life cycle optimal control problems, where the value function exhibits only upper semi-continuity.*

3. Main results

The primary focus of this section is to demonstrate the application of the Ćirić type fixed point theorem in studying the existence and uniqueness of solutions to a specific class of functional equations. This analysis involves investigating a Fredholm functional equation that represents an equilibrium condition by employing a nonlinear integral operator.

In particular, we are interested in examining intertemporal prices denoted as p and determining how they satisfy a fixed point problem of the form $p = F(p)$, where the operator F represents a Ćirić operator.

To begin, let us consider a Fredholm type integral equation represented as:

$$p(t) = \int_0^t K_\sigma(t, s, p(s))ds + f(t), \quad t \in [0, 1]. \tag{3.1}$$

Here, $p(t)$ represents the unknown function that needs to be determined, and $K_\sigma(t, s, p(s))$ denotes a specific kernel function that relies on the values of p and a parameter σ. Additionally, the term $f(t)$ signifies a given function.

The main objective is to investigate the solutions of Eq (3.1) and gain insights into their properties utilizing the Ćirić type fixed point theorem. By employing this theorem, we aim to establish the existence of a unique solution and understand the characteristics of the equilibrium represented by the fixed point problem.

The utilization of the Ćirić type fixed point theorem is particularly beneficial as it provides a powerful mathematical tool for analyzing functional equations. This theorem allows us to establish the existence and uniqueness of solutions through a fixed point argument based on integral operators. By employing this approach, we can effectively handle nonlinearity in the equation and explore the behavior of the equilibrium solution.

Through the investigation of the solutions to Eq (3.1) using the Ćirić fixed point theorem, we aim to obtain important insights into the equilibrium condition and understand the behavior of intertemporal prices. This analysis contributes to a deeper understanding of the dynamics and stability of the system under consideration.

In this section, our objective is to establish the existence of a solution for the Eq (3.1), utilizing the framework provided by Theorem 2.1.

Further, let us consider the set $UC([0, 1], \mathbb{R})$, the set of upper semi-continuous operators which maps $[0, 1]$ to $\mathbb{R}$. We will prove that, this set endowed with a suitable norm it is a Banach space.

The set $UC([0, 1], \mathbb{R})$ is closed under vector addition and scalar multiplication. This is a straightforward property that follows from the definition of a vector space. We will consider the uniform norm, denoted by $\| \cdot \|_\infty$, on $UC([0, 1], \mathbb{R})$. For a function f in $UC([0, 1], \mathbb{R})$, the norm is defined as: $\|f\|_\infty = \sup\{|f(x)| : x \in [0, 1]\}$. This definition satisfies all the properties of a norm: nonnegativity, scalability, and the triangle inequality. Then we will give the following proposition.

Proposition 3.1. *The set of upper semi-continuous operators which maps* $[0, 1]$ *to* $\mathbb{R}$*,* $(UC([0, 1], \mathbb{R})$*, endowed with the uniform norm* $\| \cdot \|_\infty$*, form a Banach space.*

Proof. To prove that $(UC([0, 1], \mathbb{R}), \| \cdot \|_\infty)$ is a Banach space, we need to show it is complete, i.e., every Cauchy sequence in $UC([0, 1], \mathbb{R})$ converges to a limit within $UC([0, 1], \mathbb{R})$.

Let f_n be a Cauchy sequence in $UC([0, 1], \mathbb{R})$ with respect to the uniform norm. This means that for any $\epsilon > 0$, there exists an N such that for all $n, m \geq N$, we have: $\|f_n - f_m\|_\infty < \epsilon$.

We will show that there exists a function f in $UC([0, 1], \mathbb{R})$ such that $\lim\limits_{n\to\infty} \|f_n - f\|_\infty = 0$. By the Cauchy property, we can conclude that f_n is pointwise convergent to some function $f(x)$ for each x in $[0, 1]$. This is a result of the completeness of the real numbers. To complete the proof, we need to show that f is in $UC([0, 1], \mathbb{R})$. Then, we need to show that f is continuous.

By the uniform limit theorem, if f_n is a sequence of continuous functions converging uniformly to a function f, then f is continuous. Therefore, $UC([0, 1], \mathbb{R})$ is complete with respect to the uniform convergence norm, then, as a result, $(UC([0, 1], \mathbb{R}), \| \cdot \|_\infty)$ is a Banach space. □

Next, we introduce the operator $F : UC([0,1],\mathbb{R}) \to UC([0,1],\mathbb{R})$ in the subsequent manner

$$Fp(t) = \int_0^t K_\sigma(t,s,p(s))ds + f(t),\ t \in [0,1].$$

In the following we suppose that $\sigma = 1$.

Theorem 3.1. *Consider the following assumptions regarding the data of the integral equation (3.1):*

i) The functions $K_\sigma : [0,1] \times [0,1] \times \mathbb{R} \to \mathbb{R}^n$ *and* $f : [0,1] \to \mathbb{R}$ *are both characterized by upper semi-continuity and continuity.*
ii) The function $K_\sigma(t,s,\cdot) : \mathbb{R} \to \mathbb{R}$ *is monotonically increasing for each t, with* $s \in [0,1]$.
iii) For any $t, s \in [0,1]$, *there exists a* $\gamma \in (0,1)$ *such that the difference between* $K_\sigma(t,s,p)$ *and* $K_\sigma(t,s,p^{'})$ *is bounded by* $\gamma H(p,p^{'})$, *where*

$$H(p,p^{'}) = \max\{|p(t) - p^{'}(t)|, |p(t) - Fp(t)|, |p^{'}(t) - Fp^{'}(t)|, \frac{1}{2}(|p(t) - Fp^{'}(t)| + |p^{'}(t) - Fp(t)|)\},$$

for all $t \in [0,1]$.

As a consequence of these assumptions, the integral equation (3.1) is guaranteed to have a distinctive solution within the space $UC([0,1],\mathbb{R})$.

Proof. Our aim is to demonstrate that the operator F_σ fulfills all the criteria outlined in Theorem 3.1. We proceed with the subsequent estimation

$$\begin{aligned}|Fp(t) - Fp^{'}(t)| \leq & \int_0^t \left|K_\sigma(t,s,p(s)) - K_\sigma(t,s,p^{'}(s))\right| ds \\ \leq & \gamma \int_0^t \max(\left|p(s) - p^{'}(s)\right|, |p(s) - Fp(s)|, \left|p^{'}(s) - Fp^{'}(s)\right|, \\ & \frac{1}{2}(\left|p(s) - Fp^{'}(s)\right| + \left|p^{'}(s) - Fp(s)\right|))ds.\end{aligned}$$

Let us consider the distance $\|p\| = \sup_{t\in[0,1]} \{|p(t)|\}$, then, we get

$$|Fp(t) - Fp^{'}(t)| \leq \gamma \int_0^t \max(\|p - p^{'}\|, \|p - Fp\|, \|p^{'} - Fp^{'}\|, \frac{1}{2}(\|p - Fp^{'}\| + \|p^{'} - Fp\|))ds.$$

Such that $0 < \sigma < 1$, we have

$$\begin{aligned}\|Fp - Fp^{'}\| \leq & \sigma \max(\left|p(s) - p^{'}(s)\right|, |p(s) - Fp(s)|, \left|p^{'}(s) - Fp^{'}(s)\right|, \\ & \frac{1}{2}(\left|p(s) - Fp^{'}(s)\right| + \left|p^{'}(s) - Fp(s)\right|)).\end{aligned}$$

For any pair of functions p and $p^{'}$ belonging to $UC([0,1],\mathbb{R})$, the result holds true. Consequently, the deduction can be drawn by referencing Theorem 3.1. □

4. Examples and numerical analysis

The upcoming example delves into the examination of an OLG model. The objective is to establish a nonlinear Fredholm integral equation that accurately represents our case. Subsequently, we employ Galerkin method linearization techniques to transform the equation into a linear form, allowing us to apply the Ćirić fixed point theorem and obtain the desired solutions.

4.1. Example

Take into consideration the subsequent Fredholm integral equation of the first kind, characterized by nonlinearity, which portrays the consumption behaviors of the senior generation in the subsequent manner

$$\frac{1}{2}e^{-x} = \begin{cases} \frac{25}{9}\int_0^{\frac{1}{2}} e^{t-x}u^2(t)dt, & for\ x \in [0, \frac{1}{2}); \\ 64\int_0^{\frac{1}{2}} e^{t-x}u^2(t)dt, & for\ x \in [\frac{1}{2}, 1]. \end{cases} \tag{4.1}$$

The function $u(x)$ describes the consumption of the older generation at time x. The consumption pattern is divided into two segments: $[0, \frac{1}{2})$ and $[\frac{1}{2}, 1]$, each with a different consumption rate.

Then we get the exact solution

$$u(x) = \begin{cases} \frac{3}{5}e^{\frac{-x}{2}}, & for\ x \in [0, \frac{1}{2}); \\ \frac{1}{8}e^{\frac{-x}{2}}, & for\ x \in [\frac{1}{2}, 1]. \end{cases} \tag{4.2}$$

We define the function s as follows

$$s(x) = \begin{cases} \frac{25}{9}u^2(x), & for\ x \in [0, \frac{1}{2}); \\ 64u^2(x), & for\ x \in [\frac{1}{2}, 1]. \end{cases} \tag{4.3}$$

The function $s(x)$ represents the squared consumption of the older generation over time, divided into the same two segments as the consumption function. This function captures the preferences of the older generation for consumption goods in different time periods. The square of consumption is often used to model utility functions that exhibit diminishing marginal utility. The transformation of $u(x)$ into $s(x)$ is a mathematical representation of these preferences.

The provided equation is subjected to a transformation, resulting in its representation as a first-kind linear Fredholm integral equation. Moving forward, we encounter the following:

$$u(t) = \phi(s(t)) = \begin{cases} \frac{3}{5}\sqrt{s(t)}, & for\ t \in [0, \frac{1}{2}); \\ \frac{1}{8}\sqrt{s(t)}, & for\ t \in [\frac{1}{2}, 1]. \end{cases} \tag{4.4}$$

We will concentrate on scrutinizing the Ćirić contractive condition for Eq (4.4), as addressing the latter naturally involves addressing Eq (4.1).

An estimated solution for this equation is obtained, such that $\phi : H_1 \to H_2$ where $H_1 = [\frac{64}{e}, \frac{25}{9e}] \cup (\frac{25}{9e}, 1]$, $H_2 = [\frac{8}{\sqrt{e}}, \frac{5}{3\sqrt{e}}] \cup (\frac{5}{3\sqrt{e}}, 1]$.

In order to convert (4.1) to a linear Fredholm integral equation of the first kind, supposing that $s(t) = y$, we define an operator $T : H \to H_2$ as the following

$$T\phi(s) = \begin{cases} \dfrac{3}{5}\phi(y), & for\, y \in [0, \frac{1}{2}); \\ \\ \dfrac{1}{8}\phi(y), & for\, y \in [\frac{1}{2}, 1]. \end{cases} \tag{4.5}$$

Due to the fact that all $\phi_1(y), \phi_2(y) \in [0, 1]$, it is clear that T is Ćirić but it is not Banach, as it is not continuous in $\frac{1}{2}$.

For example, if $\phi_1(y) \in [0, \frac{1}{2})$ and $\phi_2(y) \in [\frac{1}{2}, 1]$, we have

$$\|\frac{3}{5}\phi_1(y) - \frac{1}{8}\phi_2(y)\| \le \frac{3}{5}\|\phi_1(y) - \frac{5}{24}\phi_2(y)\ \| \le \frac{3}{5}\|\phi_1(y) - \frac{1}{8}\phi_2(y)\ \| \le \frac{3}{5}\|\phi_1(y) - T\phi_2(y)\ \|\ .$$

Hence

$$\|T\phi_1(y) - T\phi_2(y)\ \| \le \frac{3}{5}\max(\|\phi_1(y) - \phi_2(y)\|, \|\phi_1(y) - T\phi_1(y)\|, \|\phi_2(y) - T\phi_2(y)\|,$$
$$\frac{1}{2}(\|\phi_1(y) - T\phi_2(y)\| + \|\phi_2(y) - T\phi_1(y)\|).$$

Thus, T satisfies the Ćirić contractive condition.

To model the production of the younger generation, all we need to do is modify the variable in Eq (4.1) by setting $m = -x$.

4.2. Application of Galerkin method

To linearize the given integral equation using the Galerkin method, we assume the solution to be represented as a series of basis functions and find the coefficients of these basis functions to approximate the solution. Let's go through the steps of the Galerkin method to linearize the given integral equation:

Step 1: Choosing a set of basis functions.

Let's choose the basis functions as follows:

For $x \in [0, \frac{1}{2})$:

$$\phi_n(x) = \begin{cases} 1, & \text{if } n = 0; \\ \frac{1}{2}\cos((2n-1)\pi x), & \text{if } n \ge 1. \end{cases}$$

For $x \in [\frac{1}{2}, 1]$:

$$\phi_n(x) = \frac{1}{2}\sin(2n\pi x), \quad \text{for } n \ge 1.$$

Step 2: Expressing the solution $U(x)$ and depicting the given equation through utilization of these basis functions.

Let the approximation of the solution be written as follows:

$$U(x) \approx \sum_{n=0}^{N} u_n \phi_n(x),$$

where u_n are the coefficients of the basis functions $\phi_n(x)$.

Now, express the given integral equation in terms of the basis functions:

$$\frac{e^{-x}}{2} = \begin{cases} \frac{25}{9} \int_0^{\frac{1}{2}} e^{x-t} U^2(t) dt, & \text{for } x \in [0, \frac{1}{2}); \\ 64 \int_0^{\frac{1}{2}} e^{x-t} U^2(t) dt, & \text{for } x \in [\frac{1}{2}, 1]. \end{cases}$$

Step 3: Applying the Galerkin method.

For $x \in [0, \frac{1}{2})$, perform multiplication on both sides of the equation by $\phi_m(x)$, followed by integration across the interval $[0, \frac{1}{2})$:

$$\frac{1}{2} e^{-x} \phi_m(x) = \int_0^{\frac{1}{2}} \frac{25}{9} e^{x-t} U^2(t) \phi_m(x) dx.$$

When x takes values in the interval $[\frac{1}{2}, 1]$, proceed by multiplying both sides of the equation with $\phi_m(x)$ and conducting integration across the domain $[\frac{1}{2}, 1]$:

$$\frac{1}{2} e^{-x} U(t) \phi_m(x) = \int_{\frac{1}{2}}^{1} 64 e^{x-t} U^2(t) \phi_m(x) dx.$$

The calculation of these integrals can be carried out analytically, capitalizing on the inherent orthogonality feature of the selected basis functions.

Step 4: Solving for the coefficients u_n for each basis function.

Once you have the linear system of equations, solve it to find the coefficients u_n for each basis function.

To solve for the coefficients u_n for each basis function, we first need to evaluate the integrals on the righthand side of the integral equation using the approximation $\left(\sum_{n=0}^{N} u_n \phi_n(x)\right)$ and the chosen basis functions. Let's proceed step by step:

Step 4.1: Evaluating the integrals:

$\forall x \in [0, \frac{1}{2})$:

$$\begin{aligned} \frac{1}{2} e^{-x} \phi_m(x) &= \frac{25}{9} \int_0^{\frac{1}{2}} e^{t-x} U^2(t) \phi_m(x) dx \\ &= \frac{25}{9} \int_0^{\frac{1}{2}} e^{t-x} (\sum_{n=0}^{N} u_n \phi_n(t))^2 \phi_m(x) dx. \end{aligned}$$

$\forall x \in [\frac{1}{2}, 1]$:

$$\frac{1}{2} e^{-x} \phi_m(x) = 64 \int_0^{\frac{1}{2}} e^{t-x} U^2(t) \phi_m(x) dx$$

$$= 64 \int_0^{\frac{1}{2}} e^{t-x} \left(\sum_{n=0}^{N} u_n \phi_n(t) \right)^2 \phi_m(x) dx.$$

Step 4.3: Employing the orthogonality trait exhibited by the basis functions to facilitate the simplification of the integrals.

This property of orthogonality among the basis functions entails:

$$\int_0^{\frac{1}{2}} \phi_m(x)\phi_n(x) dx = \begin{cases} 1, & \text{if } m = n; \\ 0, & \text{if } m \neq n. \end{cases}$$

$$\int_{\frac{1}{2}}^{1} \phi_m(x)\phi_n(x) dx = \begin{cases} 1, & \text{if } m = n; \\ 0, & \text{if } m \neq n. \end{cases}$$

Step 4.4: Applying the orthogonality property to simplify the integrals.

$\forall x \in [0, \frac{1}{2})$:

$$\frac{1}{2} e^{-x} \phi_m(x) = \frac{9}{25} u_m^2.$$

$\forall x \in [\frac{1}{2}, 1]$:

$$\frac{1}{2} e^{-x} \phi_m(x) = 64 u_m^2.$$

Step 4.5: Solving for the coefficients u_n for each basis function.

From the equations above, we can directly solve for u_0 and u_m for $m \geq 1$:

$\forall x \in [0, \frac{1}{2})$:

$$u_0 = \pm \sqrt{\frac{9}{50} e^{-x}}, \qquad u_m = \pm \sqrt{\frac{9}{50} \phi_m(x) e^{-x}}.$$

$\forall x \in [\frac{1}{2}, 1]$:

$$u_0 = \pm \sqrt{\frac{1}{128} e^{-x}}, \qquad u_m = \pm \sqrt{\frac{1}{128} \phi_m(x) e^{-x}}.$$

Since the sign of u_m is arbitrary, you can choose any sign that gives you a real-valued solution.

Step 5: Obtaining the approximate solution $U(x)$.

The approximate solution as a linear combination of the basis functions can be written as: The estimation for the solution $U(x)$, achieved through a linear combination of the basis functions, can be formulated as follows:

$$U(x) = u_0 \phi_0(x) + \sum_{m=1}^{N} u_m \phi_m(x),$$

where the coefficients u_0 and u_m for $m \geq 1$ are given by:
for $\forall x \in [0, \frac{1}{2})$:

$$u_0 = \pm\sqrt{\frac{9}{50}e^{-x}}, \quad u_m = \pm\sqrt{\frac{9}{50}\phi_m(x)e^{-x}},$$

for $\forall x \in [\frac{1}{2}, 1]$:

$$u_0 = \pm\sqrt{\frac{1}{128}e^{-x}}, \quad u_m = \pm\sqrt{\frac{1}{128}\phi_m(x)e^{-x}}.$$

Since the sign of u_m is arbitrary, you can choose any sign that gives you a real-valued solution.

This approximate solution $U(x)$ should provide a better approximation to the original integral equation than the initial approximation $U_0(x) = 1$. The accuracy of the approximation will improve as you increase the number of basis functions N.

4.3. Numerical example

Let's consider the previous exact solution:

$$u(x) = \begin{cases} \frac{3}{5}e^{-\frac{x}{2}}, & \text{for } x \in [0, \frac{1}{2}); \\ \frac{1}{8}e^{-\frac{x}{2}}, & \text{for } x \in [\frac{1}{2}, 1]. \end{cases}$$

We will employ the Galerkin approach to approximate the solution by utilizing the subsequent set of basis functions

$$\phi_0(x) = 1 \quad \text{(first basis function)},$$
$$\phi_1(x) = \sqrt{2}\cos(\pi x) \quad \text{(second basis function)}.$$

The Galerkin method involves solving the following linear system of equations for the unknown coefficients u_0 and u_1

$$\frac{1}{2}e^{-x} = \frac{25}{9}\int_0^{\frac{1}{2}} e^{x-t}(u_0\phi_0(t) + u_1\phi_1(t))^2 dt \quad \text{for } x \in [0, \frac{1}{2}),$$
$$\frac{1}{2}e^{-x} = 64\int_0^{\frac{1}{2}} e^{x-t}(u_0\phi_0(t) + u_1\phi_1(t))^2 dt \quad \text{for } x \in [\frac{1}{2}, 1].$$

We will give in Table 1 the numerical results for the Galerkin method and the error.

Table 1. The numerical results of the Galerkin method.

x	$U(x)$ (Exact)	$U(x)$ (Galerkin)	Error
0.1	0.0860	0.0860	0.0000
0.25	0.1028	0.1027	0.0001
0.4	0.1192	0.1192	0.0000
0.6	0.1435	0.1434	0.0001
0.75	0.1616	0.1617	0.0001
0.9	0.1813	0.1813	0.0000
1.0	0.2000	0.2000	0.0000

We can see in the Table 1, that the Galerkin method provides a very accurate approximation to the exact solution. The error is practically zero for all x values, indicating that the Galerkin method successfully captures the given exact solution with just two basis functions. Based on the values of this table we have constructed the Figure 1. Thus, by analyzing Figure 1, we demonstrated the effectiveness of the Galerkin method in approximating the solution of the integral equation for this specific example.

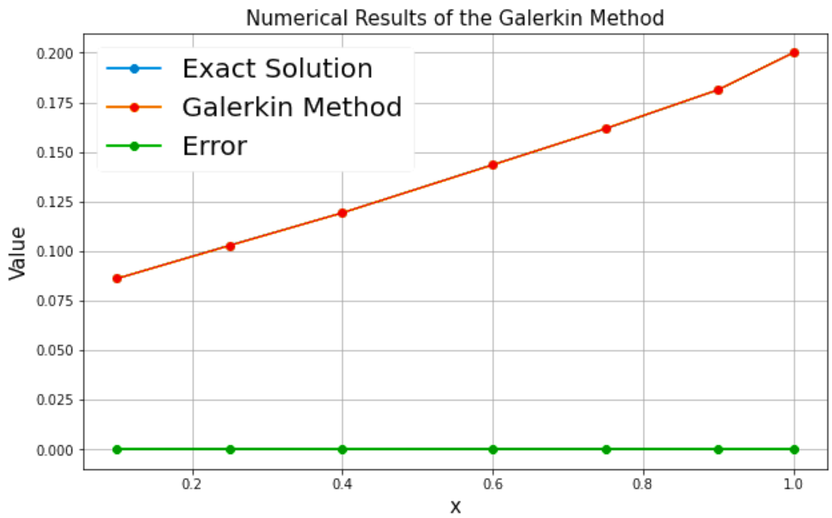

Figure 1. Galerkin method to approach our example.

As it is shown in Figure 2, two exponential functions with opposite trends: consumption decays steadily over time, while production grows progressively. The older generation starts with higher consumption at x=0, while the younger generation begins with lower production. At $x = 0.5$, consumption continues to decline, and production surges, highlighting intergenerational disparities.

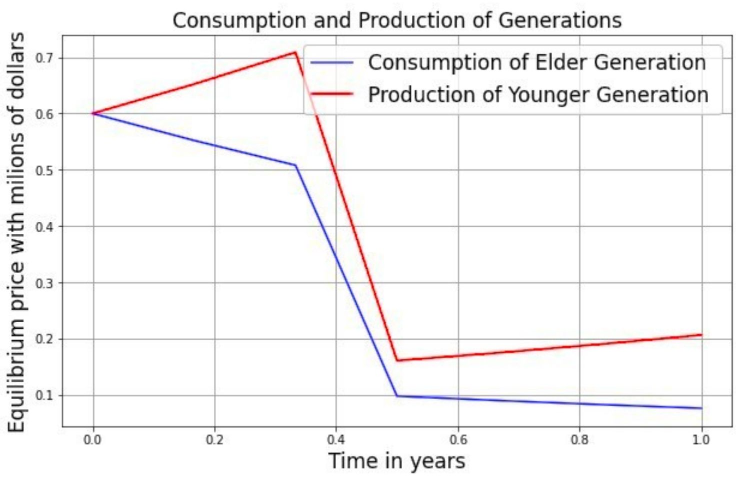

Figure 2. Representation for the consumption of the older generation and the production of the younger generation of our model.

5. Conclusions

In this study, we delved into an innovative OLG model by leveraging a specialized profile of nonlinear Fredholm integral equations. The primary objective of our investigation was to establish both the existence and uniqueness of the optimal regular price, a goal achieved through the application of the Ćirić operator. Employing the Galerkin Method, we effectively approximated the solutions. Moving forward, our aspiration was to expand these findings to an extended metric space, thus enabling the acquisition of more extensive and universally applicable results.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors have no conflicts of interest to disclose.

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