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Research article

On the generalized proximate order of functions analytic on the unit disc

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Abstract: The concept of Lindelöf proximate order has been used extensively to study the functions of completely regular growth. The main drawback of this approach is that it completely ignores the value of lower order. To overcome this problem, Chyzykov et al. introduced the concept of generalized proximate order for irregular growth. In this paper we studied the existence of generalized proximate order for every functions analytic on the unit disc with some new results for functions having irregular growth.

Keywords: analytic functions; irregular growth; generalized and quasi proximate orders and unit disc
Mathematics Subject Classification: 30E15, 30B10

1. Introduction

The concept of proximate order was used to obtain a more refined measure of growth of analytic/ entire functions. Lindelöf proximate order $\rho(r)$ has been extensively used in the setting of such problems [5,6,10,11]. They estimate $\log M(r, f)$, $M(r, f) = \max\{|f(z)| : |z| = r\}$ by the flexible function $v(r) = r^{\rho(r)}$, where $\rho(r) \to \rho_M(f)$, as r approaches one in the case of functions analytic in the unit disc. It is known by Valiron's theorem [5,6,11] that for every entire function of finite order there exists a proximate order $\rho(r)$, such that $\log M(r, f) \leq V(r)$ for all r and $\log M(r_n, f) = V(r_n)$ for some sequence $\{r_n\} \to \infty$. This concept has been used to study the functions of completely regular growth [11]. The main drawback of this approach is that it completely ignores the value of lower order $\lambda_M(f)$. There is a notion of lower proximate order $\lambda(r)$ [5,11] corresponding to finite lower order $\lambda_M(f)$. Now the question arises about how to construct a majorant $V(r)$ for $\log M(r, f)$ such that, on one hand, it keeps the information about both the order $\rho_M(f)$ and the lower order $\lambda_M(f)$ sufficiently flexible. To solve this problem, Chyzhykov et al. [3] introduced the concept of the generalized proximate order by defining quasi proximate order for $\rho_M(f) \neq \lambda_M(f)(0 \leq \lambda_M(f) < \rho_M(f) < \infty)$ and studied the existence of generalized proximate order for functions analytic in unit disc . In this paper we have obtained some new results concerning generalized proximate order of functions analytic in unit disc having irregular growth i.e., $\rho_M(f) \neq \lambda_M(f)$ with the existence of generalized proximate order for these functions, but our results and methods are different from those of Chyzhykov et al. [3].

In their scientific literature, Chyzhykov and Semochko [4] have given a general definition of growth for an entire function f in the complex plane that covers arbitrary growth. According to Chyzhykov and Semochko [4], let Φ be the class of positive unbounded increasing function on $[1, +\infty)$ such that $\varphi(e^t)$ is slowly growing, i.e.,

$$\lim_{t\to+\infty} \frac{\varphi(e^{ct})}{\varphi(e^t)} = 1, 0 < c < +\infty.$$

If $\varphi \in \Phi$, then

$$\lim_{x\to+\infty} \frac{\varphi^{-1}(\log x^m)}{x^k} = +\infty, \forall m > 0, \forall k \geq 0. \tag{1.1}$$

$$\lim_{x\to+\infty} \frac{\log \varphi^{-1}((1+\delta)x)}{\log \varphi^{-1}(x)} = +\infty, \forall \delta > 0. \tag{1.2}$$

If φ is nondecreasing, then (1.2) is equivalent to the class Φ.

Definition 1.1. [4] Let φ be an increasing unbounded function on $[1, +\infty)$, then the orders of growth of an entire function f are defined by

$$\rho_\varphi^{-0}(f) = \limsup_{r\to+\infty} \frac{\varphi(M(r, f))}{\log r}, \rho_\varphi^{-1}(f) = \limsup_{r\to+\infty} \frac{\varphi(\log M(r, f))}{\log r}.$$

Remark 1.1. If $\varphi(r) = \log\log r$, then it is clear that $\varphi \in \Phi$. In this case, the above definition of orders coincide with definitions of usual order and hyper-order, i.e., if f is entire, then

$$\rho_{\log\log(f)}^{-0} = \limsup_{r\to+\infty} \frac{\log\log M(r, f)}{\log r} = \rho(f),$$

$$\rho_{\log\log\log(f)}^{-0} = \limsup_{r\to+\infty} \frac{\log\log\log M(r, f)}{\log r} = \rho_2(f).$$

It has been shown [4] that if $\varphi \in \Phi$ and f is an entire function, then

$$\rho_\varphi^{j}(f) = \rho_\varphi^{-j}(f), j = 0, 1.$$

Chyzhykov and Semochko [4] used the concept of ρ_φ-orders in order to investigate the growth of solutions of linear differential equations in the complex plane and in the unit disc.

The concept of (p, q)-order and (p, q)-type ($p \geq q \geq 1$) was introduced by Juneja et al. [7,8] for classifications of order $\rho = 0$ and $\rho = \infty$. This concept is a modification of the classical definition of order and type obtained by replacing logarithms by iterated logarithms, where the degree of iteration is determined by p and q.

According to Sheremeta [12] we have the following definitions.

Let $\phi : [a, +\infty) \to R$ be a real valued function such that $\phi(x)$ is positive, differentiable $\forall x \in [a, +\infty)$,

strictly increasing and $\phi(x) \to \infty$ as $x \to \infty$.
For every real valued function $\gamma(x)$ such that $\gamma(x) \to 0$ as $x \to \infty, \phi$ satisfies

$$\lim_{x\to\infty} \frac{\phi[(1+\gamma(x))x]}{\phi(x)} = 1, \tag{1.3}$$

then ϕ belongs to class L^0. The function $\phi(x)$ is said to belong to the class Λ if $\phi(x) \in L^0$ and, in place of (1.3), satisfies the stronger condition

$$\lim_{x\to\infty} \frac{\phi(cx)}{\phi(x)} = 1, \tag{1.4}$$

for all $c, 0 < c < \infty$. Functions $\phi(x)$ satisfying (1.4) are called slowly increasing functions (see [12]). Using the generalized functions α,β from classes L^0 and Λ, Sheremeta introduced the generalized (α,β)-order and generalized lower (α,β)-lower order of entire functions by equalities

$$\rho(\alpha,\beta) = \limsup_{r\to\infty} \frac{\alpha(\log M(r,f))}{\beta(r)},$$

$$\lambda(\alpha,\beta) = \liminf_{r\to\infty} \frac{\alpha(\log M(r,f))}{\beta(r)},$$

where $M(r,f) = \max_{|z|=r} |f(z)|$.
For $\alpha(x) = \beta(x) = \log x$, $\rho(\alpha,\beta)$ gives the formula for $(p,q) = (2,1)$ of Juneja et al. [7]. For $\alpha(x) = \log^{[p-1]} x$ and $\beta(x) = \log^{[q]} x$, $\rho(\alpha,\beta)$ and $\lambda(\alpha,\beta)$ give the (p,q)-order and lower (p,q)-order introduced by Juneja et al. [7].

An entire function f of order ρ is said to be of completely regular growth if there exists a 2π-periodic function $h : R \to R$ which does not vanish identically such that

$$\log|f(re^{i\theta})| = h(\theta)r^\rho + o(r^\rho) \tag{1.5}$$

as $r \to \infty$, for $re^{i\theta}$ outside a union of discs $\{z : |z - z_j| < r_j\}$ satisfying

$$\sum_{|z_j|\le r} r_j = o(r)$$

as $r \to \infty$. One may replace the r^ρ in (1.5) by $r^{\rho(r)}$ with a proximate order $\rho(r)$. If the order and lower order of function f are different, then function f cannot be of completely regular growth. Bergweiler and Chyzhykov [2] gave conditions ensuring that the Julia set and the escaping set of an entire function of completely regular growth have positive Lebesgue measure. Bandura and Skaskiv [1] studied the relationship between the class of entire functions of completely regular growth of order ρ and the class of entire function with bounded l-index. Possible applications of these functions in the analytic theory of differential equations have been considered.

For an analytic function f in the unit disc $D = \{z : |z| < 1\}$, the order and lower order are defined as

$$\rho_M(f) = \limsup_{r\to1^-} \frac{\log^+\log^+ M(r,f)}{-\log(1-r)}, \lambda_M(f) = \liminf_{r\to1^-} \frac{\log^+\log^+ M(r,f)}{-\log(1-r)}, \tag{1.6}$$

$(0 \le \lambda_M(f) < \rho_M(f) < \infty)$. Chyzhykov et al. [3] defined quasi proximate order as:
For given $\eta \in (0, \rho_M(f) - \lambda_M(f))$, there exists λ and its associated function $A^* = A^*_\lambda$ on [0,1) such that

(1) $\lambda \in C^1[0,1)$;
(2) $\limsup_{r\to 1^-} \lambda(r) = \rho_M(f)$;
(3) $\liminf_{r\to 1^-} \lambda(r) = \lambda_M(f) + \eta$;
(4) $\limsup_{r\to 1^-} -|\lambda'(r)|(1-r)\log(1-r) < \infty$;
(5) $A^*(r) \le (1-r)^{-\lambda(r)} \le (1+o(1))A^*(r)$ as $r \to 1^-$;
(6) A^* is nondecreasing and $A^*(\frac{1+r}{2}) \lesssim A^*(r)$ for all $0 \le r < 1$;
(7) $\log M(r,f) \le (1-r)^{-\lambda(r)}$ for all $0 \le r < 1$.
Further if
(4′) $\limsup_{r\to 1^-} -|\lambda'(r)|(1-r)\log(1-r) = 0$;
then λ is a generalized proximate order of f.

It is noted that in condition (3) we cannot replace $\lambda_M(f)+\eta$ by $\lambda_M(f)$ without violating the condition $\limsup_{r\to 1^-} -|\lambda'(r)|(1-r)\log(1-r) < \infty$; [3, pp. 456]. Every generalized proximate order is a quasi proximate order.

For an analytic function f in the unit disc $D = \{z : |z| < 1\}$, we define

$$T^* = \limsup_{r\to 1^-} \frac{\log^+ M(r,f)}{(1-r)^{-\rho(r)}}; t^* = \liminf_{r\to 1^-} \frac{\log^+ M(r,f)}{(1-r)^{-\rho(r)}}.$$

The numbers T^* and t^* are called the type and lower type of functions analytic in D with respect to the proximate order $\rho(r)$. The lower type t^* completely ignores the value of lower proximate order $\lambda(r)$. If $0 < t^* < \infty$, then the function $\lambda(r)$ satisfying (1)-(4′)-(7) is called the generalized proximate order of f.

2. Existence of generalized proximate order for functions analytic in the unit disc

The following theorem shows that there exists a generalized proximate order for every function, analytic in D and having nonzero finite order.

Theorem 2.1. Let f be a function analytic in D having order $\rho_M(f)$ and lower order $\lambda_M(f)$ such that $0 < \lambda_M(f) < \rho_M(f) < \infty$, then for every $t^*, 0 < t^* < \infty$, there exists a generalized proximate order of f satisfying (1)-(7)-(4′).

Proof. We first assume

$$\xi(r) = \frac{(1-r)^{(\lambda_M(f)+\eta)(1-\frac{1}{\rho_M(f)})^+ + 1} \log M(r,f)}{t^*}.$$

Put $x = -\log(1-r)$ and $\xi_1(x) = \log \xi(1-e^x)$, then

$$\lim_{x\to\infty} \frac{\xi_1(x)}{x} = -(\lambda_M(f)+\eta)(1-\frac{1}{\rho_M(f)})^+ - 1 + \rho_M(f).$$

Let $\limsup_{x\to\infty} \xi_1(x) = \infty$. Let $y = S(x)$ be the boundary curve of the smallest convex domain containing the curve $y = \xi_1(x)$ and the positive ray of the x-axis. After doing suitable modifications in the small neighborhoods of the vertices in this curve, we may assume that the function $S(x)$ is differentiable in $0 \le x \le \infty$. The curve $S(x)$ is concave in the sense that a chord joining any two points of the curve lies below the curve. The curve $\frac{S(x)}{x}$ is monotonic decreasing and nonnegative, and this implies that function $\frac{S(x)}{x}$must tend to a limit as $x \to \infty$. Since the curve $y = S(x)$ and $y = \xi_1(x)$ have infinitely many common points $\{x_n\}$ such that $x_n \to \infty$, then

$$\lim_{x\to\infty} \frac{S(x)}{x} = -(\lambda_M(f) + \eta)(1 - \frac{1}{\rho_M(f)})^+ - 1 + \rho_M(f) \tag{2.1}$$

and

$$\xi_1(x) \leq S(x) \ \ for \ \ all \ \ x \geq 0. \tag{2.2}$$

In view of (2.1), we obtain

$$\lim_{x\to\infty} S'(x) = -(\lambda_M(f) + \eta)(1 - \frac{1}{\rho_M(f)})^+ - 1 + \rho_M(f). \tag{2.3}$$

Using (2.2), we get

$$\log M(r, f) \leq t^*(1 - r)^{-[(\lambda_M(f)+\eta)(1-\frac{1}{\rho_M(f)})^+-1]-\frac{S(-\log(1-r))}{-\log(1-r)}}.$$

Set

$$\lambda(r)(1 - \frac{1}{\rho_M(f)})^+ + 1 = (\lambda_M(f) + \eta)(1 - \frac{1}{\rho_M(f)})^+ - 1] + \frac{S(-\log(1 - r))}{-\log(1 - r)}. \tag{2.4}$$

Since $\lambda(r)$ is positive and differentiable in $0 \leq r_o < r < 1$, it follows that

$$\lambda(r)(1 - \frac{1}{\rho_M(f)})^+ + 1 \to (\lambda_M(f) + \eta)(1 - \frac{1}{\rho_M(f)})^+ + 1 \ \ as \ \ r \to 1^- \ \ by \ \ (2.2).$$

Further,

$$-(1 - r)|\lambda'(r)| \log(1 - r) = S'(-\log(1 - r)) - \frac{S(-\log(1 - r))}{-\log(1 - r)}.$$

Using (2.2) and (2.3), we obtain

$$-(1 - r)|\lambda'(r)| \log(1 - r) \to 0 \ \ as \ \ r \to 1^-.$$

Finally, by (2.4) and (2.2), we get

$$\log M(r, f) \leq t^*(1 - r)^{-\lambda(r)(1-\frac{1}{\rho_M(f)})^+-1}$$

for all r in $0 \leq r_o < r < 1$, and there exists a sequence $r_n \to 1$ as $n \to \infty$ in which

$$\log M(r_n, f) = t^*(1 - r)^{-\lambda(r)(1-\frac{1}{\rho_M(f)})^+-1}.$$

Thus, $\lambda(r)$ defined by (2.4) is a generalized proximate order.

Theorem 2.2. For every generalized proximate order $\lambda(r) \in C^1(0, \infty)$, there exists a generalized proximate order $\lambda_1(r) \in C^2(0, \infty)$ such that

$$|\log \frac{\lambda(r)}{\lambda_1(r)}| = o\Big[- \log(1 - r)^{-1}\Big] \ \ as \ \ r \to 1^-, \tag{2.5}$$

where $C^{\nu}(0,\infty), \nu = 1,2$ is the space of all functions defined on $[0,\infty)$ whose ν^{th} derivatives are continuous.

Proof. Suppose that $\lambda_1(r)$ and $\lambda(r)$ are the generalized proximate orders coinciding on the sequence $\{r_n\}$ such that

$$\lambda(r) = \lambda_1(r), \ \ r_n = 1 - \frac{1}{4^n}, \ \ n = 0,1,2,\ldots. \tag{2.6}$$

In this case, for $r \in [r_n, r_{n+1})$, we have

$$\begin{aligned}|\log\frac{\lambda(r)}{\lambda_1(r)}| =& |\int_{r_n}^{r}\Big[\frac{\lambda'(x)}{\lambda(x)} - \frac{\lambda_1'(x)}{\lambda_1(x)}\Big]dx| = |\int_{r_n}^{r} o\Big[\frac{1}{(1-x)\log(1-x)}\Big]dx| \\ =& o\Big[\log\frac{\log(1-r_n)}{\log(1-r)}\Big] = o\Big[-\log(1-r)^{-1}\Big] \ \ as \ \ r \to 1^-.\end{aligned}$$

To study the properties of generalized proximate order of a function analytic in unit disc, we need the concept of the slowly increasing function. A real valued function $L(r), 0 < r < 1$ is said to be slowly increasing if for every $k, 1 < k < \infty$,

$$\lim_{r\to 1}\frac{L(r+\frac{1-r}{k})}{L(r)} = 1. \tag{2.7}$$

Theorem 2.3. Let $\lambda(r)$ be a generalized proximate order of a function f analytic in unit disc and having generalized order $\lambda_M(f)+\eta$, then

$$L(r) = (1-r)^{-\lambda(r)+\lambda_M(f)+\eta} \ \ is \ a \ slowly \ increasing \ function \ of \ r \ in \ 0 < r < 1 \tag{2.8}$$

and

$$\begin{aligned}&(1-r)^{-\lambda(r)(1-\frac{1}{\rho_M(f)})^{+}-1} \ \ is \ a \ monotonically \ increasing \ function \ of \ r \ in \\ &0 \le r_o < r < 1 \ \ and \ tends \ to \ \infty \ as \ r \to 1^-.\end{aligned} \tag{2.9}$$

Proof. We have

$$L(r) = (1-r)^{-\lambda(r)+\lambda_M(f)+\eta},$$

$$\begin{aligned}\log L(r) =& (-\lambda(r) + \lambda_M(f) + \eta)\log(1-r) \\ =& -\lambda(r)\log(1-r) + (\lambda_M(f)+\eta)\log(1-r),\end{aligned}$$

so

$$\begin{aligned}\frac{L'(r)}{L(r)} =& \lambda'(r)\log(1-r) + \lambda(r)\frac{1}{1-r} - (\lambda_M(f)+\eta)\frac{1}{1-r} \\ =& \lambda'(r)\log(1-r) + \frac{\lambda(r)-\lambda_M(f)-\eta}{1-r} \\ =& \frac{(1-r)\lambda'(r)\log(1-r) + \lambda(r) - \lambda_M(f) - \eta}{(1-r)} \\ =& o(\frac{1}{1-r}), \ \ for \ all \ values \ of \ r \ sufficiently \ close \ to \ one.\end{aligned}$$

Therefore,

$$\lim_{r\to 1}\log\frac{L(r+\frac{1-r}{k})}{L(r)}=0.$$

Hence, (2.8) is proved.

In order to prove (2.9), we have

$$\begin{aligned}\frac{d}{dr}\Big[(1-r)^{-\lambda(r)(1-\frac{1}{\rho_M(f)})^+-1}\Big] =&\lambda(r)(1-r)^{-\lambda(r)(1-\frac{1}{\rho_M(f)})^+}-\lambda'(r)(1-\frac{1}{\rho_M(f)})^+\times\\ &(1-r)^{-\lambda(r)(1-\frac{1}{\rho_M(f)})^+-1}\log(1-r)\\ &>(\lambda_M(f)+\eta-\varepsilon)(1-r)^{-\lambda(r)(1-\frac{1}{\rho_M(f)})^+}>0,\end{aligned}$$

since (4′) is satisfied. This proves (2.9).

Theorem 2.4. For $(\lambda_M(f)+\eta)(1-\frac{1}{\rho_M(f)})^+>\alpha, 0<(\lambda_M(f)+\eta)(1-\frac{1}{\rho_M(f)})^+<\infty$ and $0<\beta<r<1$,

$$\begin{aligned}\int_\beta^r(1-t)^{-\lambda(t)(1-\frac{1}{\rho_M(f)})^+-1+\alpha}dt =&\frac{(1-r)^{-\lambda(r)(1-\frac{1}{\rho_M(f)})^++\alpha}}{(\lambda_M(f)+\eta)(1-\frac{1}{\rho_M(f)})^+-\alpha}\\ &+o(1-r)^{-\lambda(r)(1-\frac{1}{\rho_M(f)})^++\alpha}.\end{aligned}$$

Proof. Integrating by parts with $(\lambda_M(f)+\eta)(1-\frac{1}{\rho_M(f)})^+-1>0$ as

$$\begin{aligned}\int_\beta^r(1-t)^{-\lambda(t)(1-\frac{1}{\rho_M(f)})^+-1+\alpha}dt =&\int_\beta^r(1-t)^{\alpha-[(\lambda_M(f)+\eta)(1-\frac{1}{\rho_M(f)})^++1]}\times\\ &(1-t)^{[\lambda_M(f)+\eta-\lambda(t)](1-\frac{1}{\rho_M(f)})^++1}dt\\ &=\frac{(1-t)^{-\lambda(t)(1-\frac{1}{\rho_M(f)})^++\alpha}}{(\lambda_M(f)+\eta)(1-\frac{1}{\rho_M(f)})^+-\alpha}\Big|_\beta^r\\ &-\int_\beta^r(1-t)^{-\lambda(t)(1-\frac{1}{\rho_M(f)})^+-1+\alpha}\times\\ &\{-(1-t)\lambda'(t)((1-\frac{1}{\rho_M(f)})^++1)\log(1-t)\\ &+(\lambda(t)-\lambda_M(f)-\eta)((1-\frac{1}{\rho_M(f)})^++1)\}dt.\end{aligned}$$

From (3) and (4), we have

$$|\lambda(t)-\lambda_M(f)-\eta|<\frac{\varepsilon}{2}\ \ as\ \ r\to 1^-$$

and

$$|-(1-t)\lambda'(t)\log(1-t)|<\frac{\varepsilon}{2}.$$

Hence,

$$\int_{\beta}^{r}(1-t)^{-\lambda(t)(1-\frac{1}{\rho_M(f)})^{+}-1+\alpha}dt = \frac{(1-r)^{-\lambda(r)(1-\frac{1}{\rho_M(f)})^{+}+\alpha}}{(\lambda_M(f)+\eta)(1-\frac{1}{\rho_M(f)})^{+}-\alpha}(1+o(1)) - o(1)\int_{\beta}^{r}(1-t)^{-\lambda(t)(1-\frac{1}{\rho_M(f)})^{+}-1+\alpha}dt.$$

This implies that

$$\int_{\beta}^{r}(1-t)^{-\lambda(t)(1-\frac{1}{\rho_M(f)})^{+}-1+\alpha}dt = \frac{(1-r)^{-\lambda(r)(1-\frac{1}{\rho_M(f)})^{+}+\alpha}}{(\lambda_M(f)+\eta)(1-\frac{1}{\rho_M(f)})^{+}-\alpha} + o(1-r)^{-\lambda(r)(1-\frac{1}{\rho_M(f)})^{+}+\alpha}.$$

Hence, the proof is completed.

Let $\phi(r)$ be a bounded function defined on $(0,\infty)$ and $\lambda(r)$ be a generalized proximate order such that

$$\limsup_{r\to 1^-}\frac{\phi(r)}{(1-r)^{-\lambda(t)(1-\frac{1}{\rho_M(f)})^{+}-1}} = p,$$

$$\liminf_{r\to 1^-}\frac{\phi(r)}{(1-r)^{-\lambda(t)(1-\frac{1}{\rho_M(f)})^{+}-1}} = q,$$

and for $\alpha \geq 1$,

$$\limsup_{r\to 1^-}\{(1-r)^{\lambda(r)(1-\frac{1}{\rho_M(f)})^{+}-\alpha}\int_{\beta}^{r}\frac{\phi(r)}{(1-r)^{-\alpha}}dr = s_1,$$

$$\liminf_{r\to 1^-}\{(1-r)^{\lambda(r)(1-\frac{1}{\rho_M(f)})^{+}-\alpha}\int_{\beta}^{r}\frac{\phi(r)}{(1-r)^{-\alpha}}dr = s_2.$$

Theorem 2.5. For the constants p, q, s_1, s_2 defined above, the following inequalities hold.

$$\frac{q}{(\lambda_M(f)+\eta)(1-\frac{1}{\rho_M(f)})^{+}-\alpha} \leq s_2 \leq s_1 \leq \frac{p}{(\lambda_M(f)+\eta)(1-\frac{1}{\rho_M(f)})^{+}-\alpha}. \tag{2.10}$$

Proof. For given $\varepsilon > 0$ and $r > r_o > \beta > 0$,

$$\phi(r) < (p+\varepsilon)(1-r)^{-\lambda(r)(1-\frac{1}{\rho_M(f)})^{+}-1}$$

and

$$\int_{\beta}^{r}\frac{\phi(t)}{(1-t)^{-\alpha}}dt \leq o(1) + (p+\varepsilon)\int_{r_o}^{t}(1-t)^{\alpha-\lambda(t)}.$$

Using Theorem 2.4, we get

$$\int_{\beta}^{r} \frac{\phi(t)}{(1-t)^{-\alpha}} dt \leq o(1) + \frac{(p+\varepsilon)(1-r)^{-\lambda(r)(1-\frac{1}{\rho_M(f)})^{+}+\alpha}}{(\lambda_M(f)+\eta)(1-\frac{1}{\rho_M(f)})^{+}-\alpha} + o(1-r)^{-\lambda(r)(1-\frac{1}{\rho_M(f)})^{+}+\alpha}.$$

This implies

$$\limsup_{r\to 1^-}\{(1-r)^{\lambda(r)(1-\frac{1}{\rho_M(f)})^{+}-\alpha} \int_{\beta}^{r} \frac{\phi(t)}{(1-t)^{-\alpha}} dt \leq \frac{p}{(\lambda_M(f)+\eta)(1-\frac{1}{\rho_M(f)})^{+}-\alpha},$$

and it follows the third part of the inequality in (2.10). Similarly, it can be seen that

$$\liminf_{r\to 1^-}\{(1-r)^{\lambda(r)(1-\frac{1}{\rho_M(f)})^{+}-\alpha} \int_{\beta}^{r} \frac{\phi(t)}{(1-t)^{-\alpha}} dt \geq \frac{q}{(\lambda_M(f)+\eta)(1-\frac{1}{\rho_M(f)})^{+}-\alpha}.$$

Hence, the proof is completed.

Let

$$\gamma = \limsup_{r\to 1^-} \frac{\nu(r)}{r(1-r)^{-\lambda(r)(1-\frac{1}{\rho_M(f)})^{+}}}; \delta = \liminf_{r\to 1^-} \frac{\nu(r)}{r(1-r)^{-\lambda(r)(1-\frac{1}{\rho_M(f)})^{+}}}. \tag{2.11}$$

Lemma 2.1. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be analytic in D, having order $\rho_M(f)$ and lower order $\lambda_M(f)$ such that either $1 < \lambda_M(f) < \rho_M(f) < \infty$ or $0 \leq \lambda_M(f) < \rho_M(f) \leq 1$ with generalized proximate order $\lambda(r)$, then

$$T^* = \limsup_{r\to 1^-} \frac{\mu(r)}{(1-r)^{-\lambda(r)(1-\frac{1}{\rho_M(f)})^{+}+1}}; t^* = \liminf_{r\to 1^-} \frac{\mu(r)}{(1-r)^{-\lambda(r)(1-\frac{1}{\rho_M(f)})^{+}+1}}, \tag{2.12}$$

where $\mu(r) = \max_{n\geq 0}\{|a_n|r^n\}$.

Proof. Using inequality 1.4.11 of [6, pp. 31] for $0 < r_o < r < 1$, we have

$$\log M(r,f) \leq \log \mu(r) + \log[\{1 + 2\nu(r + \frac{(1-r)}{\nu(r)})\}\frac{1}{1-r}].$$

Further, for any $\varepsilon > 0$ from (4.5.9) of [6, pp. 45], we have

$$\nu(r), (1-r)^{-(1+\rho_M(f)+\varepsilon)},$$

for all r in $0 < r_1 < r < 1$. Let $r' \in \max(r_o, r_1)$, then for $0 < r' < r < 1$,

$$\log M(r,f) < \log \mu(r) + (1+\rho_M(f)+\varepsilon)\log \frac{\nu(r)}{\nu(r)-1} - (2+\rho_M(f)+\varepsilon).$$
$$\log(1-r) + o(1).$$

Now dividing by $(1-r)^{-\lambda(r)(1-\frac{1}{\rho_M(f)})^{+}+1}$ and proceeding the limits as $r \to 1^-$, we get

$$T^* \leq \limsup_{r\to 1^-} \frac{\mu(r)}{(1-r)^{-\lambda(r)(1-\frac{1}{\rho_M(f)})^{+}+1}}; t^* \leq \liminf_{r\to 1^-} \frac{\mu(r)}{(1-r)^{-\lambda(r)(1-\frac{1}{\rho_M(f)})^{+}+1}}.$$

The reverse inequalities follows from the relation

$$\mu(r) \leq M(r, f).$$

Now we prove

Theorem 2.6. Let f be a function analytic in unit disc having generalized proximate order $\lambda(r)$ and either $1 < \lambda_M(f) < \rho_M(f) < \infty$ or $0 \leq \lambda_M(f) < \rho_M(f) \leq 1$. Let γ, δ and T^*, t^* be defined by (2.11) and (2.12), respectively, then

$$\begin{aligned} T^* \geq & \frac{\delta}{(\lambda_M(f)+\eta)(1-\frac{1}{\rho_M(f)})^+ + 1}(\frac{k-1}{k})^{(\lambda_M(f)+\eta)(1-\frac{1}{\rho_M(f)})^+ + 1} \\ & + \frac{\gamma}{k}(\frac{k-1}{k})^{(\lambda_M(f)+\eta)(1-\frac{1}{\rho_M(f)})^+ + 1}. \end{aligned}$$

$$\begin{aligned} t^* \geq & \frac{\delta}{(\lambda_M(f)+\eta)(1-\frac{1}{\rho_M(f)})^+ + 1}(\frac{k-1}{k})^{(\lambda_M(f)+\eta)(1-\frac{1}{\rho_M(f)})^+ + 1} \\ & + \frac{\delta}{k}(\frac{k-1}{k})^{(\lambda_M(f)+\eta)(1-\frac{1}{\rho_M(f)})^+ + 1}. \end{aligned}$$

Proof. Using (2.11) for given $\varepsilon > 0$, we have

$$\nu(r) > (\delta - \varepsilon) r (1-r)^{-\lambda(r)(1-\frac{1}{\rho_M(f)})^+}$$

for all r in $0 < r_o(\varepsilon) < r < 1$. For $k > 1$, we have

$$\int_r^{r+\frac{(1-r)}{k}} \frac{\nu(t)}{t} dt > \nu(r) \log(1 + \frac{(1-r)}{kr} > \nu(r)\frac{(1-r)}{k}.$$

From [6, Eq 1.4.10], we get

$$\begin{aligned} \log \mu(r + \frac{(1-r)}{k}) &= \log \mu(r_o) + \int_{r_o}^{r} \frac{\nu(t)}{t} dt + \int_r^{r+\frac{(1-r)}{k}} \frac{\nu(t)}{t} dt \\ &> \log \mu(r_o) + (\delta - \varepsilon) \int_{r_o}^{r} (1-r)^{-\lambda(r)(1-\frac{1}{\rho_M(f)})^+} dt \\ &+ \nu(r) \frac{(1-r)}{k}. \end{aligned}$$

For $\alpha = -1$, we get from the above inequality

$$\begin{aligned} \log \mu(r + \frac{(1-r)}{k}) > \log \mu(r_o) + & \frac{(\delta-\varepsilon)(1-r)^{-\lambda(r)(1-\frac{1}{\rho_M(f)})^+ + 1}}{(\lambda_M(f)+\eta)(1-\frac{1}{\rho_M(f)})^+ + 1} \\ & + o(1-r)^{-\lambda(r)(1-\frac{1}{\rho_M(f)})^+ + 1}. \end{aligned}$$

Dividing by $(\frac{k}{k-1})^{(\lambda(r))(1-\frac{1}{\rho_M(f)})^+ + 1}$ and proceeding to limits, we get with Lemma 1.1 that

$$\begin{aligned} T^* \geq & \frac{\delta}{(\lambda_M(f)+\eta)(1-\frac{1}{\rho_M(f)})^+ + 1}(\frac{k-1}{k})^{(\lambda_M(f)+\eta)(1-\frac{1}{\rho_M(f)})^+ + 1} \\ & + \frac{\gamma}{k}(\frac{k-1}{k})^{(\lambda_M(f)+\eta)(1-\frac{1}{\rho_M(f)})^+ + 1}. \end{aligned}$$

$$t^* \geq \frac{\delta}{(\lambda_M(f)+\eta)(1-\frac{1}{\rho_M(f)})^+ + 1}(\frac{k-1}{k})^{(\lambda_M(f)+\eta)(1-\frac{1}{\rho_M(f)})^+ + 1}$$
$$+ \frac{\delta}{k}(\frac{k-1}{k})^{(\lambda_M(f)+\eta)(1-\frac{1}{\rho_M(f)})^+ + 1}.$$

Hence, the proof is completed.

Example 2.1. As an example of Theorem 2.6 and following Kapoor [9], we can find the following inequalities.

For a function f analytic in unit disc having nonzero finite order $\rho_M(f)$, we have

$$\gamma + \delta \leq \frac{((\lambda_M(f)+\eta)(1-\frac{1}{\rho_M(f)}))^+ + 2)^{((\lambda_M(f)+\eta)(1-\frac{1}{\rho_M(f)}))^+ + 2)}}{((\lambda_M(f)+\eta)(1-\frac{1}{\rho_M(f)}))^+ + 1)^{((\lambda_M(f)+\eta)(1-\frac{1}{\rho_M(f)}))^+ + 1)}} T^*,$$

$$\delta \leq ((\lambda_M(f)+\eta)(1-\frac{1}{\rho_M(f)}))^+ + 1)T^*,$$

and the equality cannot simultaneously hold in the above two inequalities. If the equality holds in the first inequality, then $t^* = 0$.

3. Conclusions

The existence of generalized proximate order for every functions analytic in the unit disc has been proved. Also, to obtain refined measure of growth of analytic/entire functions of irregular growth some new results have been obtained.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author states no conflict of interest.

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