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Research article

Some stronger forms of mean sensitivity

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Abstract: The equivalence between multi-transitive mean sensitivity and multi-transitive mean n-sensitivity for linear dynamical systems was demonstrated in this study. Furthermore, this paper presented examples that highlighted the disparities among mean sensitivity, multi-transitive mean sensitivity, and syndetically multi-transitive mean sensitivity.

Keywords: backward shift; multi-transitive mean n-sensitivity; multi-transitive mean sensitivity
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1. Introduction

Let W_1 and W_2 be two Banach spaces over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. The map R: $W_1 \to W_2$ is called a *linear operator* if

$$R(\alpha x + \beta y) = \alpha Rx + \beta Ry$$

for any $\alpha, \beta \in \mathbb{F}$ and any $x, y \in W_1$. A linear operator R: $W_1 \to W_2$ is said to be *bounded* if there exists a positive constant M such that $\|Rx\| \leq M\|x\|$ for all $x \in X$, where $\|.\|$ denotes the norm of the vectors. The set of all bounded linear operators R: $W_1 \to W_2$ is denoted by $B(W_1, W_2)$.

A linear dynamical system means a pair (W, R), where W is a Banach space and R: $W \to W$ is a bounded linear operator. Throughout the whole paper, 0_W denotes the zero element of the Banach space W. I denotes the identity operator. $\mathbb{Z}_+$, $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{C}$ denote the set of all nonnegative numbers, positive numbers, real numbers, and complex numbers, respectively.

A linear operator R: $X \to Y$ is continuous if, and only if, R: $X \to Y$ is bounded. A linear dynamical system (W, R) is *hypercyclic* if there is some $x \in W$ such that

$$orb(x, R) = \{x, Rx, R^2x, \cdots\}$$

is dense in W. If W is separable, then (W, R) is transitive if, and only if, (W, R) is hypercyclic [1, Theorem 2.19].

By [2], a linear dynamical system (W, R) is absolutely Cesàro bounded if there is $M > 0$ such that,

$$\sup_{n\in\mathbb{N}} \frac{1}{n} \sum_{k=1}^{n} \|R^k w\| \le M\|w\|$$

for any $w \in W$.

The number

$$\|R\| = \sup_{x\in W, x\neq 0_X} \frac{\|Rx\|}{\|x\|}$$

is called *the norm of the operator* R, and

$$\|R\| = \sup_{\|x\|=1} \|Rx\| = \sup_{\|x\|\le 1} \|Rx\|$$

(see for instance [3]).

This study examines some stronger forms of mean sensitive for linear dynamical systems. The concepts and properties related to sensitivity are recalled in Section 2. Section 3 establishes the equivalence between multi-transitive mean sensitivity and multi-transitive mean n-sensitivity for linear dynamical systems (Theorem 3.1). An example is built to demonstrate the existence of a linear dynamical system (W, R) that exhibits multi-transitive mean sensitivity but not syndetically multi-transitive mean sensitivity (Example 3.1). In Section 4, a perturbative result concerning syndetically multi-transitive mean sensitive systems is derived (Theorem 4.1). Conclusions propose areas for future research.

2. Preliminaries

This section recalls some concepts related to sensitivity and defines three stronger forms of mean sensitivity.

Li et al. [4] introduced the notion of mean sensitivity for the topological dynamical system (i.e., the space considered is compact and metrizable and the map involved is continuous onto). For any $x \in W$ and any $\varepsilon > 0$, denote

$$B(x, \varepsilon) = \{y \in W : \|x - y\| < \varepsilon\}.$$

A linear dynamical system (W, R) is called *mean sensitive* if there is a $\delta > 0$ such that for any $x \in W$ and any $\varepsilon > 0$, there exists a $y \in B(x, \varepsilon)$ such that

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \|R^i x - R^i y\| > \delta.$$

Several scholars have studied different properties related to mean sensitivity (see [5–11]).

Now, let us recall some concepts of positive integer sets. According to [12], a subset

$$S = \{n_1 < n_2 < \cdots\} \subset \mathbb{Z}_+$$

is *syndetic* if there exists an $M \in \mathbb{Z}_+$ such that $n_{k+1} - n_k \leq M$ for each $k \in \mathbb{N}$. S is *thickly syndetic* if for any $k \in \mathbb{Z}_+$ there is a syndetic set $\left\{m_1^k < m_2^k < \cdots\right\}$ such that

$$\bigcup_{j\in\mathbb{N}} \left\{m_j^k + 1, m_j^k + 2, \cdots, m_j^k + k\right\} \subset S.$$

S is *cofinite* if $\{u, u+1, u+2, \cdots\} \subset S$ for some $u \in \mathbb{N}$. Combining these concepts, syndetic sensitivity, cofinite sensitivity, and multi-sensitivity for the topological dynamical system were introduced by Moothathu [12].

The set of all subsets of $\mathbb{Z}_+$ is denoted by $\mathcal{P} = \mathcal{P}(\mathbb{Z}_+)$. A subset $\mathcal{G}$ of $\mathcal{P}$ is called a *Furstenberg family*, if $G_1 \subset G_2$ and $G_1 \in \mathcal{G}$, then $G_2 \in \mathcal{G}$. Subsequently, many scholars discussed various notions of $\mathscr{F}$-sensitivity in [13–18].

Let $U, V \subset W$ and denote

$$N_R(U, V) = \{n \in \mathbb{Z}_+ : R^n(U) \cap V \neq \emptyset\}.$$

The system (W, R) is called *topologically ergodic* if the set $N_R(U, V)$ is syndetic for every open subsets $U, V \subset W$; It is called *thickly systic* if the set $N_R(U, V)$ is thickly syndetic for every open subsets $U, V \subset W$; It is called *mixing* if the set $N_R(U, V)$ is cofinite for every open subsets $U, V \subset W$.

The product system of k copies of (W, R) is represented as $\left(W^k, R^{(k)}\right)$. Recall that (W, R) is *transitive* if $N_R(U, V) \neq \emptyset$ for any open subsets $U, V \subset W$ and is called *weakly mixing* if $\left(W^2, R^{(2)}\right)$ is transitive.

Let $\delta > 0$. For any $x, y \in W$, denote

$$F_R(x, y, \delta) = \left\{n \in \mathbb{N} : \frac{1}{n}\sum_{i=0}^{n-1} \|R^i x - R^i y\| > \delta\right\}.$$

Inspirited by [12, 19, 20], the following concepts are introduced.

Definition 2.1. A linear dynamical system (W, R) is *multi-transitively mean sensitive*, if there is a $\delta > 0$ such that for any finitely many open subsets $P_1, \cdots, P_k \subset W$, there exist $x_1, y_1 \in P_1; \cdots; x_k, y_k \in P_k$ such that

$$\left(\bigcap_{i=1}^{k} N_R(G_i, H_i)\right) \cap \left(\bigcap_{i=1}^{k} F_R(x_i, y_i, \delta)\right) \neq \emptyset$$

for all open subsets $G_1, \cdots, G_k, H_1, \cdots, H_k \subset W$.

Definition 2.2. A linear dynamical system (W, R) is *syndetically multi-transitive mean sensitive*, if there is a $\delta > 0$ such that for any finitely many open subsets $P_1, \cdots, P_k \subset W$, there exist $x_1, y_1 \in P_1$; $\cdots; x_k, y_k \in P_k$ such that the set

$$\left(\bigcap_{i=1}^{k} N_R(G_i, H_i)\right) \cap \left(\bigcap_{i=1}^{k} F_R(x_i, y_i, \delta)\right)$$

is syndetic for any open subsets $G_1, \cdots, G_k, H_1, \cdots, H_k \subset W$.

In fact, if (W, R) exhibits multi-transitively mean sensitive, then (W, R) is considered weakly mixing. Furthermore, if (W, R) displays multi-transitively mean sensitive, it is also classified as topologically ergodic. This can be inferred from [1, Exercise 2.5.4], which establishes that (W, R) is thickly syndetic transitive. Specifically, it is simple to confirm that a linear dynamical system (W, R) is absolutely Cesàro bounded if, and only if, it demonstrates mean sensitivity. However, it is worth noting that there exists a linear dynamical system (W, R) that is mixing but does not possess mean sensitivity (refer to [21, Example 23]).

The concept of n-sensitivity for the topological dynamical system was first introduced by Xiong [22]. Subsequently, Shao et al. [23] highlighted the distinction between n-sensitivity and $(n + 1)$-sensitivity for the minimal system (see also [24]). More recently, Li et al. [25] proposed the concept of mean n-sensitivity for the topological dynamical system. The system (W, R) is called *mean n-sensitive* if there exists a $\delta > 0$ such that for any open subset $U \subset W$, there are n distinct points $x_1, \cdots, x_n \in U$ satisfying

$$\limsup_{m\to\infty} \frac{1}{m} \sum_{k=0}^{m-1} \min_{1\le i\ne j\le n} \|R^k x_i - R^k x_j\| > \delta.$$

For any $x_1, \cdots, x_n \in W$ and $\delta > 0$, denote

$$F_R^{\min}(x_1, \cdots, x_n, \delta) = \left\{ n \in \mathbb{N} : \frac{1}{n} \sum_{k=0}^{n-1} \min_{1\le i\ne j\le n} \|R^k x_i - R^k x_j\| > \delta \right\}.$$

An other new and stronger version of n-sensitivity is as follow.

Definition 2.3. A linear dynamical system (W, R) is *multi-transitively mean n-sensitive*, if there is a $\delta > 0$ such that for finitely many open subsets $P_1, \cdots, P_k \subset W$, there exist $x_1^1, \cdots, x_n^1 \in P_1$; $\cdots$; $x_1^k, \cdots, x_n^k \in P_k$ such that

$$\left(\bigcap_{i=1}^{k} N_R(U_i, V_i)\right) \cap \left(\bigcap_{i=1}^{k} F_R^{\min}(x_1^i, \cdots, x_n^i, \delta)\right) \ne \emptyset$$

for any open subsets $G_1, \cdots, G_k, H_1, \cdots, H_k \subset W$.

3. Multi-transitive mean sensitivity

The following proof is arose by [26, Theorem 4].

Theorem 3.1. Let (W, R) be a linear dynamical system, then the following conditions are equivalent.

(1) (W, R) is multi-transitively mean sensitive;

(2) (W, R) is multi-transitively mean n-sensitive.

Proof. (1) $\Rightarrow$ (2) Since (W, R) is multi-transitively mean sensitive, for any $\sigma > 0$, $C > 0$, $k \in \mathbb{N}$ and open subsets $U_1, \cdots, U_k, V_1, \cdots, V_k \subset W$, there exist an $x \in W$ and an

$$n \in \bigcap_{i=1}^{k} N_R(U_i, V_i),$$

such that

$$\|x\| < \sigma \quad \text{and} \quad \frac{1}{n}\sum_{i=0}^{n-1}\left\|R^i x\right\| > C.$$

This means that there is an $x_0 \in W$, which causes

$$\sup_{n\in\bigcap_{i=1}^{k} N_R(U_i,V_i)} \frac{1}{n}\sum_{i=0}^{n-1}\left\|R^i x_0\right\| = \infty.$$

In fact, assume that for any $x \in W$,

$$\sup_{n\in\bigcap_{i=1}^{k} N_R(U_i,V_i)} \frac{1}{n}\sum_{i=0}^{n-1}\left\|R^i x\right\| < \infty.$$

Therefore, one can select a sequence $\{y_n\}_{n\in\mathbb{N}} \subset W$ and a sequence

$$\{M_n\}_{n\in\mathbb{N}} \subset \bigcap_{i=1}^{k} N_R(U_i, V_i),$$

satisfying that

$$\|y_n\| < \frac{1}{2^n}$$

and

$$\frac{1}{M_p}\sum_{j=0}^{M_p-1}\left\|R^j(y_1 + \cdots + y_n)\right\| > p$$

for every $1 \leq p \leq n$. Let

$$x = \sum_{n=1}^{\infty} y_n \in W,$$

then for all $p \in \mathbb{N}$,

$$\frac{1}{M_p}\sum_{j=0}^{M_p-1}\left\|R^j x\right\| \geq p,$$

a contradiction to the assumption. Thus, there exists an $x_0 \in W$ satisfying

$$\sup_{r\in\bigcap_{i=1}^{k} N_R(U_i,V_i)} \frac{1}{r}\sum_{l=0}^{r-1}\left\|R^l x_0\right\| = \infty. \tag{3.1}$$

Let $n \geq 2$ and $\varepsilon > 0$. By (3.1), there is a sequence

$$\{m_r\}_{r\in\mathbb{N}} \subset \bigcap_{i=1}^{k} N_R(U_i, V_i),$$

such that

$$\frac{1}{m_r}\sum_{l=0}^{m_r-1}\left\|R^l x_0\right\| > \frac{n(n+1)\|x_0\|}{\varepsilon}.$$

Since

$$\begin{aligned}\min_{2\leq i\neq j\leq n+1}\left\|R^l\left(\frac{x_0}{\|x_0\|}\frac{\varepsilon}{i}\right)-R^l\left(\frac{x_0}{\|x_0\|}\frac{\varepsilon}{j}\right)\right\| &= \frac{\varepsilon}{\|x_0\|}\left\|R^l x_0\right\|\min_{2\leq i\neq j\leq n+1}\left|\frac{1}{i}-\frac{1}{j}\right| \\ &\geq \frac{\varepsilon}{\|x_0\|}\left\|R^l x_0\right\|\frac{1}{n(n+1)}\end{aligned}$$

for every $l \geq 0$, then one has

$$\frac{1}{m_r}\sum_{l=0}^{m_r-1}\min_{2\leq i\neq j\leq n+1}\left\|R^l\left(\frac{x_0}{\|x_0\|}\frac{\varepsilon}{i}\right)-R^l\left(\frac{x_0}{\|x_0\|}\frac{\varepsilon}{j}\right)\right\| > 1$$

for every $r \in \mathbb{N}$. Let $x \in W$. By linearity of W,

$$x + \frac{x_0}{\|x_0\|}\frac{\varepsilon}{i} \in B(x, \varepsilon)$$

for each $2 \leq i \leq n+1$ and

$$\begin{aligned}&\frac{1}{m_r}\sum_{l=0}^{m_r-1}\min_{2\leq i\neq j\leq n+1}\left\|R^l\left(x+\frac{x_0}{\|x_0\|}\frac{\varepsilon}{i}\right)-R^l\left(x+\frac{x_0}{\|x_0\|}\frac{\varepsilon}{j}\right)\right\| \\ &= \frac{1}{m_r}\sum_{l=0}^{m_r-1}\min_{2\leq i\neq j\leq n+1}\left\|R^l\left(\frac{x_0}{\|x_0\|}\frac{\varepsilon}{i}\right)-R^l\left(\frac{x_0}{\|x_0\|}\frac{\varepsilon}{j}\right)\right\| \\ &> 1\end{aligned}$$

for every $r \in \mathbb{N}$, which implies that

$$\{m_r\}_{r\in\mathbb{N}} \subset \left(\bigcap_{i=1}^{k} N_R(U_i, V_i)\right) \cap \left(F_R^{\min}\left(x+\frac{x_0}{\|x_0\|}\frac{\varepsilon}{2}, \cdots, x+\frac{x_0}{\|x_0\|}\frac{\varepsilon}{n+1}, 1\right)\right).$$

Thus, for $\delta = 1 > 0$, any $y \in W$, and any $\sigma > 0$, there exist

$$y+\frac{x_0}{\|x_0\|}\frac{\sigma}{2}, \cdots, y+\frac{x_0}{\|x_0\|}\frac{\sigma}{n+1} \in B(y, \sigma),$$

such that

$$\left(\bigcap_{i=1}^{k} N_R(G_i, H_i)\right) \cap \left(F_R^{\min}\left(y+\frac{x_0}{\|x_0\|}\frac{\sigma}{2}, \cdots, y+\frac{x_0}{\|x_0\|}\frac{\sigma}{n+1}, 1\right)\right) \neq \emptyset$$

for finitely many open subsets $G_1, \cdots, G_k, H_1, \cdots, H_k \subset W$.

(2) ⇒ (1) The proof is trivial. □

Corollary 3.1. Let (W, R) be a linear dynamical system, then the following conditions are equivalent:

(1) (W, R) is multi-transitively mean sensitive.

(2) There is a $\delta_0 > 0$ such that, for every $\varepsilon > 0$, there exists a $y \in B(0_W, \varepsilon)$ satisfying

$$\left(\bigcap_{i=1}^{k} N_R(G_i, H_i)\right) \cap F_R(0_W, y, \delta_0) \neq \emptyset$$

for any finitely many open subsets $G_1, \cdots, G_k, H_1, \cdots, H_k \subset W$.

(3) Let $\delta > 0$. For any $\varepsilon > 0$, there exists a $y \in B(0_W, \varepsilon)$ such that

$$\left(\bigcap_{i=1}^{k} N_R(R_i, S_i)\right) \cap F_R(0_W, y, \delta) \neq \emptyset$$

for any finitely many open subsets $G_1, \cdots, G_k, H_1, \cdots, H_k \subset W$.

Proof. (1) $\Leftrightarrow$ (2) The proof is directly from the linearity of the operator.

(2) $\Rightarrow$ (3) Let $k \in \mathbb{N}$ and nonempty open subsets $G_1, \cdots, G_k, H_1, \cdots, H_k \subset W$. By the proof of Theorem 3.1, there exists an $x_0 \in W$ such that

$$\sup_{n \in \bigcap_{i=1}^{k} N_R(G_i, H_i)} \frac{1}{n} \sum_{i=0}^{n-1} \|R^i x_0\| = \infty.$$

Let $\delta > 0$ and $\varepsilon > 0$, then there exists a sequence

$$\{m_r\}_{r \in \mathbb{N}} \subset \bigcap_{i=1}^{k} N_R(G_i, H_i),$$

such that

$$\frac{1}{m_r} \sum_{i=0}^{m_r - 1} \left\| R^i \left(\frac{x_0}{\|x_0\|} \cdot \frac{\varepsilon}{2} \right) \right\| > \delta.$$

In other words,

$$\{m_r\}_{r \in \mathbb{N}} \subset \left(\bigcap_{i=1}^{k} N_R(G_i, H_i)\right) \cap F_R\left(\frac{x_0}{\|x_0\|} \frac{\varepsilon}{2}, 0_W, \delta \right).$$

This finishes the proof.

(3) $\Rightarrow$ (2) The proof is trivial. □

Note that a syndetically multi-transitive mean sensitive system is multi-transitive mean sensitive. Using Corollary 3.1, one can get that the converse is not true; see the following Example 3.1.

Before starting Example 3.1, let us recall the Hilbert space

$$l^2(\mathbb{Z}) = \left\{ x = (x_n)_{n \in \mathbb{Z}} \in \mathbb{R}^{\mathbb{Z}} : \sum_{n=-\infty}^{\infty} |x_n|^2 < \infty \right\}$$

with the inner product defined by

$$< u, v >= \sum_{k=-\infty}^{\infty} u_k v_k$$

for all

$$u = (u_k)_{k\in\mathbb{Z}}, \quad v = (v_k)_{k\in\mathbb{Z}} \in l^2(\mathbb{Z}).$$

This inner product generates the norm

$$||u|| = \sqrt{< u, u >} = \left(\sum_{k=-\infty}^{\infty} |u_k|^2\right)^{\frac{1}{2}}.$$

Let (W, R) be a linear dynamical system. If $w^* \in W^*$, then write

$$w^*(w) =< w, w^* >, \quad w \in W.$$

Define the *adjoint operator* R^*: $W^* \to W^*$ as $R^*u^* = u^* \circ R$; that is to say

$$< u, R^*u^* >=< Ru, u^* >, \ u \in W, \ u^* \in W^*.$$

Example 3.1. Let $W = l^2(\mathbb{Z})$. Define R: $W \to W$ as

$$(x_n)_{n\in\mathbb{Z}} \in W \mapsto (\lambda_{n+1}x_{n+1})_{n\in\mathbb{Z}} \in W,$$

where $\lambda = (\lambda_n)_{n\in\mathbb{Z}}$ satisfies three conditions:

(1) $t_n = \left(\prod_{u=1}^{n} \lambda_u\right)^{-1}, n \geq 1; t_n = \prod_{u=n+1}^{0} \lambda_u, n \leq -1; t_0 = 1.$

(2) $(t_n)_{n\geq 0} = (1, 1, 2, 1, \frac{1}{2}, 1, 2, 2^2, 2, 1, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2}, 1, 2, 4, 8, 4, 2, 1, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^2}, \cdots)$.

(3) $t_{-n} = \upsilon_n$ for all $n \geq 1$.

The following will show that (W, R) is multi-transitive mean sensitive but not syndetically multi-transitive mean sensitive.

Claim 3.1. (W, R) is multi-transitively mean sensitive.

Proof of Claim 3.1. Using the construction of $(t_n)_{n\geq 0}$, one can select a sequence $\{n_m\}_{m\in\mathbb{N}}$ satisfying $n_m > m$ and

$$\begin{cases} t_{n_m - i} = \frac{1}{2^{m-i}}, 0 \leq i \leq m, \\ t_{n_m + i} = \frac{1}{2^{m-i}}, 0 < i \leq m. \end{cases}$$

Let $\varepsilon > 0$. There is an $N > 0$ such that $\frac{1}{2^n} < \varepsilon$ for any $n \geq N$. Take $x_\varepsilon = (x_\varepsilon^i)_{i\in\mathbb{Z}}$ with

$$x_\varepsilon^i = \begin{cases} \frac{1}{2^m}, & i = n_m, \ m \geq N + 1, \\ 0, & otherwise, \end{cases}$$

then, $||x_\varepsilon|| = \frac{1}{2^N} < \varepsilon$.

Let $m \geq N+1$. Since

$$\frac{1}{t_{n_m}} = \prod_{u=1}^{n_m} \lambda_u = 2^m, \quad \frac{1}{t_{n_m-m}} = \prod_{u=1}^{n_m-m} \lambda_u = 2^{m-m} = 1,$$

one has

$$\prod_{u=n_m-m+1}^{n_m} \lambda_u = 2^m,$$

and then

$$R^m(x_\varepsilon) = \sum_{i=-\infty}^{\infty} \left(\prod_{j=i-m+1}^{i} |\lambda_j| \right) |x_\varepsilon^i| \geq \left(\prod_{j=n_m-m+1}^{n_m} |\lambda_j| \right) |x_\varepsilon^{n_m}| = 1.$$

This means that there is an $M > 0$ such that

$$\frac{1}{m} \sum_{i=0}^{m-1} \|R^i(x_\varepsilon)\| > \frac{1}{2}$$

for any $m \geq M$. In other words, $F_R\left(0_W, x_\varepsilon, \frac{1}{2}\right)$ is cofinite. Since (W, R) is weakly mixing by [1, Proposition 4.16], one has

$$\bigcap_{i=1}^{k} N_R(U_i, V_i) \neq \emptyset$$

for any finitely many open subsets $U_1, \cdots, U_k, V_1, \cdots, V_k \subset W$, then

$$\left(\bigcap_{i=1}^{k} N_R(U_i, V_i) \right) \cap F_R(0_W, x_\varepsilon, \frac{1}{2}) \neq \emptyset$$

for every $k \in \mathbb{N}$ and open subsets $U_1, \cdots, U_k, V_1, \cdots, V_k \subset W$. Thus, (W, R) is multi-transitively mean sensitive by Corollary 3.1.

Claim 3.2. (W, R) is not syndetically multi-transitive mean sensitive.

Proof of Claim 3.2. By [1, Remark 4.17], $(W \times W^*, R \times R^*)$ is not hypercyclic. Notice that $W \times W^*$ is separable. By Theorem 3.1, one can obtain that $(W \times W^*, R \times R^*)$ has no transitivity. Since (W^*, R^*) is weakly mixing by [1, Proposition 4.16], then, (W, R) is not topologically ergodic by [1, Exercise 1.5.6(iii)]. This means that (W, R) has no syndetic multi-transitive mean sensitivity.

In addition, by the proof of Theorem 3.1, one can obtain that if a system (W, R) is multi-transitively mean sensitive, then, (W, R) is mean sensitive. The following example indicates that the converse is not true.

Example 3.2. Let

$$W = \left\{ x = (x_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}} : \sum_{n=1}^{\infty} |x_n| < \infty \right\}$$

with the norm

$$\|x\| = \sum_{n=1}^{\infty} |x_n|.$$

Define R: $W \to W$ as

$$R(x_1, x_2, x_3, \cdots) = (0, 2x_1, 2x_2, 2x_3, \cdots)$$

for any $(x_1, x_2, x_3, \cdots) \in W$. Let $x = (x_m)_{m\in\mathbb{N}} \in W$ and $n \in \mathbb{N}$, then

$$\|R^n x\| = \sum_{i=1}^{\infty} 2^n |x_i| = 2^n \sum_{i=1}^{\infty} |x_i| = 2^n \|x\|$$

and

$$\lim_{n\to\infty} \|R^n x\| = \infty.$$

This implies that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \|R^i x\| = \infty$$

for any $x \in X$ with $x \neq 0_W$. Thus, by the linearity of W, (W, R) is mean sensitive. Notice that (W, R) has no hypercyclicity by [1, Remark 4.10] and W is separable, then by [1, Theorem 2.19]), (W, R) is not transitive. Thus, (W, R) is not multi-transitively mean sensitive.

4. Sensitive perturbations of the identity

Affected by the methods in [27, Theorem 3.3] and [1, Corollary 8.3], the following result (Theorem 4.1) can be obtained.

Let $1 \leq p < \infty$. Recall the Banach space

$$l^p = \left\{ u = (u_n)_{n\in\mathbb{N}} \in \mathbb{F}^{\mathbb{N}} : \sum_{n=1}^{\infty} |u_n|^p < \infty \right\}$$

with the norm

$$\|u\| = \left(\sum_{n=1}^{\infty} |u_n|^p \right)^{\frac{1}{p}}$$

and the Banach space

$$c_0 = \{u = (u_n)_{n\in\mathbb{N}} \in \mathbb{F}^{\mathbb{N}} : \lim_{n\to\infty} u_n = 0\}$$

with the norm

$$\|u\| = \sup_{n\in\mathbb{N}} |u_n|.$$

Define weight shift operator B_ω: $W \to W$ as

$$B_\omega(y_1, y_2, y_3, \cdots) = (\omega_2 y_2, \omega_3 y_3, \omega_4 y_4, \cdots)$$

for all $y = (y_1, y_2, y_3, \cdots) \in W$, where $\omega = (\omega_n)_{n\in\mathbb{N}}$ is a bounded sequence.

Theorem 4.1. Let $W = l^p, 1 \leq p < \infty$, and let $\omega = (\omega_n)_{n\in\mathbb{N}}$ such that $\sup_{n\in\mathbb{N}} |\omega_n| < \infty$, then $(W, I + B_\omega)$ is syndetically multi-transitive mean sensitive.

Proof. Let $\varepsilon > 0$. There is an $N > 0$ such that $\frac{1}{n} < \varepsilon$ for any $n \geq N$. Take

$$x_\varepsilon = \left(0, \frac{1}{N}, 0, 0, \cdots\right) \in W,$$

then $\|x_\varepsilon\| = \frac{1}{N} < \varepsilon$ and

$$\|(I + B_\omega)^n(x_\varepsilon)\| = \left\|\left(\sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} B_\omega^k\right)(x_\varepsilon)\right\| = \left(\frac{1}{N^p} + \left(\frac{n|\omega_2|}{N}\right)^p\right)^{1/p} \geq \frac{n|\omega_2|}{N} > 1$$

for any $n > \frac{N}{|\omega_2|}$, which means that there is an $M > 0$, satisfying

$$\frac{1}{m}\sum_{i=0}^{m-1} \|(I + B_\omega)^i x_\varepsilon\| > \frac{1}{2}$$

for any $m \geq M$. Thus, $F_{I+B_\omega}(0_W, x_\varepsilon, 1)$ is cofinite.

Since $(W, I + B_\omega)$ is mixing by [1, Corollary 8.3], then $\bigcap_{i=1}^{k} N_{I+B_\omega}(G_i, H_i)$ is cofinite for any finitely many open subsets $G_1, \cdots, G_k, H_1, \cdots, H_k \subset W$. Therefore

$$\left(\bigcap_{i=1}^{k} N_{I+B_\omega}(G_i, H_i)\right) \cap F_{I+B_\omega}(0_W, x_\varepsilon, \frac{1}{2}) \neq \emptyset$$

for any finitely many open subsets $G_1, \cdots, G_k, H_1, \cdots, H_k \subset W$. Thus, $(W, I + B_\omega)$ is syndetically multi-transitive mean sensitive by Corollary 3.1. □

Similarly, one can get the following result.

Theorem 4.2. Let $W = c_0$ and let $\omega = (\omega_n)_{n\in\mathbb{N}}$ such that $\sup_{n\in\mathbb{N}} |\omega_n| < \infty$, then $(W, I + B_\omega)$ is syndetically multi-transitive mean sensitive.

5. Conclusions

In this research, it was demonstrated that there is an equivalence between multi-transitive mean sensitivity and multi-transitive mean n-sensitivity in the context of linear dynamical systems. Additionally, it was proven that there is the existence of a system (W, R) that is multi-transitive mean sensitive but not syndetically multi-transitive mean sensitive. This study provided evidence of the relation between these different types of system sensitivities. Whether similar conclusions hold in ergodic theory will be investigated in the future.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest regarding the publication of this paper.

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