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Research article

Algebraic invariants of edge ideals of some circulant graphs

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Abstract: Let S be a polynomial ring over a field and I be an edge ideal associated with some classes of circulant graphs. We discussed the algebraic invariants, namely, regularity, projective dimension, depth, and the Stanley depth of S/I.

Keywords: monomial ideal; regularity; projective dimension; depth; Stanley depth; circulant graphs
Mathematics Subject Classification: Primary 13C15; Secondary 13P10, 13F20

1. Introduction

Let $S = K[x_1, \ldots, x_l]$ be a polynomial ring over the field K with standard grading and $\mathbf{N}$ be a finitely generated graded S-module. Suppose that $\mathbf{N}$ admits the following minimal free resolution:

$$0 \longrightarrow \bigoplus_{j\in\mathbb{Z}} S(-j)^{\beta_{r,j}(\mathbf{N})} \longrightarrow \bigoplus_{j\in\mathbb{Z}} S(-j)^{\beta_{r-1,j}(\mathbf{N})} \longrightarrow \cdots \longrightarrow \bigoplus_{j\in\mathbb{Z}} S(-j)^{\beta_{0,j}(\mathbf{N})} \longrightarrow \mathbf{N} \longrightarrow 0.$$

If pdim($\mathbf{N}$) denotes the *projective dimension* of $\mathbf{N}$, then

$$\text{pdim}(\mathbf{N}) = \max\left\{i : \beta_{i,j}(\mathbf{N}) \neq 0\right\}.$$

If reg($\mathbf{N}$) denotes the *Castelnuovo-Mumford regularity* (or simply *regularity*) of $\mathbf{N}$, then

$$\text{reg}(\mathbf{N}) = \max\left\{j - i : \beta_{i,j}(\mathbf{N}) \neq 0\right\}.$$

The regularity measures the complexity of a module, and the projective dimension measures how far a module is from being projective. We refer the readers to [1–4] for a more detailed study of these two invariants of $\mathbf{N}$. If $\mathfrak{m} := (x_1, \ldots, x_l)$ is the unique maximal graded ideal of S, then the *depth* of $\mathbf{N}$ is defined to be the common length of all maximal $\mathbf{N}$-sequences in $\mathfrak{m}$. For more details about invariant depth, we refer the readers to [5].

In 1982, Stanley defined an invariant called the Stanley depth of a graded module over a graded commutative ring. Let $\mathbf{N}$ be a finitely generated $\mathbb{Z}^l$-graded S-module. The K-subspace $vK[W]$ is generated by all elements of the form vf, where v is a homogeneous element in $\mathbf{N}$, f is a monomial in $K[W]$, and $W \subseteq \{x_1, \ldots, x_l\}$. If $vK[W]$ is a free $K[W]$-module then it is called a Stanley space of dimension $|W|$. A decomposition $\mathcal{P}$ of K-vector space $\mathbf{N}$ as a finite direct sum of Stanley spaces is called a Stanley decomposition of $\mathbf{N}$. Let $\mathcal{P} : \mathbf{N} = \oplus_{j=1}^{m} v_j K[W_j]$, and the Stanley depth of $\mathcal{P}$ is $\operatorname{sdepth}(\mathcal{P}) = \min\left\{|W_j| : j = 1, 2, \ldots, m\right\}$. The number

$$\operatorname{sdepth}(\mathbf{N}) := \max\left\{\operatorname{sdepth}(\mathcal{P}) : \mathcal{P} \text{ is a Stanley decomposition of } \mathbf{N}\right\},$$

is called the *Stanley depth* of $\mathbf{N}$. Stanley decompositions have applications in the normal form theory for systems of differential equations (see [6–8]). Herzog et al. [9] gave the method for computing the Stanley depth of monomial ideals. After that, Ichim et al. [10], introduced an algorithm for computing the Stanley depth of a finitely generated module over a polynomial ring. Although the algorithms exist, it is still hard to compute the Stanley depth. Therefore, it is crucial to give values and bounds for Stanley depth of some classes of modules. We refer the readers to [11–13] for some known results on Stanley depth. Stanley conjectured [14] that $\operatorname{sdepth}(\mathbf{N}) \geq \operatorname{depth}(\mathbf{N})$. Duval et al. disproved this conjecture in [15]. However, it is still interesting to determine the classes of the $\mathbb{Z}^l$-graded S-module that satisfy this inequality. For some recent results regarding this inequality, known as Stanley's inequality, see [3, 16–18].

Let $G := (V(G), E(G))$ be a graph with vertex set $V(G) = \{x_1, \ldots, x_l\}$ and edge set $E(G)$. Throughout this work, all graphs are finite and simple. The *edge ideal* $I(G)$ associated with G is a squarefree monomial ideal; that is, $I(G) = \left(x_i x_j : \{x_i, x_j\} \in E(G)\right)$. A graph G is *l-regular* if every vertex of G has degree l. Fix an integer $n \geq 2$ and a subset $\mathbb{S} \subset \left\{1, \ldots, \left\lfloor \frac{n}{2} \right\rfloor\right\}$. The *circulant graph* $C_n(\mathbb{S})$ is defined to be a graph with vertex set $\{x_1, ..., x_n\}$ and edge set

$$E(C_n(\mathbb{S})) = \left\{\{x_i, x_j\} : |i - j| \text{ or } n - |i - j| \in \mathbb{S}\right\}.$$

For convenience, the representation $C_n(a_1, \ldots, a_q)$ is used for $C_n\left(\{a_1, \ldots, a_q\}\right)$. Generally, a circulant graph $C_n(a_1, \ldots, a_q)$ is $2q$-regular, except if $2a_q = n$, in which case, it is $(2q - 1)$-regular. See Figure 1 for examples of circulant graphs.

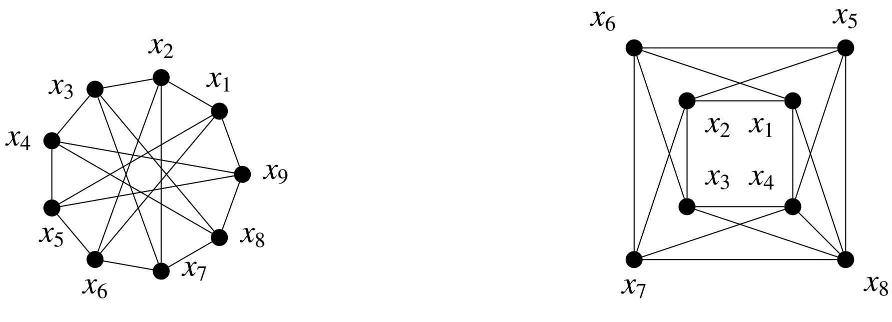

Figure 1. From left to right, $C_9(1, 4)$ and $C_8(1, 3)$.

Circulant graphs are sometimes viewed as generalized cycles as $\mathbb{C}_n = C_n(1)$. Circulant graphs were introduced in 1846, and they have a number of applications in computer network design, telecommunication networks, data connection networks, group theory, and others [19–22]. Several papers have been written on the aforementioned algebraic invariants of edge ideals associated with circulant graphs; see [23–25]. Uribe-Paczka et al. [4] computed regularity of all cubic circulant graphs. Later, Shaukat et al. [26] gave the exact values of depth, projective dimension, and lower bounds of Stanley depth of the quotient rings of the edge ideals associated with cubic circulant graphs. Unlike cubic circulant graphs [27], there is no simple characterization or formula to uniquely represent all four and five regular circulant graphs. The classification of all four and five regular circulant graphs is a topic of ongoing research, and many mathematicians and computer scientists are working to gain deeper insights into the properties of these graphs [22, 25]. In practice, researchers often focus on specific subclasses of circulant graphs to make progress in their study.

Motivated by the above-mentioned works on the algebraic invariants of edge ideals associated with circulant graphs, our aim is to extend the study of cubic circulant graphs. In particular, we study the above-mentioned invariants of the quotient rings of the edge ideals associated with some families of four and five regular circulant graphs, which include $C_{2n}(1, n-1)$, $C_{2n}(1, 2)$, and $C_{2n}(1, n-1, n)$, where $n \geq 3$. These graphs are depicted in Figures 2 and 3.

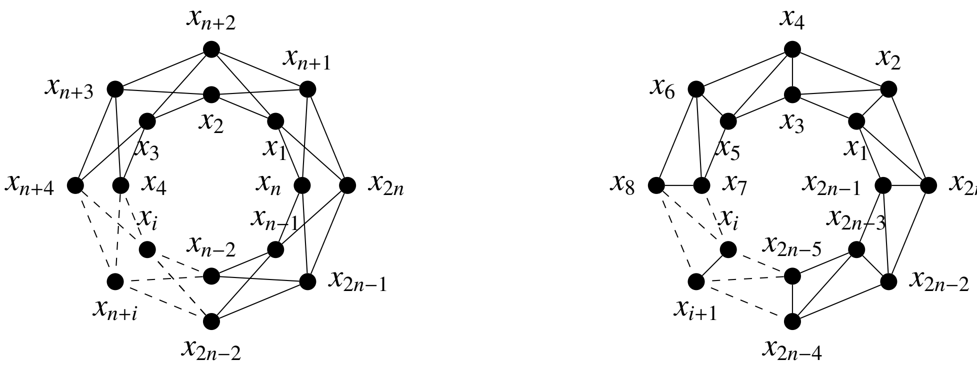

Figure 2. From left to right, $C_{2n}(1, n-1)$ and $C_{2n}(1, 2)$.

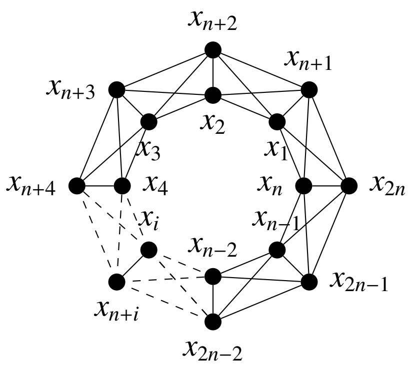

Figure 3. $C_{2n}(1, n-1, n)$.

We give the exact values of depth, projective dimension, and bounds for the Stanley depth of $K[V(C_{2n}(1,n-1))]/I(C_{2n}(1,n-1))$, (see Theorem 4.1, Corollary 4.2, and Theorem 4.3). In Theorem 4.5, we give a formula for the regularity of the edge ideal associated with $C_{2n}(1,n-1)$ when $n \equiv 0,1 \pmod 3$, and sharp bounds when $n \equiv 2 \pmod 3$. Zahid et al. gave values and sharp bounds in [12, Corollaries 3.6 and 3.8] for depth and the Stanley depth of module $K[V(C_{2n}(1,2))]/I(C_{2n}(1,2))$. We give the exact values of the regularity of the edge ideal associated with $C_{2n}(1,2)$ when n is even and tight bounds when n is odd, see Theorem 4.6. Also, the exact values for depth and sharp bounds for Stanley depth of the module $K[V(C_{2n}(1,n-1,n))]/I(C_{2n}(1,n-1,n))$ were given by Zahid et al. in [28, Theorem 3.3, and Corollary 3.4]. Our Theorem 4.7 gives the exact value for the regularity of edge ideal associated with $C_{2n}(1,n-1,n)$. It is worth mentioning that for computation of the said algebraic invariants for four and five regular circulant graphs, the algebraic invariants associated with certain subgraphs of $C_{2n}(1,n-1), C_{2n}(1,2)$ and $C_{2n}(1,n-1,n)$ play a significant role; see, for instance, Lemmas 3.4 and 3.6–3.9. We acknowledge the use of CoCoA [29] and Macaulay2 [30] for calculations.

2. Preliminaries

In this section, we recall some results and definitions that will be used throughout the paper.

Lemma 2.1 ([18]). *Let* $0 \to \mathbf{X} \to \mathbf{Y} \to \mathbf{Z} \to 0$ *be a short exact sequence of* $\mathbb{Z}^l$*-graded* S*-modules, then*

$$\operatorname{sdepth}(\mathbf{Y}) \geq \min\{\operatorname{sdepth}(\mathbf{Z}), \operatorname{sdepth}(\mathbf{X})\}.$$

Lemma 2.2 (Depth Lemma). *If* $0 \to \mathbf{X} \to \mathbf{Y} \to \mathbf{Z} \to 0$ *is a short exact sequence of modules over a local ring* S*, or a Noetherian graded ring with* S_0 *local, then*

(a) $\operatorname{depth}(\mathbf{Y}) \geq \min\{\operatorname{depth}(\mathbf{X}), \operatorname{depth}(\mathbf{Z})\}$;
(b) $\operatorname{depth}(\mathbf{X}) \geq \min\{\operatorname{depth}(\mathbf{Y}), \operatorname{depth}(\mathbf{Z})+1\}$;
(c) $\operatorname{depth}(\mathbf{Z}) \geq \min\{\operatorname{depth}(\mathbf{Y}), \operatorname{depth}(\mathbf{X})-1\}$.

Lemma 2.3 ([18, Corollary 1.3]). *Let* $J \subset S$ *be a monomial ideal and* z *be a monomial such that* $z \notin J$*, then* $\operatorname{depth}(S/(J:z)) \geq \operatorname{depth}(S/J)$.

Lemma 2.4 ([16, Proposition 2.7]). *Let* $J \subset S$ *be a monomial ideal and* z *be a monomial such that* $z \notin J$*, then* $\operatorname{sdepth}(S/(J:z)) \geq \operatorname{sdepth}(S/J)$.

When we introduce new variables into the ring, depth and Stanley depth will likewise increase [9, Lemma 3.6], while regularity will not change [31, Lemma 3.5]. The subsequent lemma provides a summary of these findings.

Lemma 2.5. *Let* J *be a monomial ideal of* S*, and* $\bar{R} = S \otimes_K K[x_{l+1}]$ *a polynomial ring in* $l+1$ *variables, then* $\operatorname{depth}(\bar{R}/J) = \operatorname{depth}(S/J)+1, \operatorname{sdepth}(\bar{R}/J) = \operatorname{sdepth}(S/J)+1$ *and* $\operatorname{reg}(\bar{R}/J) = \operatorname{reg}(S/J)$.

We also recall the following useful lemmas.

Lemma 2.6 ([32, Proposition 2.2.20]). *For* $1 \leq r < l$*, let* $S = \mathcal{R}_1 \otimes_K \mathcal{R}_2$*, where* $\mathcal{R}_1 = K[x_1,\dots,x_r]$ *and* $\mathcal{R}_2 = K[x_{r+1},\dots,x_l]$*, then* $S/(I+J) \cong \mathcal{R}_1/I \otimes_K \mathcal{R}_2/J$.

By using Lemma 2.6 and combining it with [32, Proposition 2.2.21] and [18, Theorem 3.1] for depth and Stanley depth, respectively, we get the following useful result.

Lemma 2.7. *For* $1 \leq r < l$, *let* $S = \mathcal{R}_1 \otimes_K \mathcal{R}_2$, *where* $\mathcal{R}_1 = K[x_1, \ldots, x_r]$ *and* $\mathcal{R}_2 = K[x_{r+1}, \ldots, x_l]$, *then* $\operatorname{depth}_S(\mathcal{R}_1/I \otimes_K \mathcal{R}_2/J) = \operatorname{depth}_S(S/(I+J)) = \operatorname{depth}_{\mathcal{R}_1}(\mathcal{R}_1/I) + \operatorname{depth}_{\mathcal{R}_2}(\mathcal{R}_2/J)$ *and we have* $\operatorname{sdepth}_S(\mathcal{R}_1/I \otimes_K \mathcal{R}_2/J) \geq \operatorname{sdepth}_{\mathcal{R}_1}(\mathcal{R}_1/I) + \operatorname{sdepth}_{\mathcal{R}_2}(\mathcal{R}_2/J)$.

Let $l \geq 2$. A graph G on vertex set $\{x_1, \ldots, x_l\}$ is said to be a *path* of length $l-1$ if

$$E(G) = \Big\{\{x_i, x_{i+1}\} : i \in \{1, \ldots, l-1\}\Big\}.$$

We represent the path of length $l-1$ by $\mathbb{P}_l$. A graph G on vertex set $\{x_1, \ldots, x_l\}$ is said to be a *cycle* of length l if $E(G) = E(\mathbb{P}_l) \cup \{\{x_l, x_1\}\}$. We represent the cycle of length l by $\mathbb{C}_l$. A *bipartite graph* is a graph in which the set of vertices is partitioned into two disjoint sets called partite sets such that no two vertices of a graph within the same partite set are adjacent. Let $l \geq 1$, a *complete graph* $\mathbb{K}_l$ on l vertices is a graph in which each pair of vertices is connected by an edge. A *complete bipartite graph* is a bipartite graph such that every vertex of one partite set is connected to every vertex of the other partite set. Let $\mathcal{K}_{u,v}$ denote the complete bipartite graph with partite sets $\mathcal{K}_u = \{x_1, \ldots, x_u\}$ and $\mathcal{K}_v = \{x_{u+1}, \ldots, x_{u+v}\}$. A vertex x_j is a *neighbor* of a vertex x_i in a graph G if $\{x_i, x_j\} \in E(G)$. The *neighborhood* $N_G(x_i)$ of a vertex x_i is the set of all neighbors of x_i, that is, $N_G(x_i) := \Big\{x_j \in V(G) \ : \ \{x_i, x_j\} \in E(G)\Big\}$. A *subgraph* $\mathcal{H}$ of a graph G, denoted by $\mathcal{H} \subseteq G$, is a graph such that $V(\mathcal{H}) \subseteq V(G)$ and $E(\mathcal{H}) \subseteq E(G)$. For a subset $\mathcal{T} \subseteq V(G)$, an *induced subgraph* of G is a graph $G' := (\mathcal{T}, E(G'))$, such that $E(G') = \Big\{\{x_i, x_j\} \in E(G) : \{x_i, x_j\} \subseteq \mathcal{T}\Big\}$. A *matching* M in a graph G is a subset of $E(G)$ in which no two edges are adjacent. An *induced matching* in G is a matching that forms an induced subgraph of G. An *induced matching number* of G denoted by $\operatorname{indmat}(G)$ is defined as

$$\operatorname{indmat}(G) = \max\Big\{|M| : \ M \text{ is an induced matching in } G\Big\}.$$

Katzman showed in [33, Lemma 2.2] that $\operatorname{indmat}(G)$ is a lower bound for the regularity of $S/I(G)$. Afterward, Hà et al. showed in [34, Corollary 6.9] that the regularity of $S/I(G)$ is equal to the $\operatorname{indmat}(G)$ if G is a chordal graph. The following lemma combines these results.

Lemma 2.8. *If G is a finite simple graph, then* $\operatorname{reg}(S/I(G)) \geq \operatorname{indmat}(G)$. *Moreover, if G is a chordal graph, then* $\operatorname{reg}(S/I(G)) = \operatorname{indmat}(G)$.

Lemma 2.9 ([35, Lemma 3.2]). *Let* $1 \leq r < l$, $\mathcal{R}_1 = K[x_1, \ldots, x_r]$ *and* $\mathcal{R}_2 = K[x_{r+1}, \ldots, x_l]$. *If I and J are monomial ideals such that* $I \subset \mathcal{R}_1$, $J \subset \mathcal{R}_2$, *and* $S = \mathcal{R}_1 \otimes_K \mathcal{R}_2$, *then*

$$\operatorname{reg}(S/I+J) = \operatorname{reg}(\mathcal{R}_1/I) + \operatorname{reg}(\mathcal{R}_2/J).$$

The following result was proved by Kalai et al. in [36, Theorem 1.4] for squarefree monomial ideals and was later generalized for arbitrary monomial ideals by Herzog in [37, Corollary 3.2].

Lemma 2.10. *If I and J are the monomial ideals of S, then* $\operatorname{reg}(S/(I+J)) \leq \operatorname{reg}(S/I) + \operatorname{reg}(S/J)$.

In the following lemma, proof of parts (a) and (c) follows from Corollary 20.19 and Proposition 20.20 of [38], while part (b) comes from [39, Lemma 2.10].

Lemma 2.11 ([2, Theorem 4.7]). *Let I be a monomial ideal and z be a variable of S, then*

(a) $\operatorname{reg}(S/I) = 1 + \operatorname{reg}(S/(I:z))$ *if* $\operatorname{reg}(S/(I,z)) < \operatorname{reg}(S/(I:z))$,

(b) $\operatorname{reg}(S/I) \in \{\operatorname{reg}(S/(I,z)), \operatorname{reg}(S/(I,z)) + 1\}$ *if* $\operatorname{reg}(S/(I:z)) = \operatorname{reg}(S/(I,z))$,
(c) $\operatorname{reg}(S/I) = \operatorname{reg}(S/(I,z))$ *if* $\operatorname{reg}(S/(I:z)) < \operatorname{reg}(S/(I,z))$.

It is clear and well known that $\operatorname{depth}(S) = \operatorname{sdepth}(S) = l$ and $\operatorname{reg}(S) = 0$.

Lemma 2.12 ([5, Theorems 1.3.3]). *(Auslander–Buchsbaum formula) Let R be a commutative Noetherian local ring and* $\mathbf{N}$ *be a non-zero finitely generated R-module of finite projective dimension, then*

$$\operatorname{pdim}(\mathbf{N}) + \operatorname{depth}(\mathbf{N}) = \operatorname{depth}(R).$$

Now, we recall the results that were proved in [40, Lemma 2.8], [41, Lemma 4], and [1, Lemma 3.1.1] for depth, Stanley depth and regularity, respectively.

Lemma 2.13. *If* $l \geq 2$, *then*

(a) $\operatorname{depth}(S/I(\mathbb{P}_l)) = \operatorname{sdepth}(S/I(\mathbb{P}_l)) = \left\lceil \frac{l}{3} \right\rceil$,
(b) $\operatorname{reg}(S/I(\mathbb{P}_l)) = \left\lceil \frac{l-1}{3} \right\rceil$.

Lemma 2.14 ([42, Proposition 1.3, Proposition 1.8 and Theorems 1.9]). *If* $l \geq 3$, *then*

(a) $\operatorname{depth}(S/I(\mathbb{C}_l)) = \left\lceil \frac{l-1}{3} \right\rceil$,
(b) $\operatorname{sdepth}(S/I(\mathbb{C}_l)) = \left\lceil \frac{l-1}{3} \right\rceil$, *for* $l \equiv 0, 2 (\operatorname{mod} 3)$ *and*

$$\left\lceil \frac{l-1}{3} \right\rceil \leq \operatorname{sdepth}(S/I(\mathbb{C}_l)) \leq \left\lceil \frac{l}{3} \right\rceil, \text{ for } l \equiv 1 (\operatorname{mod} 3).$$

The value of regularity of the cycle can be deduced from the work of Jacques [43, Theorem 7.6.28] and the required following form is given in [44, Theorem 5.2].

Lemma 2.15. *If* $l \geq 3$, *then*

$$\operatorname{reg}(S/I(\mathbb{C}_l)) = \begin{cases} \left\lfloor \frac{l}{3} \right\rfloor, & \text{if } l \equiv 0, 1 \ (\operatorname{mod}\ 3); \\ \left\lfloor \frac{l}{3} \right\rfloor + 1, & \text{if } l \equiv 2 \ (\operatorname{mod}\ 3). \end{cases}$$

Lemma 2.16 ([45, Theorems 1.4]). *Let* $u, v \geq 1$ *and* $S = K[V(\mathcal{K}_{u,v})]$, *then*

$$\operatorname{depth}(S/I(\mathcal{K}_{u,v})) = 1 \leq \operatorname{sdepth}(S/I(\mathcal{K}_{u,v})).$$

The following result proved by Shaukat et al. [26, Lemma 3.1] is helpful in the computation of depth of edge ideals. We will use this result in subsequent proofs of some formulas for the depth.

Lemma 2.17. *Let G be a connected graph with* $V(G) = \{x_1, \ldots, x_l\}$. *If* $N_G(x_i) = \left\{x_{i_1}, \ldots, x_{i_q}\right\}$, *then*

$$(I(G) : x_i)/I(G) \cong \bigoplus_{t=1}^{q} S_t/J_t[x_{i_t}],$$

where $S_1 = K[V(G) \backslash N_G(x_{i_1})]$, $S_t = K[V(G) \backslash (N_G(x_{i_t}) \cup \{x_{i_1}, x_{i_2}, \ldots, x_{i_{t-1}}\})]$, *for* $t \geq 2$, *and* $J_t = (S_t \cap I(G))$ *for* $t \geq 1$.

3. Invariants of cyclic modules associated with certain subgraphs of $C_{2n}(1, n-1), C_{2n}(1, 2)$ and $C_{2n}(1, n-1, n)$

For $n \geq 2$, we introduce some families of subgraphs, namely E_n, F_n and G_n of $C_{2n}(1, n-1), C_{2n}(1, 2)$ and $C_{2n}(1, n-1, n)$, respectively as given in Figures 4 and 5. The vertex sets of these subgraphs are $V(E_n) = V(F_n) = V(G_n) = \bigcup_{i=1}^{n} \{x_i, y_i\}$ and the edge sets are as follows:

- $E(E_n) = \bigcup_{i=1}^{n-1} \Big\{ \{x_i, x_{i+1}\}, \{y_i, y_{i+1}\}, \{x_i, y_{i+1}\}, \{x_{i+1}, y_i\} \Big\}$,
- $E(F_n) = \bigcup_{i=1}^{n-1} \Big\{ \{x_i, y_i\}, \{x_i, x_{i+1}\}, \{y_i, y_{i+1}\}, \{x_i, y_{i+1}\} \Big\} \bigcup \{x_n, y_n\}$,
- $E(G_n) = \bigcup_{i=1}^{n-1} \Big\{ \{x_i, y_i\}, \{x_i, x_{i+1}\}, \{y_i, y_{i+1}\}, \{x_i, y_{i+1}\}, \{x_{i+1}, y_i\} \Big\} \bigcup \{x_n, y_n\}$.

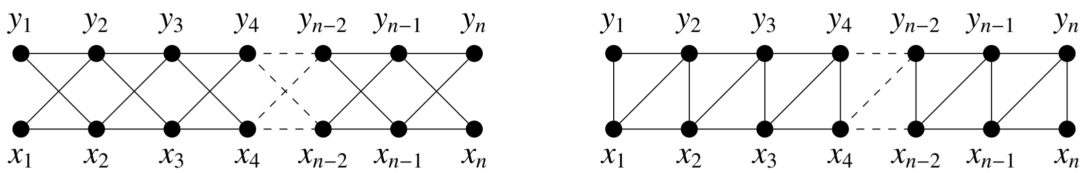

Figure 4. From left to right, E_n and F_n.

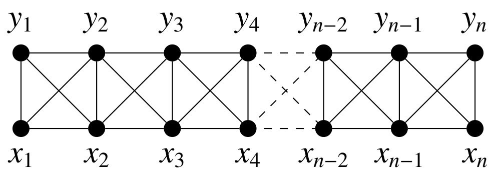

Figure 5. G_n.

In this section, we give exact values of depth, projective dimension, and regularity of the cyclic module $K[V(E_n)]/I(E_n)$. We also give bounds for the Stanley depth of such a module. Moreover, we compute the exact values of regularity of cyclic modules $K[V(F_n)]/I(F_n)$ and $K[V(G_n)]/I(G_n)$. It is worth mentioning that these findings are helpful in the subsequent section for proving our main results.

Remark 3.1. *To cater some special cases in the proofs of subsequent results, the quotient rings associated with E_n, G_n and F_n for $n \leq 1$, are described as follows:*

- $K[V(E_{-1})]/I(E_{-1}) \cong K[V(E_0)]/I(E_0) \cong K[V(F_0)]/I(F_0) \cong K[V(G_0)]/I(G_0) \cong K$ *and* $\operatorname{depth}(K) = \operatorname{sdepth}(K) = \operatorname{reg}(K) = 0$;
- $K[V(E_1)]/I(E_1) \cong K[x, y]$, *we have* $\operatorname{depth}(K[x, y]) = \operatorname{sdepth}(K[x, y]) = 2$ *and* $\operatorname{reg}(K[x, y]) = 0$;

- $K[V(F_1)]/I(F_1) \cong K[V(G_1)]/I(G_1) \cong K[V(\mathbb{P}_2)]/I(\mathbb{P}_2)$, *then by Lemma 2.13, we get* $\operatorname{depth}(K[V(\mathbb{P}_2)]/I(\mathbb{P}_2)) = \operatorname{sdepth}(K[V(\mathbb{P}_2)]/I(\mathbb{P}_2)) = \operatorname{reg}(K[V(\mathbb{P}_2)]/I(\mathbb{P}_2)) = 1$.

Remark 3.2. Let $i \in \mathbb{Z}^+$. If $k < i$ then we consider $\cup_i^k \{x_i y_{i+1}, x_i x_{i+1}, y_i y_{i+1}, x_{i+1} y_i\} = \emptyset$. Also we take $x_a y_b = 0$, whenever a or b is not positive.

For a monomial ideal I, $\mathbb{G}(I)$ denotes the minimal set of monomial generators of monomial ideal I and $\operatorname{supp}(I) := \{x_i : x_i | v \text{ for some } v \in \mathbb{G}(I)\}$.

Remark 3.3. *Let $I \subset S = K[x_1, \dots, x_l]$ be a squarefree monomial ideal minimally generated by monomials of a degree of at most 2. We associate a graph G_I with ideal I such that $V(G_I) = \operatorname{supp}(I)$ and $E(G_I) = \left\{\{x_i, x_j\} : x_i x_j \in \mathbb{G}(I)\right\}$. Let $x_t, x_r \in S$ be the variables of polynomial ring S such that $x_t, x_r \notin I$, then $(I : x_t)$, (I, x_t), $((I, x_t), x_r)$ and $((I, x_t) : x_r)$ are the monomial ideals of S such that $G_{(I:x_t)}$, $G_{(I,x_t)}$, $G_{((I,x_t),x_r)}$ and $G_{((I,x_t):x_r)}$ are subgraphs of G_I.*

By using Remark 3.3, see Figures 6 and 7 as examples of $G_{(I(E_7):y_6)}, G_{(I(E_7),y_6)}, G_{((I(E_7),y_6),x_6)}$, and $G_{((I(E_7),y_6):x_6)}$. From Figures 6 and 7, we have the following isomorphisms:

$$K[V(E_7)]/(I(E_7) : y_6) \cong K[V(E_4)]/I(E_4) \otimes_K K[y_6, x_6],$$

$$K[V(E_7)]/(I(E_7), y_6) \cong K[V(E_5)]/(I(E_5), x_5x_6, x_6y_5, x_6y_7, x_6x_7),$$

$$K[V(E_7)]/((I(E_7), y_6), x_6) \cong K[V(E_5)]/I(E_5) \otimes_K K[y_7, x_7],$$

$$K[V(E_7)]/((I(E_7), y_6) : x_6) \cong K[V(E_4)]/I(E_4) \otimes_K K[x_6].$$

Figure 6. From left to right, $G_{(I(E_7):y_6)}$ and $G_{(I(E_7),y_6)}$.

Figure 7. From left to right, $G_{((I(E_7),y_6),x_6)}$ and $G_{((I(E_7),y_6):x_6)}$.

First, we find the exact value of the depth and lower bound of the Stanley depth for $K[V(E_n)/I(E_n))$.

Lemma 3.4. *Let $n \geq 2$. If $S = K[V(E_n)]$, then*

$$\operatorname{sdepth}(S/I(E_n)) \geq \operatorname{depth}(S/I(E_n)) = \begin{cases} \left\lceil \frac{n+4}{3} \right\rceil, & \text{if } n \equiv 1 \pmod 3; \\ \left\lceil \frac{n}{3} \right\rceil, & \text{otherwise}. \end{cases}$$

Proof. We first prove the result for depth. If $n = 2$, then $E_2 \cong \mathbb{C}_4$. It is clear that the result holds by using Lemma 2.14. If $n = 3$, we have $E_3 \cong \mathcal{K}_{4,2}$, then from Lemma 2.16, we have depth $(S/I(E_n)) = 1$. Let $n \geq 4$. We consider the following cases:

Case 1. Let $n \equiv 1(\text{mod}\, 3)$. Consider the following short exact sequences

$$0 \longrightarrow S/\left(I(E_n) : y_{n-1}\right) \xrightarrow{\cdot y_{n-1}} S/I(E_n) \longrightarrow S/\left(I(E_n), y_{n-1}\right) \longrightarrow 0,$$

$$0 \longrightarrow S/\left((I(E_n), y_{n-1}) : x_{n-1}\right) \xrightarrow{\cdot x_{n-1}} S/\left(I(E_n), y_{n-1}\right) \longrightarrow S/\left((I(E_n), y_{n-1}), x_{n-1}\right) \longrightarrow 0.$$

By Lemma 2.2,

$$\operatorname{depth}\Big(S/(I(E_n)\Big) \geq \min\Big\{\operatorname{depth}\Big(S/(I(E_n) : y_{n-1})\Big), \operatorname{depth}\Big(S/(I(E_n), y_{n-1})\Big)\Big\}, \tag{3.1}$$

$$\operatorname{depth}\Big(S/(I(E_n), y_{n-1})\Big) \geq \min\Big\{\operatorname{depth}\Big(S/((I(E_n), y_{n-1}) : x_{n-1})\Big), \operatorname{depth}\Big(S/((I(E_n), y_{n-1}), x_{n-1})\Big)\Big\}. \tag{3.2}$$

We have

$$S/(I(E_n) : y_{n-1}) \cong K[V(E_{n-3})]/I(E_{n-3}) \otimes_K K[y_{n-1}, x_{n-1}], \tag{3.3}$$

$$S/((I(E_n), y_{n-1}), x_{n-1}) \cong K[V(E_{n-2})]/I(E_{n-2}) \otimes_K K[y_n, x_n], \tag{3.4}$$

$$S/((I(E_n), y_{n-1}) : x_{n-1}) \cong K[V(E_{n-3})]/I(E_{n-3}) \otimes_K K[x_{n-1}]. \tag{3.5}$$

As $n - 3 \equiv 1\,(\text{mod}\, 3)$, by applying Lemma 2.5 and Remark 3.1 on Eq (3.3) and using induction on n, we get

$$\operatorname{depth}\Big(S/(I(E_n) : y_{n-1})\Big) = \left\lceil \frac{n-3+4}{3} \right\rceil + 2 = \left\lceil \frac{n+4}{3} \right\rceil + 1.$$

Since $n - 2 \equiv 2\,(\text{mod}\, 3)$, by using Lemma 2.5 on Eq (3.4) and induction on n, it follows that

$$\operatorname{depth}\Big(S/((I(E_n), y_{n-1}), x_{n-1})\Big) = \left\lceil \frac{n-2}{3} \right\rceil + 2 = \left\lceil \frac{n+4}{2} \right\rceil.$$

Now, by Eq (3.5) and applying induction on n, Lemma 2.5 and Remark 3.1, we get

$$\operatorname{depth}\Big(S/((I(E_n), y_{n-1}) : x_{n-1})\Big) = \left\lceil \frac{n-3+4}{3} \right\rceil + 1 = \left\lceil \frac{n+4}{3} \right\rceil.$$

Here,

$$\operatorname{depth}\Big(S/((I(E_n), y_{n-1}), x_{n-1})\Big) = \operatorname{depth}\Big(S/((I(E_n), y_{n-1}) : x_{n-1})\Big),$$

and by using Eq (3.2),

$$\operatorname{depth}\Big(S/(I(E_n), y_{n-1})\Big) \geq \left\lceil \frac{n+4}{3} \right\rceil.$$

By Eq (3.1), we get

$$\operatorname{depth}(S/(I(E_n)) \geq \left\lceil \frac{n+4}{3} \right\rceil. \tag{3.6}$$

For the other inequality, if $y_n \notin I(E_n)$, then

$$S/(I(E_n):y_n) \cong K[V(E_{n-2})]/I(E_{n-2}) \otimes_K K[y_n, x_n].$$

Since $n-2 \equiv 2 \pmod 3$, by Lemmas 2.3, 2.5 and induction on n,

$$\operatorname{depth}(S/I(E_n)) \leq \operatorname{depth}(S/(I(E_n):y_n)) = \left\lceil \frac{n-2}{3} \right\rceil + 2 = \left\lceil \frac{n+4}{3} \right\rceil. \tag{3.7}$$

We get the required result by combining Eqs (3.6) and (3.7).

Case 2. Let $n \equiv 2 (\bmod 3)$. Consider the short exact sequence

$$0 \longrightarrow (I(E_n):y_{n-1})/I(E_n) \xrightarrow{\cdot y_{n-1}} S/I(E_n) \longrightarrow S/(I(E_n):y_{n-1}) \longrightarrow 0. \tag{3.8}$$

Note that here we have

$$N_{E_n}(y_{n-1}) = \{y_{n-2}, x_{n-2}, y_n, x_n\},$$
$$S_1 = K[V(E_n)\backslash N_{E_n}(y_{n-2})],$$
$$S_2 = K\big[V(E_n)\backslash(N_{E_n}(x_{n-2}) \cup \{y_{n-2}\})\big],$$
$$S_3 = K\big[V(E_n)\backslash(N_{E_n}(y_n) \cup \{y_{n-2}, x_{n-2}\})\big],$$
$$S_4 = K\big[V(E_n)\backslash(N_{E_n}(x_n) \cup \{y_{n-2}, x_{n-2}, y_n\})\big],$$
$$J_1 = (S_1 \cap I(E_n)), \quad J_2 = (S_2 \cap I(E_n)),$$
$$J_3 = (S_3 \cap I(E_n)), \quad J_4 = (S_4 \cap I(E_n)),$$

then by using Lemma 2.17, we get

$$\begin{aligned}
(I(E_n):y_{n-1})/I(E_n) &\cong S_1/J_1[y_{n-2}] \oplus S_2/J_2[x_{n-2}] \oplus S_3/J_3[y_n] \oplus S_4/J_4[x_n] \\
&\cong \frac{K[x_1,\ldots,x_{n-4},x_{n-2},x_n,y_1,\ldots,y_{n-4},y_n]}{(\cup_{i=1}^{n-5}\{x_iy_{i+1},x_ix_{i+1},y_iy_{i+1},x_{i+1}y_i\})}[y_{n-2}] \\
&\oplus \frac{K[x_1,\ldots,x_{n-4},x_n,y_1,\ldots,y_{n-4},y_n]}{(\cup_{i=1}^{n-5}\{x_iy_{i+1},x_ix_{i+1},y_iy_{i+1},x_{i+1}y_i\})}[x_{n-2}] \\
&\oplus \frac{K[x_1,\ldots,x_{n-3},x_n,y_1,\ldots,y_{n-3}]}{(\cup_{i=1}^{n-4}\{x_iy_{i+1},x_ix_{i+1},y_iy_{i+1},x_{i+1}y_i\})}[y_n] \\
&\oplus \frac{K[x_1,\ldots,x_{n-3},y_1,\ldots,y_{n-3}]}{(\cup_{i=1}^{n-4}\{x_iy_{i+1},x_ix_{i+1},y_iy_{i+1},x_{i+1}y_i\})}[x_n] \\
&\cong \Big(K[V(E_{n-4})]/I(E_{n-4}) \otimes_K K[x_{n-2},x_n,y_n,y_{n-2}]\Big) \\
&\oplus \Big(K[V(E_{n-4})]/I(E_{n-4}) \otimes_K K[x_n,y_n,x_{n-2}]\Big) \\
&\oplus \Big(K[V(E_{n-3})]/I(E_{n-3}) \otimes_K K[x_n,y_n]\Big) \\
&\oplus \Big(K[V(E_{n-3})]/I(E_{n-3}) \otimes_K K[x_n]\Big).
\end{aligned}$$

By Lemma 2.5 on Eq (3.3),

$$\operatorname{depth}(S/(I(E_n):y_{n-1})) = \operatorname{depth} K[V(E_{n-3})]/I(E_{n-3}) + \operatorname{depth} K[y_{n-1}, x_{n-1}]. \tag{3.9}$$

Also,

$$\begin{aligned} &\operatorname{depth}((I(E_n):y_{n-1})/I(E_n)) \\ &= \min\Big\{ \operatorname{depth}(K[V(E_{n-4})]/I(E_{n-4})) + 4, \operatorname{depth}(K[V(E_{n-4})]/I(E_{n-4})) + 3, \\ &\qquad \operatorname{depth}(K[V(E_{n-3})]/I(E_{n-3})) + 2, \operatorname{depth}(K[V(E_{n-3})]/I(E_{n-3})) + 1\Big\}. \end{aligned} \tag{3.10}$$

Here, $n-4 \equiv 1 \pmod 3$ and $n-3 \equiv 2 \pmod 3$. We apply induction on Eq (3.9) and get

$$\operatorname{depth}(S/(I(E_n):y_{n-1})) = \left\lceil \frac{n-3}{3} \right\rceil + 2 = \left\lceil \frac{n}{3} \right\rceil + 1. \tag{3.11}$$

Using induction on n and Remark 3.1 on Eq (3.10),

$$\begin{aligned} &\operatorname{depth}((I(E_n):y_{n-1})/I(E_n)) \\ &= \min\left\{ \left\lceil \frac{n-4+4}{3} \right\rceil + 4, \left\lceil \frac{n-4+4}{3} \right\rceil + 3, \left\lceil \frac{n-3}{3} \right\rceil + 2, \left\lceil \frac{n-3}{3} \right\rceil + 1 \right\} = \left\lceil \frac{n}{3} \right\rceil. \end{aligned} \tag{3.12}$$

We get the required result by applying Lemma 2.2 on Eq (3.8).

Case 3. If $n \equiv 0 \pmod 3$, then $n-4 \equiv 2 \pmod 3$ and $n-3 \equiv 0 \pmod 3$. By applying induction on Eq (3.9),

$$\operatorname{depth}(S/(I(E_n):y_{n-1})) = \left\lceil \frac{n-3}{3} \right\rceil + 2 = \left\lceil \frac{n}{3} \right\rceil + 1. \tag{3.13}$$

By using Eq (3.10) and applying induction on n, we get

$$\begin{aligned} &\operatorname{depth}((I(E_n):y_{n-1})/I(E_n)) \\ &= \min\left\{ \left\lceil \frac{n-4}{3} \right\rceil + 4, \left\lceil \frac{n-4}{3} \right\rceil + 3, \left\lceil \frac{n-3}{3} \right\rceil + 2, \left\lceil \frac{n-3}{3} \right\rceil + 1 \right\} = \left\lceil \frac{n}{3} \right\rceil. \end{aligned} \tag{3.14}$$

The required result is obtained by applying Lemma 2.2 on Eq (3.8).

This completes the proof for depth. Next, we prove the result for the lower bound of Stanley depth. If $n = 2$, then $E_2 \cong \mathbb{C}_4$ and the result holds by Lemma 2.14. If $n = 3$, we get the required result from Lemma 2.16. Let $n \geq 4$. We get the lower bound for Stanley depth in a similar way to the depth just by replacing Lemmas 2.2 and 2.3 with Lemmas 2.1 and 2.4, respectively. □

By using the Auslander Buchsbaum formula, we have the following result.

Corollary 3.5. *Let $n \geq 2$ and $S = K[V(E_n)]$, then*

$$\operatorname{pdim}(S/I(E_n)) = \begin{cases} 2n - \left\lceil \frac{n+4}{3} \right\rceil, & \text{if } n \equiv 1 \pmod 3; \\ 2n - \left\lceil \frac{n}{3} \right\rceil, & \text{otherwise}. \end{cases}$$

Proof. The required result follows from Lemmas 2.12 and 3.4. □

Now, we will find the upper bound for Stanley depth of $K[V(E_n)/I(E_n)]$.

Lemma 3.6. *Let* $n \geq 2$ *and* $S = K[V(E_n)]$*, then*

$$\operatorname{sdepth}(S/I(E_n)) \leq \begin{cases} \frac{2n}{3}, & \text{if } n \equiv 0 \pmod 3; \\ \frac{2n+2}{3}, & \text{if } n \equiv 2 \pmod 3; \\ \frac{2n+4}{3}, & \text{if } n \equiv 1 \pmod 3. \end{cases}$$

Proof. If $n = 2$, then $E_2 \cong \mathbb{C}_4$ and we get the required result by Lemma 2.14. If $n = 3$ and since $y_2 \notin I(E_3)$, then $S/(I(E_3) : y_2) \cong K[x_2, y_2]/(0)$. Thus we have by Lemma 2.4,

$$\operatorname{sdepth}(S/I(E_3)) \leq \operatorname{sdepth}(S/(I(E_3) : y_2)) = \operatorname{sdepth}(K[x_2, y_2]) = 2.$$

Let $n \geq 4$. If $n \equiv 0 \pmod 3$, then $n-3 \equiv 0 \pmod 3$. Since $x_{n-1}y_{n-1} \notin I(E_n)$, we have

$$S/\big(I(E_n) : x_{n-1}y_{n-1}\big) \cong K[V(E_{n-3})]/I(E_{n-3}) \otimes_K K[x_{n-1}, y_{n-1}].$$

By using Lemma 2.5 and applying induction on n,

$$\operatorname{sdepth}\big(S/(I(E_n) : x_{n-1}y_{n-1})\big) = \operatorname{sdepth}(K[V(E_{n-3})]/I(E_{n-3})) + 2 \leq \frac{2(n-3)}{3} + 2 = \frac{2n}{3}.$$

Therefore, by applying Lemma 2.4, we get

$$\operatorname{sdepth}(S/I(E_n)) \leq \operatorname{sdepth}\big(S/(I(E_n) : x_{n-1}y_{n-1})\big) \leq \frac{2n}{3}.$$

Let $n \equiv 2 \pmod 3$. Since $y_n \notin I(E_n)$,

$$S/(I(E_n) : y_n) \cong K[V(E_{n-2})]/I(E_{n-2}) \otimes_K K[y_n, x_n].$$

Since $n-2 \equiv 0 \pmod 3$, by using Lemmas 2.4, 2.5 and induction on n, we get

$$\begin{aligned} \operatorname{sdepth}(S/I(E_n)) &\leq \operatorname{sdepth}(S/(I(E_n) : y_n)) = \operatorname{sdepth}(K[V(E_{n-2})]/I(E_{n-2})) + 2 \\ &\leq \frac{2(n-2)}{3} + 2 = \frac{2n+2}{3}. \end{aligned}$$

If $n \equiv 1 \pmod 3$, then $n-2 \equiv 2 \pmod 3$. The proof follows a similar strategy and we get

$$\begin{aligned} \operatorname{sdepth}((S/I(E_n))) &\leq \operatorname{sdepth}(S/(I(E_n) : y_n)) = \operatorname{sdepth}(K[V(E_{n-2})]/I(E_{n-2})) + 2 \\ &\leq \frac{2(n-2)+2}{3} + 2 = \frac{2n+4}{3}. \end{aligned}$$

This completes the proof. □

In the following lemmas we will find the exact values for regularity of the cyclic modules $K[V(E_n)]/I(E_n)$, $K[V(F_n)]/I(F_n)$ and $K[V(G_n)]/I(G_n)$.

Lemma 3.7. *Let $n \geq 2$ and $S = K[V(E_n)]$, then* $\operatorname{reg}(S/I(E_n)) = \left\lceil \frac{n-1}{3} \right\rceil$.

Proof. Let $S = K[V(E_n)]$. If $n = 2$, then clearly by Lemma 2.15, and we get $\operatorname{reg}(K[V(E_2)]/I(E_2)) = \operatorname{reg}(K[V(\mathbb{C}_4)]/I(\mathbb{C}_4)) = 1$. Let $n \geq 3$, we have the following K-algebra isomorphisms:

$$S/(I(E_n) : x_{n-2}) \cong K[V(E_{n-4})]/I(E_{n-4}) \otimes_K K[x_{n-2}, y_{n-2}, x_n, y_n], \tag{3.15}$$

$$S/((I(E_n), x_{n-2}), y_{n-2}) \cong K[V(E_{n-3})]/I(E_{n-3}) \otimes_K K[V(E_2)]/I(E_2), \tag{3.16}$$

$$S/((I(E_n), x_{n-2}) : y_{n-2}) \cong K[V(E_{n-4})]/I(E_{n-4}) \otimes_K K[y_{n-2}, x_n, y_n]. \tag{3.17}$$

If $n = 3$, by using Eq (3.15) we get $S/(I(E_3) : x_1) \cong K[V(E_{-1})]/I(E_{-1}) \otimes_K K[x_1, y_1, x_3, y_3]$. Moreover, by Eq (3.16), we have $S/((I(E_3), x_1), y_1) \cong K[V(E_0)]/I(E_0) \otimes_K K[V(E_2)]/I(E_2)$, and by Eq (3.17), $S/((I(E_3), x_1) : y_1) \cong K[V(E_{-1})]/I(E_{-1}) \otimes_K K[y_1, x_3, y_3]$. By Remark 3.1 and Lemma 2.5, we get

$$\operatorname{reg}\left(S/(I(E_3) : x_1)\right) = \operatorname{reg}\left(K[V(E_{-1})]/I(E_{-1})\right) = \operatorname{reg}(K) = 0,$$
$$\operatorname{reg}\left(S/((I(E_3), x_1) : y_1)\right) = \operatorname{reg}\left(K[V(E_{-1})]/I(E_{-1})\right) = \operatorname{reg}(K) = 0,$$

and

$$\operatorname{reg}\left(S/((I(E_3), x_1), y_1)\right) = \operatorname{reg}\left(K[V(E_0)]/I(E_0)\right) + \operatorname{reg}\left(K[V(E_2)]/I(E_2)\right) = 0 + 1 = 1.$$

Since $\operatorname{reg}\left(S/((I(E_3), x_1) : y_1)\right) < \operatorname{reg}\left(S/((I(E_3), x_1), y_1)\right)$, by Lemma 2.11(c), we have $\operatorname{reg}(S/(I(E_3), x_1)) = 1$. Also, $\operatorname{reg}(S/(I(E_3) : x_1)) < \operatorname{reg}(S/(I(E_3), x_1)$, and again by Lemma 2.11(c), $\operatorname{reg}(S/I(E_3)) = 1$. If $n = 4$, by using a similar strategy, one can get $\operatorname{reg}(S/I(E_4)) = 1$. Let $n \geq 5$. By using induction on n, Remark 3.1, Lemma 2.5 and Eqs (3.15)–(3.17), we get

$$\operatorname{reg}\left(S/(I(E_n) : x_{n-2})\right) = \operatorname{reg}\left(K[V(E_{n-4})]/I(E_{n-4})\right) = \left\lceil \frac{n-5}{3} \right\rceil,$$

$$\operatorname{reg}\left(S/((I(E_n), x_{n-2}) : y_{n-2})\right) = \operatorname{reg}\left(K[V(E_{n-4})]/I(E_{n-4})\right) = \left\lceil \frac{n-5}{3} \right\rceil,$$

and by Lemma 2.9,

$$\begin{aligned}\operatorname{reg}\left(S/((I(E_n), x_{n-2}), y_{n-2})\right) &= \operatorname{reg}\left(K[V(E_{n-3})]/I(E_{n-3})\right) + \operatorname{reg}\left(K[V(E_2)]/I(E_2)\right)\\ &= \left\lceil \frac{n-4}{3} \right\rceil + 1 = \left\lceil \frac{n-1}{3} \right\rceil.\end{aligned}$$

Since

$$\operatorname{reg}\left(S/((I(E_n), x_{n-2}) : y_{n-2})\right) < \operatorname{reg}\left(S/((I(E_n), x_{n-2}), y_{n-1})\right),$$

by Lemma 2.11(c) we get $\operatorname{reg}\left(S/(I(E_n), x_{n-2})\right) = \left\lceil \frac{n-1}{3} \right\rceil$. Also we have

$$\operatorname{reg}\left(S/(I(E_n) : x_{n-2})\right) < \operatorname{reg}\left(S/(I(E_n), x_{n-2}\right).$$

Again by Lemma 2.11(c), the required result follows. □

Lemma 3.8. *Let $n \geq 2$ and $S = K[V(F_n)]$, then* $\operatorname{reg}(S/I(F_n)) = \left\lceil \frac{n}{2} \right\rceil$.

Proof. If $n = 2$, then we have $S/(I(F_2) : y_1) \cong K[y_1, x_2]$, and $S/(I(F_2), y_1) \cong K[V(\mathbb{C}_3)]/I(\mathbb{C}_3)$. By Lemmas 2.5 and 2.15, $\operatorname{reg}\left(S/(I(F_2) : y_1)\right) = 0$ and $\operatorname{reg}\left(S/(I(F_2), y_1)\right) = K[V(\mathbb{C}_3)]/I(\mathbb{C}_3) = 1$. Since $\operatorname{reg}\left(S/(I(F_2) : y_1)\right) < \operatorname{reg}\left(S/(I(F_2), y_1)\right)$, therefore by using Lemma 2.11(c) we get $\operatorname{reg}(K[V(F_2)]/I(F_2)) = 1$. Let $n = 3$ and $F_3 = H_1 \cup H_2$, where $H_1 \cong H_2 \cong F_2$ and $H_1 \cap H_2 \neq \emptyset$. By Lemma 2.10, we get

$$\operatorname{reg}\left(S/I(F_3)\right) \leq \operatorname{reg}\left(K[V(H_1)]/I(H_1)\right) + \operatorname{reg}\left(K[V(H_2)]/I(H_2)\right) = 2.$$

For the other inequality, let $M = \left\{\{x_1, y_1\}, \{x_3, y_3\}\right\}$. It is clear that M is an induced matching, therefore, $\operatorname{indmat}(F_n) \geq |M| = 2$. By combining the two inequalities, we get $\operatorname{reg}(S/I(F_3)) = 2$. Let $n \geq 4$. Here we consider the following two cases:

Case 1. If n is even, we have the following K-algebra isomorphisms:

$$S/(I(F_n) : y_{n-1}) \cong K[V(F_{n-3})]/I(F_{n-3}) \otimes_K K[y_{n-1}, x_n], \tag{3.18}$$

$$S/((I(F_n), y_{n-1}), x_{n-1}) \cong K[V(F_{n-2})]/I(F_{n-2}) \otimes_K K[V(\mathbb{P}_2)]/I(\mathbb{P}_2), \tag{3.19}$$

$$S/(((I(F_n), y_{n-1}) : x_{n-1}), y_{n-2}) \cong K[V(F_{n-3})]/I(F_{n-3}) \otimes_K K[x_{n-1}], \tag{3.20}$$

$$S/(((I(F_n), y_{n-1}) : x_{n-1}) : y_{n-2}) \cong K[V(F_{n-4})]/I(F_{n-4}) \otimes_K K[x_{n-1}, y_{n-2}]. \tag{3.21}$$

If $n = 4$, we have

$$S/(I(F_4) : y_3) \cong K[V(F_1)]/I(F_1) \otimes_K K[y_3, x_4],$$
$$S/((I(F_4), y_3), x_3) \cong K[V(F_2)]/I(F_2) \otimes_K K[V(\mathbb{P}_2)]/I(\mathbb{P}_2),$$
$$S/((I(F_4), y_3) : x_3) \cong K[V(\mathbb{C}_3)]/I(\mathbb{C}_3) \otimes_K K[x_3].$$

By Lemma 2.5 and Remark 3.1 we have $\operatorname{reg}\left(S/(I(F_4) : y_3)\right) = \operatorname{reg}\left(K[V(F_1)]/I(F_1)\right) = 1$, and by Lemmas 2.13 and 2.15, we have

$$\operatorname{reg}\left(S/((I(F_4), y_3), x_3)\right) = \operatorname{reg}\left(K[V(F_2)]/I(F_2)\right) + \operatorname{reg}\left(K[V(\mathbb{P}_2)]/I(\mathbb{P}_2)\right) = 2$$

and

$$\operatorname{reg}\left(S/((I(F_4), y_3) : x_3)\right) = \operatorname{reg}\left(K[V(\mathbb{C}_3)]/I(\mathbb{C}_3)\right) = 1.$$

Since $\operatorname{reg}\left(S/((I(F_4), y_3) : x_3)\right) < \operatorname{reg}\left(S/((I(F_4), y_3), x_3)\right)$, by using Lemma 2.11(c), we get $\operatorname{reg}\left(S/(I(F_4), y_3)\right) = 2$. Moreover, $\operatorname{reg}\left(S/(I(F_4) : y_3)\right) < \operatorname{reg}\left(S/(I(F_4), y_3\right)$, and again by Lemma 2.11(c), we get $\operatorname{reg}(S/(I(F_4)) = 2$. Let $n \geq 6$. By using induction on n, Lemmas 2.5 and 2.9 on Eqs (3.18)–(3.21), we get

$$\operatorname{reg}\left(S/(I(F_n) : y_{n-1})\right) = \operatorname{reg}\left(K[V(F_{n-3})]/I(F_{n-3})\right) = \left\lceil \frac{n-3}{2} \right\rceil,$$

$$\begin{aligned}\operatorname{reg}\left(S/((I(F_n), y_{n-1}), x_{n-1})\right) &= \operatorname{reg}\left(K[V(F_{n-2})]/I(F_{n-2})\right) + \operatorname{reg}\left(K[V(\mathbb{P}_2)]/I(\mathbb{P}_2)\right) \\ &= \left\lceil \frac{n-2}{2} \right\rceil + 1 = \left\lceil \frac{n}{2} \right\rceil,\end{aligned}$$

$$\operatorname{reg}\Big(S/(((I(F_n),y_{n-1}):x_{n-1}),y_{n-2})\Big)=\operatorname{reg}\Big(K[V(F_{n-3})]/I(F_{n-3})\Big)=\left\lceil\frac{n-3}{2}\right\rceil,$$

$$\operatorname{reg}\Big(S/(((I(F_n),y_{n-1}):x_{n-1}):y_{n-2})\Big)=\operatorname{reg}\Big(K[V(F_{n-4})]/I(F_{n-4})\Big)=\left\lceil\frac{n-4}{2}\right\rceil.$$

Since n is even,

$$\operatorname{reg}\Big(S/(((I(F_n),y_{n-1}):x_{n-1}):y_{n-2})\Big)<\operatorname{reg}\Big(S/(((I(F_n),y_{n-1}):x_{n-1}),y_{n-2})\Big),$$

and by Lemma 2.11(c), we get

$$\operatorname{reg}\Big(S/((I(F_n),y_{n-1}):x_{n-1})\Big)=\left\lceil\frac{n-3}{2}\right\rceil.$$

Also, $\operatorname{reg}\Big(S/((I(F_n),y_{n-1}):x_{n-1})\Big)<\operatorname{reg}\Big(S/((I(F_n),y_{n-1}),x_{n-1})\Big)$, and again by Lemma 2.11(c), $\operatorname{reg}\Big(S/(I(F_n),y_{n-1})\Big)=\left\lceil\frac{n}{2}\right\rceil$. We have $\operatorname{reg}\Big(S/(I(F_n):y_{n-1})\Big)<\operatorname{reg}\Big(S/(I(F_n),y_{n-1})\Big)$; therefore, by Lemma 2.11(c), the required result follows.

Case 2. If n is odd, then $F_n=F_2\cup H$, where $H\cong F_{n-1}$ and $F_2\cap H\neq\emptyset$. By induction on n and Lemma 2.10, we get

$$\begin{aligned}\operatorname{reg}(S/I(F_n))&\le\operatorname{reg}(K[V(F_2)]/I(F_2))+\operatorname{reg}(K[V(F_{n-1})]/I(F_{n-1}))\\&=1+\left\lceil\frac{n-1}{2}\right\rceil=\left\lceil\frac{n+1}{2}\right\rceil=\left\lceil\frac{n}{2}\right\rceil.\end{aligned}$$

For the second inequality, we define $M=\Big\{\{x_1,y_1\},\{x_3,y_3\},\dots,\{x_n,y_n\}\Big\}$. M is clearly an induced matching and $|M|=\left\lceil\frac{n}{2}\right\rceil$, thus, $\operatorname{indmat}(F_n)\ge\left\lceil\frac{n}{2}\right\rceil$. By Lemma 2.8, we have $\operatorname{reg}(S/I(F_n))\ge\left\lceil\frac{n}{2}\right\rceil$.

□

Lemma 3.9. *If* $n\ge 2$ *and* $S=K[V(G_n)]$, *then* $\operatorname{reg}(S/I(G_n))=\left\lceil\frac{n}{2}\right\rceil$.

Proof. If $n=2$, then clearly $G_2\cong\mathbb{K}_4$; therefore, by Lemma 2.8, $\operatorname{indmat}(G_2)=1$, and we get $\operatorname{reg}(K[V(G_2)]/I(G_2))=1$. Let $n\ge 3$, and we have the following K-algebra isomorphisms:

$$S/(I(G_n):y_{n-1})\cong K[V(G_{n-3})]/I(G_{n-3})\otimes_K K[y_{n-1}],\tag{3.22}$$

$$S/((I(G_n),y_{n-1}),x_{n-1})\cong K[V(G_{n-2})]/I(G_{n-2})\otimes_K K[V(\mathbb{P}_2)]/I(\mathbb{P}_2),\tag{3.23}$$

$$S/((I(G_n),y_{n-1}):x_{n-1})\cong K[V(G_{n-3})]/I(G_{n-3})\otimes_K K[x_{n-1}].\tag{3.24}$$

If $n=3$, we have

$$S/(I(G_3):y_2)\cong K[V(G_0)]/I(G_0)\otimes_K K[y_2],$$

$$S/((I(G_3),y_2),x_2)\cong K[V(G_1)]/I(G_1)\otimes_K K[V(\mathbb{P}_2)]/I(\mathbb{P}_2),$$

$$S/((I(G_3),y_2):x_2)\cong K[V(G_0)]/I(G_0)\otimes_K K[x_2].$$

By Remark 3.1, Lemmas 2.5 and 2.13, $\operatorname{reg}\Big(S/(I(G_3):y_2)\Big)=0$, $\operatorname{reg}\Big(S/((I(G_3),y_2),x_2)\Big)=\operatorname{reg}\Big(K[V(G_1)]/I(G_1)\Big)+\operatorname{reg}\Big(K[V(\mathbb{P}_2)]/I(\mathbb{P}_2)\Big)=2$ and $\operatorname{reg}\Big(S/((I(G_3),y_2):x_2)\Big)=0$. Since we have

$\operatorname{reg}\big(S/((I(G_3), y_2) : x_2)\big) < \operatorname{reg}\big(S/((I(G_3), y_2), x_2)\big)$, by Lemma 2.11(c), $\operatorname{reg}\big(S/(I(G_3), y_2)\big) = \lceil\frac{3}{2}\rceil = 2$. Also, we have $\operatorname{reg}\big(S/(I(G_3) : y_2)\big) < \operatorname{reg}\big(S/(I(G_3), y_2\big)$, and again by Lemma 2.11(c), we get $\operatorname{reg}(S/I(G_3)) = 2$. Let $n \geq 4$. By induction on n, Lemmas 2.5, 2.13 and using Eqs (3.22)–(3.24), we get

$$\operatorname{reg}\big(S/(I(G_n) : y_{n-1})\big) = \operatorname{reg}\big(K[V(G_{n-3})]/I(G_{n-3})\big) = \left\lceil\frac{n-3}{2}\right\rceil,$$

$$\operatorname{reg}\big(S/((I(G_n), y_{n-1}) : x_{n-1})\big) = \operatorname{reg}\big(K[V(G_{n-3})]/I(G_{n-3})\big) = \left\lceil\frac{n-3}{2}\right\rceil,$$

and by Lemma 2.9,

$$\begin{aligned}\operatorname{reg}\big(S/((I(G_n), y_{n-1}), x_{n-1})\big) &= \operatorname{reg}\big(K[V(G_{n-2})]/I(G_{n-2})\big) + \operatorname{reg}\big(K[V(\mathbb{P}_2)]/I(\mathbb{P}_2)\big)\\ &= \left\lceil\frac{n-2}{2}\right\rceil + 1 = \left\lceil\frac{n}{2}\right\rceil.\end{aligned}$$

Since $\operatorname{reg}\big(S/((I(G_n), y_{n-1}) : x_{n-1})\big) < \operatorname{reg}\big(S/((I(G_n), y_{n-1}), x_{n-1})\big)$, by Lemma 2.11(c) we get $\operatorname{reg}\big(S/(I(G_n), y_{n-1})\big) = \left\lceil\frac{n}{2}\right\rceil$. Also we have $\operatorname{reg}\big(S/(I(G_n) : y_{n-1})\big) < \operatorname{reg}\big(S/(I(G_n), y_{n-1}\big)$, and again by Lemma 2.11(c), the required result follows. This completes the proof. □

4. Invariants of cyclic modules associated with $C_{2n}(1, n-1), C_{2n}(1, 2)$ and $C_{2n}(1, n-1, n)$

In this section, we find some invariants of the edge ideals of some families of 4-regular and 5-regular circulant graphs. We compute depth and projective dimension of the cyclic module $K[V(C_{2n}(1, n-1))]/I(C_{2n}(1, n-1))$. Moreover, bounds for the Stanley depth of such module are also given. When $n \equiv 0, 1 (\mathrm{mod}\, 3)$, we give the exact value for the regularity of such a module; otherwise, we have sharp bounds. For cyclic module $K[V(C_{2n}(1, 2))]/I(C_{2n}(1, 2))$, we give the exact value of regularity if n is even and sharp bounds if n is odd. Also, we find the exact value of the regularity of $K[V(C_{2n}(1, n-1, n))]/I(C_{2n}(1, n-1, n))$. It will be convenient to use the labeling of the vertices of the graphs, as shown in Figures 8 and 9.

Before proving the main results, we give the following example by using Remark 3.3, which will be helpful in understanding the strategy of the proofs. See for instance, Figures 10 and 11 for subgraphs $G_{(I(C_{16}(1,7)):x_8)}, G_{(I(C_{16}(1,7)),x_8)}, G_{((I(C_{16}(1,7)),x_8),y_8)}$ and $G_{((I(C_{16}(1,7)),x_8):y_8)}$ of circulant graph $G_{I(C_{16}(1,7))}$. It is clear from the Figures 10 and 11 that we have the following isomorphisms:

$$K[V(C_{16}(1, 7))]/(I(C_{16}(1, 7)) : x_8) \cong K[V(E_5)]/I(E_5) \otimes_K K[x_8, y_8],$$

$$K[V(C_{16}(1, 7))]/(I(C_{16}(1, 7)), x_8) \cong K[V(E_7), y_8]/(I(E_7), x_1y_8, y_1y_8, x_7y_8, y_7y_8),$$

$$K[V(C_{16}(1, 7))]/((I(C_{16}(1, 7)), x_8), y_8) \cong K[V(E_7)]/I(E_7),$$

and

$$K[V(C_{16}(1, 7))]/((I(C_{16}(1, 7)), x_8) : y_8) \cong K[V(E_5)]/I(E_5) \otimes_K K[y_8].$$

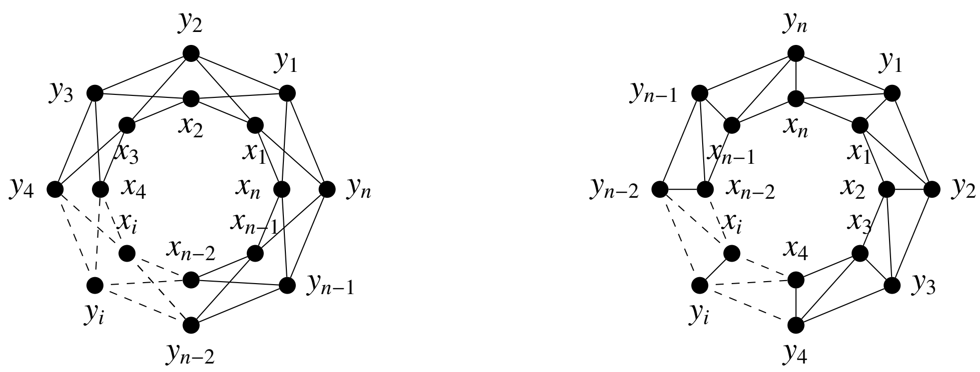

Figure 8. From left to right, $C_{2n}(1, n-1)$ and $C_{2n}(1, 2)$.

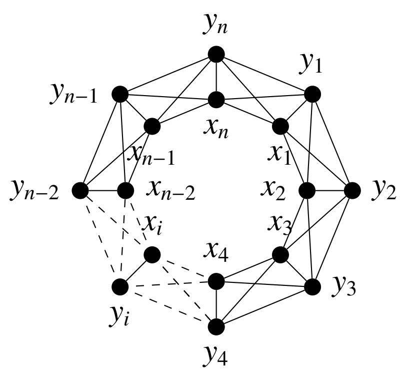

Figure 9. $C_{2n}(1, n-1, n)$.

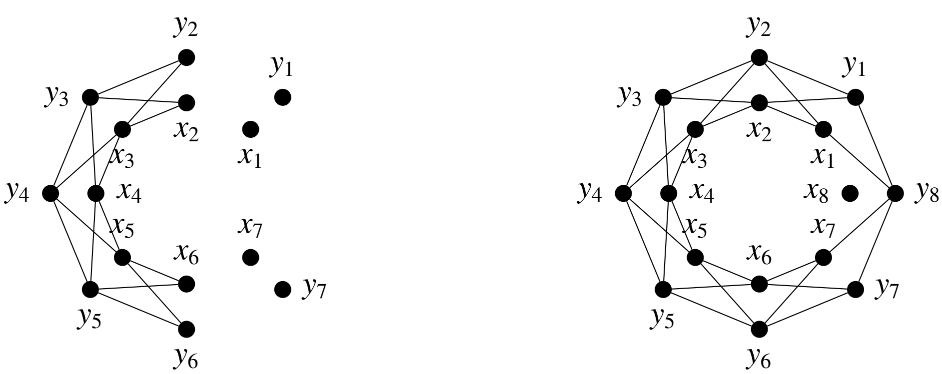

Figure 10. From left to right, $G_{(I(C_{16}(1,7)):x_8)}$ and $G_{(I(C_{16}(1,7)),x_8)}$.

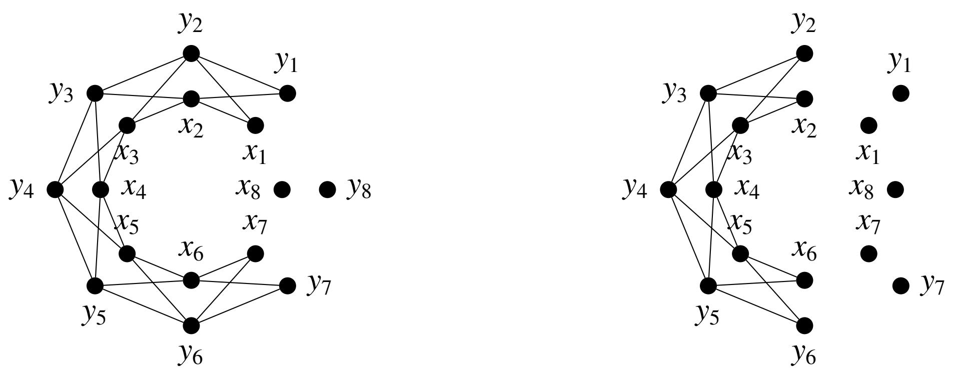

Figure 11. From left to right, $G_{((I(C_{16}(1,7)),x_8),y_8)}$ and $G_{((I(C_{16}(1,7)),x_8):y_8)}$.

First, we will compute the exact value of depth and lower bound of the Stanley depth for the cyclic module $K[V(C_{2n}(1, n-1))]/I(C_{2n}(1, n-1))$.

Theorem 4.1. *Let $n \geq 3$, $G = C_{2n}(1, n-1)$ and $S = K[V(G)]$, then*

$$\operatorname{sdepth}(S/I(G)) \geq \operatorname{depth}(S/I(G)) = \begin{cases} \left\lceil \frac{n-1}{3} \right\rceil, & \text{if } n \equiv 0, 1 \pmod 3; \\ \left\lceil \frac{n}{3} \right\rceil, & \text{otherwise.} \end{cases}$$

Proof. First we prove the result for depth. If $n = 3$, we consider the following short exact sequence

$$0 \longrightarrow (I(G) : x_3)/I(G) \xrightarrow{\cdot x_3} S/I(G) \longrightarrow S/(I(G) : x_3) \longrightarrow 0. \tag{4.1}$$

We have

$$K[V(G)]/(I(G) : x_3) \cong \frac{K[y_3]}{(0)}[x_3], \tag{4.2}$$

and

$$N_G(x_3) = \{y_2, x_2, y_1, x_1\},$$

$$S_1 = K[V(G)\backslash N_G(y_2)], \quad S_2 = K[V(G)\backslash(N_G(x_2) \cup \{y_2\})],$$

$$S_3 = K[V(G)\backslash(N_G(y_1) \cup \{y_2, x_2\})], \quad S_4 = K[V(G)\backslash(N_G(x_1) \cup \{y_2, x_2, y_1\})],$$

$$J_1 = (S_1 \cap I(G)), \quad J_2 = (S_2 \cap I(G)),$$

$$J_3 = (S_3 \cap I(G)), \quad J_4 = (S_4 \cap I(G)),$$

then by using Lemma 2.17, we have

$$\begin{aligned} (I(G) : x_3)/I(G) &\cong S_1/J_1[y_2] \oplus S_2/J_2[x_2] \oplus S_3/J_3[y_1] \oplus S_4/J_4[x_1] \\ &\cong \frac{K[x_2]}{(0)}[y_2] \oplus \frac{K}{(0)}[x_2] \oplus \frac{K[x_1]}{(0)}[y_1] \oplus \frac{K}{(0)}[x_1]. \end{aligned} \tag{4.3}$$

We apply Lemma 2.5 on Eq (4.2), $\text{depth}(K[V(G)]/(I(G):x_3)) = \text{depth}(K[y_3, x_3]) = 2$ and by Eq (4.3)

$$\begin{aligned}&\text{depth}((I(G):x_3)/I(G))\\&\quad= \min\Big\{\text{depth}(K[x_2]) + 1, \text{depth}(K[x_2]), \text{depth}(K[x_1]) + 1, \text{depth}(K[x_1])\Big\} = 1.\end{aligned}$$

By using Lemma 2.2 on Eq (4.1), $\text{depth}(S/I(G)) = 1$. If $n = 4$, we consider the following short exact sequence

$$0 \longrightarrow (I(G):x_4)/I(G) \xrightarrow{\cdot x_4} S/I(G) \longrightarrow S/(I(G):x_4) \longrightarrow 0. \tag{4.4}$$

We have

$$K[V(G)]/(I(G):x_4) \cong K[x_2, x_4, y_2, y_4], \tag{4.5}$$

and

$$N_G(x_4) = \{y_3, x_3, y_1, x_1\},$$
$$S_1 = K[V(G)\backslash N_G(y_3)],\quad S_2 = K[V(G)\backslash(N_G(x_3) \cup \{y_3\})],$$
$$S_3 = K[V(G)\backslash(N_G(y_1) \cup \{y_3, x_3\})],\quad S_4 = K[V(G)\backslash(N_G(x_1) \cup \{y_3, x_3, y_1\})],$$
$$J_1 = (S_1 \cap I(G)),\quad J_2 = (S_2 \cap I(G)),$$
$$J_3 = (S_3 \cap I(G)),\quad J_4 = (S_4 \cap I(G)),$$

then by using Lemma 2.17, we have

$$\begin{aligned}(I(G):x_4)/I(G) &\cong S_1/J_1[y_3] \oplus S_2/J_2[x_3] \oplus S_3/J_3[y_1] \oplus S_4/J_4[x_1]\\&\cong \frac{K[x_1, x_3, y_1]}{(0)}[y_3] \oplus \frac{K[x_1, y_1]}{(0)}[x_3] \oplus \frac{K[x_1]}{(0)}[y_1] \oplus \frac{K}{(0)}[x_1].\end{aligned} \tag{4.6}$$

By applying Lemma 2.5 on Eq (4.5),

$$\text{depth}\Big(K[V(G)]/(I(G):x_4)\Big) = \text{depth}\Big(K[x_2, x_4, y_2, y_4]\Big) = 4$$

and by using Eq (4.6) we get

$$\begin{aligned}&\text{depth}\Big((I(G):x_4)/I(G)\Big)\\&\quad= \min\Big\{\text{depth}(K[x_1, x_3, y_1, y_3]), \text{depth}(K[x_1, y_1, x_3]), \text{depth}(K[x_1, y_1]), \text{depth}(K[x_1])\Big\} = 1.\end{aligned}$$

By using Lemma 2.2 on Eq (4.4), we get $\text{depth}(S/I(G)) = 1$. Let $n \geq 5$. Consider the short exact sequence

$$0 \longrightarrow (I(G):x_n)/I(G) \xrightarrow{\cdot x_n} S/I(G) \longrightarrow S/(I(G):x_n) \longrightarrow 0. \tag{4.7}$$

We have the following K-algebra isomorphisms:

$$S/(I(G):x_n) \cong K[V(E_{n-3})]/I(E_{n-3}) \otimes_K K[y_n, x_n], \tag{4.8}$$

and

$$N_G(x_n) = \{y_{n-1}, x_{n-1}, y_1, x_1\},$$
$$S_1 = K[V(G)\backslash N_G(y_{n-1})],\quad S_2 = K[V(G)\backslash(N_G(x_{n-1}) \cup \{y_{n-1}\})],$$

$$S_3 = K[V(G)\backslash(N_G(y_1) \cup \{y_{n-1}, x_{n-1}\})], \quad S_4 = K[V(G)\backslash(N_G(x_1) \cup \{y_{n-1}, x_{n-1}, y_1\})],$$
$$J_1 = (S_1 \cap I(G)), \quad J_2 = (S_2 \cap I(G)),$$
$$J_3 = (S_3 \cap I(G)), \quad J_4 = (S_4 \cap I(G)),$$

then by Lemma 2.17,

$$\begin{aligned}
(I(G) : x_n)/I(G) &\cong S_1/J_1[y_{n-1}] \oplus S_2/J_2[x_{n-1}] \oplus S_3/J_3[y_1] \oplus S_4/J_4[x_1] \\
&\cong \frac{K[x_1, \dots, x_{n-3}, x_{n-1}, y_1, \dots, y_{n-3}]}{(\cup_{i=1}^{n-4} \{x_i y_{i+1}, x_i x_{i+1}, y_i y_{i+1}, x_{i+1} y_i\})}[y_{n-1}] \\
&\quad \oplus \frac{K[x_1, \dots, x_{n-3}, y_1, \dots, y_{n-3}]}{(\cup_{i=1}^{n-4} \{x_i y_{i+1}, x_i x_{i+1}, y_i y_{i+1}, x_{i+1} y_i\})}[x_{n-1}] \\
&\quad \oplus \frac{K[x_1, x_3, \dots, x_{n-2}, y_3, \dots, y_{n-2}]}{(\cup_{i=3}^{n-3} \{x_i y_{i+1}, x_i x_{i+1}, y_i y_{i+1}, x_{i+1} y_i\})}[y_1] \\
&\quad \oplus \frac{K[x_3, \dots, x_{n-2}, y_3, \dots, y_{n-2}]}{(\cup_{i=3}^{n-3} \{x_i y_{i+1}, x_i x_{i+1}, y_i y_{i+1}, x_{i+1} y_i\})}[x_1] \\
&\cong \Big(K[V(E_{n-3})]/I(E_{n-3}) \otimes_K K[x_{n-1}, y_{n-1}]\Big) \oplus \big(K[V(E_{n-3})]/I(E_{n-3}) \otimes_K K[x_{n-1}]\big) \\
&\quad \oplus \Big(K[V(E_{n-4})]/I(E_{n-4}) \otimes_K K[x_1, y_1]\Big) \oplus \Big(K[V(E_{n-4})]/I(E_{n-4}) \otimes_K K[x_1]\Big).
\end{aligned} \tag{4.9}$$

By Lemma 2.5, we have

$$\operatorname{depth}(S/(I(G) : x_n)) = \operatorname{depth} K[V(E_{n-3})]/I(E_{n-3}) + \operatorname{depth} K[y_n, x_n], \tag{4.10}$$

$$\begin{aligned}
&\operatorname{depth}((I(G) : x_n)/I(G)) \\
&= \min\Big\{\operatorname{depth}(K[V(E_{n-3})]/I(E_{n-3})) + 2, \operatorname{depth}(K[V(E_{n-3})]/I(E_{n-3})) + 1, \\
&\qquad \operatorname{depth}(K[V(E_{n-4})]/I(E_{n-4})) + 2, \operatorname{depth}(K[V(E_{n-4})]/I(E_{n-4})) + 1\Big\}.
\end{aligned} \tag{4.11}$$

If $n \equiv 1 \pmod 3$, then $n-3 \equiv 1 \pmod 3$ and $n-4 \equiv 0 \pmod 3$. By using Lemma 3.4 in Eq (4.10), we get

$$\operatorname{depth}(S/(I(G) : x_n)) = \left\lceil \frac{n-3+4}{3} \right\rceil + 2 = \left\lceil \frac{n+4}{3} \right\rceil + 1.$$

By applying Lemma 3.4 on Eq (4.11), we get

$$\begin{aligned}
\operatorname{depth}((I(G) : x_n)/I(G)) &= \min\left\{\left\lceil \frac{n-3+4}{3} \right\rceil + 2, \left\lceil \frac{n-3+4}{3} \right\rceil + 1, \left\lceil \frac{n-4}{3} \right\rceil + 2, \left\lceil \frac{n-4}{3} \right\rceil + 1\right\} \\
&= \min\left\{\left\lceil \frac{n+4}{3} \right\rceil + 1, \left\lceil \frac{n+4}{3} \right\rceil, \left\lceil \frac{n-1}{3} \right\rceil + 1, \left\lceil \frac{n-1}{3} \right\rceil\right\} \\
&= \left\lceil \frac{n-1}{3} \right\rceil.
\end{aligned}$$

We obtain the required result by applying Lemma 2.2 on Eq (4.7). If $n \equiv 0 \pmod 3$, the proof is similar. If $n \equiv 2 \pmod 3$, then $n-3 \equiv 2 \pmod 3$ and $n-4 \equiv 1 \pmod 3$. By using a similar strategy and Remark 3.1, we get $\operatorname{depth}(S/I(G)) = \left\lceil \frac{n}{3} \right\rceil$. For the lower bound of the Stanley depth, the proof is similar to depth one and has to replace Lemma 2.2 with Lemma 2.1. This completes the proof. □

Corollary 4.2. *Let* $n \geq 3$, $G = C_{2n}(1, n-1)$ *and* $S = K[V(G)]$, *then*

$$\operatorname{pdim}(S/I(G)) = \begin{cases} 2n - \left\lceil \frac{n-1}{3} \right\rceil, & \text{if } n \equiv 0, 1 \pmod 3; \\ 2n - \left\lceil \frac{n}{3} \right\rceil, & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

Proof. The required result follows by Lemma 2.12 and Theorem 4.1. □

Now we give an upper bound for the Stanley depth of the $K[V(C_{2n}(1, n-1))]/I(C_{2n}(1, n-1))$.

Proposition 4.3. *Let* $n \geq 3$, $G = C_{2n}(1, n-1)$ *and* $S = K[V(G)]$, *then*

$$\operatorname{sdepth}(S/I(G)) \leq \begin{cases} \frac{2n}{3}, & \text{if } n \equiv 0 \pmod 3; \\ \frac{2n+2}{3}, & \text{if } n \equiv 2 \pmod 3; \\ \frac{2n+4}{3}, & \text{if } n \equiv 1 \pmod 3. \end{cases}$$

Proof. If $n = 3$, by Lemma 2.4, we have $\operatorname{sdepth}(S/I(G)) \leq \operatorname{sdepth}(S/(I(G) : x_3))$. By Eq (4.2), Lemma 2.5, $\operatorname{sdepth}(S/I(G)) \leq 2$, if $n = 4$ and $x_4 \notin I(G)$, by using Lemma 2.4 we have $\operatorname{sdepth}(S/I(G)) \leq \operatorname{sdepth}(S/(I(G) : x_4))$. By Eq (4.5), Lemma 2.5 and $\operatorname{sdepth}(S/I(G)) \leq 4$, let $n \geq 5$. If $n \equiv 1 \pmod 3$, then $n - 3 \equiv 1 \pmod 3$. By using Lemmas 2.4 and 2.5 on Eq (4.8), we get $\operatorname{sdepth}(S/I(G)) \leq \operatorname{sdepth}(S/(I(G) : x_n)) = \operatorname{sdepth}(K[V(E_{n-3})]/I(E_{n-3})) + 2$. Therefore, by Lemma 3.6, we get $\operatorname{sdepth}(S/(I(G) : x_n)) \leq \frac{2(n-3)+4}{3} + 2 = \frac{2n+4}{3}$. The required result follows that is $\operatorname{sdepth}(S/I(G)) \leq \frac{2n+4}{3}$. For $n \equiv 0, 2 \pmod 3$, the proof is similar. □

Remark 4.4. Let $n \geq 3$, then Stanley's inequality holds for $K[V(C_{2n}(1, n-1))]/I(C_{2n}(1, n-1))$.

The next two results provide the values and bounds for regularity of modules $K[V(C_{2n}(1, n-1))]/I(C_{2n}(1, n-1))$ and $K[V(C_{2n}(1, 2))]/I(C_{2n}(1, 2))$.

Theorem 4.5. *Let* $n \geq 3$ *and* $S = K[V(C_{2n}(1, n-1))]$. *If* $n \equiv 0, 1 \pmod 3$, *then*

$$\operatorname{reg}\Big(S/I(C_{2n}(1, n-1)\Big) = \left\lceil \frac{n-2}{3} \right\rceil.$$

Otherwise

$$\left\lceil \frac{n-2}{3} \right\rceil \leq \operatorname{reg}\Big(S/I(C_{2n}(1, n-1)\Big) \leq \left\lceil \frac{n-2}{3} \right\rceil + 1.$$

Proof. We have the following K-algebra isomorphisms:

$$S/(I(C_{2n}(1, n-1)) : x_n) \cong K[V(E_{n-3})]/I(E_{n-3}) \otimes_K K[x_n, y_n], \tag{4.12}$$

$$S/((I(C_{2n}(1, n-1)), x_n), y_n) \cong K[V(E_{n-1})]/I(E_{n-1}), \tag{4.13}$$

$$S/((I(C_{2n}(1, n-1)), x_n) : y_n) \cong K[V(E_{n-3})]/I(E_{n-3}) \otimes_K K[y_n]. \tag{4.14}$$

If $n = 3$, we have

$$S/(I(C_6(1, 2)) : x_3) \cong K[V(E_0)]/I(E_0) \otimes_K K[x_3, y_3],$$

$$S/((I(C_6(1,2)),x_3),y_3) \cong K[V(E_2)]/I(E_2),$$

$$S/((I(C_6(1,2)),x_3):y_3) \cong K[V(E_0)]/I(E_0) \otimes_K K[y_3].$$

By applying Lemmas 2.5, 3.7 and Remark 3.1, we get

$$\operatorname{reg}\left(S/(I(C_6(1,2)):x_3)\right) = \operatorname{reg}\left(K[V(E_0)]/I(E_0)\right) = 0,$$

$$\operatorname{reg}\left(S/((I(C_6(1,2)),x_3):y_3)\right) = \operatorname{reg}\left(K[V(E_0)]/I(E_0)\right) = 0,$$

$$\operatorname{reg}\left(S/((I(C_6(1,2)),x_3),y_3)\right) = \operatorname{reg}\left(K[V(E_2)]/I(E_2)\right) = 1.$$

Since $\operatorname{reg}\left(S/((I(C_6(1,2)),x_3):y_3)\right) < \operatorname{reg}\left(S/((I(C_6(1,2)),x_3),y_3)\right)$, by Lemma 2.11(c), we get $\operatorname{reg}\left(S/(I(C_6(1,2)),x_3)\right) = 1$. Also we have $\operatorname{reg}\left(S/(I(C_6(1,2)):x_3)\right) < \operatorname{reg}\left(S/(I(C_6(1,2)),x_3)\right)$, and again by Lemma 2.11(c), $\operatorname{reg}(S/I(C_6(1,2))) = 1$. For $n = 4$, by using the similar strategy, we get $\operatorname{reg}(S/I(C_8(1,3))) = 1$. Let $n \geq 5$. If $n \equiv 0 \pmod 3$, then $n-3 \equiv 0 \pmod 3$ and $n-1 \equiv 2 \pmod 3$. By applying Lemmas 3.7 and 2.5 on Eqs (4.12)–(4.14), we get

$$\operatorname{reg}\left(S/(I(C_{2n}(1,n-1)):x_n)\right) = \operatorname{reg}\left(K[V(E_{n-3})]/I(E_{n-3})\right) = \left\lceil \frac{n-4}{3} \right\rceil,$$

$$\operatorname{reg}\left(S/((I(C_{2n}(1,n-1)),x_n):y_n)\right) = \operatorname{reg}\left(K[V(E_{n-3})]/I(E_{n-3})\right) = \left\lceil \frac{n-4}{3} \right\rceil,$$

and

$$\operatorname{reg}\left(S/((I(C_{2n}(1,n-1)),x_n),y_n)\right) = \operatorname{reg}\left(K[V(E_{n-1})]/I(E_{n-1})\right) = \left\lceil \frac{n-2}{3} \right\rceil.$$

Since $\left\lceil \frac{n-4}{3} \right\rceil < \left\lceil \frac{n-2}{3} \right\rceil$, by Lemma 2.11(c) we get $\operatorname{reg}\left(S/(I(C_{2n}(1,n-1)),x_n)\right) = \left\lceil \frac{n-2}{3} \right\rceil$. Also $\operatorname{reg}\left(S/(I(C_{2n}(1,n-1)):x_n)\right) < \operatorname{reg}\left(S/(I(C_{2n}(1,n-1)),x_n)\right)$, and again by Lemma 2.11(c), we get the required result. If $n \equiv 1 \pmod 3$, then $n-3 \equiv 1 \pmod 3$ and $n-1 \equiv 0 \pmod 3$. By applying the similar strategy, we get the desired result. Let $n \equiv 2 \pmod 3$. Here $C_{2n}(1,n-1) = E_3 \cup H$, where $H \cong E_{n-1}$ and $E_3 \cap H \neq \emptyset$. In this case $n-1 \equiv 1 \pmod 3$ as $\operatorname{reg}(S/I(E_3)) = 1$, by Lemmas 3.7 and 2.10,

$$\operatorname{reg}(S/I(C_{2n}(1,n-1))) \leq \operatorname{reg}(K[V(E_3)]/I(E_3)) + \operatorname{reg}(K[V(E_{n-1})]/I(E_{n-1})) = 1 + \left\lceil \frac{n-2}{3} \right\rceil.$$

For the other inequality, define $M = \left\{\{x_1,x_2\},\{x_4,x_5\},\ldots,\{x_{n-3},x_{n-4}\}\right\}$. Since M is an induced matching and $|M| = \left\lceil \frac{n-2}{3} \right\rceil$, then, $\operatorname{indmat}(C_{2n}(1,n-1)) \geq \left\lceil \frac{n-2}{3} \right\rceil$. By Lemma 2.8, we have $\operatorname{reg}(S/I(C_{2n}(1,n-1))) \geq \left\lceil \frac{n-2}{3} \right\rceil$. This completes the proof. □

Theorem 4.6. *Let $n \geq 3$. If n is even, then*

$$\operatorname{reg}\left(K[V(C_{2n}(1,2))]/I(C_{2n}(1,2))\right) = \left\lceil \frac{n-1}{2} \right\rceil.$$

If n is odd, we have

$$\frac{n-1}{2} \leq \operatorname{reg}\left(K[V(C_{2n}(1,2))]/I(C_{2n}(1,2))\right) \leq \left\lceil \frac{n-1}{2} \right\rceil + 2.$$

Proof. Let $S = K[V(C_{2n}(1,2))]$. If $n = 3$, then $C_6(1,2) = F_3 \cup H$, where $H \cong F_2$ and $F_3 \cap H \neq \emptyset$. By Lemmas 2.10 and 3.8, we get

$$\operatorname{reg}(K[V(C_6(1,2))]/I(C_6(1,2))) \leq \operatorname{reg}(K[V(F_3)]/I(F_3)) + \operatorname{reg}(K[V(H)]/I(H)) = 3.$$

For the second inequality, let $M = \{\{x_1, y_1\}\}$. Here M is an induced matching; thus, $\operatorname{indmat}(C_6(1,2)) \geq |M| = 1$ and we have $1 \leq \operatorname{reg}(K[V(C_6(1,2))]/I(C_6(1,2)) \leq 3$. If $n = 4$,

$$K[V(C_8(1,2))]/(I(C_8(1,2)) : x_3) \cong K[V(\mathbb{C}_3)]/I(\mathbb{C}_3) \otimes_K K[x_3],$$

$$K[V(C_8(1,2))]/((I(C_8(1,2)), x_3), y_3) \cong K[V(F_3)]/I(F_3),$$

$$K[V(C_8(1,2))]/((I(C_8(1,2)), x_3) : y_3) \cong K[V(\mathbb{C}_3)]/I(\mathbb{C}_3) \otimes_K K[y_3].$$

By using Lemmas 2.5, 3.8 and 2.15, we get

$$\operatorname{reg}\left(K[V(C_8(1,2))]/(I(C_8(1,2)) : x_3)\right) = \operatorname{reg}\left(K[V(\mathbb{C}_3)]/I(\mathbb{C}_3)\right) = 1,$$

$$\operatorname{reg}\left(K[V(C_8(1,2))]/((I(C_8(1,2)), x_3), y_3)\right) = \operatorname{reg}\left(K[V(F_3)]/I(F_3)\right) = 2,$$

$$\operatorname{reg}\left(K[V(C_8(1,2))]/((I(C_8(1,2)), x_3) : y_3)\right) = \operatorname{reg}\left(K[V(\mathbb{C}_3)]/I(\mathbb{C}_3)\right) = 1,$$

as we have

$$\operatorname{reg}\left(K[V(C_8(1,2))]/((I(C_8(1,2)), x_3) : y_3)\right) < \operatorname{reg}\left(K[V(C_8(1,2))]/((I(C_8(1,2)), x_3), y_3)\right).$$

By Lemma 2.11(c),

$$K[V(C_8(1,2))]/(I(C_8(1,2)), x_3) = 2 > K[V(C_8(1,2))]/(I(C_8(1,2)) : x_3),$$

and again by Lemma 2.11(c), $\operatorname{reg}\left(K[V(C_8(1,2))]/I(C_8(1,2)\right) = 2$. Let $n \geq 5$. Here we consider the following two cases:

Case 1. If n is even, by Lemma 2.11(c), $\operatorname{reg}\left(S/I(C_{2n}(1,2))\right) = \operatorname{reg}\left(S/(I(C_{2n}(1,2)), x_{n-1})\right)$ if $\operatorname{reg}\left(S/(I(C_{2n}(1,2)) : x_{n-1})\right) < \operatorname{reg}\left(S/(I(C_{2n}(1,2)), x_{n-1})\right)$. We have the following isomorphisms:

$$S/((I(C_{2n}(1,2)) : x_{n-1}) : y_{n-2}) \cong K[V(F_{n-4})]/I(F_{n-4}) \otimes_K K[y_{n-2}, x_{n-1}],$$

$$S/((I(C_{2n}(1,2)) : x_{n-1}), y_{n-2}) \cong K[V(F_{n-3})]/I(F_{n-3}) \otimes_K K[x_{n-1}],$$

$$S/((I(C_{2n}(1,2)), x_{n-1}), y_{n-1}) \cong K[V(F_{n-1})]/I(F_{n-1}),$$

$$S/(((I(C_{2n}(1,2)), x_{n-1}) : y_{n-1}), x_n) \cong K[V(F_{n-3})]/I(F_{n-3}) \otimes_K K[y_{n-1}],$$

$$S/(((I(C_{2n}(1,2)), x_{n-1}) : y_{n-1}) : x_n) \cong K[V(F_{n-4})]/I(F_{n-4}) \otimes_K K[y_{n-1}, x_n].$$

By using Lemmas 2.5 and 3.8 on the above isomorphisms, we get

$$\operatorname{reg}\left(S/((I(C_{2n}(1,2)) : x_{n-1}) : y_{n-2})\right) = \operatorname{reg}\left(K[V(F_{n-4})]/I(F_{n-4})\right) = \left\lceil \frac{n-4}{2} \right\rceil,$$

$$\operatorname{reg}\Big(S/((I(C_{2n}(1,2)) : x_{n-1}), y_{n-2})\Big) = \operatorname{reg}\Big(K[V(F_{n-3})]/I(F_{n-3})\Big) = \left\lceil \frac{n-3}{2} \right\rceil,$$

$$\operatorname{reg}\Big(S/((I(C_{2n}(1,2)), x_{n-1}), y_{n-1})\Big) = \operatorname{reg}\Big(K[V(F_{n-1})]/I(F_{n-1})\Big) = \left\lceil \frac{n-1}{2} \right\rceil,$$

$$\operatorname{reg}\Big(S/(((I(C_{2n}(1,2)), x_{n-1}) : y_{n-1}), x_n)\Big) = \operatorname{reg}\Big(K[V(F_{n-3})]/I(F_{n-3})\Big) = \left\lceil \frac{n-3}{2} \right\rceil,$$

$$\operatorname{reg}\Big(S/(((I(C_{2n}(1,2)), x_{n-1}) : y_{n-1}) : x_n)\Big) = \operatorname{reg}\Big(K[V(F_{n-4})]/I(F_{n-4})\Big) = \left\lceil \frac{n-4}{2} \right\rceil.$$

Since $\operatorname{reg}\Big(S/((I(C_{2n}(1,2)) : x_{n-1}) : y_{n-2})\Big) < \operatorname{reg}\Big(S/((I(C_{2n}(1,2)) : x_{n-1}), y_{n-2})\Big)$, by Lemma 2.11(c), $\operatorname{reg}\Big(S/(I(C_{2n}(1,2)) : x_{n-1})\Big) = \left\lceil \frac{n-3}{2} \right\rceil$. Also, $\operatorname{reg}\Big(S/(((I(C_{2n}(1,2)), x_{n-1}) : y_{n-1}) : x_n)\Big) < \operatorname{reg}\Big(S/(((I(C_{2n}(1,2)), x_{n-1}) : y_{n-1}), x_n)\Big)$, and again by Lemma 2.11(c) we get $\operatorname{reg}\Big(S/((I(C_{2n}(1,2)), x_{n-1}) : y_{n-1})\Big) = \left\lceil \frac{n-3}{2} \right\rceil$. This implies $\operatorname{reg}\Big(S/((I(C_{2n}(1,2)), x_{n-1}) : y_{n-1})\Big) < \operatorname{reg}\Big(S/((I(C_{2n}(1,2)), x_{n-1}), y_{n-1})\Big)$ by using Lemma 2.11(c), and we have that $\operatorname{reg}\Big(S/(I(C_{2n}(1,2)), x_{n-1})\Big) = \left\lceil \frac{n-1}{2} \right\rceil$. Thus, the required result follows as

$$\operatorname{reg}\Big(S/(I(C_{2n}(1,2)) : x_{n-1})\Big) < \operatorname{reg}\Big(S/(I(C_{2n}(1,2)), x_{n-1})\Big) = \left\lceil \frac{n-1}{2} \right\rceil.$$

Case 2. If n is odd, here $C_{2n}(1,2) = F_3 \cup H$, where $H \cong F_{n-1}$ and $F_3 \cap H \neq \emptyset$. By Lemmas 2.10 and 3.8, we get

$$\operatorname{reg}(S/I(C_{2n}(1,2))) \leq \operatorname{reg}(K[V(F_3)]/I(F_3)) + \operatorname{reg}(K[V(F_{n-1})]/I(F_{n-1})) = 2 + \left\lceil \frac{n-1}{2} \right\rceil.$$

In the case of the second inequality, we define $M = \Big\{\{x_1, y_1\}, \{x_3, y_3\}, \ldots, \{x_{n-2}, y_{n-2}\}\Big\}$. Clearly, M is an induced matching and it follows that $\operatorname{indmat}(C_{2n}(1,2)) \geq |M| = \frac{n-1}{2}$. By Lemma 2.8, we have $\operatorname{reg}(S/I(C_{2n}(1,2))) \geq \frac{n-1}{2}$.

□

In the following result, we find the exact value for the regularity of cyclic module $K[V(C_{2n}(1, n-1, n))]/I(C_{2n}(1, n-1, n))$.

Theorem 4.7. *If* $n \geq 3$, *then* $\operatorname{reg}\Big(K[V(C_{2n}(1, n-1, n))]/I(C_{2n}(1, n-1, n))\Big) = \left\lceil \frac{n-1}{2} \right\rceil$.

Proof. Let $S = K[V(C_{2n}(1, n-1, n))]$. We have the following K-algebra isomorphisms:

$$S/(I(C_{2n}(1, n-1, n)) : y_n) \cong K[V(G_{n-3})]/I(G_{n-3}) \otimes_K K[y_n], \tag{4.15}$$

$$S/((I(C_{2n}(1, n-1, n)), y_n), x_n) \cong K[V(G_{n-1})]/I(G_{n-1}), \tag{4.16}$$

$$S/((I(C_{2n}(1, n-1, n)), y_n) : x_n) \cong K[V(G_{n-3})]/I(G_{n-3}) \otimes_K K[x_n]. \tag{4.17}$$

By applying Lemmas 2.5 and 3.9 and Remark 3.1 on Eqs (4.15)–(4.17), we get

$$\operatorname{reg}\left(S/\,(I(C_{2n}(1,n-1,n)) : y_n)\,\right) = \operatorname{reg}\left(K[V(G_{n-3})]/I(G_{n-3})\right) = \left\lceil \frac{n-3}{2} \right\rceil,$$

$$\operatorname{reg}\left(S/((I(C_{2n}(1,n-1,n)), y_n) : x_n)\right) = \operatorname{reg}\left(K[V(G_{n-3})]/I(G_{n-3})\right) = \left\lceil \frac{n-3}{2} \right\rceil,$$

and

$$\operatorname{reg}\left(S/((I(C_{2n}(1,n-1,n)), y_n), x_n)\right) = \operatorname{reg}\left(K[V(G_{n-1})]/I(G_{n-1})\right) = \left\lceil \frac{n-1}{2} \right\rceil.$$

Since $\left\lceil \frac{n-3}{2} \right\rceil < \left\lceil \frac{n-1}{2} \right\rceil$, by Lemma 2.11(c) we get $\operatorname{reg}\left(S/(I(C_{2n}(1,n-1,n)), y_n)\right) = \left\lceil \frac{n-1}{2} \right\rceil$. Also,

$$\operatorname{reg}\left(S/(I(C_{2n}(1,n-1,n)) : y_n)\right) < \operatorname{reg}\left(S/(I(C_{2n}(1,n-1,n)), y_n)\right),$$

and again by Lemma 2.11(c), the required result follows. □

5. Conclusions

In this paper we compute the algebraic invariants namely regularity, projective dimension, depth, and the Stanley depth of the quotient rings of the edge ideals associated with some classes of circulant graphs. It will be interesting but seems challenging to find these algebraic invariants for the quotient rings of the edge ideals of all four and five regular circulant graph.

Use of AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare that there is no conflict of interest in this article .

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