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Research article

Efficient picture fuzzy soft CRITIC-CoCoSo framework for supplier selection under uncertainties in Industry 4.0

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Abstract: The picture fuzzy soft set (PiFSS) is a new hybrid model to address complex and uncertain information in Industry 4.0. Topological structure on PiFSS develops an innovative approach for topological data analysis to seek an optimal and unanimous decision in decision-making processes. This conception combines the advantages of a picture fuzzy set (PiFS) and a soft set (SS), allowing for a more comprehensive representation of the ambiguity in the supplier selection. Moreover, the criteria importance through intercriteria correlation (CRITIC) and the combined compromise solution (CoCoSo) technique is applied to the proposed framework to determine the relative importance of the evaluation parameter and to select the most suitable supplier in the context of sustainable development. The suggested technique was implemented and evaluated by applying it to a manufacturing company as a case study. The outcomes reveal that the approach is practical, efficient and produces favorable results when used for decision-making purposes. Evaluating and ranking of efficient suppliers based on their sustainability performance can be effectively accomplished through the use of PiFS-topology, thus facilitating the decision-making process in the CE and Industry 4.0 era.

Keywords: picture fuzzy soft topology; supplier selection; circular economy; Industry 4.0; CRITIC-CoCoSo; decision making
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1. Introduction

The notions of circular economy (CE) and Industry 4.0 have become more popular in recent times. Due to their potential for fostering sustainable development, they show great prospects for growth. In this context, the sustainable suppliers selection (SSS) has become a crucial decision-making process for companies. In the context of CE and the Industry 4.0 era, the process of SSS has acquired paramount significance within the realm of decision-making. As global awareness grows regarding the importance of sustainability and conscientious utilization of resources, there is an increasingly significant focus on fostering responsible resource consumption, and it is essential for organizations to select suppliers that align with their sustainability goals. The CE aspires to cut waste and maximize resource efficiency, and Industry 4.0 supports the adoption of modern technology for more effective manufacturing and administration of supply chains. Therefore, decision-making applications of SSS can assist organizations in selecting suppliers that contribute to their sustainability goals and align with the principles of CE and Industry 4.0. By utilizing such applications, organizations can improve their overall sustainability performance, minimize waste and promote responsible resource consumption. This introduction lays the foundation for exploring the significance of SSS within the domain of CE and Industry 4.0 and the potential benefits of employing decision-making applications in this area.

1.1. Literature review

1.1.1. Multi criteria decision-making based uncertain data modeling

Multi criteria decision-making (MCDM) is a predetermined method for finding the optimal solution from a range of accessible alternatives. Hence, the MCDM is essential for solving global concerns. The proper decision has the ability to completely transform one's lifestyle. The decision expert (DE) evaluates the advantages, characteristics and constraints of universal components to arrive at a sound conclusion and facilitate optimal decision-making. To find a solution, Zadeh [1] introduced an innovative concept that subsequently led to the development and evolution of the fuzzy set (FS). This novel technique offers the highest possible degree of gyration over a wide range of diverse technological domains, making it a very useful tool. In the context of an FS, numerical values within the range of 0 to 1, known as membership values (MVs), are assigned to different alternatives. In some situations, however, DEs will express their assessments in terms of positive membership value (PMV) and non-membership value (NMVs), respectively. Because of this, Atanassov [2] came up with the idea for an extension of FS that he called intuitionistic fuzzy set (IFS). It comprises MV and NMV, representing satisfaction levels from good to unacceptable. Therefore, it is a very useful tool to explain complicated and uncertain data. Pythagorean fuzzy set [3] and q-Rung orthopair fuzzy set [4] are robust extensions of FSs and IFSs. Deli et al. [5] defined intuitionistic fuzzy parameterized soft sets (IFPSS) and introduced an adaptable MCDM technique. While the use of IFS has proven effective in tackling complex problems across various domains, there are instances where this particular framework may not be applicable. Al-Shami and Mhemdi [6] introduced the concept of (m,n)-Fuzzy sets, to address limitations in existing fuzzy set theories. Al-Shami et al. [7] introduced the concept of (a, b)-fuzzy soft sets, enabling the modeling of varying importance between membership and non-membership degrees. Assume that the standard FS and IFS are unable to adequately explain human voting responses such as "yes", "no", "abstain" and "refuse". Cuong [8] proposed a new idea called a picture fuzzy set (PiFS) to address the problems of this nature, and DEs will express their assessments in terms of degree of positive membership (DPM), degree of neutral membership (DNuM) and degree of negative membership (DNM), respectively. In real life situations, picture fuzzy set theory offers multiple options for decision-making. For instance:

- Consider an individual facing a health issue. Here, we can link positive, neutral and negative membership functions to the likelihood of recovery, various treatment options and the severity of the illness, respectively. Refusal might stem from the individual's inadequate financial situation, leading to an inability to cover medical costs and a consequent refusal to seek hospitalization.

- Imagine a scenario where someone is accused of a crime. In this situation, positive, neutral and negative membership functions could be associated with the potential for maximum punishment a moderate level of punishment, or the release of the accused, respectively. Refusal, in this context, might involve the dismissal of the case due to a reconciliation or agreement between the involved parties.

The PiFSS was introduced by Yang et al. [9]. The PiFSS is an effective model for dealing with ambiguity as it categorizes parameters into accurate, uncertain, and undependable levels across a broad range. Cuong and Hai [10] established the foundational logic operators, as well as their implications on PiFSs. Thong [11] created a novel algorithm for automatic fuzzy clustering by employing composite cardinality and particle swarm optimization. Neutrosophy and Neutrosophic set [12] and spherical fuzzy sets [13–15] are innovative models for computational intelligence and fuzzy modeling. Akram et al. [16, 17] proposed threshold graphs under picture Dombi fuzzy information and extended MULTIMOORA method for MAGDM.

1.1.2. Fuzzy topology based uncertain data modeling

The versatility of topology offers a robust approach to handling imprecise, uncertain, or incomplete information in practical, real-world scenarios, making it an invaluable tool across diverse applications. Chang [18] introduced fuzzy topological spaces. Cooker [19] established the foundational principles of intuitionistic fuzzy topological spaces. Shabir and Naz [20] introduced the concept of soft topological spaces and explored properties of soft open sets, closures, interior points, neighborhoods and separation axioms. Tanay and Kandemir [21] worked on fuzzy soft topological spaces. Razzzq and Riaz [22, 23] introduced the concept of the M-parameterized N-soft (MPNS) set and MPNS-topology with applications in MCDM techniques. Picture fuzzy soft topology plays a pivotal role in addressing real-world problems across various domains. Its applications are widespread and impactful, especially in fields like medical imaging, geographical information systems, and pattern recognition. In medical applications, picture fuzzy soft topology aids in the analysis and interpretation of complex, multi-dimensional medical data, facilitating accurate diagnoses and treatment planning. Additionally, in geographical information systems, it enables the efficient analysis of spatial data, such as remote sensing and land-use patterns, contributing to urban planning and environmental studies. Moreover, in pattern recognition, this framework assists in recognizing patterns in data sets with uncertainties, enhancing machine learning algorithms and artificial intelligence systems. Numerous authors have contributed to the study of topological structures, as detailed in Table 1.

Table 1. Fuzzy topology and related work.

Benchmarks	Researchers	References
Fuzzy topology	Lowen	[24]
Fuzzy topology	Lowen	[25]
Fuzzy topology	Chaudhuri & Das	[26]
IF topology	Ozcag & Coker	[27]
Soft topology	Cagman et al.	[28]
Soft topological spaces	Shabir and Naz	[29]
N-Soft topology	Riaz et al.	[30]
FPFS topology	Riaz & Hashmi	[31]
HFS topological spaces	Riaz et al.	[32]
PyF topology	Haydar et al.	[33]
PiF topology	Razzaq et al.	[34]

1.1.3. Industry 4.0 and CE

In recent years, researchers, practitioners and academics have acknowledged the significance of industrial concepts such as CE practices and Industry 4.0 technologies [35]. This perspective has been evolving gradually over the course of several years. It has been known for a while that the CE is a wide term that incorporates environmental well being into economic activity through the implementation of a regeneration or restorative economic system [36]. This procedure entails consciously modifying the perception that the product has reached the culmination of its lifespan. Despite important conceptual limitations, the idea of a CE is being put into practice with the goal of achieving sustainability via the recycling of resources and the elimination of waste and harmful substances. According to [37], the attainment of the CE may not be regarded as the ultimate objective, but rather as a constituent part of a broader strategic approach aimed at enhancing the efficiency of resource allocation and utilization. However, the term "Industry 4.0" refers to economic infrastructures that are governed by information technology [38]. The combination of CE principles with the technological advancements of Industry 4.0 is becoming an increasingly essential component in sustainable supply chain management. According to [39], the process of transforming a linear supply chain into a circular supply chain is impeded by inconsistencies in the data that come from a variety of sources. Stock et al. [40] conducted research that demonstrated modern organizations need the resilience, flexibility and discernibility that Industry 4.0 offers in order to prevent the failure of sustainable reusing, recycling and re manufacturing. As a consequence of this, the present research has discovered that technological advancements such as Industry 4.0 might pave the door for the implementation of CE techniques [41]. In addition, an adequate number of criteria has to be defined in order to completely integrate the technology of industry 4.0 into the practices of the CE throughout the green supplier selection process [42]. Identifying the Industry 4.0 criteria, which are based on CE methods, are critical in the selection since they establish the framework for choosing the best supplier. Most of the previous studies drew conclusions by considering the conventional aspects of sustainability, such as environmental, social and economic factors [43]. As CE trends and the Fourth Industrial Revolution manifests, it is now imperative for organizations and governments to take into account all facets of sustainability while making decisions. As a result, Industry 4.0 and CE principles must be accounted for in procurement practices. This study is the inaugural attempt to use CE methodologies to incorporate Industry 4.0 technology into SSC operations via a unified framework.

1.1.4. Sustainable supply chain management

As the environment continues to deteriorate and the gap between social classes grows wider, the most crucial concern in recent times is the emergence of sustainability. As a consequence of this, putting sustainability into practice requires a significant change away from maximizing of profits and toward the environmental performance and social objectives of businesses [44]. Companies are aware of the significance of integrating sustainable practices across their supply chains [45]. As a consequence of this, circular supply chain management (CSCM) mandates the implementation of environmentally favorable business procedures and the encouragement of employees to operate into a responsible manner [46]. The CSCM strategy aims to reduce waste, minimize its environmental impact and save enterprises money. The CSCM refers to the management of supplies, financial transactions, as well as the cooperation between companies, while considering the financial, environmental and social aspects of sustainable development objectives [47]. Stricter regulations [48] from the government, social activism, increased public knowledge, pressures, organizational image, business brand and consumer demands are the driving forces for the incorporation of sustainability into supply chains. The literature review of CE and Industry 4.0 shown in Table 2.

Table 2. MCDM Techniques.

Industry 4.0	Method	Application	Reference
Automated guided vehicles 3D, robotics printing number	Single-valued neutrosophic	Automotive sector	Das et al. [49]
Delivery delay, Rate of product return, cost, and the adoption of Industry 4.0	FTOPSIS & PFAHP	-	Gul & AK [50]
Cloud computing, cognitive computing, Cyber-physical systems &	FVIKOR	Cement Factory	Gul [51]
Waste treatment, product assembly, product selling, waste separation, product printing, product life cycle, customer service, operational procedures	WASPAS, TOPSIS, AHP	Automotive industry	Gupta and Barua [52]
company culture, governance or management structure, technology, organizational structure and quality control	MASs	Manufacturing firms	Ghadimi Approach et al. [53]
Artificial intelligence, big data, blockchain, cloud computing, cybersecurity, Internet of Things, additive manufacturing, augmented reality, autonomous robots, and automatic vehicles	GRA, TOPSIS, VIKOR	Agri-food industry	Banaeian et al. [54]
Tracking ability, concentration of supply chain activity, management of cybersecurity threats	Multi-Choice Goal Programming	Hypothetical case study, Decision Support systems	Chen et al. [55]

1.2. *Highlights and contribution*

Choosing an alternative is a critical aspect of MCDM. By employing the PiFSS in the MCDM approach, decision makers can make informed decisions because they can assess the degree of positive membership, neutral membership and negative membership values using a parameterized approach. The current research has contributed to the examination of MCDM under uncertainty in various aspects:

- Our proposal to tackle decision making problems involves utilizing the PiFS-topology and its fundamental characteristics.
- A comprehensive, step-by-step explanation of the picture fuzzy soft CRITIC-CoCoSo method is provided, along with all of its important formulas.
- To demonstrate the accuracy and reliability, a comparison is done between the CRITIC-CoCoSo and existing methods, revealing that both propose the same optimal solution.
- A numerical case study of supplier selection in CE is presented and the exhaustive analysis proves the practicality and rationality of the proposed techniques.

1.3. Structure of paper

The remaining sections of the article are structured as follows: In Section 2, the introductory concepts, including PiFS, PiFSS, score function (SF), accuracy function (AF), as well as some operational laws of PiFS, are described in a concise manner. Section 3 outlines the primary findings regarding PiFS-topology. In Section 4 of the paper, the CRITIC-CoCoSo model is introduced. This model utilizes attribute weights to solve MCDM problems and provides an example of the model's practical application in selecting sustainable suppliers for CE and Industry 4.0. Section 5 includes sensitivity analysis, and Section 6 comprises comparative analysis. Finally, in Section 7, the work is summarized, and future research plans are outlined.

2. Preliminaries

To study the rest of the paper, we begin by introducing fundamental ideas of PiFSs that are essential. Some rudimentary concepts related to this research work can be studied in [8, 9, 56–60].

Definition 2.1. [8] In PiFS $\widehat{\mathcal{T}}$ with the universal set M, the DPM ($0 \leq \mu(\alpha) \leq 1$), DNuM ($0 \leq \delta(\alpha) \leq 1$), and DNM ($0 \leq \zeta(\alpha) \leq 1$) are assigned to each alternative $\alpha \in M$. A PiFS can be presented as

$$\widehat{\mathcal{T}} = \Big\{(\alpha, \mu(\alpha), \delta(\alpha), \zeta(\alpha)) : \alpha \in M\Big\}$$

with the condition

$$0 \leq \mu(\alpha) + \delta(\alpha) + \zeta(\alpha) \leq 1$$

Then, a picture fuzzy number (PiFN) can be written as, $\aleph = (\mu(\alpha), \delta(\alpha), \zeta(\alpha))$.

Definition 2.2. [57] Let $\aleph = (\mu, \delta, \zeta)$ be a PiFN, then SF of PiFN is defined as

$$S(\aleph) = \mu - \delta - \zeta.$$

where $-1 \leq \mathcal{S}(\aleph) \leq 1$. If $\aleph_i$ and $\aleph_j$ are two PiFNs, then

(1) If $\mathcal{S}(\aleph_i) < \mathcal{S}(\aleph_j)$ then $\aleph_i$ precedes $\aleph_j$ i.e. $\aleph_i \prec \aleph_j$,

(2) If $\mathcal{S}(\aleph_i) > \mathcal{S}(\aleph_j)$ then $\aleph_i$ succeeds $\aleph_j$ i.e. $\aleph_i \succ \aleph_j$,

(3) If $\mathcal{S}(\aleph_i) = \mathcal{S}(\aleph_j)$ then $\aleph_i \sim \aleph_j$.

Definition 2.3. [57] Let $\aleph = (\mu, \delta, \zeta)$ be a PiFN, then AF of PiFN is defined as,

$$\mathcal{A}(\aleph) = \mu + \delta + \zeta.$$

If $\aleph_i$ and $\aleph_j$ are two PiFNs, then

(1) If $\mathcal{S}(\aleph_i)$ and $\mathcal{S}(\aleph_j)$ coincide and $\mathcal{A}(\aleph_i)$ exceeds $\mathcal{A}(\aleph_j)$ then $\aleph_i > \aleph_j$,

(2) If $\mathcal{S}(\aleph_i)$ and $\mathcal{S}(\aleph_j)$ coincide and $\mathcal{A}(\aleph_i)$ precedes $\mathcal{A}(\aleph_j)$ then $\aleph_i > \aleph_j$,

(3) If both $\mathcal{S}(\aleph_i), \mathcal{S}(\aleph_j)$ and $\mathcal{A}(\aleph_i), \mathcal{A}(\aleph_j)$ coincide then $\aleph_i \sim \aleph_j$.

Cuong [8] proposed fundamental operations on PiFSs including union, intersection, inclusion, complement and equality. Now we review these concepts in the following definition.

Definition 2.4. Let $\widehat{\mathcal{T}}_A$ and $\widehat{\mathcal{T}}_B$ be two PiFSs. Then

(1) $\widehat{\mathcal{T}}_A \cup \widehat{\mathcal{T}}_B = \left\{\left\langle \alpha,\ \Big(\max(\mu_A(\alpha),\ \mu_B(\alpha)),\ \min(\delta_A(\alpha),\ \delta_B(\alpha)),\ \min(\zeta_A(\alpha),\ \zeta_B(\alpha))\Big)\right\rangle : \alpha \in M\right\}$.

(2) $\widehat{\mathcal{T}}_A \cap \widehat{\mathcal{T}}_B = \left\{\left\langle \alpha,\ \Big(\min(\mu_A(\alpha),\ \mu_B(\alpha)),\ \min(\delta_A(\alpha),\ \delta_B(\alpha)),\ \max(\zeta_A(\alpha),\ \zeta_B(\alpha))\Big)\right\rangle : \alpha \in M\right\}$.

(3) $\widehat{\mathcal{T}}_A \subseteq \widehat{\mathcal{T}}_B \iff \mu_A(\alpha) \leq \mu_B(\alpha),\ \delta_A(\alpha) \leq \delta_B(\alpha),\ \zeta_A(\alpha) \geq \zeta_B(\alpha),\ \forall \alpha \in M$.

(4) $\widehat{\mathcal{T}}_A = \widehat{\mathcal{T}}_B \iff \widehat{\mathcal{T}}_A \subseteq \widehat{\mathcal{T}}_B$ and $\widehat{\mathcal{T}}_A \supseteq \widehat{\mathcal{T}}_B$.

(5) $\widehat{\mathcal{T}}_A^c = \Big\{\alpha,\ \Big(\zeta_A(\alpha),\ \delta_A(\alpha),\ \mu_A(\alpha)\Big) : \alpha \in M\Big\}$.

Riaz *et al.* [56] proposed topological data analysis for spherical fuzzy soft information. We extend this idea towards picture fuzzy soft information. Fundamental operation on PiFSs and information aggregation was proposed in [57–59]. Riaz *et al.* [60] proposed some new operations on PiFNs to address the limitations of existing operations.

Definition 2.5. [60] Let $\aleph_A = (\mu_A, \delta_A, \zeta_A)$ and $\aleph_B = (\mu_B, \delta_B, \zeta_B)$ be two PiFNs, then

(1) $\aleph_A \oplus \aleph_B = \Big(1 - (1-\mu_A)(1-\mu_B),\ \delta_A\delta_B,\ (\zeta_A + \delta_A)(\zeta_B + \delta_B) - \delta_A\delta_B\Big)$.

(2) $\aleph_A \otimes \aleph_B = \Big((\mu_A + \delta_A)(\mu_B + \delta_B) - \delta_A\delta_B,\ \delta_A\delta_B,\ 1 - (1-\zeta_A)(1-\zeta_B)\Big)$.

(3) $\lambda\aleph_A = \Big(1 - (1-\mu_A)^\lambda,\ \delta_A^\lambda,\ (\delta_A + \zeta_A)^\lambda - \zeta_A^\lambda\Big)$.

(4) $\aleph_A^\lambda = \Big((\mu_A + \delta_A)^\lambda) - \delta_A^\lambda,\ \delta_A^\lambda,\ 1 - (1-\zeta_A)^\lambda\Big)$.

Example 2.6. Consider a universal set $X = \{\alpha_1, \alpha_2, \alpha_3\}$ be two PiFSs $\widehat{\mathcal{T}}_T$ and $\widehat{\mathcal{T}}_J$ as follows:

$$\widehat{\mathcal{T}}_T = \Big\{\Big\langle\alpha_1, (0.214, 0.234, 0.120)\Big\rangle, \Big\langle\alpha_2, (0.213, 0.415, 0.003)\Big\rangle, \Big\langle\alpha_3, (0.129, 0.515, 0.124)\Big\rangle\Big\},$$

$$\widehat{\mathcal{T}}_J = \Big\{\Big\langle\alpha_1, (0.304, 0.321, 0.112)\Big\rangle, \Big\langle\alpha_2, (0.419, 0.430, 0.001)\Big\rangle, \Big\langle\alpha_3, (0.239, 0.535, 0.121)\Big\rangle\Big\}.$$

According to the operations on PiFSs, given in Definition 2.4, we see that

$$\widehat{\mathcal{T}}_T \subseteq \widehat{\mathcal{T}}_J \Rightarrow \widehat{\mathcal{T}}_T \cup \widehat{\mathcal{T}}_J \neq \widehat{\mathcal{T}}_J, \widehat{\mathcal{T}}_T \cap \widehat{\mathcal{T}}_J = \widehat{\mathcal{T}}_T.$$

Now, we modify inclusion and intersection operations on PiFSs as follows:

- Inclusion: If $\mu_T(\alpha) \leq \mu_J(\alpha),\ \delta_T(\alpha) \geq \delta_J(\alpha),\ \zeta_T(\alpha) \geq \zeta_J(\alpha)$.
- Intersection: $\Big\{(\alpha, \min\{\mu_T(\alpha), \mu_J(\alpha)\}, \max\{\delta_T(\alpha), \delta_J(\alpha)\}, \max\{\zeta_T(\alpha), \zeta_J(\alpha)\})|\ \alpha \in M\Big\}$.

We now employ altered forms of the inclusion and intersection operations and introduce the union and complement operations as defined in Definition 2.4. We demonstrate the use of these operations through an example.

Example 2.7. Let $M = \{\alpha_1, \alpha_2, \alpha_3\}$ be the universe of discourse with $\widehat{\mathcal{T}}_T$ and $\widehat{\mathcal{T}}_J$, two PiFSs in M.

$$\widehat{\mathcal{T}}_T = \{\langle\alpha_1, (0.214, 0.334, 0.112)\rangle, \langle\alpha_2, (0.213, 0.430, 0.003)\rangle, \langle\alpha_3, (0.129, 0.535, 0.124)\rangle\},$$

$$\widehat{\mathcal{T}}_J = \{\langle\alpha_1, (0.304, 0.321, 0.022)\rangle, \langle\alpha_2, (0.419, 0.415, 0.001)\rangle, \langle\alpha_3, (0.239, 0.515, 0.121)\rangle\}.$$

By employing the newly defined operations on PiFSs, it is evident that, $\widehat{\mathcal{T}}_T \subseteq \widehat{\mathcal{T}}_J$ implies that $\widehat{\mathcal{T}}_T \cup \widehat{\mathcal{T}}_J = \widehat{\mathcal{T}}_J$ and $\widehat{\mathcal{T}}_T \cap \widehat{\mathcal{T}}_J = \widehat{\mathcal{T}}_T$.

Definition 2.8. [57–59] The null PiFS is represented by the symbol $\widehat{\Phi}$ and is formally defined as follows:

$$\widehat{\mathcal{T}}_{\widehat{\Phi}} = \Big\{\Big\langle\alpha, (0, 0, 1)\Big\rangle : \alpha \in M\Big\}.$$

Definition 2.9. [57–59] The absolute PiFS is symbolized as $\widehat{\chi}$ and is formally delineated as follows:

$$\widehat{\mathcal{T}}_{\widehat{\chi}} = \Big\{\Big\langle\alpha, (1, 0, 0)\Big\rangle : \alpha \in M\Big\}.$$

Example 2.10. Using PiFSs as described above, we observe that

$$\widehat{\mathcal{T}}_{\widehat{\Phi}} \nsubseteq \widehat{\mathcal{T}}_T, \quad \widehat{\mathcal{T}}_{\widehat{\Phi}} \cup \widehat{\mathcal{T}}_T \neq \widehat{\mathcal{T}}_T, \quad \widehat{\mathcal{T}}_{\widehat{\Phi}} \cap \widehat{\mathcal{T}}_T \neq \widehat{\Phi},$$

$$\widehat{\mathcal{T}}_T \subseteq \widehat{\mathcal{T}}_{\widehat{\chi}}, \quad \widehat{\mathcal{T}}_T \cup \widehat{\mathcal{T}}_{\widehat{\chi}} = \widehat{\mathcal{T}}_{\widehat{\chi}}, \quad \widehat{\mathcal{T}}_T \cap \widehat{\mathcal{T}}_{\widehat{\chi}} = \widehat{\mathcal{T}}_T.$$

Suppose we have a collection $\beta(M)$ consisting of all PiFSs on a universal set M. Unfortunately, defining a topological structure on this collection is challenging due to certain limitations. In order to address these constraints, we present a novel assemblage denoted as $\beta_{(\delta)}(M)$, comprising PiFSs defined on M, that have a fixed degree of neutral membership δ between 0 and 1. With this new collection, we can define the null set and absolute set of PiFS as follows.

Definition 2.11. A PiFS in $\beta_{(\delta)}(M)$, $0 \leq \delta \leq 1$, is called a null PiFS, if

$$\phi_E = \Big\{\Big\langle\alpha, (0,\ \delta,\ 1-\delta)\Big\rangle : \alpha \in M\Big\}.$$

In the case where $\delta = 0$, the null PiFS becomes $\phi_E = \Big\{\Big\langle\alpha, (0,\ 0,\ 1)\Big\rangle : \alpha \in M\Big\}$.

Definition 2.12. A PiFS in $\beta_{(\delta)}(M)$, $0 \leq \delta \leq 1$, is called an absolute PiFS, if

$$\breve{M}_E = \Big\{\Big\langle\alpha, (1-\delta,\ \delta,\ 0)\Big\rangle : \alpha \in M\Big\}.$$

In the case where $\delta = 0$, the absolute PiFS becomes $\breve{M}_E = \Big\{\Big\langle\alpha, (1,\ 0,\ 0)\Big\rangle : \alpha \in M\Big\}$.

Within the manuscript, the supposition is made that M denotes the entirety of the universe, whereas E signifies the assemblage of properties. Additionally, A is delineated as a subset of E, 2^M represents the set encompassing all conceivable subsets of M and, finally, $PiFS^M$ is the category containing all PiFSs in M.

Definition 2.13. The score function $F = \left(\gamma_{ij}\right)_{m\times n}$ of each $F\left(\xi^{\varphi}{}_j\right)(\alpha_i) = (\mu_F\left(\xi^{\varphi}{}_j\right)(\alpha_i), \delta_F\left(\xi^{\varphi}{}_j\right)(\alpha_i), \zeta_F\left(\xi^{\varphi}{}_j\right)(\alpha_i))$ can be defined as

$$S(\aleph) = \frac{1+\mu-\delta-\zeta}{2}. \tag{1}$$

Definition 2.14. [61] The soft set, denoted as SS, within the context of a mapping $\mathfrak{F} : A \to 2^M$, can be formally represented either as $(\mathfrak{F}, A)$ or $\mathfrak{F}_A$. The formal definition of an SS is as follows:

$$(\mathfrak{F}, A) = \left\{(\alpha,\ \mathfrak{F}(\alpha)) :\ \alpha \in A\right\}.$$

Definition 2.15. [9] Let $\mathcal{P} : A \to PiFS^M$ be a mapping, then the picture fuzzy soft set (PiFSS) is denoted as $(\mathcal{P}, A)$ or $\mathcal{P}_A$, and defined by

$$\begin{aligned}
(\mathcal{P}, A) &= \left\{(\xi^{\varphi},\ \mathcal{P}(\xi^{\varphi})) : \xi^{\varphi} \in A,\ \alpha \in M\right\} \\
&= \left\{\left(\xi^{\varphi}, \{\alpha,\ \mu(\alpha),\ \delta(\alpha),\ \zeta(\alpha)\}\right) : \xi^{\varphi} \in A,\ \alpha \in M\right\} \\
&= \left\{\left(\xi^{\varphi},\ \left\{\frac{\alpha}{(\mu(\alpha),\ \delta(\alpha),\ \zeta(\alpha))}\right\}\right) : \xi^{\varphi} \in A,\ \alpha \in M\right\} \\
&= \left\{\left(\xi^{\varphi},\ \left\{\frac{(\mu(\alpha),\ \delta(\alpha),\ \zeta(\alpha))}{\alpha}\right\}\right) : \xi^{\varphi} \in A,\ \alpha \in M\right\}.
\end{aligned}$$

The group of all PiFSs in M is known as a picture fuzzy soft class (PiFSS-Class), and it can be represented as PiFS(M, E). A PiFSS, denoted as $\mathcal{P}_A$, can be represented in Table 3, by utilizing two sets, $M = \{\alpha_1,\ \cdots,\ \alpha_m\}$ and $A = \{\xi^{\varphi}{}_1,\ \cdots,\ \xi^{\varphi}{}_n\}$.

Table 3. PiFSS.

$\mathcal{P}_A$	$\xi^{\varphi}{}_1$	$\xi^{\varphi}{}_2$	$\cdots$	$\xi^{\varphi}{}_n$
α_1	$(\mu_{11}, \delta_{11}, \zeta_{11})$	$(\mu_{12}, \delta_{12}, \zeta_{12})$	$\cdots$	$(\mu_{1n}, \delta_{1n}, \zeta_{1n})$
α_2	$(\mu_{21}, \delta_{21}, \zeta_{21})$	$(\mu_{22}, \delta_{22}, \zeta_{22})$	$\cdots$	$(\mu_{2n}, \delta_{2n}, \zeta_{2n})$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
α_m	$(\mu_{m1}, \delta_{m1}, \zeta_{m1})$	$(\mu_{m2}, \delta_{m2}, \zeta_{m2})$	$\cdots$	$(\mu_{mn}, \delta_{mn}, \zeta_{mn})$

And its PiFS matrix is

$$\begin{aligned}
\mathcal{P}_A &= [(\mu_{ij}, \delta_{ij}, \zeta_{ij})]_{m\times n} \\
&= \begin{pmatrix}
(\mu_{11}, \delta_{11}, \zeta_{11}) & (\mu_{12}, \delta_{12}, \zeta_{12}) & \cdots & (\mu_{1n}, \delta_{1n}, \zeta_{1n}) \\
(\mu_{21}, \delta_{21}, \zeta_{21}) & (\mu_{22}, \delta_{22}, \zeta_{22}) & \cdots & (\mu_{2n}, \delta_{2n}, \zeta_{2n}) \\
\vdots & \vdots & \ddots & \vdots \\
(\mu_{m1}, \delta_{m1}, \zeta_{m1}) & (\mu_{m2}, \delta_{m2}, \zeta_{m2}) & \cdots & (\mu_{mn}, \delta_{mn}, \zeta_{mn})
\end{pmatrix}
\end{aligned}$$

Example 2.16. Consider a set of hostels $M = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$. Let $A = \{\xi^{\varphi}{}_1, \xi^{\varphi}{}_2, \xi^{\varphi}{}_3, \xi^{\varphi}{}_4, \xi^{\varphi}{}_5\}$ be the set of attributes, where

$\xi^{\varphi}{}_1$ represents affordability,
$\xi^{\varphi}{}_2$ represents cleanliness,
$\xi^{\varphi}{}_3$ represents good food,
$\xi^{\varphi}{}_4$ represents capaciousness.
$\xi^{\varphi}{}_5$ represents good location.

On the premise of the aforementioned attributes, a decision expert examined the alternatives and documented their assessment in the form of PiFSS as given in Table 4.

Table 4. PiFSS.

$\mathcal{P}_A$	$\xi^{\varphi}{}_1$	$\xi^{\varphi}{}_2$	$\xi^{\varphi}{}_3$	$\xi^{\varphi}{}_4$	$\xi^{\varphi}{}_5$
α_1	(0.40, 0.30, 0.10)	(0.10, 0.30, , 0.20)	(0.50, 0.20, 0.30)	(0.10, 0.50, 0.20)	(0.20, 0.10, 0.60)
α_2	(0.20, 0.10, 0.30)	(0.40, 0.10, 0.30)	(0.60, 0.10, 0.10)	(0.10, 0.50, 0.10)	(0.30, 0.20, 0.30)
α_3	(0.10, 0.20, 0.60)	(0.50, 0.10, 0.30)	(0.60, 0.10, 0.20)	(0.40, 0.20, 0.10)	(0.20, 0.40, 0.10)
α_4	(0.20, 0.50, 0.10)	(0.20, 0.40, 0.10)	(0.10, 0.50, 0.20)	(0.60, 0.10, 0.20)	(0.40, 0.20, 0.20)
α_5	(0.10, 0.50, 0.40)	(0.20, 0.10, 0.30)	(0.20, 0.50, 0.10)	(0.20, 0.10, 0.20)	(0.50, 0.10, 0.30)

Definition 2.17. Let $\mathcal{P}_A$ be a PiFSS, the complement of the PiFSS is denoted as $\mathcal{P}^c_A$ or $\mathcal{P}'_A$, and is formally defined as,

$$\mathcal{P}^c_A = \left\{\left(\langle\alpha,\ \zeta_{\xi^{\varphi}}(\alpha),\ \delta_{\xi^{\varphi}}(\alpha),\ \mu_{\xi^{\varphi}}(\alpha)\rangle\right) : \xi^{\varphi} \in A,\ \alpha \in M\right\}.$$

Definition 2.18. Let $\mathcal{P}_{A_1}$ and $\mathcal{P}_{A_2}$ be two PiFSSs over M. Then, $\mathcal{P}_{A_1}$ is a PiFS-subset of $\mathcal{P}_{A_2}$, i.e. $\mathcal{P}_{A_1}\widetilde{\subseteq}\mathcal{P}_{A_2}$, if

(i) $A_1 \subseteq A_2$, and

(ii) $\mathcal{P}^{(1)}(e)$ is PiFSS-subset of $\mathcal{P}^{(2)}(e)$ for all $e \in A_1$.

Definition 2.19. Extended Union: Let $\mathcal{P}_{A_1}$ and $\mathcal{P}_{A_2}$ be two PiFSSs defined on M. The extended union (EU) is defined as $\mathcal{P}_{\mathcal{E}_{\mathcal{U}}} = \mathcal{P}_{A_1}\widetilde{\cup}_{\mathcal{E}}\,\mathcal{P}_{A_2}$, where $\mathcal{E}_{\mathcal{U}} = A_1 \cup A_2$, and for all $\xi^{\varphi} \in T$,

$$\mathcal{P}_{\mathcal{E}_{\mathcal{U}}}(\xi^{\varphi}) = \begin{cases} \mathcal{P}_1(\xi^{\varphi}), & \text{if } \xi^{\varphi} \in A_1 \backslash A_2 \\ \mathcal{P}_2(\xi^{\varphi}), & \text{if } \xi^{\varphi} \in A_2 \backslash A_1 \\ \mathcal{P}_1(\xi^{\varphi}) \cup \mathcal{P}_2(\xi^{\varphi}), & \text{if } \xi^{\varphi} \in A_1 \cap A_2, \end{cases}$$

where $\mathcal{P}_1(\xi^{\varphi}) \cup \mathcal{P}_2(\xi^{\varphi})$ is the union of two PiFSSs.

Definition 2.20. Restricted Union: Let $\mathcal{P}_{A_1}$ and $\mathcal{P}_{A_2}$ be two PiFSSs defined on M. The restricted union (RU) is defined as $\mathcal{P}_{\mathcal{R}_{\mathcal{U}}} = \mathcal{P}_{A_1}\widetilde{\cup}_{\mathcal{R}}\,\mathcal{P}_{A_2}$, where $\mathcal{R}_{\mathcal{U}} = A_1 \cap A_2$ and, for all $\xi^{\varphi} \in U$, then $\mathcal{P}_{\mathcal{R}_{\mathcal{U}}}(\xi^{\varphi}) = \mathcal{P}_1(\xi^{\varphi}) \cup \mathcal{P}_2(\xi^{\varphi})$ is the union of two PiFSSs.

Definition 2.21. Extended intersection: Let $\mathcal{P}_{A_1}$ and $\mathcal{P}_{A_2}$ be two PiFSSs defined on M. The extended intersection (EI) is defined as $\mathcal{P}_{\mathcal{E}_{\mathcal{I}}} = \mathcal{P}_{A_1}\widetilde{\cap}_{\mathcal{E}}\,\mathcal{P}_{A_2}$,, where $\mathcal{E}_{\mathcal{I}} = A_1 \cup A_2$ and

$$\mathcal{P}_{\mathcal{E}_{\mathcal{I}}}(\xi^{\varphi}) = \begin{cases} \mathcal{P}_1(\xi^{\varphi}), & \text{if } \xi^{\varphi} \in A_1 \backslash A_2 \\ \mathcal{P}_2(\xi^{\varphi}), & \text{if } \xi^{\varphi} \in A_2 \backslash A_1 \\ \mathcal{P}_1(\xi^{\varphi}) \cap \mathcal{P}_2(\xi^{\varphi}), & \text{if } \xi^{\varphi} \in A_1 \cap A_2. \end{cases}$$

Definition 2.22. Restricted intersection: Let $\mathcal{P}_{A_1}$ and $\mathcal{P}_{A_2}$ be two PiFSSs defined on M. The restricted intersection (RI) is defined as $\mathcal{P}_{\mathcal{R}_I} = \mathcal{P}_{A_1} \widetilde{\cap}_{\mathcal{R}} \mathcal{P}_{A_2}$, where $\mathcal{R}_I = A_1 \cap A_2$, then $\mathcal{P}(\xi^\varphi) = \mathcal{P}_1(\xi^\varphi) \cap \mathcal{P}_2(\xi^\varphi)$ is the intersection of two PiFSSs.

Example 2.23. Let $M = \{\alpha_j : j = 1,2,3,4\}$ and $E = \{\xi^\varphi{}_i : i = 1,2,\cdots,5\}$. Suppose that $A = \{\xi^\varphi{}_1, \xi^\varphi{}_2, \xi^\varphi{}_5\}, \xi^\varphi{}_3\}$ and $D = \{\xi^\varphi{}_1, \xi^\varphi{}_2, \xi^\varphi{}_5, \xi^\varphi{}_6\}$. Then, consider two PiFSSs $\mathcal{P}_A$ and $\mathcal{P}_D$ in M defined as

$$\mathcal{P}_A = \begin{pmatrix} & \xi^\varphi{}_1 & \xi^\varphi{}_2 & \xi^\varphi{}_5 \\ \alpha_1 & (0.54,0.36,0.41) & (0.62,0.21,0.54) & (0.36,0.25,0.78) \\ \alpha_2 & (0.81,0.23,0.18) & (0.72,0.31,0.11) & (0.45,0.18,0.36) \\ \alpha_3 & (0.72,0.20,0.17) & (0.52,0.13,0.48) & (0.90,0.12,0.11) \\ \alpha_4 & (0.89,0.15,0.24) & (0.45,0.32,0.57) & (0.52,0.31,0.46) \end{pmatrix}$$

$$\mathcal{P}_D = \begin{pmatrix} & \xi^\varphi{}_1 & \xi^\varphi{}_2 & \xi^\varphi{}_5 & \xi^\varphi{}_6 \\ \alpha_1 & (0.25,0.18,0.56) & (0.78,0.16,0.41) & (0.54,0.25,0.31) & (0.63,0.24,0.28) \\ \alpha_2 & (0.78,0.11,0.12) & (0.72,0.30,0.19) & (0.38,0.45,0.23) & (0.91,0.11,0.12) \\ \alpha_3 & (0.39,0.42,0.25) & (0.61,0.32,0.48) & (0.72,0.18,0.24) & (0.58,0.13,0.35) \\ \alpha_4 & (0.89,0.10,0.12) & (0.52,0.31,0.38) & (0.48,0.36,0.41) & (0.63,0.24,0.18) \end{pmatrix}$$

$\mathcal{P}_A \cup_{\mathcal{E}} \mathcal{P}_D = \mathcal{P}_{\mathcal{E}_{\mathcal{U}}}$
where, $\mathcal{E}_{\mathcal{U}} = A \cup D = \{\xi^\varphi{}_1, \xi^\varphi{}_2, \xi^\varphi{}_5, \xi^\varphi{}_6\}$

$$\mathcal{P}_{\mathcal{E}_{\mathcal{U}}} = \begin{pmatrix} (0.54,0.18,0.41) & (0.78,0.16,0.41) & (0.54,0.25,0.31) & (0.63,0.24,0.28) \\ (0.81,0.11,0.18) & (0.72,0.30,0.11) & (0.45,0.18,0.23) & (0.91,0.11,0.12) \\ (0.72,0.20,0.17) & (0.61,0.13,0.48) & (0.90,0.12,0.11) & (0.58,0.13,0.35) \\ (0.89,0.10,0.12) & (0.54,0.31,0.38) & (0.52,0.31,0.41) & (0.63,0.24,0.18) \end{pmatrix}$$

$\mathcal{P}_A \cup_{\mathcal{R}} \mathcal{P}_D = \mathcal{P}_{\mathcal{R}_{\mathcal{U}}}$
where, $\mathcal{R}_{\mathcal{U}} = A \cap D = \{\xi^\varphi{}_1, \xi^\varphi{}_2, \xi^\varphi{}_5\}$

$$\mathcal{P}_{\mathcal{R}_{\mathcal{U}}} = \begin{pmatrix} (0.54,0.18,0.41) & (0.78,0.16,0.41) & (0.54,0.25,0.31) \\ (0.81,0.11,0.18) & (0.72,0.30,0.11) & (0.45,0.18,0.23) \\ (0.72,0.20,0.17) & (0.61,0.13,0.48) & (0.90,0.12,0.11) \\ (0.89,0.10,0.12) & (0.54,0.31,0.38) & (0.52,0.31,0.41) \end{pmatrix}$$

$\mathcal{P}_A \cap_{\mathcal{E}} \mathcal{P}_D = \mathcal{P}_{\mathcal{E}_I}$
where, $\mathcal{E}_I = A \cup D = \{\xi^\varphi{}_1, \xi^\varphi{}_2, \xi^\varphi{}_5, \xi^\varphi{}_6\}$

$$\mathcal{P}_{\mathcal{E}_I} = \begin{pmatrix} (0.25,0.18,0.56) & (0.62,0.16,0.54) & (0.36,0.25,0.78) & (0.63,0.24,0.28) \\ (0.78,0.11,0.18) & (0.72,0.30,0.19) & (0.38,0.18,0.36) & (0.91,0.11,0.12) \\ (0.39,0.20,0.25) & (0.52,0.13,0.48) & (0.72,0.12,0.24) & (0.58,0.13,0.35) \\ (0.89,0.10,0.24) & (0.45,0.31,0.57) & (0.48,0.31,0.46) & (0.63,0.24,0.18) \end{pmatrix}$$

$\mathcal{P}_A \cap_{\mathcal{R}} \mathcal{P}_D = \mathcal{P}_{\mathcal{R}_I}$
where, $\mathcal{R}_I = A \cap D = \{\xi^\varphi{}_1, \xi^\varphi{}_2, \xi^\varphi{}_5, \xi^\varphi{}_6\}$

$$\mathcal{P}_{\mathcal{R}_I} = \begin{pmatrix} (0.25,0.18,0.56) & (0.62,0.16,0.54) & (0.36,0.25,0.78) \\ (0.78,0.11,0.18) & (0.72,0.30,0.19) & (0.38,0.18,0.36) \\ (0.39,0.20,0.25) & (0.52,0.13,0.48) & (0.72,0.12,0.24) \\ (0.89,0.10,0.24) & (0.45,0.31,0.57) & (0.48,0.31,0.46) \end{pmatrix}$$

3. Picture fuzzy soft topology

PiFS-topology, an advanced subfield of fuzzy topology, has been introduced to enhance the modeling capabilities of fuzzy sets (FSs) for the representation of intricate spatial relationships. It combines the principles of PiFS, which facilitate the representation of data in a more detailed and flexible manner, with the concept of SS, which are designed to handle uncertain and incomplete information. By integrating PiFS and SS into the framework of topology, PiFS-topology provides a robust mathematical tool for the analysis of data across various domains, including but not limited to image processing, pattern recognition, geographical information systems, and decision-making processes that involve imprecise spatial information. In this section, we elucidate the concept of PiFS-topology by exploring the null set, absolute set, EU, and RI of PiFSS.

Definition 3.1. Let $\beta_\delta(M, E)$ denote the set of all PiFSSs in M, where the DNuM δ is fixed between 0 and 1. If A and B are subsets of E, then a subset $\widetilde{\tau}$ of $\beta_\delta(M, E)$ is called a PiFS-topology if it satisfies the following properties:

(i) $\phi_E, \breve{M}_E \,\widetilde{\in}\, \widetilde{\tau}$,

(ii) $\mathcal{P}_A, \mathcal{P}_B \widetilde{\in} \widetilde{\tau}$ then $\mathcal{P}_A \widetilde{\cap} \mathcal{P}_B \,\widetilde{\in}\, \widetilde{\tau}$,

(iii) If $\mathcal{P}_i \widetilde{\in} \widetilde{\tau}, \forall\ i \in I$, then $\widetilde{\cup}_{i \in I}\, \mathcal{P}_i \,\widetilde{\in}\, \widetilde{\tau}$.

The pair $(\tilde{M}_E, \widetilde{\tau})$, or simply $\tilde{M}_E$, is called a *PiFSS-topological space*.

Definition 3.2. Consider a topological space $(\tilde{M}_E, \widetilde{\tau})$, where the collection of sets in $\widetilde{\tau}$ are referred to as PiFS-open sets. The sets that are not in $\widetilde{\tau}$ but whose complements are in $\widetilde{\tau}$ are called PiFS-closed sets. Specifically, the complements of the PiFS-open sets are known as PiFS-closed sets.

Definition 3.3. Suppose we have a PiFS-topology denoted by $(\tilde{M}_E, \widetilde{\tau}_M)$. Let Y be a subset of M and $\breve{Y}_E$ be an absolute PiFSS in Y. In this case, we can define the PiFS-relative topology on Y as follows:

$$\widetilde{\tau}_Y = \left\{ \mathcal{P}_B : \mathcal{P}_B = \mathcal{P}_A \widetilde{\cap} \breve{Y}_E, \mathcal{P}_A \in \widetilde{\tau}_M \right\}.$$

That is, PiFS-open sets of PiFS-relative topology are $\mathcal{P}_B = \mathcal{P}_A \widetilde{\cap} \breve{Y}_E$, where $\mathcal{P}_A$ are PiFS-open sets of $\widetilde{\tau}_M$.

Example 3.4. Let $M = \{\alpha_1, \alpha_2, \alpha_3\}$ and $E = \{\xi^\varphi{}_i : i = 1, 2, 3, 4\}$. Take two sub-collections $A = \{\xi^\varphi{}_1, \xi^\varphi{}_2\}$ and $B = \{\xi^\varphi{}_1, \xi^\varphi{}_2, \xi^\varphi{}_3\}$ of E. Let δ denote the DNuM, which remains constant within the interval [0,1], (say) $\delta = 0.1$.

$$\begin{aligned}
\mathcal{P}^{(1)}_A &= \left\{ \left(\xi^\varphi{}_1, \left\{ \frac{(0.241, 0.1, 0.365)}{\alpha_1} \right\}, \left\{ \frac{(0.154, 0.1, 0.567)}{\alpha_2} \right\} \right), \left(\xi^\varphi{}_2, \left\{ \frac{(0.231, 0.1, 0.48)}{\alpha_2} \right\}, \left\{ \frac{(0.240, 0.1, 0.488)}{\alpha_3} \right\} \right) \right\}, \\
\mathcal{P}^{(2)}_B &= \left\{ \left(\xi^\varphi{}_1, \left\{ \frac{(0.337, 0.1, 0.331)}{\alpha_1} \right\}, \left\{ \frac{(0.620, 0.1, 0.107)}{\alpha_2} \right\} \right), \left(\xi^\varphi{}_2, \left\{ \frac{(0.238, 0.1, 0.132)}{\alpha_2} \right\}, \left\{ \frac{(0.287, 0.1, 0.478)}{\alpha_3} \right\} \right), \right. \\
&\qquad \left. \left(\xi^\varphi{}_3, \left\{ \frac{(0.235, 0.1, 0.184)}{\alpha_1} \right\}, \left\{ \frac{(0.113, 0.1, 0.016)}{\alpha_2} \right\} \right) \right\}.
\end{aligned}$$

Then,

$$\widetilde{\tau}_M = \left\{\phi_E,\ \check{M}_E,\ \mathcal{P}_A^{(1)},\ \mathcal{P}_B^{(2)}\right\}$$

is a PiFS-topology on M.

We consider an absolute PiFSS on $Y = \{\alpha_2, \alpha_3\} \subseteq M$ to be

$$\check{Y}_E = \left\{\left({\xi^{\varphi}}_i, \left\{\frac{(0.9, 0.1, 0)}{\alpha_2}\right\}, \left\{\frac{(0.9, 0.1, 0)}{\alpha_3}\right\}\right) : 1 \leq i \leq 4\right\}.$$

Since

$$\begin{aligned}
\check{Y}_E \widetilde{\cap} \phi_E &= \phi_E, \\
\check{Y}_E \widetilde{\cap} \mathcal{P}_A^{(1)} &= \left\{\left({\xi^{\varphi}}_1, \left\{\frac{(0.234, 0.100, 0.789)}{\alpha_2}\right\}\right), \left({\xi^{\varphi}}_2, \left\{\frac{(0.342, 0.100, 0.546)}{\alpha_2}\right\}, \left\{\frac{(0.351, 0.100, 0.569)}{\alpha_3}\right\}\right)\right\} \\
&= \mathcal{P}_A^{(1)}, \\
\check{Y}_E \widetilde{\cap} \mathcal{P}_B^{(2)} &= \left\{\left({\xi^{\varphi}}_1, \left\{\frac{(0.731, 0.100, 0.218)}{\alpha_2}\right\}\right), \left({\xi^{\varphi}}_2, \left\{\frac{(0.349, 0.100, 0.242)}{\alpha_2}\right\}, \left\{\frac{(0.398, 0.100, 0.321)}{\alpha_3}\right\}\right),\right. \\
&\quad \left.\left({\xi^{\varphi}}_3, \left\{\frac{(0.323, 0.100, 0.216)}{\alpha_2}\right\}\right)\right\} \\
&= \mathcal{P}_B^{(2)}, \\
\check{Y}_E \widetilde{\cap} \check{M}_E &= \check{Y}_E
\end{aligned}$$

so

$$\widetilde{\tau}_Y = \left\{\phi_E,\ \mathcal{P}_A^{(1)},\ \mathcal{P}_B^{(2)},\ \check{Y}_E\right\}.$$

is a PiFS-relative topology of $\widetilde{\tau}_M$.

Definition 3.5. Let $(\tilde{M}_E, \tilde{\tau})$ be a PiFS-topological space and $\mathcal{P}_A \widetilde{\subseteq} \tilde{X}_E$.

(1) PiFS-interior:
The *interior* $\mathcal{P}_A^{\circ}$ of $\mathcal{P}_A$ is an EU of all PiFS-open subsets of $\mathcal{P}_A$. Note that $\mathcal{P}_A^{\circ}$ is the largest PiFS-open subset of $\mathcal{P}_A$.

(2) PiFS-closure:
The *closure* $\overline{\mathcal{P}_A}$ of $\mathcal{P}_A$ is an RI of all PiFS-closed supersets of $\mathcal{P}_A$. Note that $\overline{\mathcal{P}_A}$ is the smallest PiFS-closed superset of $\mathcal{P}_A$.

(3) PiFS-frontier:
The boundary or frontier, denoted as $Fr(\mathcal{P}_A)$, of the set $\mathcal{P}_A$ is characterized as follows:

$$Fr(\mathcal{P}_A) = \overline{\mathcal{P}_A} \,\widetilde{\cap}\, \overline{P_A^c}.$$

(4) PiFSS exterior:
The *exterior* $Ext(\mathcal{P}_A)$ of $\mathcal{P}_A$ is defined as

$$Ext(\mathcal{P}_A) = (\mathcal{P}_A^c)^{\circ}.$$

In the forthcoming Example 3.6, we exemplify the fundamental notions of the interior, exterior, closure and frontier of PiFSS.

Example 3.6. Let $M = \{\alpha_1, \alpha_2, \alpha_3\}$ be any crisp set and $E = \{\xi^{\varphi}{}_1, \xi^{\varphi}{}_2\}$ be the set of attributes. We will examine subsets of $\tilde{M}_E$, with a fixed value of δ, which we will set to be $\delta = 0.100$.

$$\begin{aligned}
\mathcal{P}_E^{(1)} &= \Big\{\Big(\xi^{\varphi}{}_1, \Big\{\frac{(0.32, 0.10, 0.12)}{\alpha_1}\Big\}, \Big\{\frac{(0.13, 0.10, 0.28)}{\alpha_2}\Big\}, \Big\{\frac{(0.32, 0.10, 0.22)}{\alpha_3}\Big\}\Big), \\
&\quad \Big(\xi^{\varphi}{}_2, \Big\{\frac{(0.23, 0.10, 0.21)}{\alpha_1}\Big\}, \Big\{\frac{(0.38, 0.10, 0.12)}{\alpha_2}\Big\}, \Big\{\frac{(0.29, 0.10, 0.24)}{\alpha_3}\Big\}\Big)\Big\}, \\
\mathcal{P}_E^{(2)} &= \Big\{\Big(\xi^{\varphi}{}_1, \Big\{\frac{(0.42, 0.10, 0.11)}{\alpha_1}\Big\}, \Big\{\frac{(0.34, 0.10, 0.22)}{\alpha_2}\Big\}, \Big\{\frac{(0.51, 0.10, 0.16)}{\alpha_3}\Big\}\Big), \\
&\quad \Big(\xi^{\varphi}{}_2, \Big\{\frac{(0.34, 0.10, 0.11)}{\alpha_1}\Big\}, \Big\{\frac{(0.48, 0.10, 0.32)}{\alpha_2}\Big\}, \Big\{\frac{(0.30, 0.10, 0.09)}{\alpha_3}\Big\}\Big)\Big\}, \\
\mathcal{P}_E^{(3)} &= \Big\{\Big(\xi^{\varphi}{}_1, \Big\{\frac{(0.51, 0.10, 0.01)}{\alpha_1}\Big\}, \Big\{\frac{(0.41, 0.10, 0.12)}{\alpha_2}\Big\}, \Big\{\frac{(0.51, 0.10, 0.05)}{\alpha_3}\Big\}\Big), \\
&\quad \Big(\xi^{\varphi}{}_2, \Big\{\frac{(0.42, 0.10, 0.10)}{\alpha_1}\Big\}, \Big\{\frac{(0.61, 0.10, 0.25)}{\alpha_2}\Big\}, \Big\{\frac{(0.40, 0.10, 0.07)}{\alpha_3}\Big\}\Big)\Big\}.
\end{aligned}$$

Then, by Definition 3.1, the collection $\tilde{\tau} = \Big\{\phi_E, \breve{M}_E, \mathcal{P}_E^{(1)}, \mathcal{P}_E^{(2)}, \mathcal{P}_E^{(3)}\Big\}$ is a PiFS-topology. Consider a PiFSS $\mathcal{P}_E$ given by

$$\begin{aligned}
\mathcal{P}_E &= \Big\{\Big(\xi^{\varphi}{}_1, \Big\{\frac{(0.44, 0.10, 0.02)}{\alpha_1}\Big\}, \Big\{\frac{(0.36, 0.10, 0.21)}{\alpha_2}\Big\}, \Big\{\frac{(0.52, 0.10, 0.14)}{\alpha_3}\Big\}\Big), \\
&= \Big(\xi^{\varphi}{}_2, \Big\{\frac{(0.44, 0.10, 0.09)}{\alpha_1}\Big\}, \Big\{\frac{(0.50, 0.10, 0.1)}{\alpha_2}\Big\}, \Big\{\frac{(0.40, 0.10, 0.06)}{\alpha_3}\Big\}\Big)\Big\}.
\end{aligned}$$

(i) PiFS-interior of $\mathcal{P}_E$:

The members of $\tilde{\tau}$ are obviously PiFS open sets and ϕ_E , $\mathcal{P}_E^{(1)}$ and $\mathcal{P}_E^{(2)}$ are the open subsets of $\mathcal{P}_E$. So,

$$\begin{aligned}
\mathcal{P}_E^{\circ} &= \phi_E \,\tilde{\cup}\, \mathcal{P}_E^{(1)} \,\tilde{\cup}\, \mathcal{P}_E^{(2)} \\
&= \mathcal{P}_E^{(2)}.
\end{aligned}$$

(ii) PiFS-closure of $\mathcal{P}_E$:

To ascertain the closure of $\mathcal{P}_E$, one needs to examine the closed PiFSSs that are associated with it.

$$\begin{aligned}
\big(\phi_E\big)^c &= \breve{M}_E \\
\big(\breve{M}_E\big)^c &= \phi_E \\
(\mathcal{P}_E^{(1)})^c &= \Big\{\Big(\xi^{\varphi}{}_1, \Big\{\frac{(0.12, 0.10, 0.32)}{\alpha_1}\Big\}, \Big\{\frac{(0.28, 0.10, 0.13)}{\alpha_2}\Big\}, \Big\{\frac{(0.22, 0.10, 0.32)}{\alpha_3}\Big\}\Big),
\end{aligned}$$

$$\left(\xi^{\varphi}{}_2,\left\{\frac{(0.21,0.10,0.23)}{\alpha_1}\right\},\left\{\frac{(0.12,0.10,0.38)}{\alpha_2}\right\},\left\{\frac{(0.29,0.10,0.24)}{\alpha_3}\right\}\right)\Bigg\},$$

$$(\mathcal{P}^{(2)}_E)^c = \Bigg\{\left(\xi^{\varphi}{}_1,\left\{\frac{(0.11,0.10,0.42)}{\alpha_1}\right\},\left\{\frac{(0.22,0.10,0.34)}{\alpha_2}\right\},\left\{\frac{(0.16,0.10,0.51)}{\alpha_3}\right\}\right),$$
$$\left(\xi^{\varphi}{}_2,\left\{\frac{(0.11,0.10,0.34)}{\alpha_1}\right\},\left\{\frac{(0.32,0.10,0.48)}{\alpha_2}\right\},\left\{\frac{(0.09,0.10,0.30)}{\alpha_3}\right\}\right)\Bigg\},$$

$$(\mathcal{P}^{(3)}_E)^c = \Bigg\{\left(\xi^{\varphi}{}_1,\left\{\frac{(0.01,0.10,0.51)}{\alpha_1}\right\},\left\{\frac{(0.12,0.10,0.41)}{\alpha_2}\right\},\left\{\frac{(0.05,0.10,0.51)}{\alpha_3}\right\}\right),$$
$$\left(\xi^{\varphi}{}_2,\left\{\frac{(0.10,0.10,0.42)}{\alpha_1}\right\},\left\{\frac{(0.25,0.10,0.61)}{\alpha_2}\right\},\left\{\frac{(0.07,0.10,0.40)}{\alpha_3}\right\}\right)\Bigg\}.$$

The closed superset $\breve{M}_E$ is uniquely determined as the smallest set that contains the set $\mathcal{P}_E$. So,

$$\overline{\mathcal{P}_E} = \breve{M}_E.$$

(iii) PiFS-frontier of $\mathcal{P}_E$:
For the purpose of finding a frontier of $\mathcal{P}_E$, we need

$$(\mathcal{P}_E)^c = \Bigg\{\left(\xi^{\varphi}{}_1,\left\{\frac{(0.02,0.10,0.44)}{\alpha_1}\right\},\left\{\frac{(0.21,0.10,0.36)}{\alpha_2}\right\},\left\{\frac{(0.14,0.10,0.52)}{\alpha_3}\right\}\right),$$
$$= \left(\xi^{\varphi}{}_2,\left\{\frac{(0.09,0.10,0.44)}{\alpha_1}\right\},\left\{\frac{(0.1,0.10,0.50)}{\alpha_2}\right\},\left\{\frac{(0.06,0.10,0.40)}{\alpha_3}\right\}\right)\Bigg\}.$$

where $\breve{M}_E$ denote the unique closed superset that contains the complement of the set $\mathcal{P}_E$

$$\begin{aligned}\overline{\mathcal{P}^c_E} &= \breve{M}_E\\ Fr(\mathcal{P}_E) &= \overline{\mathcal{P}_E}\,\tilde{\cap}\,\overline{\mathcal{P}^c_E}\\ &= \breve{M}_E\,\tilde{\cap}\,\breve{M}_E\\ &= \breve{M}_E.\end{aligned}$$

(iv) PiFS-exterior of $\mathcal{P}_E$:
ϕ_E is the only open subset of $(\mathcal{P}_E)^c$. Thus, the $Int(\mathcal{P}^c_E)$ is ϕ_E.

$$\begin{aligned}Ext(\mathcal{P}_E) &= (\mathcal{P}^c_E)^\circ\\ &= \phi_E.\end{aligned}$$

Theorem 3.7. Let $(\tilde{M}_E, \tilde{\tau})$ be a picture fuzzy soft topological space over M, and $\mathcal{P}_A, \mathcal{P}_B$ are PiFSSs in M. Then

(1) $(\phi_E)^\circ = \phi_E$ and $(\breve{M}_E)^\circ = \breve{M}_E$.

(2) $(\mathcal{P}_A)^\circ \subseteq \mathcal{P}_A$.

(3) A is a PiFS-open set $\Leftrightarrow \mathcal{P}_A = (\mathcal{P}_A)^\circ$.

(4) $((\mathcal{P}_A)^\circ)^\circ = (\mathcal{P}_A)^\circ$.

(5) $\mathcal{P}_A \subseteq \mathcal{P}_B \Rightarrow (\mathcal{P}_A)^\circ \subseteq (\mathcal{P}_B)^\circ$.

(6) $(\mathcal{P}_A)^\circ \cup (\mathcal{P}_B)^\circ \subseteq (\mathcal{P}_A \cup \mathcal{P}_B)^\circ$.

(7) $(\mathcal{P}_A \cap \mathcal{P}_B)^\circ = (\mathcal{P}_A)^\circ \cap (\mathcal{P}_B)^\circ$.

Proof.

(1) It is obvious by Definition 3.5.

(2) It is obvious by Definition 3.5.

(3) If $\mathcal{P}_A$ is a PiFSS open set in M, then $\mathcal{P}_A$ is itself a PiFSS open set in M which contains $\mathcal{P}_A$. So, $\mathcal{P}_A$ itself is the largest PiFSS open set contained in $\mathcal{P}_A$ and $int(\mathcal{P}_A) = \mathcal{P}_A$. Conversely, suppose that $(\mathcal{P}_A)^\circ = \mathcal{P}_A$. Since $(\mathcal{P}_A)^\circ$ is always PiFSS open, to $\mathcal{P}_A$ must be PiFSS open.

(4) Let $(\mathcal{P}_A)^\circ = \mathcal{P}_B$. Then, $(\mathcal{P}_B)^\circ = \mathcal{P}_B$ from (3), and then $((\mathcal{P}_A)^\circ)^\circ = (\mathcal{P}_A)^\circ$.

(5) Consider $\mathcal{P}_A \subseteq \mathcal{P}_B$, as $(\mathcal{P}_A)^\circ \subseteq \mathcal{P}_A \subseteq \mathcal{P}_B$, $(\mathcal{P}_A)^\circ$ is a PiFSS open subset of $\mathcal{P}_B$, then by the definition we have that $(\mathcal{P}_A)^\circ \subseteq (\mathcal{P}_B)^\circ$.

(6) It is clear that $\mathcal{P}_A \subseteq \mathcal{P}_A \cup \mathcal{P}_B$ and $\mathcal{P}_B \subseteq \mathcal{P}_A \cup \mathcal{P}_B$. Thus, $(\mathcal{P}_A)^\circ \subseteq (\mathcal{P}_A \cup \mathcal{P}_B)^\circ$ and $(\mathcal{P}_B)^\circ \subseteq (\mathcal{P}_A \cup \mathcal{P}_B)^\circ$. So, we have that $(\mathcal{P}_A)^\circ \cup (\mathcal{P}_B)^\circ \subseteq (\mathcal{P}_A \cup \mathcal{P}_B)^\circ$ using 5.

(7) It is known that $(\mathcal{P}_A \cap \mathcal{P}_B)^\circ \subseteq (\mathcal{P}_A)^\circ$ and $(\mathcal{P}_A \cap \mathcal{P}_B)^\circ \subseteq (\mathcal{P}_B)^\circ$ by 5 so that $(\mathcal{P}_A \cap \mathcal{P}_B)^\circ \subseteq (\mathcal{P}_A)^\circ \cap (\mathcal{P}_B)^\circ$. Also, from $(\mathcal{P}_A)^\circ \subseteq \mathcal{P}_A$ and $(\mathcal{P}_B)^\circ \subseteq \mathcal{P}_B$, we have $(\mathcal{P}_A)^\circ \cap (\mathcal{P}_B)^\circ \subseteq \mathcal{P}_A \cap \mathcal{P}_B$. These imply that $(\mathcal{P}_A \cap \mathcal{P}_B)^\circ = (\mathcal{P}_A)^\circ \cap (\mathcal{P}_B)^\circ$.

□

Theorem 3.8. Let $(\tilde{M}_E, \tilde{\tau})$ be a PiFSS topological space over X and $\mathcal{P}_A, \mathcal{P}_B$ are PiFSSs in M. Then,

(1) $\overline{(\phi_E)} = \phi_E$ and $\overline{(\check{M}_E)} = \check{M}_E$,

(2) $\mathcal{P}_A \subseteq \overline{(\mathcal{P}_A)}$,

(3) A is a PiFSS closed set $\Leftrightarrow \mathcal{P}_A = \overline{(\mathcal{P}_A)}$,

(4) $\overline{(\overline{(\mathcal{P}_A)})} = \overline{(\mathcal{P}_A)}$,

(5) $\mathcal{P}_A \subseteq \mathcal{P}_B \Rightarrow \overline{(\mathcal{P}_A)} \subseteq \overline{(\mathcal{P}_B)}$,

(6) $\overline{(\mathcal{P}_A)} \cup (\mathcal{P}_B)^\circ = \overline{(\mathcal{P}_A)} \cup \overline{(\mathcal{P}_B)}$,

(7) $\overline{(\mathcal{P}_A \cap \mathcal{P}_B)} \subseteq \overline{(\mathcal{P}_A)} \cap \overline{(\mathcal{P}_B)}$.

Proof. The proof is obvious by Definition 3.5. □

Theorem 3.9. *Let $(\tilde{M}_E, \tilde{\tau})$ be a PiFS-topological space and $\mathcal{P}_A \widetilde{\subseteq} \tilde{M}_E$, then*

(1) $(\mathcal{P}^\circ_A)^c = \overline{(\mathcal{P}^c_A)}$, *and*

(2) $(\overline{\mathcal{P}_A})^c = (\mathcal{P}_A^c)^\circ$.

Theorem 3.10. *Let $(\tilde{M}_E, \tilde{\tau})$ be a PiFS-topological space and $\mathcal{P}_A \widetilde{\subseteq} \tilde{M}_E$, then $Fr(\mathcal{P}_A) = Fr(\mathcal{P}_A^c)$.*

Proof. By Definition 3.5, we see that

$$\begin{aligned} Fr(\mathcal{P}_A) &= \overline{(\mathcal{P}_A)} \,\widetilde{\cap}\, \overline{(\mathcal{P}_A^c)} \\ &= \overline{(\mathcal{P}_A^c)} \,\widetilde{\cap}\, \overline{(\mathcal{P}_A)} \\ &= \overline{(\mathcal{P}_A^c)} \,\widetilde{\cap}\, \overline{[(\mathcal{P}_A^c)]^c} \\ &= Fr(\mathcal{P}_A^c). \end{aligned}$$

□

Remark. The intersection of two PiFS-topological spaces is always a PiFS-topological space, but their union need not be so. The counter example of the result is given below.

Example 3.11. Let $M = \{\alpha_1, \alpha_2\}$ and $E = \{\xi^{\varphi}{}_1, \xi^{\varphi}{}_2, \xi^{\varphi}{}_3, \xi^{\varphi}{}_4\}$. Let $A = \{\xi^{\varphi}{}_1, \xi^{\varphi}{}_2\}, B = \{\xi^{\varphi}{}_3, \xi^{\varphi}{}_4\} \subseteq E$ and δ as a fixed value in $[0, 1]$, (say) $\delta = 0.100$. Consider PiFSSs, given as

$$\begin{aligned} \mathcal{P}_A &= \Bigg\{\Bigg(\xi^{\varphi}{}_1, \Big\{\frac{(0.400, 0.100, 0.300)}{\alpha_1}\Big\}, \Big\{\frac{(0.189, 0.100, 0.342)}{\alpha_2}\Big\}\Bigg), \\ &\quad \Bigg(\xi^{\varphi}{}_2, \Big\{\frac{(0.291, 0.100, 0.169)}{\alpha_1}\Big\}, \Big\{\frac{(0.300, 0.100, 0.269)}{\alpha_2}\Big\}\Bigg)\Bigg\} \\ \mathcal{P}_B &= \Bigg\{\Bigg(\xi^{\varphi}{}_3, \Big\{\frac{(0.189, 0.100, 0.231)}{\alpha_1}\Big\}, \Big\{\frac{(0.401, 0.100, 0.189)}{\alpha_2}\Big\}\Bigg), \\ &\quad \Bigg(\xi^{\varphi}{}_4, \Big\{\frac{(0.845, 0.100, 0.207)}{\alpha_1}\Big\}, \Big\{\frac{(0.784, 0.100, 0.200)}{\alpha_2}\Big\}\Bigg)\Bigg\} \end{aligned}$$

then

$$\widetilde{\tau}_1 = \{\phi_E,\ \mathcal{P}_A,\ \breve{M}_E\},$$

$$\widetilde{\tau}_2 = \{\phi_E,\ \mathcal{P}_B,\ \breve{M}_E\}$$

are PiFS-topologies on M, but

$$\widetilde{\tau}_1 \,\widetilde{\cup}\, \widetilde{\tau}_2 = \{\phi_E,\ \breve{M}_E,\ \mathcal{P}_A,\ \mathcal{P}_B\}$$

is not so.

Example 3.12. By Example 3.6, it can be seen that

$$\begin{aligned} \tilde{\tau}_1 &= \{\phi_E,\ \mathcal{P}_E^{(1)},\ \breve{M}_E\}, \\ \tilde{\tau}_2 &= \{\phi_E,\ \mathcal{P}_E^{(1)},\ \mathcal{P}_E^{(2)},\ \mathcal{P}_E^{(3)},\ \breve{M}_E\} \end{aligned}$$

are two PiFS-topologies in M. Since $\tilde{\tau}_1 \,\tilde{\subset}\, \tilde{\tau}_2$, $\tilde{\tau}_1$ is coarser than $\tilde{\tau}_2$. Then, $\tilde{\tau}_2$ is called finer than $\tilde{\tau}_1$.

Example 3.13. Let $M = \{\alpha_1, \alpha_2, \alpha_3\}$ be the universe of discourse and $E = \{\xi^{\varphi}{}_i : i = 1,2,3,4\}$ be the group of attributes with $A = \{\xi^{\varphi}{}_1, \xi^{\varphi}{}_2\} \subseteq E$, and take $\delta = 0.1$. Assume that

$$\begin{aligned}\mathcal{P}_A \;=\; & \Big\{\Big(\xi^{\varphi}{}_1, \Big\{\frac{(0.309, 0.100, 0.469)}{\alpha_1}\Big\}, \Big\{\frac{(0.718, 0.100, 0.367)}{\alpha_2}\Big\}\Big), \\ & \Big(\xi^{\varphi}{}_2, \Big\{\frac{(0.321, 0.100, 0.348)}{\alpha_1}\Big\}, \Big\{\frac{(0.691, 0.100, 0.278)}{\alpha_2}\Big\}\Big)\Big\}\end{aligned}$$

Clearly,

$$\widetilde{\tau} = \{\phi_E,\ \breve{M}_E,\ \mathcal{P}_A,\ P^c_A\}$$

fails to be a PiFS-topology in M for neither $\mathcal{P}_A \widetilde{\cap} P^c_A \widetilde{\in} \widetilde{\tau}$ nor $\mathcal{P}_A \widetilde{\cup} P^c_A \widetilde{\in} \widetilde{\tau}$.

Definition 3.14. Let $(M_E, \widetilde{\tau})$ be a PiFSS-topological space. Then, $\mathbb{B} \widetilde{\subseteq} \widetilde{\tau}$ is a *PiFSS-basis* for $\widetilde{\tau}$ if, each $\mathcal{P}_A \widetilde{\in} \widetilde{\tau}$ is a PiFSS union of members of PiFS-topology, that is, $\mathcal{P}_A = \widetilde{\cup}\mathcal{B}$.

Example 3.15. From Example 1, the collection

$$\mathbb{B} = \left\{P^{(1)}_A, P^{(2)}_A, P^{(3)}_A, \breve{M}_E\right\}$$

is a PiFSS-basis for the PiFS-topology

$$\widetilde{\tau} = \left\{\phi_E, \breve{M}_E, P^{(1)}_A, P^{(2)}_A, P^{(3)}_A\right\}.$$

4. Picture fuzzy Soft CRITIC-CoCoSo technique

Let M be a set consisting of elements $\{\alpha_1, \alpha_2, \ldots, \alpha_m\}$, and let E be a set containing elements $\{\xi^{\varphi}{}_1, \xi^{\varphi}{}_2, \ldots, \xi^{\varphi}{}_n\}$. These sets represent a collection of alternatives and parameters, respectively. The weight vector (WV) is defined as $W = \{w_1, w_2, \ldots, w_n\}$, where each w_j is a real number satisfying the range $w_j \in [0,1]$ and the condition $\sum_{j=1}^{n} w_j = 1$. Let the representation of the preference value of alternative α_i concerning parameter $\xi^{\varphi}{}_j$ be designated as a PiFSN $F\left(\xi^{\varphi}{}_j\right)(\alpha_i) = \left(\mu_F\left(\xi^{\varphi}{}_j\right)(\alpha_i), \delta_F\left(\xi^{\varphi}{}_j\right)(\alpha_i), \zeta_F\left(\xi^{\varphi}{}_j\right)(\alpha_i)\right)$, which can be displayed in Table 5.

Table 5. Preference values.

$\mathcal{P}_A$	$\xi^{\varphi}{}_1$	$\xi^{\varphi}{}_2$	$\cdots$	$\xi^{\varphi}{}_n$
α_1	$F(\xi^{\varphi}{}_1)(\alpha_1)$	$F(\xi^{\varphi}{}_2)(\alpha_1)$	$\cdots$	$F(\xi^{\varphi}{}_n)(\alpha_1)$
α_2	$F(\xi^{\varphi}{}_1)(\alpha_2)$	$F(\xi^{\varphi}{}_2)(\alpha_2)$	$\cdots$	$F(\xi^{\varphi}{}_n)(\alpha_2)$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
α_m	$F(\xi^{\varphi}{}_1)(\alpha_m)$	$F(\xi^{\varphi}{}_2)(\alpha_m)$	$\cdots$	$F(\xi^{\varphi}{}_n)(\alpha_m)$

The evaluation standards can provide valuable insights into decision-making matters. The weightage assigned to each standard signifies the degree of significance it holds, known as "objective weights". The CRITIC technique, introduced by Diakoulaki et al. [62], is used to determine the objective weight of a particular criterion in an MCDM problem. This approach considers both the relative strength of each criterion and the possible conflicts between them. In order to tackle the preference information represented by PiFSN, we employ this methodology within a framework characterized by a PiFS environment, as elucidated subsequently. Let us consider a scenario where $F\left(\xi^{\varphi}{}_{j}\right)(\alpha_i)$ (where i ranges from 1 to m and j ranges from 1 to n) represents the PiFSN of the ith choice relative to the jth parameter. Furthermore, let w_j^o indicate the fuzzy objective weight of the jth parameter, with C representing a group of cost parameters and B representing a group of benefit parameters. The steps for calculating PiFS objective weights using CRITIC are provided below. The flowchart of the proposed technique is shown in Figure 1.

Algorithm: (CRITIC Technique)

Step 1:
Determine score function $F = \left(\gamma_{ij}\right)_{m\times n}$ of each $F\left(\xi^{\varphi}{}_{j}\right)(\alpha_i) = \left(\mu_F\left(\xi^{\varphi}{}_{j}\right)(\alpha_i), \delta_F\left(\xi^{\varphi}{}_{j}\right)(\alpha_i), \zeta_F\left(\xi^{\varphi}{}_{j}\right)(\alpha_i)\right)$ by Eq1.

Step 2:
Transform the score matrix F into a standard picture fuzzy soft matrix $F' = \left(\gamma'_{ij}\right)_{m\times n}$ by Eq 2.

$$\gamma'_{ij} = \begin{cases} \frac{\gamma_{ij}-\gamma_j^-}{\gamma_j^+-\gamma_j^-}, \text{if } j \in B, \\ \frac{\gamma_j^+-\gamma_{ij}}{\gamma_j^+-\gamma_j^-}, \text{if } j \in C, \end{cases} \tag{2}$$

where $\gamma_j^- = \min_i \gamma_{ij}$ and $\gamma_j^+ = \max_i \gamma_{ij}$.

Step 3:
Calculate the standard deviations of the criteria by utilizing Eq 3.

$$\sigma_j = \sqrt{\frac{\sum_{i=1}^m \left(\gamma'_{ij} - \bar{\gamma}_j\right)^2}{m}} \tag{3}$$

where $\bar{\gamma}_j = \frac{\sum_{i=1}^m \gamma'_{ij}}{m}$.

Step 4:
Determine the correlation between pairs of criteria by employing Eq 4 for computational analysis.

$$\rho_{jk} = \frac{\sum_{i=1}^m \left(\gamma'_{ij} - \bar{\gamma}_j\right)\left(\gamma'_{ik} - \bar{\gamma}_k\right)}{\sqrt{\sum_{i=1}^m \left(\gamma'_{ij} - \bar{\gamma}_j\right)^2 \sum_{i=1}^m \left(\gamma'_{ik} - \bar{\gamma}_k\right)^2}}. \tag{4}$$

Step 5:
Determine the quantity of information of each criterion as follows:

$$c_j = \sigma_j \sum_{k=1}^n \left(1 - \rho_{jk}\right). \tag{5}$$

Step 6:
Calculate the objective weight of each criterion,

$$\omega_j = \frac{c_j}{\sum_{j=1}^n c_j}. \tag{6}$$

Step 7:

Calcuation of combined weights: linear weighted comprehensive method

Let it be assumed that the subjective weight assigned by decision makers is denoted as $w = \{w_1, w_2, \ldots, w_n\}$, where $\sum_{j=1}^{n} w_j = 1$ and $0 \leq w_j \leq 1$. The objective weight, determined through the computation defined in Eq 11, is represented as $\omega = \{\omega_1, \omega_2, \ldots, \omega_n\}$, with $\sum_{j=1}^{n} \omega_j = 1$ and $0 \leq \omega_j \leq 1$. Consequently, the composite weight $\varpi = \{\varpi_1, \varpi_2, \ldots, \varpi_n\}$ can be formally defined as follows:

$$\varpi_j = \frac{w_j * \omega_j}{\sum_{j=1}^{n} w_j * \omega_j}. \tag{7}$$

The PiFS-CoCoSo method, is a novel MCDM approach devised by Yazdani et al. in 2019. This method employs an integrated EWP (Extenics Weighted Product) and SAW (Simple Additive Weighting) model, providing a collection of compromise solutions. To address MCDM problems, we propose the PiFS-CoCoSo approach, which can be described in the following manner:

Algorithm: (CRITIC-CoCoSo)

Step 1:

Attain the provided decision matrix $G = \left(G_{ij}\right)_{m\times n}$ utilizing linguistic terms outlined in Table 7.

Step 2:

Convert the linguistic matrix into a PiFSS denoted as (F, A), as illustrated in Table 8.

Step 3:

Calculate the score function $F = \left(\gamma_{ij}\right)_{m\times n}$ of each PiFSN $F\left(\xi^{\varphi}{}_{j}\right)(\alpha_i) = (\mu_F\left(\xi^{\varphi}{}_{j}\right)(\alpha_i), \delta_F\left(\xi^{\varphi}{}_{j}\right)(\alpha_i),$
$\zeta_F\left(\xi^{\varphi}{}_{j}\right)(\alpha_i))$ by Eq 1.

Step 4:

Transform the matrix $F = \left(\gamma_{ij}\right)_{m\times n}$ into a standard picture fuzzy soft matrix $F' = \left(\gamma'_{ij}\right)_{m\times n}$ by Eq 2.

Step 5:

Calculate the standard deviations of the criteria using Eq 3.

Step 6:

Determine the correlation between pairs of criteria by utilizing Eq 4 for calculation.

Step 7:

Calculate the informational content of each criterion using Eq 5.

Step 8:

Determine the weight assigned to each criterion in accordance with Eq 6 in order to calculate the objective weight.,

Step 9:

Determine the aggregate weight ϖ using Eq 7.

Step 10:

Determine the comprehensive sequence of weighted comparability, denoted as S_i, through mathematical calculation.

$$S_i = \sum_{j=1}^{n} \varpi_j * \gamma'_{ij}. \tag{8}$$

Step 11:

Determine the total power weight of the comparability sequence as denoted by the variable P_i.

$$P_i = \sum_{j=1}^{n} \left(\gamma'_{ij}\right)^{\varpi_j}. \tag{9}$$

Step 12:

Three methods of appraisal score strategies are employed to ascertain the relative weights of alternative options according to Eqs (15)–(17).

$$k_{ia} = \frac{P_i + S_i}{\sum_{i=1}^{m}(P_i + S_i)}, \tag{10}$$

$$k_{ib} = \frac{S_i}{\min_i S_i} + \frac{P_i}{\min_i P_i}, \tag{11}$$

$$k_{ic} = \frac{\lambda S_i + (1-\lambda)P_i}{\lambda \max_i S_i + (1-\lambda)\max_i P_i}, 0 \le \lambda \le 1 \tag{12}$$

Step 13:

Determine the valuation k_i through the utilization of Eq 13 .

$$k_i = \sqrt[3]{k_{ia}k_{ib}k_{ic}} + \frac{k_{ia} + k_{ib} + k_{ic}}{3}. \tag{13}$$

Step 14:

Arrange the alternatives based on their evaluated value, denoted as k_i (where i ranges from 1 to m).

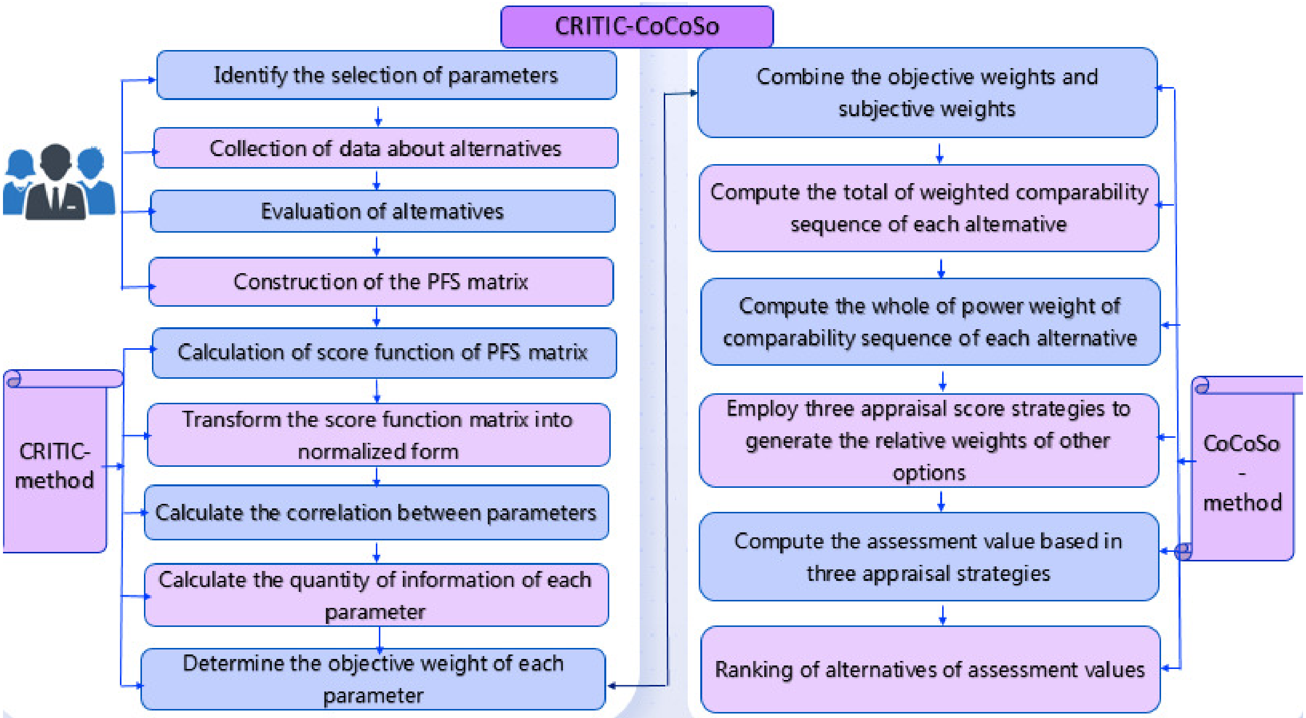

Figure 1. Flowchart of proposed technique.

4.1. Case study

The SSS has become an essential aspect of business operations in the era of CE and Industry 4.0. With the increasing emphasis on reducing waste, conserving resources and mitigating environmental impact, companies must carefully evaluate their suppliers to ensure they align with their sustainability goals. In the CE, suppliers must adopt a closed-loop approach to product design, production and disposal. This involves creating products that can be recycled, reused or repurposed, and minimizing waste in the production process. Therefore, companies must choose suppliers that prioritize sustainable practices, such as using renewable energy sources, minimizing water usage and reducing carbon emissions. In the era of Industry 4.0, suppliers must also embrace digitalization and automation to improve efficiency and reduce waste. This involves using advanced technologies such as artificial intelligence, internet of things and robotics to optimize production processes and reduce energy consumption. Companies must therefore choose suppliers that have embraced these technologies and have a track record of implementing sustainable practices in their operations. SSS is a critical component of CE and Industry 4.0 strategies. Through meticulous assessment of a supplier's sustainability performance, organizations have the opportunity to diminish their ecological footprint, bolster their brand standing and actively participate in fostering a more sustainable trajectory for the future. To ensure SSS, companies can use a variety of tools and frameworks, such as the sustainability assessment framework (SAF) and the global reporting initiative (GRI). These frameworks provide a standardized approach to evaluating suppliers based on their environmental, social and governance (ESG) performance. Companies can also collaborate with industry associations and non-governmental organizations (NGOs) to identify suppliers that have demonstrated a commitment to sustainability. Sustainable supplier selection is crucial for companies aiming to operate in the CE and Industry 4.0 era. Companies are expected to select suppliers that comply with environmental, social, and economic standards to achieve sustainable operations. To address this challenge, an MCDM framework was proposed to select sustainable suppliers based on their performance in various criteria. A company in the automotive industry, aiming to operate in the CE and Industry 4.0 era, used the proposed MCDM framework to select sustainable suppliers. The company identified 20 potential suppliers and selected the top five suppliers based on their performance in the selected criteria. The results of the MCDM framework showed that environmental performance was the most important criterion followed by social responsibility, economic viability, innovation, and digitalization. The top five suppliers were selected based on their overall performance in the selected criteria. Industry 4.0 is transforming the way manufacturers operate, and CE is becoming increasingly important. The CE is a business model that emphasizes the reuse, repair and recycling of resources, rather than the traditional model of extracting, producing and disposing of them. In this context, circular supplier selection is critical to ensuring that businesses operate sustainably. This case study explores how the CRITIC-CoCoSo technique was applied in Industry 4.0. The CRITIC-CoCoSo technique is an MCDM approach that combines the CRITIC (criteria importance through inter criteria correlation) method and the CoCoSo (combined compromise solution) method. It is a powerful tool for supplier selection that can help businesses make informed decisions by considering multiple criteria simultaneously. The CRITIC-CoCoSo technique was applied to a case study of a manufacturing company that was looking to select suppliers for its CE initiatives. The company had identified several potential suppliers, but it needed a way to evaluate them objectively based on multiple criteria. The first step in applying the CRITIC-CoCoSo technique was to identify the criteria that would be used to evaluate the suppliers. The company identified the following criteria: product quality, environmental impact, social responsibility, innovation, and cost.

The subsequent phase involved ascertaining the comparative significance of each criterion. This was done using the CRITIC method, which considers the intercriteria correlations between the criteria. The results showed that product quality was the most important criterion, followed by environmental impact, social responsibility, innovation, and cost.

Table 6. Types of criteria and sub-criteria.

Criteria	Sub-Criteria	References	Criteria Type
Delivery	Delivery reliability	Cavalcante et al. [63]	Benefit
	The precision of delivery	Expert Feedback	Benefit
	Network reliability	Parkouhi & Ghadikolaei [64]	Benefit
Quality	Rate of rejection	Feng & Gong [65]	Benefit
	Quality assurance	Hou & Xie [66]	Benefit
	Quality control	Hou & Xie [66]	Benefit
Environmental	Sustainable packaging	Mina et al. [67]	Benefit
	Environmental standards	Govindanet al. [68]	Benefit
	Response research and development	Lee et al. [69]	Benefit
	Verification of environmental safety	Feng & Gong [65]	Benefit
Cost	Information cost	Expert feedback	Cost
	Material cost	Govindanet al. [68]	Cost
	Cost of inspection	Feng & Gong [65]	Cost
Capability	Developmental research capacity	Kannan [70]	Benefit
	Technical capability	Haeriet al. [71]	Benefit
	Strong monetary resources	Yazdaniet al. [72]	Benefit
	Executive ability	Kannan [70]	Benefit
	Recycling ability	Goren [73]	Benefit
Flexibility	Quantity of order	Feng and Gong [74]	Benefit
	Goods exchange	Expert opinion	Benefit
	Responsiveness	Hou & Xie [66]	Benefit
	Variety in products	Feng and Gong [65]	Benefit

The third step was to evaluate the potential suppliers based on the criteria. This was done using the CoCoSo method, which calculates a compromise solution that balances the criteria. The results showed that Supplier A had the highest score, followed by Supplier B, Supplier C and Supplier D. The proposed MCDM framework can be used to select sustainable suppliers in the CE and industry 4.0 era. The framework helps companies to identify and evaluate suppliers based on their performance in various criteria, which leads to the selection of suppliers that comply with environmental, social, and economic standards. The framework provides a systematic and transparent approach for supplier selection, which is essential for companies aiming to operate in the CE and industry 4.0 era. The explanation of criterions and sub-criterions is given in Table 6.

Suppose that there is set of seven suppliers $S = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7\}$, the parameter set $E = \{\xi^{\varphi}{}_1$ = Delivery, $\xi^{\varphi}{}_2$ = Quality, $\xi^{\varphi}{}_3$ = Enviromental, $\xi^{\varphi}{}_4$ = Cost, $\xi^{\varphi}{}_5$ = Capability, $\xi^{\varphi}{}_6$ = Flexibility$\}$ is employed in assessing the best supplier by domain expert, in which $\xi^{\varphi}{}_4$ is a cost type attribute, while $\xi^{\varphi}{}_1, \xi^{\varphi}{}_2, \xi^{\varphi}{}_3, \xi^{\varphi}{}_5$ and $\xi^{\varphi}{}_6$ are benefit type attributes. The evaluations for supplier selection arise from decision experts, and the form of linguistic terms is shown in the Table 7.

Table 7. Linguistic Terms.

Linguistic terms	Abbrevation	PiFNs
Highly effective	HE	(1.0, 0.0, 0.0)
Very effective	VE	(0.6, 0.2, 0.2)
Effective	E	(0.3, 0.4, 0.3)
Moderately effective	ME	(0.2, 0.5, 0.3)
Less effective	LE	(0.1, 0.3, 0.6)
Not effective	NE	(0.0, 0.0, 1.0)

Table 8. PiFSS.

	$\xi^{\varphi}{}_1$	$\xi^{\varphi}{}_2$	$\xi^{\varphi}{}_3$	$\xi^{\varphi}{}_4$	$\xi^{\varphi}{}_5$	$\xi^{\varphi}{}_6$	$\xi^{\varphi}{}_7$
α_1	HE	E	NE	LE	E	E	ME
α_2	VE	NE	ME	E	HE	NE	LE
α_3	NE	E	ME	LE	HE	E	VE
α_4	E	ME	LE	VE	E	NE	HE
α_5	VE	E	VE	E	ME	NE	HE
α_6	E	LE	VE	HE	E	NE	LE
α_7	ME	E	HE	NE	LE	VE	LE

Table 9. Assessment by linguistic terms.

	$\xi^{\varphi}{}_1$	$\xi^{\varphi}{}_2$	$\xi^{\varphi}{}_3$	$\xi^{\varphi}{}_4$	$\xi^{\varphi}{}_5$	$\xi^{\varphi}{}_6$
α_1	(1.0,0.0,0.0)	(0.3,0.4,0.3)	(0.0,1.0,1.0)	(0.1,0.3,0.6)	(0.3,0.4,0.3)	(0.3,0.4,0.3)
α_2	(0.6,0.2,0.2)	(0.0,0.0,1.0)	(0.2,0.5,0.3)	(0.3,0.4,0.3)	(1.0,0.0,0.0)	(0.0,0.0,1.0)
α_3	(0.0,0.0,1.0)	(0.3,0.4,0.3)	(0.2,0.5,0.3)	(0.1,0.3,0.6)	(1.0,0.0,0.0)	(0.3,0.4,0.3)
α_4	(0.3,0.4,0.3)	(0.2,0.5,0.3)	(0.1,0.3,0.6)	(0.6,0.2,0.2)	(0.3,0.4,0.3)	(0.0,0.0,1.0)
α_5	(0.6,0.2,0.2)	(0.3,0.4,0.3)	(0.6,0.2,0.2)	(0.3,0.4,0.3)	(0.2,0.5,0.3)	(0.0,0.0,1.0)
α_6	(0.3,0.4,0.3)	(0.1,0.3,0.6)	(0.6,0.2,0.2)	(1.0,0.0,0.0)	(0.3,0.4,0.3)	(0.0,0.0,1.0)
α_7	(0.2,0.5,0.3)	(0.3,0.4,0.3)	(1.0,0.0,0.0)	(0.0,0.0,1.0)	(0.1,0.3,0.6)	(0.6,0.2,0.2)

Next, we use the presented algorithm ($\lambda = 0.5$) to select the optimal supplier under a PiFS environment.

Step 1: Achieve the given assessment Table 8 by linguistic terms as given in Table 7.
Step 2: Convert linguistic term table into the PiFSS, which is shown in Table 9.
Step 3: Compute SF $F = \left(\gamma_{ij}\right)_{5\times4}$ of each PiFSN $F\left(\xi^{\varphi}{}_{j}\right)(\alpha_i) = \left(\mu_F\left(\xi^{\varphi}{}_{j}\right)(\alpha_i), \nu_F\left(\xi^{\varphi}{}_{j}\right)(\alpha_i)\right)$ by Eq 1.

$$F = \left(\gamma_{ij}\right)_{5\times4} = \begin{pmatrix} 1.0 & 0.3 & 0.0 & 0.1 & 0.3 & 0.3 \\ 0.6 & 0.0 & 0.2 & 0.3 & 1.0 & 0.0 \\ 0.0 & 0.3 & 0.2 & 0.1 & 1.0 & 0.3 \\ 0.3 & 0.2 & 0.1 & 0.6 & 0.3 & 0.0 \\ 0.6 & 0.3 & 0.6 & 0.3 & 0.2 & 0.0 \\ 0.3 & 0.1 & 0.6 & 1.0 & 0.3 & 0.0 \\ 0.2 & 0.3 & 1.0 & 0.0 & 0.1 & 0.6 \end{pmatrix}.$$

Step 4: Transform the matrix $F = \left(\gamma_{ij}\right)_{5\times4}$ into a standard picture fuzzy soft matrix $F' = \left(\gamma'_{ij}\right)_{5\times4}$ by Eq 2.

$$F' = \left(\gamma_{ij}\right)_{5\times4} = \begin{pmatrix} 1 & 1 & 0 & 0.9 & 0.2222 & 0.5 \\ 0.6 & 0 & 0.2 & 0.7 & 1 & 0 \\ 0 & 1 & 0.2 & 0.9 & 1 & 0.5 \\ 0.3 & 0.6667 & 0.1 & 0.4 & 0.2222 & 0 \\ 0.6 & 1 & 0.6 & 0.7 & 0.1111 & 0 \\ 0.3 & 0.3333 & 0.6 & 0 & 0.2222 & 0 \\ 0.2 & 1 & 1 & 1 & 0 & 1 \end{pmatrix}.$$

Step 5:
Determine the criteria standard deviations by Eq 3, given in Table 10.

Table 10. Standard Deviations.

σ_1	σ_2	σ_3	σ_4	σ_5	σ_6
0.3057	0.375	0.3314	0.3245	0.3888	0.3642

Step 6:
Calculate the correlation between criteria pairs by using Eq 4.

$$\rho_{jk} = \begin{pmatrix} 1 & 0.8728 & -1.4469 & 0.8344 & -1.0176 & 0.6605 \\ 0.8728 & 1 & -1.0683 & 0.9986 & -1.2595 & 0.9420 \\ -1.4469 & -1.0683 & 1 & -1.0338 & 0.5712 & -1.0124 \\ 0.8344 & 0.9986 & -1.0338 & 1 & -1.2636 & 0.9571 \\ -1.0176 & -1.2595 & 0.5712 & -1.2636 & 1 & -1.9865 \\ 0.6605 & 0.9420 & -1.0124 & 0.9571 & -1.9865 & 1 \end{pmatrix}.$$

Step 7:
Determine the quantity of information of each criterion by Eq 5, given in Table 11.

Table 11. Quantity of information.

c_1	c_2	c_3	c_4	c_5	c_6
1.5582	1.6927	2.9789	1.4627	3.8710	1.9811

Step 8:
Calculate the objective weights of each criteria by Eq 6, given in Table 12.

Table 12. Objective Weights.

ω_1	ω_2	ω_3	ω_4	ω_5	ω_6
0.115	0.125	0.2199	0.108	0.2858	0.1463

Step 9: Let us consider the subjective weight, as directly presented by decision-makers, is $\xi = \{0.1135, 0.1242, 0.1674,\ 0.3728,\ 0.0696,\ 0.1524\}$, where $\sum_{j=1}^{n} \xi_j = 1, 0 \leq \xi_j \leq 1$. The objective weight, computed by Eq. (11), is $\omega = \{0.115, 0.125, 0.12199,\ 0.108,\ 0.2858,\ 0.1463\}$, where $\sum_{j=1}^{n} \omega_j = 1, 0 \leq \omega_j \leq 1$. Calculate the combined weight ϖ by Eq 7, given in Table 13.

Table 13. Combined Weights.

ϖ_1	ϖ_2	ϖ_3	ϖ_4	ϖ_5	ϖ_6
0.0883	0.105	0.249	0.2723	0.1346	0.1508

Step 10, 11: Calculate the entire weighted comparability sequence for every supplier as S_i and entire power weight of comparability sequences for each supplier as P_i, given in Table 14.

Table 14. Comparability Sequence.

S_1	S_2	S_3	S_4	S_5	S_6	S_7
0.5437	0.428	0.6098	0.2602	0.513	0.2408	0.7948
P_1	P_2	P_3	P_4	P_5	P_6	P_7
4.6893	3.5331	4.5423	4.017	4.4879	3.4875	3.4875

Step 12: Derive three appraisal score strategies as given in Table 15.

Table 15. Score Strategies.

k_{1a}	k_{2a}	k_{3a}	k_{4a}	k_{5a}	k_{6a}	k_{7a}
0.1585	0.12	0.1561	0.1296	0.1515	0.1129	0.1715
k_{1b}	k_{2b}	k_{3b}	k_{4b}	k_{5b}	k_{6b}	k_{7b}
3.6022	2.7902	3.8347	2.2324	3.417	2.000	4.6961
k_{1c}	k_{2c}	k_{3c}	k_{4c}	k_{5c}	k_{6c}	k_{7c}
0.9242	0.6996	0.9099	0.7554	0.8832	0.6584	1.000

Step 13: Calculate the evaluation value k_i by Eq 13, given in Table 16.

Table 16. Final Ranking.

k_1	k_2	k_3	k_4	k_5	k_6	k_7
2.3697	1.8196	2.4501	1.6414	2.2542	1.4536	2.8863

Step 14: Rank alternatives α_i by the assessed value $k_i (i = 1, 2, \cdots, m)$.

$$\alpha_7 \succ \alpha_6 \succ \alpha_3 \succ \alpha_1 \succ \alpha_5 \succ \alpha_2 \succ \alpha_4.$$

The results of the CRITIC-CoCoSo analysis showed that seventh supplier was the best choice for the manufacturing company's CE initiatives. Seventh supplier scored high on product quality, environmental impact, social responsibility and innovation, and its cost was competitive with the other suppliers. The results also showed that the company could use the CRITIC-CoCoSo technique to evaluate suppliers objectively and make informed decisions based on multiple criteria.

Table 17. The impact of the values $\curlyvee$ on the decision result.

$\curlyvee$	k_1	k_2	k_3	k_4	k_5	k_6	k_7	Result
$\curlyvee = -10$	2.4239	1.8561	2.4829	1.7327	2.3078	1.5298	2.8863	α_7
$\curlyvee = -6$	2.4219	1.8548	2.4817	1.7294	2.3058	1.5271	2.8863	α_7
$\curlyvee = -5.5$	2.4215	1.8545	2.4814	1.7287	2.3054	1.5265	2.8863	α_7
$\curlyvee = -5$	2.4210	1.8542	2.4811	1.7279	2.3049	1.5258	2.8863	α_7
$\curlyvee = -4.5$	2.4205	1.8538	2.4829	1.7270	2.3044	1.5250	2.8863	α_7
$\curlyvee = -4$	2.4198	1.8534	2.4804	1.7258	2.3037	1.5241	2.8863	α_7
$\curlyvee = -3.5$	2.4190	1.8528	2.4799	1.7245	2.3029	1.5230	2.8863	α_7
$\curlyvee = -3$	2.4180	1.8522	2.4793	1.7228	2.3019	1.5216	2.8863	α_7
$\curlyvee = -2.5$	2.4167	1.8513	2.4785	1.7207	2.3007	1.5198	2.8863	α_7
$\curlyvee = -2$	2.4150	1.8502	2.4775	1.7179	2.2990	1.5174	2.8863	α_7
$\curlyvee = -1.5$	2.4127	1.8486	2.4761	1.7140	2.2967	1.5142	2.8863	α_7
$\curlyvee = -1$	2.4093	1.8463	2.4740	1.7084	2.2934	1.5095	2.8863	α_7
$\curlyvee = 0$	2.3941	1.8360	2.4648	1.6827	2.2783	1.4881	2.8863	α_7
$\curlyvee = 1$	2.2126	1.7145	2.3575	1.3522	2.0985	1.2143	2.8863	α_7
$\curlyvee = 1.5$	2.5534	1.9438	2.5623	1.9441	2.4356	1.7067	2.8863	α_7
$\curlyvee = 2$	2.4759	1.8914	2.5146	1.8186	2.3592	1.6016	2.8863	α_7
$\curlyvee = 2.5$	2.4575	1.8789	2.5033	1.7883	2.3410	1.5763	2.8863	α_7
$\curlyvee = 3$	2.4492	1.8733	2.4983	1.7746	2.3328	1.5649	2.8863	α_7
$\curlyvee = 3.5$	2.4445	1.8701	2.4954	1.7668	2.3281	1.5584	2.8863	α_7
$\curlyvee = 4$	2.4414	1.8680	2.4936	1.7618	2.3251	1.5542	2.8863	α_7
$\curlyvee = 4.5$	2.4393	1.8666	2.4923	1.7583	2.3230	1.5512	2.8863	α_7
$\curlyvee = 5$	2.4378	1.8655	2.4913	1.7557	2.3215	1.5491	2.8863	α_7
$\curlyvee = 5.5$	2.4366	1.8647	2.4906	1.7537	2.3203	1.5474	2.8863	α_7
$\curlyvee = 6$	2.4356	1.8641	2.4900	1.7522	2.3194	1.5461	2.8863	α_7
$\curlyvee = 10$	2.4319	1.8616	2.4877	1.7460	2.3157	1.5409	2.8863	α_7

5. Sensitivity analysis

Sensitivity analysis, is a powerful tool used in various fields, including optimization and decision-making processes. It involves examining how changes in input variables impact the output or optimal solutions of a particular model or technique. In the context of the CRITIC-CoCoCo technique, a fascinating observation can be made regarding the sensitivity of the optimal solution to the value of $\vee$. The value of $\vee$ represents a weight or importance factor assigned to the criteria being evaluated. It reflects the decision-maker's preferences and influences the outcome of the decision process. Interestingly, in the CRITIC-CoCoCo technique, when the value of lambda is changed, the optimal solution remains the same. This implies that the relative importance or weight assigned to different criteria does not alter the final decision outcome. Despite variations in $\vee$, the selection of the best alternative remains consistent. Note that the stability in the optimal solution regardless of value of $\vee$ proves the efficiency of the proposed CRITIC-CoCoCo technique. It suggests that this technique provides a robust and reliable decision-making framework as it ensures that the chosen alternative remains unaffected by changes in the importance assigned to individual criteria. Pictorial representation is given in Figure 2.

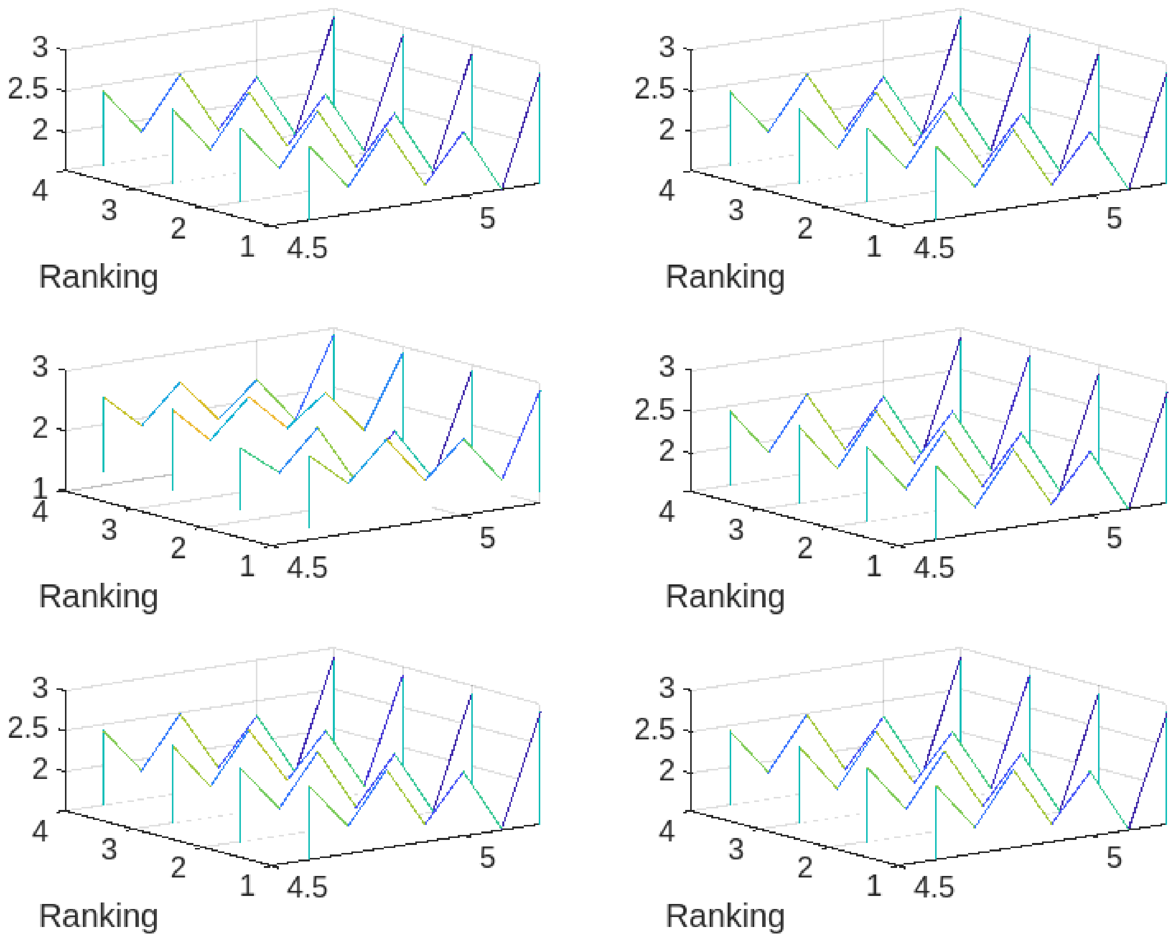

Figure 2. Sensitive analysis.

6. Comparative analysis and discussion

As we delve into the discussion and comparison analysis of CRITIC-CoCoSo technique with other already proposed techniques such as BWM, CRITIC-COPRAS, CPT-CoCoSo, LBWA-CoCoSo, and SWARA-COPRAS, it is important to keep an open mind and evaluate each method on its own merits.

- Compared to traditional statistical methods, CRITIC-CoCoSo offers a more systematic approach to identifying critical factors and analyzing their interrelationships in complex systems.
- First, let us consider the BWM technique. While it is a widely used method for decision-making, it has some limitations when it comes to handling complex decision-making problems with a large number of criteria and alternatives. On the other hand, the CRITIC-CoCoSo technique addresses these limitations and provides a more comprehensive approach for decision-making, especially when dealing with complex and large-scale decision-making problems.
- Similarly, the CRITIC-COPRAS technique is a valuable approach for decision-making, but it has some limitations when it comes to handling uncertainties and subjective preferences. In contrast, the CRITIC-CoCoSo technique takes into account subjective preferences and uncertainties and provides a more robust approach for decision-making.
- The CPT-CoCoSo and LBWA-CoCoSo techniques are also valuable approaches for decision-making, but they have limitations when it comes to handling incomplete and uncertain information. The CRITIC-CoCoSo technique, however, provides a more comprehensive approach for decision-making by considering both complete and incomplete information.
- Finally, the SWARA-COPRAS technique is a useful approach for decision-making, but it has limitations when it comes to handling conflicting criteria and preferences. In contrast, the CRITIC-CoCoSo technique provides a more effective approach for dealing with conflicting criteria and preferences, thereby making it a more suitable approach for complex decision-making problems.
- Unlike some existing techniques that require domain experts to provide subjective weights for different factors, CRITIC-CoCoSo uses a data-driven approach that automatically generates objective weights based on the available data.
- While some approaches, such as decision trees or regression models, can provide insights into the relationships between factors and outcomes, they may not be able to capture the nonlinear and interactive effects that CRITIC-CoCoSo can reveal.
- In contrast to some machine learning techniques that require large amounts of training data, CRITIC-CoCoSo can handle small or sparse datasets and still produce meaningful results.
- Compared to some other multicriteria decision analysis techniques, such as analytic hierarchy process (AHP) or technique for order of preference by similarity to ideal solution (TOPSIS), CRITIC-CoCoSo can handle more complex and diverse decision scenarios with multiple conflicting objectives and uncertain or incomplete information.
- While some approaches focus solely on identifying the most important factors or criteria, CRITIC-CoCoSo also provides a comprehensive analysis of the interrelationships and dependencies among factors, which can help decision-makers understand the underlying mechanisms and potential trade-offs in the system.

Overall, while each of these techniques has its own strengths and limitations, the CRITIC-CoCoSo technique stands out as a comprehensive, robust, and effective approach for decision-making, especially when dealing with complex and large-scale decision-making problems.

Table 18. Comparison of optimal decision.

Authors	Techniques	Ranking of alternatives	Result
Haktanir and Kahraman [75]	CRITIC-REGIME	$\alpha_7 > \alpha_6 > \alpha_3 > \alpha_1 > \alpha_5 > \alpha_2 > \alpha_4$	α_7
Zhang and Wei [76]	CPT-CoCoSo	$\alpha_7 > \alpha_6 > \alpha_3 > \alpha_1 > \alpha_5 > \alpha_2 > \alpha_4$	α_7
Peng & Luo [77]	CoCoSo	$\alpha_7 > \alpha_6 > \alpha_3 > \alpha_1 > \alpha_5 > \alpha_2 > \alpha_4$	α_7
Korucuk et al. [78]	LBWA-CoCoSo	$\alpha_7 > \alpha_6 > \alpha_3 > \alpha_1 > \alpha_5 > \alpha_2 > \alpha_4$	α_7
Qiyas [79]	CoCoSo	$\alpha_7 > \alpha_6 > \alpha_3 > \alpha_1 > \alpha_5 > \alpha_2 > \alpha_4$	α_7
Mohata et al. [80]	CRITIC-COPRAS	$\alpha_7 > \alpha_6 > \alpha_3 > \alpha_1 > \alpha_5 > \alpha_2 > \alpha_4$	α_7
Kamali et al. [81]	CRITIC-COPRAS	$\alpha_7 > \alpha_6 > \alpha_3 > \alpha_1 > \alpha_5 > \alpha_2 > \alpha_4$	α_7
Saraji et al. [82]	SWARA-CRITIC-COPRAS	$\alpha_7 > \alpha_6 > \alpha_3 > \alpha_1 > \alpha_5 > \alpha_2 > \alpha_4$	α_7
YILMAZ [83]	BWM-CoCoSo	$\alpha_7 > \alpha_6 > \alpha_3 > \alpha_1 > \alpha_5 > \alpha_2 > \alpha_4$	α_7
Proposed	CRITIC-CoCoSo	$\alpha_7 > \alpha_6 > \alpha_3 > \alpha_1 > \alpha_5 > \alpha_2 > \alpha_4$	α_7

6.1. Managerial implications

CRITIC-CoCoSo technique is a valuable tool for supplier selection in the context of CE and Industry 4.0. This technique enables decision-makers to assess and evaluate potential suppliers based on multiple criteria, such as environmental performance, social responsibility and economic viability. The CRITIC-CoCoSo technique promotes sustainability and supports the transition towards a more circular and digital economy.

- One of the main managerial implications of CRITIC-CoCoSo technique is that it allows companies to make more informed and strategic decisions when selecting suppliers. By considering a range of criteria, companies can identify suppliers that not only offer the best value for money, but also align with their environmental and social values. This, in turn, can help to build a more sustainable and responsible supply chain, which is increasingly important to customers and other stakeholders.
- Another important implication of the CRITIC-CoCoSo technique is that it can help companies to manage risk more effectively. In a CE, supply chains are often more complex and interconnected than in traditional linear models. As a result, there are often greater risks associated with supplier selection, such as environmental and social risks. By using the CRITIC-CoCoSo technique, companies can identify and mitigate these risks, ensuring that their supply chain is more resilient and sustainable.
- Finally, the CRITIC-CoCoSo technique can help companies to drive innovation and competitiveness. By selecting suppliers based on a range of criteria, companies can identify suppliers that are at the forefront of sustainability and CE practices. This can help to stimulate innovation and collaboration within the supply chain, leading to new product and service offerings and improved competitiveness.

In conclusion, the use of the CRITIC-CoCoSo technique in supplier selection in the context of CE and Industry 4.0 has important managerial implications. By promoting sustainability, managing risk, and driving innovation and competitiveness, companies can build more resilient and responsible supply chains, which are crucial for long-term success in a rapidly changing business landscape.

7. Conclusions

A robust picture fuzzy soft CRITIC-CoCoSo approach is developed for the selection of suppliers within the context of the CE in Industry 4.0. It empowers companies to assess potential suppliers according to various criteria and arrive at well-informed decisions that strike a balance between environmental, social obligations and cost considerations. Employing this technique, companies can select suppliers who are aligned with their CE objectives and foster a sustainable future. The use of topological structure offers a new perspective for computational intelligence, fuzzy modeling and data analysis. This article provides comprehensive coverage of various aspects of PiFS-topology, including the construction of PiFS-topology through PiFS-EU, PiFS-RI, null-PiFS and absolute-PiFS. This study examines a case study that demonstrates the effectiveness of the CRITIC-CoCoSo technique in resolving supplier selection problems in the context of Industry 4.0 and revealed that the seventh supplier was the ideal option for the manufacturing company's CE initiatives due to its high scores in product quality, environmental impact, social responsibility and innovation as well as its competitive cost. A comprehensive comparison analysis and sensitive analysis of CRITIC-CoCoSo technique with existing techniques is provided. Through a systematic and rigorous examination of these methodologies, we aimed to shed light on their strengths, limitations and practical implications for managerial decision-making.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article

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Conflict of Interest

The authors have no conflict of interest.

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