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Research article

On the (total) Roman domination in Latin square graphs

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Abstract: Latin square, also known as Latin square matrix, refers to a kind of $n \times n$ matrix, in which there are exactly n different symbols and each symbol appears exactly once in each row and column. A Latin square graph $\Gamma(L)$ is a simple graph associated with a Latin square L. This paper studied the relationships between the (total) Roman domination number and (total) domination number of Latin square graph $\Gamma(L)$. We showed that $\gamma_R(\Gamma(L)) = 2\gamma(\Gamma(L))$ or $\gamma_R(\Gamma(L)) = 2\gamma(\Gamma(L)) - 1$, and $\gamma_{tR}(\Gamma(L)) \geq \frac{8\gamma_t(\Gamma(L))}{5}$ for $n \geq 2$. In 2021, Pahlavsay et al. proved $\gamma(\Gamma(L)) \geq \lceil \frac{n}{2} \rceil$ and $\gamma_t(\Gamma(L)) \geq \lceil \frac{4n-2}{7} \rceil$ for $n \geq 2$. In this paper, we showed that $\gamma_R(\Gamma(L)) \geq 2\lceil \frac{n}{2} \rceil$ (equality holds if, and only if, $\gamma(\Gamma(L)) = \lceil \frac{n}{2} \rceil$) and $\gamma_t(\Gamma(L)) > \frac{4n}{7}$ for $n \geq 2$. Since $\gamma_R(G) \leq 2\gamma(G)$ and $\gamma_{tR}(G) \leq 2\gamma_t(G)$ for any graph G, our results can deduce or improve Pahlavsay et al.'s results. Moreover, we characterized these Latin squares for $\gamma_R(\Gamma(L)) = 2\lceil \frac{n}{2} \rceil$, which is equal to $\gamma(\Gamma(L)) = \lceil \frac{n}{2} \rceil$.

Keywords: Latin square; Latin square graphs; Roman domination number; total Roman domination number
Mathematics Subject Classification: 05C69

1. Introduction

All graphs considered in this paper are undirected and simple. Let $G = (V, E)$ be a graph where $V = V(G)$ is the vertex set and $E = E(G)$ as the edge set. For any vertex $v \in V$, let $N_G(u)$ (simply, $N(u)$) be the neighborhood of v and let $N_G[u] = N_G(u) \cup \{u\}$ (simply, $N[u]$) be the closed neighborhood of v. Let $G[S]$ be the subgraph of G induced by $S \subseteq V$, $\delta(G)$ and $\Delta(G)$ be the minimum and maximum degree of G, respectively.

A set $D \subseteq V(G)$ is a dominating set if for each $v \in V(G)$ either $v \in D$ or v has at least one neighbor in D. The domination number $\gamma(G)$ is the minimum cardinality of all dominating sets of G. A set $D_t \subseteq V(G)$ is a total dominating set if each $v \in V(G)$ is adjacent to at least one vertex in D_t. The total domination number $\gamma_t(G)$ is the minimum cardinality of all total dominating sets of G. There are many variations on domination for different applications. Domination is well studied in graph theory and the literature on this subject has been surveyed and detailed in books by Haynes et al. [12–14].

A map $f : V \to \{0, 1, 2\}$ is a Roman dominating function of G, if for every vertex v with $f(v) = 0$, there exists a vertex $u \in N(v)$ such that $f(u) = 2$. The weight of a Roman function is given by $f(V) = \sum_{u \in V} f(u)$. The minimum weight of all Roman dominating functions on G is called the Roman domination number of G and it is denoted by $\gamma_R(G)$. For a Roman function f, let $V_i = \{v \in V(G) : f(v) = i\}$ for $i \in \{0, 1, 2\}$. Since these three sets can determine f, we can equivalently write $f = (V_0; V_1; V_2)$, then $f(V) = |V_1| + 2|V_2|$. Cockayne et al. [10] introduced the notion of Roman domination in graphs. A total Roman dominating function, which was presented in [2], is a Roman dominating function $f = (V_0; V_1; V_2)$ on G, satisfying that $\delta(G[V_1 \bigcup V_2]) \geq 1$. The total Roman domination number $\gamma_{tR}(G)$ is the minimum weight of all total Roman dominating functions on G. The Roman domination number and total Roman domination number have been studied widely, and many structure properties of (total) Roman dominating sets or many good bounds for these parameters are given; see, for example, [1, 3–5, 9, 11, 16–18, 20–24].

A Latin square of order n is an $n \times n$ matrix containing n symbols, which are contained in the set $[n] := \{1, 2, \ldots, n\}$ such that each row and each column contains one copy of each symbol. Usually, we use L to represent the Latin square. A Latin square graph $\Gamma(L)$ is a simple graph associated with a Latin square L. In this paper, we study the Roman domination number and total Roman of Latin square graphs.

2. Preliminary results

The notion of the Latin square was introduced by Leonhard Euler in 1783 as a new kind of magic square and plays an important role in various fields, such as in combinatorics, statistics and informatics. In recent years, together with their associated graph, they have been intensively studied because of their connections with other areas of mathematics and their practical applications. See the book "Latin squares and their applications" [15] for more applications of Latin squares.

Definition 2.1. *[19]* Let L be a Latin square of order n and let L' be the l^2 cells defined by ℓ rows and ℓ columns in L. Write L'^c as the $(n-\ell)^2$ cells defined by all the other $n-\ell$ rows and all the other $n-\ell$ columns corresponding to L' and L. If for some $1 \leq \ell \leq n$, L' forms a Latin square of order ℓ, it is called a *Latin subsquare* of L.

For a Latin square, there is only one corresponding symbol s for a specific row r and column c, which can be recorded as $s = L_{r,c}$ or represented by the triple (r, c, s).

Definition 2.2. *[19]* For a Latin square L, we define

$$E(L) = \{(r, c, s) : 1 \leq r, c, s \leq n \text{ and } s = L_{r,c}\}$$

to be the set of *entries* of L. Let $A \subseteq E(L)$. We call $\{s \in [n] : (r, c, s) \in A\}$ the symbol set of A.

Definition 2.3. *[6]* Let L be a Latin square of order n. A *partial transversal* is a subset of $E(L)$ such that no two entries share the same row, column or symbol. We say that a partial transversal is *maximal* if it is not properly contained in any other partial transversal. A *transversal* is a partial transversal with cardinality n, i.e., it is a set of entries that includes exactly one entry from each row, column, and symbol.

Note that not every Latin square possesses a transversal.

Bose in [7] introduced a kind of strongly regular graph by constructing a simple graph from a Latin square.

Definition 2.4. *[7]* A *Latin square graph* $\Gamma(L)$ is a simple graph generated by a Latin square L, with vertex set $E(L)$ and two vertices adjacent if, and only if, they are in the same row or column or have the same symbol.

Example 2.1. A Latin square L of order four and its associated Latin square graph are shown in Figure 1. Note that $T = \{(1, 1, 1), (2, 2, 3), (3, 4, 2)\}$ is a maximal partial transversal of L and L does not have a transversal.

1	2	3	4
2	3	4	1
3	4	1	2
4	1	2	3

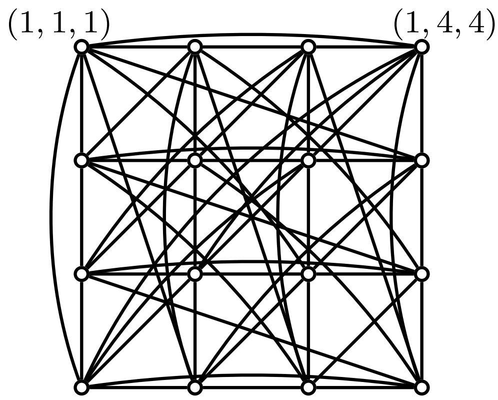

Figure 1. L, $\Gamma(L)$ and red numbers are corresponding to a maximal partial transversal of L.

Let L be a Latin square of order $n \geq 3$. The Latin square graph $\Gamma(L)$) is a $3(n-1)$-regular graph, and any two different vertices (r, c, s) and (r, c', s') have n neighbors in-common, $n-2$ vertices in the row r and two vertices in the columns c and c'. By interchanging the role of rows, columns and symbols, any two different vertices (r, c, s) and (r', c, s') have n neighbors in-common and any two different vertices (r, c, s) and (r', c', s) also have n neighbors in-common. So any two adjacent vertices of $\Gamma(L)$ have n neighbors in-common. Any two different vertices (r, c, s) and (r', c', s') have two neighbors in-common; that is, $(r, c', L_{r,c'})$ and $(r', c, L_{r',c})$.

Note that any maximal partial transversal of L corresponds to a dominating set in $\Gamma(L)$. A transversal of L is an efficient three-dominating set in $\Gamma(L)$ (and vice versa). An efficient three-dominating set D of a graph G is a dominating set of G such that for each vertex $v \notin D$, $|N_G(D) \cap D| = 3$.

Proposition 2.1. *[10]* For any graph G,

$$\gamma(G) \leq \gamma_R(G) \leq 2\gamma(G).$$

It is well known [8] that almost all graphs have diameter two.

Proposition 2.2. Let G be a graph with diameter 2. Then

$$\gamma_R(G) = 2\gamma(G) - 1 \textit{ or } \gamma_R(G) = 2\gamma(G).$$

Moreover, if $f = (V_0; V_1; V_2)$ is a minimum Roman dominating function of G with $|V_1|$ minimum, then
(1) $|V_1| \leq 1$.
(2) $\gamma_R(G) = 2\gamma(G) - 1$ if, and only if, $|V_1| = 1$.
(3) $\gamma_R(G) = 2\gamma(G)$ if, and only if, $|V_1| = 0$.

Proof. Let $f = (V_0; V_1; V_2)$ be a minimum Roman dominating function of G with a $|V_1|$ minimum. We will show $|V_1| \leq 1$. Suppose to the contrary that there exists two different vertices $u, v \in V_1$. Note that V_1 is an independent set (i.e., no edges between any two distinct vertices) of G since $|V_1|$ is minimum. Since the diameter of G is two, there exists a vertex w which is a common neighbor of u and v, then

$$f' = ((V_0 \cup \{u, v\}) \setminus \{w\}; V_1 \setminus \{u, v\}; V_2 \cup \{w\})$$

is also a minimum Roman dominating function of G, which contradicts the assumption that $|V_1|$ is minimum. Thus, $|V_1| \leq 1$. Note that $V_1 \cup V_2$ is a dominating set of G and $|V_1| + |V_2| \geq \gamma(G)$. Therefore,

$$\gamma_R(G) = f(V(G)) = |V_1| + 2|V_2| \geq 2\gamma(G) - |V_1| \geq 2\gamma(G) - 1. \tag{2.1}$$

By Proposition 2.1, $\gamma_R(G) \leq 2\gamma(G)$. Hence $\gamma_R(G) = 2\gamma(G) - 1$ or $\gamma_R(G) = 2\gamma(G)$.

If $\gamma_R(G) = 2\gamma(G) - 1$, then $|V_1| = 1$ by 2.1. If $|V_1| = 0$, then $\gamma_R(G) = 2\gamma(G)$ by 2.1. If $\gamma_R(G) = 2\gamma(G)$ is an even number, then $|V_1| = 0$ by 2.1 since $|V_1| \leq 1$. Hence, conclusions (2) and (3) hold.

Proposition 2.3. *[19]* Let L be a Latin square of order $n \geq 2$. Then

$$\gamma(\Gamma(L)) \leq n - 1.$$

Observation 2.1. Let L be a Latin square of order n. Then the order of every Latin subsquare of L, except L, is no more than $\lfloor \frac{n}{2} \rfloor$. If n is even and L' is a Latin subsquare of L of order $\frac{n}{2}$, then L'^c is also a Latin subsquare of L of order $\frac{n}{2}$ and has the same symbol set of L'.

3. Roman domination of Latin square graphs

Since any two nonadjacent vertices in a Latin square graph have two neighbors in common, the diameter of a Latin square graph is no more than two. Indeed, the diameter of a Latin square graph is two for $n \geq 2$. By Proposition 2.2, We have the following result.

Proposition 3.1. Let L be a Latin square. Then $\gamma_R(\Gamma(L)) = 2\gamma(\Gamma(L)) - 1$ or $\gamma_R(\Gamma(L)) = 2\gamma(\Gamma(L))$.

Theorem 3.1. Let L be a Latin square of order $n \geq 2$. Then

$$2\lceil \frac{n}{2} \rceil \leq \gamma_R(\Gamma(L)) \leq 2n - 2,$$

and further, we have

(1) when n is even, $\gamma_R(\Gamma(L)) = n$ if, and only if, there is a Latin subsquare of order $n/2$ which contains a transversal.

(2) when n is odd, $\gamma_R(\Gamma(L)) = n + 1$ if, and only if, L satisfies one of the following conditions:

I. there exists a Latin subsquare L' of order $\frac{n-1}{2}$ and a partial transversal T_1 with cardinality $\frac{n-1}{2}$ in L'^c of L, where L' and T_1 have the same symbol set.

II. there exists a maximal partial transversal T_2 of L with cardinality $\frac{n+1}{2}$ and all symbols of vertices in all the other rows and all the other columns corresponding to T_2 belong to the symbol set of T_2.

III. there exists a Latin subsquare L'' of order $\frac{n-1}{2}$ and a partial transversal T_3 with cardinality $\frac{n-1}{2}$ in the same rows and all the other columns (or the same columns and the other rows) corresponding to L'' of L (see Figure 2 for example).

Moreover, if $f = (V_0; V_1; V_2)$ is a minimum Roman dominating function of G with $|V_1|$ minimum and $\gamma_R(\Gamma(L)) = 2\lceil\frac{n}{2}\rceil$, then $|V_1| = 0$.

1	2	3	4	5	6
2	3	1	5	4	6
3	1	2	6	4	5
4	5	6	1	2	3
5	6	4	2	3	1
6	4	5	3	1	2

1	2	3	4	5
2	1	4	5	3
3	4	5	2	1
4	5	1	3	2
5	3	2	1	4

1	2	3	4	5
2	3	1	5	4
4	5	2	1	3
5	1	4	3	2
3	4	5	2	1

1	2	3	4	5
2	1	5	3	4
3	5	4	2	1
4	3	1	5	2
5	4	2	1	3

Figure 2. Examples for Theorem 3.

Proof. By Propositions 2.1 and 2.3, $\gamma_R(\Gamma(L)) \le 2\gamma(\Gamma(L)) \le 2n - 2$.

Let $f = (V_0; V_1; V_2)$ be a minimum Roman dominating function of $\Gamma(L)$ with $|V_1|$ minimum and $V_2 = \{(r_1, c_1, s_1), \dots, (r_k, c_k, s_k)\}$ where $|V_2| = k$. By Proposition 2.2, $|V_1| \le 1$. Since $|V_1| + 2|V_2| = \gamma_R(\Gamma(L)) \le 2n - 2$, $|V_1| + |V_2| \le n - 1$ (note that $|V_1|$ is an integer). Hence there exits one row in L such that all vertices in it belong to V_0. By symmetry, we assume all vertices of $\Gamma(L)$ in the first row belong to V_0. Since $(1, c, L_{1,c}) \in V_0$ should be adjacent to a vertex in V_2 for every $c \in [n] \setminus \{c_1, c_2, \dots, c_k\}$, $L_{1,c}$ must be in the set $\{s_1, s_2, \dots, s_k\}$. There are at least $n - |\{c_1, c_2, \dots, c_k\}| \ge n - k$ such vertices, so

$$k \ge |\{s_1, s_2, \dots, s_k\}| \ge n - |\{c_1, c_2, \dots, c_k\}| \ge n - k, \ \textit{then}\ k \ge \lceil\frac{n}{2}\rceil. \tag{3.1}$$

Hence,

$$\gamma_R(\Gamma(L)) = |V_1| + 2|V_2| = |V_1| + 2k \ge 2\lceil\frac{n}{2}\rceil. \tag{3.2}$$

If $\gamma_R(\Gamma(L)) = 2\lceil\frac{n}{2}\rceil$, then $|V_1| = 0$ since $\gamma_R(\Gamma(L))$ is even and $|V_1| \le 1$. Let

$$A = \{(r, c, L_{r,c}) : \ 1 \le r, c \le n, r \ne r_i, c \ne c_j, i, j = 1, 2, \dots, k\}.$$

(1) Suppose n is even. Assume $\gamma_R(\Gamma(L)) = n$. Then these equalities hold in Eqs (3.1) and (3.2), and $2k = n$, $|\{c_1, c_2, \dots, c_k\}| = |\{s_1, s_2, \dots, s_k\}| = k$. By symmetry, $|\{r_1, r_2, \dots, r_k\}| = k$.

Note that all vertex in A should be dominated by V_2. So, $L_{r,c} \in \{s_1, s_2, \dots, s_k\}$ for any vertices in A. Hence, the k^2 cells defined by rows $\{c_1, c_2, \dots, c_k\}$ and columns $\{r_1, r_2, \dots, r_k\}$ form a Latin square L' of order k and V_2 is a transversal of L'. Conversely, assume there is a Latin subsquare of order $n/2$, which contains a transversal B. Clearly, $f = (V_0, \emptyset, B)$ is a Roman dominating function of $\Gamma(L)$, where $V_0 = V(\Gamma(L)) \setminus B$, so $\gamma_R(\Gamma(L)) \le 2|B| = n$. Hence, $\gamma_R(\Gamma(L)) = n$.

(2) Suppose n is odd in the following. Assume $\gamma_R(\Gamma(L)) = n + 1$. Note that $|V_1| = 0$ and $n = 2k - 1$. By Eq (3.1),

$$2k \ge |\{s_1, s_2, \dots, s_k\}| + |\{c_1, c_2, \dots, c_k\}| \ge 2k - 1.$$

By symmetry,

$$2k \ge |\{s_1, s_2, \dots, s_k\}| + |\{r_1, r_2, \dots, r_k\}| \ge 2k - 1.$$

If $|\{s_1, s_2, \dots, s_k\}| = k - 1$, then $|\{r_1, r_2, \dots, r_k\}| = |\{c_1, c_2, \dots, c_k\}| = k$. Hence, all symbols of A belong to $\{s_1, s_2, \dots, s_k\}$, which implies A is a Latin subsquare of order $k - 1$. Assume without loss of generality $s_{k-1} = s_k$. Then $V_2 \setminus \{(r_k, c_k, s_k)\}$ is a partial transversal of L with cardinality $k - 1$. This implies L satisfies condition I.

Suppose now $|\{s_1, s_2, \ldots, s_k\}| = k$. If $|\{r_1, r_2, \ldots, r_k\}| = |\{c_1, c_2, \ldots, c_k\}| = k$, then $T_2 = V_2$ is a maximal partial transversal of L with cardinality k and the symbols of A belong to the symbol set of T_2. This implies that L satisfies condition II. If $|\{r_1, r_2, \ldots, r_k\}| = |\{c_1, c_2, \ldots, c_k\}| = k-1$, then all symbols of A belong to $\{s_1, s_2, \ldots, s_k\}$, which implies A is a Latin subsquare of order k, a contradiction with Observation 2. The remaining case is either $|\{r_1, r_2, \ldots, r_k\}| = k-1$ or $|\{c_1, c_2, \ldots, c_k\}| = k-1$. Assume without loss of generality $|\{r_1, r_2, \ldots, r_k\}| = k$ and $|\{c_1, c_2, \ldots, c_k\}| = k-1$. Let

$$A' = \{(r, c, L_{r,c}) : 1 \le r, c \le n, r \ne r_i, i = 1, 2, \ldots, k, c = c_j \; for \; some \; j \in [k]\}.$$

Since $|\{s_1, s_2, \ldots, s_k\}| = k$, the symbol set of A' is $[n] \setminus \{s_1, s_2, \ldots, s_k\}$, so A' is a Latin subsquare of order $n - k = k - 1$ of L. Since $|\{r_1, r_2, \ldots, r_k\}| = k$ and $|\{c_1, c_2, \ldots, c_k\}| = k-1$, there exist two distinct vertices $u, v \in V_2$ that are in the same column and $V_2 \setminus \{v\}$ is a partial transversal with cardinality $k-1$ in A^c. This implies that L satisfies condition III.

If L satisfies condition I, then let $V_2 = T_1 \cup \{(r, c, L_{r,c})\}$, where $(r, c, L_{r,c}) \in L'^c$, T_1 has no entries in row r and no entries in column c. If L satisfies condition II, then let $V_2 = T_2$. If L satisfies condition III, then let $V_2 = T_3 \cup \{(r, c, L_{r,c})\}$, where $E(L'') \cup T_3$ has no entries in column c and $L_{r,c}$ is the only entry that is not in the symbol set of $E(L'') \cup T_3$. One can check that $f = (E(L) \setminus V_2, \emptyset, V_2)$ is a Roman dominating function of L with $f(E(L)) = n + 1$.

By Proposition 2.1 and Theorem 3.1, we have the following result directly.

Corollary 3.1. *[19]* Let L be a Latin square of order $n \ge 2$. Then

$$\gamma(\Gamma(L)) \ge \lceil \frac{n}{2} \rceil,$$

and the equality holds if, and only if, $\gamma_R(\Gamma(L)) = 2\lceil \frac{n}{2} \rceil$.

By Corollary 3.1, the characterization of $\gamma_R(\Gamma(L)) = 2\lceil \frac{n}{2} \rceil$ in Theorem 3.1 is also a characterization of $\gamma(\Gamma(L)) = \lceil \frac{n}{2} \rceil$.

Proposition 3.2. For a Latin square of order $n = 2k$, according to the following block design

$$L = \begin{array}{|c|c|} \hline A & A+k \\ \hline A+k & A \\ \hline \end{array},$$

where A is a Latin square of order k its symbol set is $[k]$, and every symbol in $A + k$ is equal to the corresponding symbol in A plus k. Moreover, the symbols of the main diagonal of A are pairwise distinct. In $[k]$, choose two integers $r_0 \ne c_0$. Let L' be a Latin square of order n with $L'_{r,c} = L_{r,c}$, where $r \ne r_0$ or $c \ne c_0$ and $L'_{r_0,c_0} = L'_{r_0+k,c_0+k} = L_{r_0,c_0} + k$, $L'_{r_0+k,c_0} = L'_{r_0,c_0+k} = L_{r_0,c_0}$. Then

$$\gamma_R(\Gamma(L')) = 2\gamma(\Gamma(L')) - 1 = 2k + 1.$$

Proof. Clearly, L' does not contain a Latin subsquare of order k, and by Theorem 3.1 we know that $\gamma(\Gamma(L')) \ge k + 1$. On the other hand,

$$S = \{(1, 1, L'_{1,1}), (2, 2, L'_{2,2}), \ldots, (k, k, L'_{k,k})\} \bigcup \{(r_0 + k, c_0 + k, L_{r_0,c_0} + k)\}$$

is a dominating set of $\Gamma(L')$ with $|S| = k + 1$, then $\gamma(\Gamma(L')) \le \frac{n}{2} + 1$. In conclusion, $\gamma(\Gamma(L')) = k + 1$. Let $V_2 = S \setminus \{(r_0 + k, c_0 + k, L_{r_0,c_0} + k)\}$, $V_1 = \{(r_0 + k, c_0 + k, L_{r_0,c_0} + k)\}$ and $V_0 = V(\Gamma(L')) \setminus (V_1 \cup V_2)$, then $f = (V_0; V_1; V_2)$ is a Roman dominating function of $\Gamma(L')$ with weight $2k + 1$. By Proposition 3.1, $\gamma_R(\Gamma(L')) \ge 2\gamma(\Gamma(L')) - 1 = 2k + 2 - 1 = 2k + 1$. Hence, $\gamma_R(\Gamma(L')) = 2\gamma(\Gamma(L')) - 1 = 2k + 1$.

Example 3.1. Let L be the Latin square of order 10 that is shown in Figure 3. One can check

$$S = \{(1,1,1),(2,2,3),(3,1,8),(3,3,5),(4,4,2),(5,5,4)\}$$

is a minimum dominating set of $\Gamma(L)$. Let

$$V_1 = \{(8,6,8)\},\ V_2 = \{(1,1,1),(2,2,3),(3,3,5),(4,4,2),(5,5,4)\},$$

Then $f = (V(\Gamma(L)) \setminus (V_1 \cup V_2), V_1, V_2)$ is a Roman dominating function of $\Gamma(L)$. By Proposition 3.2, $\gamma_R(\Gamma(L)) = 11$.

1	2	3	4	5	6	7	8	9	10
2	3	4	5	1	7	8	9	10	6
8	4	5	1	2	3	9	10	6	7
4	5	1	2	3	9	10	6	7	8
5	1	2	3	4	10	6	7	8	9
6	7	8	9	10	1	2	3	4	5
7	8	9	10	6	2	3	4	5	1
3	9	10	6	7	8	4	5	1	2
9	10	6	7	8	4	5	1	2	3
10	6	7	8	9	5	1	2	3	4

Figure 3. A Latin square of order 10 where red numbers are corresponding to V_2, and blue numbers are corresponding to V_1.

4. Total Roman domination

Lemma 4.1. *[2]* If G is a graph without isolated vertices, then

$$\gamma_t(G) \le \gamma_{tR}(G) \le 2\gamma_t(G).$$

For a simple graph G, let $X \subseteq V(G)$ and $Y \subseteq X$. The *private neighborhood* of Y corresponding to X is defined as the set

$$PN(Y, X, G) = \{u \in V(G) | N(u) \cap X \subseteq Y\}.$$

Lemma 4.2. Let G be a graph without isolated vertices and each edge of which lies in a triangle. Let $f = (V_0; V_1; V_2)$ be a minimum total Roman dominating function of G with $|V_1|$ minimum. Then $\emptyset \ne PN(x, V_1 \cup V_2, G) \subseteq V_2$ for any vertex $x \in V_1$ and V_2 is a dominating set of G. Further, if any two nonadjacent vertices have a common neighbor in G and $\Delta(G) \le |V(G)| - 2$, then $\gamma_{tR}(G) \ge \frac{8\gamma_t(G)}{5}$.

Proof. Let x be any vertex in V_1. Since every vertex in V_0 is adjacent to a vertex in V_2, $PN(x, V_1 \cup V_2, G) \cap V_0 = \emptyset$. Since $f = (V_0; V_1; V_2)$ is a total Roman dominating function of G, there exists a vertex $y \in V_1 \cup V_2$ that is adjacent to x in G. If $PN(x, V_1 \cup V_2, G) = \emptyset$, then

$$f_1 = (V_0 \cup \{x\}; V_1 \setminus \{x, y\}; V_2 \cup \{y\})$$

is also a minimum total Roman dominating function with $|V_1 \setminus \{x, y\}| < |V_1|$, a contradiction. Hence $PN(x, V_1 \cup V_2, G) \neq \emptyset$.

We prove $PN(x, V_1 \cup V_2, G) \subseteq V_2$. Suppose to the contrary that there exists a vertex $y \in PN(x, V_1 \cup V_2, G) \cap V_1$. If $x \in PN(y, V_1 \cup V_2, G)$, then

$$f_2 = ((V_0 \cup \{x, y\}) \setminus \{z\}; V_1 \setminus \{x, y\}; V_2 \cup \{z\})$$

is also a minimum total Roman dominating function with $|V_1 \setminus \{x, y\}| < |V_1|$, a contradiction, where z is a common neighbor of x and y since edge xy lies in a triangle. Assume $x \notin PN(y, V_1 \cup V_2, G)$. Then

$$f_3 = (V_0 \cup \{y\}, V_1 \setminus \{x, y\}, V_2 \cup \{x\})$$

is also a minimum total Roman dominating function with $|V_1 \setminus \{x, y\}| < |V_1|$, a contradiction. Therefore, $PN(x, V_1 \cup V_2, G) \subseteq V_2$.

For any vertex $x \in V_1$, since $\emptyset \neq PN(x, V_1 \cup V_2, G) \subseteq V_2$, x has a neighbor in V_2, so V_2 is a dominating set of G.

Assume any two nonadjacent vertices have a common neighbor in G and $\Delta(G) \leq |V(G)| - 2$. The last inequality implies that $\gamma_{tR}(G) \geq 4$. If $|V_1| \leq 1$, then

$$\gamma_{tR}(G) = |V_1| + 2|V_2| \geq 2(|V_1| + |V_2|) - 1 \geq 2\gamma_t(G) - 1 \geq \frac{8\gamma_t(G)}{5}.$$

Suppose $|V_1| \geq 2$. For any two different vertices $x, y \in V_1$, we show $|PN(\{x, y\}, V_1 \cup V_2, G)| \geq 3$. Suppose to the contrary that there exist two different vertices $x, y \in V_1$ such that $PN(x, V_1 \cup V_2, G) = \{x'\}$, $PN(y, V_1 \cup V_2, G) = \{y'\}$ and $PN(\{x, y\}, V_1 \cup V_2, G) = \{x', y'\}$ (note that $|PN(x, V_1 \cup V_2, G)| \geq 1$ and $|PN(y, V_1 \cup V_2, G)| \geq 1$). Since $x', y' \in V_2$, $N_G(x') \cap (V_1 \cup V_2) = \{x\}$ and $N_G(y') \cap (V_1 \cup V_2) = \{y\}$. Let

$$f_4 = ((V_0 \cup \{x, y\}) \setminus \{z\}; (V_1 \setminus \{x, y\}) \cup \{z\}; V_2),$$

where z is a common neighbor of x' and y'. Then f_4 is also a total Roman dominating function with weight $f_4(V(G)) = f(V(G)) - 1$ contradicting the minimality of f. Since $\emptyset \neq PN(w, V_1 \cup V_2, G) \subseteq V_2$ for any vertex w in V_1 and $|V_1|$ is minimum, $PN(\{x, y\}, V_1 \cup V_2, G) \subseteq V_2$ for any two different vertices $x, y \in V_1$. Hence,

$$\frac{|V_1|}{2} \leq \frac{|V_2|}{3}.$$

Therefore,

$$\begin{aligned}
\gamma_{tR}(G) &= |V_1| + 2|V_2| \\
&= |V_1| + \frac{8|V_2|}{5} + \frac{2|V_2|}{5} \\
&\geq |V_1| + \frac{8|V_2|}{5} + \frac{3|V_1|}{5} \\
&\geq \frac{8(|V_1| + |V_2|)}{5} \\
&\geq \frac{8\gamma_t(G)}{5}.
\end{aligned}$$

A strongly regular graph with parameters (v, k, λ, μ) is a regular graph with v vertices of degree k such that every two adjacent vertices have exactly λ common neighbors and every two non-adjacent vertices have exactly μ common neighbors.

Corollary 4.1. If G is a strongly regular graph with parameters (v, k, λ, μ) and $\lambda\mu \geq 1$, then $\gamma_{tR}(G) \geq \frac{8\gamma_t(G)}{5}$. Furthermore, if L is a Latin square of order $n \geq 2$, then $\gamma_{tR}(\Gamma(L)) \geq \frac{8\gamma_t(\Gamma(L))}{5}$.

Lemma 4.3. *[19]* Let L be a Latin square of order $n \geq 3$. Then

$$\gamma_t(\Gamma(L)) \leq n - 1.$$

Theorem 4.1. Let L be a Latin square of order $n \geq 2$. Then

$$\gamma_{tR}(\Gamma(L)) > \frac{8n}{7}.$$

Proof. One can easily obtain that $\gamma_{tR}(\Gamma(L)) = 3, 4$ for $n = 2, 3$, respectively, which implies the inequality holds. Further, assume $n \geq 4$. Let $f = (V_0; V_1; V_2)$ be a minimum total Roman dominating function of $G = \Gamma(L)$ with V_1 minimum. Note that any two adjacent vertices $\Gamma(L)$ have n neighbors in common, which satisfies the conditions in Lemma 4.2. By Lemma 4.2, V_2 is a dominating set of G. Let V_{21} be the set of all isolated vertices in $G[V_2]$ and let $V_{22} = V_2 \setminus V_{21}$. By the definition of the total Roman dominating function, each vertex in V_{21} is adjacent to a vertex in V_1. By definition of Latin square graphs, $|N_G(v) \cap V_{21}| \leq 3$ for any vertex $v \in V_1$ (V_{21} is independent and any vertex in V_{21} is adjacent to a vertex in V_1). Therefore,

$$|V_{21}| \leq 3|V_1| \Rightarrow |V_{22}| \geq |V_2| - 3|V_1|. \tag{4.1}$$

For all $i = 1, 2, \ldots, n$, define $\mathfrak{r}_i = |\{(i, c, s) \in V_{22} | 1 \leq c, s \leq n\}|$, $\mathfrak{c}_i = |\{(r, i, s) \in V_{22} | 1 \leq r, s \leq n\}|$, $\mathfrak{s}_i = |\{(r, c, i) \in V_{22} | 1 \leq r, c \leq n\}|$. Moreover, define

$$\mathfrak{r} = \sum_{i=1, \mathfrak{r}_i \neq 0}^{n} (\mathfrak{r}_i - 1),\ \mathfrak{c} = \sum_{i=1, \mathfrak{c}_i \neq 0}^{n} (\mathfrak{c}_i - 1),\ \mathfrak{s} = \sum_{i=1, \mathfrak{s}_i \neq 0}^{n} (\mathfrak{s}_i - 1).$$

Let H be the graph with vertex set V_{22} and edges as follows. Two vertices in the same row $(r, c_1, s_1), (r, c_2, s_2) \in V_{22}$, with $c_1 < c_2$, are adjacent if, and only if, there does not exist $(r, c_3, s_3) \in V_{22}$ with $c_1 < c_3 < c_2$. Two vertices in the same column $(r_1, c, s_1), (r_2, c, s_2) \in V_{22}$, with $r_1 < r_2$, are adjacent if, and only if, there does not exist $(r_3, c, s_3) \in V_{22}$ with $r_1 < r_3 < r_2$. Two vertices in the same symbol $(r_1, c_1, s), (r_2, c_2, s) \in V_{22}$, with $r_1 < r_2$, are adjacent if, and only if, there does not exist $(r_3, c_3, s) \in V_{22}$ with $r_1 < r_3 < r_2$. By construction $|E(H)| = \mathfrak{r} + \mathfrak{c} + \mathfrak{s}$, since $\delta(G[V_{22}]) \geq 1$, $\delta(H) \geq 1$,

$$|E(H)| = \mathfrak{r} + \mathfrak{c} + \mathfrak{s} \geq \frac{|V_{22}|}{2}. \tag{4.2}$$

By Lemmas 4.1 and 4.3, $\gamma_{tR}(G) \leq 2\gamma_t(G) \leq 2n - 2$. We know that L has at least one row and one column whose cells do not correspond to any vertices of V_2. Let c_0 denote the column of L that does not contain any vertices of V_2 and, similarly, r_0 the row of L that do not contain any vertices of V_2. In c_0, there are at least $n - |V_2| + \mathfrak{r}$ vertices that do not share a common row with entries corresponding to vertices in V_2. Since V_2 is a dominating set of $\Gamma(L)$, all these $n - |V_2| + \mathfrak{r}$ vertices corresponding to entries in c_0 are dominated by vertices of V_2 that share the same symbols. Hence,

$$n - |V_2| + \mathfrak{r} \leq |V_2| - \mathfrak{s}. \tag{4.3}$$

Likewise,

$$n - |V_2| + \mathfrak{c} \leq |V_2| - \mathfrak{s}. \tag{4.4}$$

Plus (4.3) and (4.4), we have

$$2n + (\mathfrak{r} + \mathfrak{c} + \mathfrak{s}) \leq 4|V_2|. \tag{4.5}$$

By (4.1) and (4.2), we have

$$2n + \frac{|V_2| - 3|V_1|}{2} \leq 4|V_2|,$$

and equivalently

$$4n \leq 7|V_2| + 3|V_1| \leq \frac{7}{2}(2|V_2| + |V_1|) = \frac{7}{2}\gamma_{tR}(G). \tag{4.6}$$

Finally, we have

$$\gamma_{tR}(G) \geq \frac{8n}{7}.$$

Suppose $\gamma_{tR}(G) = \frac{8n}{7}$. Then all equalities hold in the above equations and we have

$$|V_1| = \mathfrak{s} = 0,\ |V_2| = \frac{4n}{7},\ \mathfrak{r} = \mathfrak{c} = \frac{n}{7}.$$

Hence, L has $n - |V_2| + \mathfrak{r} = \frac{4n}{7}$ rows and also $\frac{4n}{7}$ columns whose cells do not corresponding to any vertices of V_2. Let L' be the $\frac{4n}{7} \times \frac{4n}{7}$ submatrix of L obtained from L induced by these $\frac{4n}{7}$ rows and also $\frac{4n}{7}$ columns. Since all vertices correspond to L' should be dominated by vertices of V_2 that share the same symbols and $|V_2| = \frac{4n}{7}$, L' is a Latin subsquare of order $\frac{4n}{7}$, which contradicts Observation 2. Therefore,

$$\gamma_{tR}(G) > \frac{8n}{7}.$$

In [19], Pahlavsay et al. proved $\gamma_t(\Gamma(L)) \geq \frac{4n-2}{7}$ for $n \geq 2$. Since $2\gamma_t(\Gamma(L)) \geq \gamma_{tR}(\Gamma(L))$ by Lemma 4.1, we improve Pahlavsay et al.'s result to the following result by Theorem 4.1.

Corollary 4.2. Let L be a Latin square of order $n \geq 2$. Then

$$\gamma_t(\Gamma(L)) > \frac{4n}{7}.$$

Example 4.1. Figure 4 is a Latin square L of order six. Consider a total Roman dominating function $f = (V_0; V_1; V_2)$, where $V_2 = \{(1,1,1),(2,2,3),(3,3,2)\}$, $V_1 = \{(3,1,3)\}$ and all remaining vertices belong to V_0, then we have $\gamma_{tR}(\Gamma(L)) \leq 7$. Note that $V_1 \cup V_2$ is a total dominating set of $\Gamma(L)$. Hence $\gamma_t(\Gamma(L)) \leq 4$. On the other hand, by Theorem 4.1 and Corollary 4.2, $\gamma_{tR}(\Gamma(L)) \geq \lceil \frac{8\times 6}{7} \rceil = 7$ and $\gamma_t(\Gamma(L)) \geq \lceil \frac{4\times 6}{7} \rceil = 4$. Hence, $\gamma_{tR}(\Gamma(L)) = 7$ and $\gamma_t(\Gamma(L)) = 4$. This example shows all lower bounds for $\gamma_{tR}(\Gamma(L))$ in this section are tight.

1	2	3	4	5	6
2	3	1	5	6	4
3	1	2	6	4	5
4	5	6	1	2	3
5	6	4	2	3	1
6	4	5	3	1	2

Figure 4. A Latin square of order six.

5. Conclusions

In 2021, Pahlavsay et al. [19] studied the domination problems for Latin square graphs. We followed their results and studied the Roman domination problems for Latin square graphs. We obtained few tight bounds for these parameters. Our results generalized Pahlavsay et al.'s results. Domination in graphs has many variations (such as, p-domination, paired domination). The readers can study other domination problems of Latin square graphs.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

References

1. H. A. Ahangar, J. Amjadi, S. M. Sheikholeslami, M. Soroudi, On the total Roman domination number of graphs, *Ars Comb.*, **150** (2020), 225–240.
2. H. Abdollahzadeh Ahangar, M. A. Henning, V. Samodivkin, I. G. Yero, Total Roman domination in graphs, *Appl. Anal. Discr. Math.*, **10** (2016), 501–517. https://doi.org/10.2298/AADM160802017A
3. N. A. Abd Aziz, M. A. Henning, N. J. Rad, H. Kamarulhaili, Improved bounds on the k-tuple (Roman) domination number of a graph, *Graph. Combinator.*, **38** (2022), 75. https://doi.org/10.1007/s00373-022-02471-5

4. J. Amjadi, S. M. Sheikholeslami, M. Soroudi, On the total roman domination in trees, *Discuss. Math. Graph T.*, **39** (2019), 519–532. https://doi.org/10.7151/dmgt.2099
5. G. Asemian, N. J. Rad, A. Tehranian, H. Rasouli, On the total Roman domination stability in graphs, *AKCE Int. J. Graphs Co.*, **18** (2021), 166–172. https://doi.org/10.1080/09728600.2021.1992257
6. D. Best, T. Marbach, R. J. Stones, I. M. Wanless, Covers and partial transversals of Latin squares, *Des. Codes Cryptogr.*, **87** (2019), 1109–1136. https://doi.org/10.1007/s10623-018-0499-9
7. R. C. Bose, Strongly regular graphs, partial geometries and partially balanced designs, *Pac. J. Math.*, **13** (1963), 389–419. https://doi.org/10.2140/pjm.1963.13.389
8. B. Bollabás, *Graph theory*, New York: Springer-Verlag, 1979. https://doi.org/10.1007/978-1-4612-9967-7
9. H. Chen, C. Lu, Roman {2}-domination problem in graphs, *Discuss. Math. Graph T.*, **42** (2022), 641–660. https://doi.org/10.7151/dmgt.2332
10. E. J. Cockayne, P. A. Dreyer, S. M. Hedetniemi, S. T. Hedetniemi, Roman domination in graphs, *Discrete Math.*, **278** (2004), 11–22. https://doi.org/10.1016/j.disc.2003.06.004
11. H. Gao, J. Huang, Y. Yang, Double Roman domination in generalized Petersen graphs, *Bull. Iran. Math. Soc.*, **48** (2022), 885–894. https://doi.org/10.1007/s41980-021-00551-5
12. T. W. Haynes, S. T. Hedetniemi, P. J. Slater, *Fundamentals of domination in graphs*, New York: Marcel Dekker, 1998.
13. T. W. Haynes, S. T. Hedetniemi, P. J. Slater, *Domination in graphs: Advanced topics*, 1997.
14. T. W. Haynes, S. T. Hedetniemi, M. A. Henning, *Structures of domination in graphs: Developments in mathematics*, Cham: Springer, 2021. https://doi.org/10.1007/978-3-030-58892-2
15. A. D. Keedwell, J. Dénes, *Latin squares and their applications*, Elsevier, 2015.
16. Z. Li, A note on the bounds of Roman domination numbers, *AIMS Math.*, **6** (2021), 3940–3946. https://doi.org/10.3934/math.2021234
17. C. H. Liu, G. J. Chang, Roman domination on strongly chordal graphs, *J. Comb. Optim.*, **26** (2013), 608–619. https://doi.org/10.1007/s10878-012-9482-y
18. A. C. Martínez, I. Peterin, I. G. Yero, Roman domination in direct product graphs and rooted product graphs, *AIMS Math.*, **6** (2021), 11084–11096. https://doi.org/10.3934/math.2021643
19. B. Pahlavsay, E. Palezzato, M. Torielli, Domination for Latin square graphs, *Graph. Combinator.*, **37** (2021), 971–985. https://doi.org/10.1007/s00373-021-02297-7
20. A. Poureidi, N. J. Rad, Algorithmic and complexity aspects of problems related to total Roman domination for graph, *J. Comb. Optim.*, **39** (2020), 747–763. https://doi.org/10.1007/s10878-019-00514-x
21. A. Poureidi, N. J. Rad, Algorithmic aspects of the independent 2-rainbow domination number and independent Roman {2}-domination number, *Discuss. Math. Graph T.*, **42** (2022), 709–726. https://doi.org/10.7151/dmgt.2299
22. A. Poureidi, A. Abd, A. Noor, N. Jafari Rad, H. Kamarulhaili, Computing strong Roman domination of trees and unicyclic graphs in linear time, *Bull. Malays. Math. Sci. Soc.*, **45** (2022), 2509–2523. https://doi.org/10.1007/s40840-022-01301-4

23. H. M. Xing, X. Chen, X. G. Chen, A note on roman domination in graphs, *Discrete Math.*, **306** (2006), 3338–3340. https://doi.org/10.1016/j.disc.2006.06.018

24. J. Yang, Y. Chen, Z. Li, Some sufficient conditions for a tree to have its weak Roman domination number be equal to its domination number plus 1, *AIMS Math.*, **8** (2023), 17702–17718. https://doi.org/10.3934/math.2023904