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Research article

New formulas of convolved Pell polynomials

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Abstract: The article investigates a class of polynomials known as convolved Pell polynomials. This class generalizes the standard class of Pell polynomials. New formulas related to convolved Pell polynomials are established. These formulas may be useful in different applications, in particular in numerical analysis. New expressions are derived for the high-order derivatives of these polynomials, both in terms of their original polynomials and in terms of various well-known polynomials. As special cases, connection formulas linking the convolved Pell polynomials with some other polynomials can be deduced. The new moments formula of the convolved Pell polynomials that involves a terminating hypergeometric function of the unit argument is given. Then, some reduced specific moment formulas are deduced based on the reduction formulas of some hypergeometric functions. Some applications, including new specific definite and weighted definite integrals, are deduced based on some of the developed formulas. Finally, a matrix approach for this kind of polynomial is presented.

Keywords: convolved polynomials; recurrence relation; Zeilberger's algorithm; connection formulas; matrix approach
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1. Introduction

Many sequences of polynomials are described by recursive formulas of order two. The investigations of different polynomials have garnered much interest from many authors. Some different degenerate polynomials were investigated in [1–3]. Some orthogonal polynomials were studied in [4–6]. Some other contributions concerning different sequences of polynomials can be found in [7–12].

The Fibonacci and Lucas polynomials and their related numbers are of essential importance due to their various applications in biology, physics, statistics, and computer science [13]. Many authors were interested in introducing and investigating several generalizations and modifications of Fibonacci and Lucas polynomials. The authors of [14] investigated two classes that generalize Fibonacci and Lucas polynomials, and they utilized them to compute some radicals in reduced forms. In [15], Gauss-Fibonacci and Gauss-Lucas polynomials were studied along with their applications, while in [16], generalized Lucas polynomials and their links with Fibonacci and Lucas polynomials were discussed. Some other contributions regarding these polynomials and their applications can be found in [17–19].

The derivation of several formulas for special functions is of interest to several mathematicians. Some of these formulas play important roles in applied mathematics and, in particular, in numerical analysis; see for instance [20–23]. For example, it is possible to obtain spectral solutions of differential equations by expressing the different derivatives of various polynomials as combinations of their original ones (see, for example, [24] and [25]). In addition, the first-order derivatives of different polynomials may be utilized to apply a matrix approach to solve different types of differential equations (see [26–28]). Moreover, the formulas concerned with the repeated integrals of different special functions allow us to find numerical solutions to some important differential equations (see, for example, [29]).

Pell and Pell-Lucas polynomials are particular polynomials that generalize Fibonacci and Lucas polynomials. There are several contributions related to Pell and Pell-Lucas polynomials and their related numbers. For example, various studies on Pell and Pell-Lucas numbers can be found in the book by Koshy [30]. Formulas on the integration and derivative sequences of these polynomials have been given in [31]. Some Chebyshev and Pell connections have been derived in [32]. A kind of polynomials called "convolved (p, q) Fibonacci polynomials" was considered in [33]. This type of polynomial generalizes many famous types of polynomials. For some other contributions related to Pell and Pell-Lucas polynomials and related numbers; reference can be made to [34–39]. In numerical analysis, these polynomials have also been employed to solve various differential equations; see for instance, [40–42].

In this paper, we are interested in investigating a particular class of the general class established in [33]. More precisely, we will investigate a class of polynomials called convolved Pell polynomials which generalizes the well-known class of Pell polynomials. We will develop new formulas for this type of polynomials. We note here that the contributions related to these polynomials are few. This provided the motivation for our study in this paper.

The novelty of the contributions of this paper can be listed in points as follows:

- New formulas concerned with this type of polynomial are established.
- New connections between these polynomials and different celebrated orthogonal and non-orthogonal polynomials are derived.

Here are the main points of the contributions of this paper:

- We establish new derivative formulas of convolved Pell polynomials based on the connection formulas between two convolved Pell polynomials of different parameters.
- We establish new derivative formulas of convolved Pell polynomials in terms of different well-known special functions. As a result, connection formulas between convolved Pell polynomials and some other polynomials can be deduced.

- We develop moments formulas of convolved Pell polynomials. These formulas can be calculated by using the reduction formulas of certain terminating hypergeometric functions.
- We deduce some results related to the Pell polynomials as special cases of those of the convolved Pell polynomials.
- We present some applications of the derived formulas.
- We present a new approach for the convolved Pell polynomials, based on matrix calculus.

The paper has the following structure. We will discuss some elementary properties of the convolved Pell polynomials in the next section. In addition, an overview of Jacobi polynomials (JPs) and the two classes of generalized Fibonacci and generalized Lucas polynomials is given. Section 3 is devoted to developing new expressions for the derivatives of the convolved Pell polynomials. Section 4 presents further formulas for the derivatives of convolved Pell polynomials as combinations of various special functions, including some orthogonal polynomials. Moments formulas of the convolved Pell polynomials are derived in Section 5 accompanied by some reductions for some specific cases. In Section 6, some applications of the derived formulas are presented. In Section 7, a matrix-based approach for the convolved Pell polynomials is proposed. Recurrence relations and determinant forms are two examples of the fundamental results that are presented. Section 8 provides some final thoughts.

2. Some fundamental properties of the convolved Pell polynomials

This section presents an overview of the convolved Pell polynomials. In addition, an account of JPs and the generalized Fibonacci and generalized Lucas polynomials is given.

2.1. An overview of the convolved Pell polynomials

For every complex number M, the authors of [33] introduced the convolved (a, b)-Fibonacci polynomials $G_{i,M}(x)$, where $a(x)$ and $b(x)$ are polynomials with real coefficients. The polynomials $G_{i,M}(x)$ (of degree i) can be constructed by using the generating function:

$$\left(1 - a(x)\,t - b(x)\,t^2\right)^{-M} = \sum_{i=0}^{\infty} G_{i,M}(x)\,t^i,$$

and they can be explicitly expressed as

$$G_{i,M}(x) = \sum_{r=0}^{\lfloor \frac{i}{2} \rfloor} \binom{M+r-1}{r} \binom{M+i-r-1}{i-2r} a^{i-2r}(x)\,b^r(x),$$

where $\lfloor z \rfloor$ denotes the floor function.
Moreover, these polynomials satisfy the following recursive formula:

$$i\,G_{i,M}(x) - (M+i-1)\,a(x)\,G_{i-1,M}(x) - (2M+i-2)\,b(x)\,G_{i-1,M}(x) = 0, \quad i \geq 2,$$

with $G_{0.M}(x) = 1,\ G_{1,M}(x) = M\,a(x)$.

In this paper, we are interested in investigating a subclass of the convolved (a, b)-Fibonacci polynomials, that is the class of the convolved Pell polynomials $P_{i,M}(x)$. This class corresponds to the following choice:

$$a(x) = 2\,x, \quad b(x) = 1.$$

In power form, Pell polynomials are presented as follows:

$$P_{i,M}(x) = \sum_{r=0}^{\lfloor \frac{i}{2} \rfloor} \frac{2^{i-2\,r}\,(M)_{i-r}}{r!\,(i-2\,r)!}\,x^{i-2r}, \tag{2.1}$$

and the following recurrence relation applies

$$P_{i,M}(x) = \frac{2(i+M-1)}{i}\,x\,P_{i-1,M}(x) + \frac{(i+2\,M-2)}{i}\,P_{i-2,M}(x), \qquad i \geq 2, \tag{2.2}$$

with $P_{0,M}(x) = 1, P_{1,M}(x) = 2Mx$.

It can be shown that the inversion formula of $P_{i,M}(x)$ is

$$x^i = \frac{i!}{2^i} \sum_{r=0}^{\lfloor \frac{i}{2} \rfloor} \frac{(-1)^r}{r!\,(M)_{i-2r}\,(M+i-2r+1)_r}\,P_{i-2r,M}(x). \tag{2.3}$$

Remark 2.1. *It is important to notice that the convolved Pell polynomials generalize the well-known Pell polynomials. In fact, when $M = 1$, $P_{i,M}(x)$ become Pell polynomials, that is $P_i(x) = P_{i,1}(x)$.*

2.2. Classical JPs

The standard JPs $P_r^{(\gamma,\delta)}(x),\ x \in [-1,1],\ r \geq 0\ \gamma > -1, \delta > -1$, (see [43]) can be expressed in the following hypergeometric form

$$P_r^{(\gamma,\delta)}(z) = \frac{(\gamma+1)_r}{r!}\ {}_2F_1\left(\begin{matrix} -r, r+\gamma+\delta+1 \\ \gamma+1 \end{matrix}\ \middle|\ \frac{1-z}{2}\right).$$

The normalized JPs may be given by (see [44])

$$J_r^{(\gamma,\delta)}(z) = {}_2F_1\left(\begin{matrix} -r, r+\gamma+\delta+1 \\ \gamma+1 \end{matrix}\ \middle|\ \frac{1-z}{2}\right). \tag{2.4}$$

Remark 2.2. *It is to be noted that $J_r^{(\gamma,\delta)}(z)$ which are defined in* (2.4) *satisfy:*

$$J_r^{(\gamma,\delta)}(1) = 1.$$

An advantage of introducing such polynomials is that some well-known polynomials can be easily derived from them.

The following celebrated classes of polynomials are particular cases of $J_r^{(\gamma,\delta)}(z)$:

$$\begin{aligned} T_r(z) &= J_r^{(-\frac{1}{2},-\frac{1}{2})}(z), & U_r(z) &= (r+1)\,J_r^{(\frac{1}{2},\frac{1}{2})}(z), \\ V_r(z) &= J_r^{(-\frac{1}{2},\frac{1}{2})}(z), & W_r(z) &= (2r+1)\,J_r^{(\frac{1}{2},-\frac{1}{2})}(z), \\ \mathcal{L}_r(z) &= J_r^{(0,0)}(z), & C_r^{(\gamma)}(z) &= J_r^{(\gamma-\frac{1}{2},\gamma-\frac{1}{2})}(z), \end{aligned}$$

where $T_r(z), U_r(z), V_r(z), W_r(z)$, denote, respectively, the Chebyshev polynomials (CPs) of the first, second, third and fourth kinds, while $\mathcal{L}_r(z)$ and $C_r^{(\gamma)}(z)$ denote respectively, the Legendre and

ultraspherical polynomials.
The corresponding shifted polynomials on [0, 1] denoted by $\tilde{J}_r^{(\gamma,\delta)}(z)$ are defined by

$$\tilde{J}_r^{(\gamma,\delta)}(z) = J_r^{(\gamma,\delta)}(2z-1).$$

The polynomials $\tilde{J}_r^{(\gamma,\delta)}(z)$ have the following analytic form

$$\tilde{J}_r^{(\gamma,\delta)}(z) = \sum_{i=0}^{r} \frac{(-1)^i\, r!\, \Gamma(\gamma+1)\, (\delta+1)_r\, (\gamma+\delta+1)_{2r-i}}{i!\, (r-i)!\, \Gamma(r+\gamma+1)\, (\gamma+\delta+1)_r\, (\delta+1)_{r-i}}\, z^{r-i}. \tag{2.5}$$

The inversion formula of (2.5) is given by

$$z^\ell = \sum_{m=0}^{\ell} \frac{\binom{\ell}{m}(\gamma+1)_{\ell-m}\, (\ell-m+\delta+1)_m}{(2\ell-2m+\gamma+\delta+2)_m\, (\ell-m+\gamma+\delta+1)_{\ell-m}}\, \tilde{J}_{\ell-m}^{(\gamma,\delta)}(z).$$

The useful books [43, 45] regarding JPs and related special families might be consulted.

2.3. Generalized Fibonacci and generalized Lucas polynomials

Assume the nonzero real numbers A, B, R, S. The two generalized classes of Fibonacci and Lucas polynomials may be constructed, respectively, in accordance with the following recurrence relations:

$$F_0^{A,B}(z) = 1,\ F_1^{A,B}(z) = A\,z, \qquad F_k^{A,B}(x) = A\,z\,F_{k-1}^{A,B}(z) + B\,F_{k-2}^{A,B}(z), \quad k \geq 2, \tag{2.6}$$

and

$$L_0^{R,S}(z) = 2,\ L_1^{R,S}(z) = R\,z, \qquad L_k^{R,S}(z) = R\,z\,L_{k-1}^{R,S}(z) + S\,L_{k-2}^{R,S}(z), \quad k \geq 2. \tag{2.7}$$

The polynomials $F_k^{A,B}(z)$ and $L_k^{R,S}(z)$ can respectively be expressed by Binet's formulas as follows:

$$F_k^{A,B}(z) = \frac{\left(A\,z + \sqrt{A^2\,z^2 + 4\,B}\right)^k - \left(A\,z - \sqrt{A^2\,z^2 + 4\,B}\right)^k}{2^k\,\sqrt{A^2\,z^2 + 4\,B}},$$

and

$$L_k^{R,S}(z) = \frac{\left(R\,z + \sqrt{R^2\,z^2 + 4\,S}\right)^k + \left(R\,z - \sqrt{R^2\,z^2 + 4\,S}\right)^k}{2^k}.$$

The two analytic formulas of these polynomials are, respectively,

$$F_k^{A,B}(z) = \sum_{m=0}^{k} A^{k-2m} B^m \binom{k-m}{m} z^{k-2m}, \quad k \geq 0,$$

and

$$L_k^{R,S}(z) = k \sum_{\ell=0}^{\lfloor \frac{k}{2} \rfloor} \frac{R^{k-2\ell}\, S^{\ell} \binom{k-\ell}{\ell}}{k-\ell}\, z^{k-2\ell}, \quad k \geq 1.$$

Their inversion formulas are respectively given by

$$z^m = \frac{1}{A^m} \sum_{i=0}^{\lfloor \frac{m}{2} \rfloor} \frac{(-1)^i (m-2i+1)(m-i+2)_{i-1} B^i}{i!}\, F_{m-2i}^{A,B}(z), \quad m \geq 0,$$

and

$$z^m = \frac{1}{R^m} \sum_{i=0}^{\lfloor \frac{m}{2} \rfloor} \frac{\xi_{m-2\,i}(-1)^i\,(m-i+1)_i\,S^i}{i!}\,L^{R,S}_{m-2i}(z), \quad m \ge 0,$$

with

$$\xi_m = \begin{cases} \frac{1}{2}, & m = 0, \\ 0, & \text{otherwise.} \end{cases} \tag{2.8}$$

Several polynomial sequences are specific ones of the two classes of generalized Fibonacci and generalized Lucas polynomials. More precisely, we have the following:

$$\begin{aligned} F_{k+1}(x) &= F^{1,1}_k(x), & P_{k+1}(x) &= F^{2,1}_k(x), \\ \mathcal{F}_{k+1}(x) &= F^{3,-2}_k(x), & U_k(x) &= F^{2,-1}_k(x), \\ E_k(x,\mu) &= F^{1,-\mu}_k(x), \end{aligned}$$

where $F_k(x), P_k(x), \mathcal{F}_k(x)$ are, respectively, the Fibonacci, Pell and Fermat polynomials, while $U_k(x)$ and $E_k(x,\mu)$ denote, respectively, the Chebyshev and the Dickson polynomials of the second kind. Furthermore,

$$\begin{aligned} L_k(x) &= L^{1,1}_k(x), & Q_k(x) &= L^{2,1}_k(x), \\ f_k(x) &= L^{3,-2}_k(x), & 2\,T_k(x) &= L^{2,-1}_k(x), \\ D_k(t,\mu) &= L^{1,-\mu}_k(x), \end{aligned}$$

where $L_k(x)$ denotes the Lucas polynomials, $Q_k(x)$ represents the Pell-Lucas polynomials, $f_k(x)$ denotes the Fermat-Lucas polynomials, while $T_k(x)$ and $D_k(t,\mu)$ denote, respectively, the Chebyshev and the Dickson polynomials of the first kind.

3. New derivative expressions of the convolved Pell polynomials

Here we derive new expressions for the high-order derivatives of the convolved Pell polynomials in terms of their original polynomials. To do this, we will follow a new approach that relies on utilizing the connection formula between convolved Pell polynomials with two different parameters. For this purpose, we prove two preliminary lemmas.

Lemma 3.1. *Consider the non-negative integers i, q with $i \ge q$. We apply the following equation:*

$$D^q P_{i,M}(x) = \frac{2^q\,\Gamma(q+M)}{\Gamma(M)}\,P_{i-q,M+q}(x). \tag{3.1}$$

Proof. We proceed by induction. We first prove that the lemma applies for $q = 1$, that is,

$$DP_{i,M}(x) = 2\,M\,P_{i-1,M+1}(x). \tag{3.2}$$

The last relation can be easily proved by differentiating the power form representation of $P_{i,M}(x)$ in (2.1) to get

$$DP_{i,M}(x) = \sum_{r=0}^{\lfloor \frac{i}{2} \rfloor} \frac{2^{i-2\,r}\,(M)_{i-r}}{r!\,(i-2\,r-1)!}\,x^{i-2r-1}$$

$$=2\,M\sum_{r=0}^{\lfloor\frac{i-1}{2}\rfloor}\frac{2^{i-2\,r-1}\,(M+1)_{i-r-1}}{r!\,(i-2\,r-1)!}\,x^{i-2r-1}=2\,M\,P_{i-1,M+1}(x).$$

Assume now that (3.1) is valid. For the proof to be complete, we need to show that

$$D^{q+1}P_{i,M}(x)=\frac{2^{q+1}\,\Gamma(q+M+1)}{\Gamma(M)}\,P_{i-q-1,M+q+1}(x). \tag{3.3}$$

From (3.1), we have

$$D^{q+1}P_{i,M}(x)=\frac{2^{q}\,\Gamma(q+M)}{\Gamma(M)}\,D\,P_{i-q,M+q}(x).$$

If relation (3.2) is inserted into the last relation, then (3.3) can be immediately obtained. This ends the proof. □

Lemma 3.2. *Let $P_{i,M}(x)$ and $P_{j,T}(x)$ be two convolved Pell polynomials of degrees i and j, respectively. The following connection formula applies:*

$$P_{i,M}(x)=\frac{\Gamma(T)\,\Gamma(i+M)}{\Gamma(M)}\sum_{\ell=0}^{\lfloor\frac{i}{2}\rfloor}\frac{(-1)^{\ell}\,(i-2\ell+T)\,(-\ell-M+T+1)_{\ell}}{\ell!\,\Gamma(i-\ell+T+1)\,(-i-M+1)_{\ell}}\,P_{i-2\,\ell,T}(x). \tag{3.4}$$

Proof. We prove (3.4) by induction. For $i=0$, it is true. Assume that (3.4) holds for any positive number $m<i$. Now, consider the recurrence relation satisfied by $P_{i,M}(x)$ in (2.2). The application of the induction hypothesis to $P^{M}_{i-1}(x)$ and $P^{M}_{i-2}(x)$ yields

$$P_{i,M}(x)=\frac{2(i+M-1)}{i}\,x\sum_{L=0}^{\lfloor\frac{i-1}{2}\rfloor}G_{\ell,i-1}\,P_{i-2\,\ell-1,T}(x)+\frac{(i+2\,M-2)}{i}\sum_{L=0}^{\lfloor\frac{i}{2}\rfloor-1}G_{\ell,i-2}\,P_{i-2\,\ell-2,T}(x), \tag{3.5}$$

where

$$G_{\ell,i}=\frac{(-1)^{\ell}\,\Gamma(T)\,\Gamma(i+M)\,(i-2\ell+T)\,(-\ell-M+T+1)_{\ell}}{\ell!\,\Gamma(M)\,\Gamma(i-\ell+T+1)\,(-i-M+1)_{\ell}}.$$

Based on the recurrence relation (2.2), we get

$$x\,P_{i-2\ell-1,T}=\frac{-i+2\ell-2T+2}{2(i-2\ell+T-1)}\,P_{i-2\ell-2,T}(x)+\frac{i-2\ell}{2(i-2\ell+T-1)}P_{i-2\ell,T}(x).$$

If we insert the last relation into relation (3.5), we have

$$\begin{aligned}P_{i,M}(x)=&\frac{i+2M-2}{i}\sum_{\ell=0}^{\lfloor\frac{i}{2}\rfloor-1}G_{\ell,i-2}\,P_{i-2\ell-2,T}(x)+\sum_{\ell=0}^{\lfloor\frac{i-1}{2}\rfloor}G_{\ell,i-1}\Bigg(\frac{(i-2\ell)(i+M-1)}{i(i-2\ell+T-1)}\,P_{i-2\ell,T}(x)\\&-\frac{(i+M-1)(i-2\ell+2T-2)}{i(i-2\ell+T-1)}\,P_{i-2\ell-2,T}(x)\Bigg).\end{aligned} \tag{3.6}$$

Now, in order to prove (3.4), we prove it for both the two cases corresponding to i even and i odd. From (3.6), it is easy to see that

$$\begin{aligned} P_{2i,M}(x) =& \frac{\Gamma(2i+M)\,\Gamma(T)}{\Gamma(M)\,\Gamma(2i+T)}\,P_{2i,T}(x) + \sum_{\ell=1}^{i-1}\Bigg(\frac{(-1+i+M)\,G_{\ell-1,2i-2}}{i} \\ &- \frac{(-1+2i+M)(i-\ell+T)\,G_{\ell-1,2i-1}}{i\,(1+2i-2\ell+T)} + \frac{(i-\ell)(-1+2i+M)\,G_{\ell,2i-1}}{i\,(-1+2i-2\ell+T)}\Bigg)\,P_{2i-2\ell,T}(x) \\ &+ \frac{(-1+i+M)\,G_{i-1,2i-2}}{i} - \frac{(-1+2i+M)T\,G_{i-1,2i-1}}{i(1+T)}. \end{aligned} \tag{3.7}$$

It is not difficult to show the identity:

$$\begin{aligned} &\frac{(-1+i+M)\,G_{\ell-1,2i-2}}{i} - \frac{(-1+2i+M)(i-\ell+T)\,G_{\ell-1,2i-1}}{i\,(1+2i-2\ell+T)} \\ &+ \frac{(i-\ell)(-1+2i+M)\,G_{\ell,2i-1}}{i\,(-1+2i-2\ell+T)} = G_{\ell,2i}, \end{aligned}$$

and accordingly, (3.7) turns into

$$\begin{aligned} P_{2i,M}(x) =& \frac{\Gamma(T)\,\Gamma(2i+M)}{\Gamma(M)\,\Gamma(2i+T)}\,P_{2i,T}(x) + \sum_{\ell=1}^{i-1} G_{\ell,2i}\,P_{2i-2\ell,T}(x) \\ &+ \frac{(i-1+M)G_{i-1,2i-2}}{i} - \frac{(-1+2i+M)T\,G_{i-1,2i-1}}{i(1+T)}. \end{aligned} \tag{3.8}$$

If we note the two identities

$$\begin{aligned} G_{0,2i} =& \frac{\Gamma(T)\,\Gamma(2i+M)}{\Gamma(M)\,\Gamma(2i+T)}, \\ G_{i,2i} =& \frac{(-1+i+M)G_{i-1,2i-2}}{i} - \frac{(-1+2i+M)\,T\,G_{i-1,2i-1}}{i(1+T)}, \end{aligned}$$

then, from (3.8), the following formula can be obtained:

$$P_{2i,M}(x) = \sum_{\ell=0}^{i} G_{\ell,2i}\,P_{2i-2\,\ell,T}(x). \tag{3.9}$$

Performing similar algebraic computations, it can be shown that

$$P_{2i+1,M}(x) = \sum_{\ell=0}^{i} G_{\ell,2i+1}\,P_{2i-2\,\ell+1,T}(x). \tag{3.10}$$

Merging the two formulas (3.9) and (3.10) leads to (3.4). This proves Lemma 3.2. □

Now, we can prove the main theorem in which the q-th derivative of $P_{r,M}(x)$ is expressed as a combination of their original ones.

Theorem 3.1. *Assume that r and q are two positive integers with $r \geq q \geq 1$. The following formula is valid:*

$$D^q P_{r,M}(x) = 2^q \sum_{\ell=0}^{\lfloor \frac{r-q}{2} \rfloor} (-1)^\ell \binom{\ell+q-1}{\ell} (-2\ell+M-q+r)(-\ell+M-q+r+1)_{q-1}\, P_{r-q-2\ell,M}(x). \quad (3.11)$$

Proof. From Lemma 3.2, we have

$$P_{r-q,M+q}(x) = \frac{\Gamma(M)\,\Gamma(M+r)}{\Gamma(M+q)} \sum_{\ell=0}^{\lfloor \frac{r-q}{2} \rfloor} \frac{(-1)^\ell\,(-\ell-q+1)_\ell\,(-2\ell+M-q+r)}{\ell!\,(-M-r+1)_\ell\,\Gamma(-\ell+M-q+r+1)}\, P_{r-q-2\ell,M}(x). \quad (3.12)$$

Formula (3.12) together with Lemma 3.1 yields the desired result given by (3.11). □

The derivatives of the Pell polynomials can be deduced from Theorem 3.1, by setting $M = 1$.

Corollary 3.1. *Assume that r and q are positive integers with $r \geq q \geq 1$. The following expression for the Pell polynomials applies:*

$$D^q P_r(x) = 2^q \sum_{\ell=0}^{\lfloor \frac{r-q}{2} \rfloor} (-1)^\ell (1+r-2\ell-q) \binom{\ell+q-1}{\ell} (r-\ell-q+2)_{q-1}\, P_{r-q-2\ell}(x).$$

4. Derivatives of the convolved Pell polynomials in terms of various polynomials

In this section, we present new expressions of $D^q P_{j,M}(x),\ j \geq q$ as combinations of some special functions.

4.1. Derivatives in terms of some orthogonal polynomials

Theorem 4.1. *$D^q P_{j,M}(x),\ j \geq q$ can be expanded in terms of the ultraspherical polynomials $C_j^{(\lambda)}(x)$ as follows:*

$$\begin{aligned} D^q P_{j,M}(x) = &\frac{2^{q+1}\,\Gamma(j+M)\,\Gamma(1+\lambda)}{\Gamma(M)\,\Gamma(1+2\lambda)} \sum_{\ell=0}^{\lfloor \frac{j-q}{2} \rfloor} \frac{(j-2\ell-q+\lambda)\,\Gamma(j-2\ell-q+2\lambda)}{\ell!\,(j-2\ell-q)!\,\Gamma(1+j-\ell-q+\lambda)} \\ &\times {}_2F_1\left(\begin{matrix} -\ell, -j+\ell+q-\lambda \\ 1-j-M \end{matrix} \middle| -1 \right) C^{(\lambda)}_{j-q-2\ell}(x). \end{aligned} \quad (4.1)$$

Proof. The analytical expression of $P_{j,M}(x)$ leads to the following formula

$$D^q P_{j,M}(x) = \sum_{r=0}^{\lfloor \frac{j-q}{2} \rfloor} \frac{2^{j-2r}\,(M)_{j-r}(1+j-q-2r)_q}{(j-2r)!\,r!}\, x^{j-q-2r}.$$

After applying the inversion formula of $C_j^{(\lambda)}(x)$ (see [46]), the last formula turns into

$$\begin{aligned} D^q P_{j,M}(x) =& \frac{\sqrt{\pi}}{\Gamma(M)\Gamma\left(\frac{1}{2}+\lambda\right)} \sum_{r=0}^{\left\lfloor \frac{j-q}{2} \right\rfloor} \frac{2^{1-j+q+2r-2\lambda} 2^{j-2r}\, \Gamma(j+M-r)}{r!} \\ & \times \sum_{t=0}^{\left\lfloor \frac{j-q}{2} \right\rfloor - r} \frac{(j-q-2r-2t+\lambda)\Gamma(j-q-2r-2t+2\lambda)}{t!(j-q-2r-2t)!\Gamma(1+j-q-2r-t+\lambda)} C_{j-2r-q-2t}^{(\lambda)}(x). \end{aligned} \tag{4.2}$$

Relation (4.2) turns into:

$$\begin{aligned} D^q P_{j,M}(x) =& \frac{\sqrt{\pi}\, 2^{1+q-2\lambda}}{\Gamma(M)\Gamma\left(\frac{1}{2}+\lambda\right)} \sum_{\ell=0}^{\left\lfloor \frac{j-q}{2} \right\rfloor} \frac{(j-2\ell-q+\lambda)\Gamma(j-2\ell-q+2\lambda)}{(j-2\ell-q)!} \\ & \times \sum_{r=0}^{\ell} \frac{\Gamma(j+M-r)}{r!(\ell-r)!\Gamma(1+j-\ell-q-r+\lambda)} C_{j-q-2\ell}^{(\lambda)}(x), \end{aligned}$$

which can be converted to the form of (4.1). □

Some particular formulas can be obtained from (4.1) as special cases.

Corollary 4.1. *$D^q P_{j,M}(x)$, $j \geq q$ can be written as*

$$\begin{aligned} D^q P_{j,M}(x) =& \frac{2^{q-1}\ \sqrt{\pi}\, \Gamma(j+M)}{\Gamma(M)} \sum_{\ell=0}^{\left\lfloor \frac{j-q}{2} \right\rfloor} \frac{1+2j-4\ell-2q}{\ell!\, \Gamma\left(\frac{3}{2}+j-\ell-q\right)}\ {}_2F_1\left(\left. \begin{matrix} -\ell, -j+\ell+q-\frac{1}{2} \\ 1-j-M \end{matrix} \right| -1 \right) \times \\ & \mathcal{L}_{j-q-2\ell}(x), \end{aligned} \tag{4.3}$$

$$D^q P_{j,M}(x) = \frac{2^{q+1}\, \Gamma(j+M)}{\Gamma(M)} \sum_{\ell=0}^{\left\lfloor \frac{j-q}{2} \right\rfloor} \frac{1}{\ell!\, (j-\ell-q)!}\ {}_2F_1\left(\left. \begin{matrix} -\ell, -j+\ell+q \\ 1-j-M \end{matrix} \right| -1 \right) T_{j-q-2\ell}(x), \tag{4.4}$$

$$D^q P_{j,M}(x) = \frac{2^{q}\, \Gamma(j+M)}{\Gamma(M)} \sum_{\ell=0}^{\left\lfloor \frac{j-q}{2} \right\rfloor} \frac{1+j-2\ell-q}{\ell!\, (j-\ell-q+1)!}\ {}_2F_1\left(\left. \begin{matrix} -\ell, -j+\ell+q-1 \\ 1-j-M \end{matrix} \right| -1 \right) U_{j-q-2\ell}(x), \tag{4.5}$$

where $\mathcal{L}_j(x)$ denotes the Legendre polynomials and $T_j(x)$ and $U_j(x)$ denote CPs of the first and second kind, respectively.

In what follows, making steps similar to those in the proof of Theorem 4.1, we give the expressions of the derivatives of the convolved Pell polynomials as combinations of Hermite, Laguerre, shifted Jacobi, generalized Fibonacci, generalized Lucas and Bernoulli polynomials.

Theorem 4.2. *$D^q P_{j,M}(x)$, $j \geq q$ can respectively be expanded in terms of the Hermite polynomials $H_j(x)$ and Laguerre polynomials $L_j^{(\gamma)}(x)$ as follows:*

$$D^q P_{j,M}(x) = \frac{2^{q+1}\, \Gamma\left(\frac{3}{2}+j\right)}{\sqrt{\pi}} \sum_{\ell=0}^{\left\lfloor \frac{j-q}{2} \right\rfloor} \frac{1}{\ell!\, (j-2\ell-q)!}\ {}_1F_1\left(-\ell; -\frac{1}{2}-j; 1\right) H_{j-q-2\ell}(x), \tag{4.6}$$

$$D^q P_{j,M}(x) = \nu_j \left(\sum_{\ell=0}^{\left\lfloor \frac{j-q}{2} \right\rfloor} U_{\ell,j,q}\, L^{(\gamma)}_{j-q-2\ell}(x) + \sum_{\ell=0}^{\left\lfloor \frac{1}{2}(j-q-1) \right\rfloor} \bar{U}_{\ell,j,q}\, L^{(\gamma)}_{j-q-2\ell-1}(x) \right), \tag{4.7}$$

where

$$\begin{aligned}
\nu_j =& (-1)^{j-q}\, 2^j\, (M)_j\, \Gamma(1+j-q+\gamma),\\
U_{\ell,j,q} =& \frac{1}{(2\ell)!\,\Gamma(1+j-2\ell-q+\gamma)}\ {}_2F_3\left(\begin{matrix} -\ell, \frac{1}{2}-\ell \\ -\frac{3}{2}-j, -\frac{j}{2}+\frac{q}{2}-\frac{\gamma}{2}, \frac{1}{2}-\frac{j}{2}+\frac{q}{2}-\frac{\gamma}{2} \end{matrix} \middle| -\frac{1}{4} \right),\\
\bar{U}_{\ell,j,q} =& -\frac{1}{(2\ell+1)!\,\Gamma(j-2\ell-q+\gamma)}\ {}_2F_3\left(\begin{matrix} -\ell, \frac{1}{2}-\ell \\ -\frac{3}{2}-j, -\frac{j}{2}+\frac{q}{2}-\frac{\gamma}{2}, \frac{1}{2}-\frac{j}{2}+\frac{q}{2}-\frac{\gamma}{2} \end{matrix} \middle| -\frac{1}{4} \right).
\end{aligned}$$

Theorem 4.3. *$D^q P_{j,M}(x)$, $j \geq q$ can be expanded in terms of the special class of JPs. $J_r^{(\gamma,\gamma+1)}(x)$ as follows:*

$$D^q P_{j,M}(x) = \xi_j \left(\sum_{\ell=0}^{\left\lfloor \frac{j-q}{2} \right\rfloor} A_{\ell,j,q}\, J^{(\gamma,\gamma+1)}_{j-q-2\ell}(x) + \sum_{\ell=0}^{\left\lfloor \frac{1}{2}(j-q-1) \right\rfloor} B_{\ell,j,q}\, J^{(\gamma,\gamma+1)}_{j-q-2\ell-1}(x) \right),$$

where

$$\begin{aligned}
\xi_j =& \frac{2^{-1+q-2\gamma}\sqrt{\pi}\,(M)_j}{\Gamma(1+\gamma)},\\
A_{\ell,j,q} =& \frac{\Gamma(2+j-2\ell-q+2\gamma)}{\ell!\,(j-2\ell-q)!\,\Gamma\left(\frac{3}{2}+j-\ell-q+\gamma\right)}\ {}_2F_1\left(\begin{matrix} -\ell, -\frac{1}{2}-j+\ell+q-\gamma \\ 1-j-M \end{matrix} \middle| -1 \right),\\
B_{\ell,j,q} =& \frac{\Gamma(1+j-2\ell-q+2\gamma)}{\ell!\,(j-2\ell-q-1)!\,\Gamma\left(\frac{3}{2}+j-\ell-q+\gamma\right)}\ {}_2F_1\left(\begin{matrix} -\ell, -\frac{1}{2}-j+\ell+q-\gamma \\ 1-j-M \end{matrix} \middle| -1 \right).
\end{aligned}$$

Corollary 4.2. *$D^q P_{j,M}(x)$, $j \geq q$ can be expanded in terms of the special class of CPs of the third kind as follows:*

$$\begin{aligned}
D^q P_{j,M}(x) =& 2^q\,(M)_j \left(\sum_{\ell=0}^{\left\lfloor \frac{j-q}{2} \right\rfloor} \frac{1}{\ell!(j-\ell-q)!}\ {}_2F_1\left(\begin{matrix} -\ell, -j+\ell+q \\ 1-j-M \end{matrix} \middle| -1 \right) V_{j-q-2\ell}(x) \right.\\
& \left. + \sum_{\ell=0}^{\left\lfloor \frac{1}{2}(j-q-1) \right\rfloor} \frac{1}{\ell!(j-\ell-q)!}\ {}_2F_1\left(\begin{matrix} -\ell, -j+\ell+q \\ 1-j-M \end{matrix} \middle| -1 \right) V_{j-q-2\ell-1}(x) \right).
\end{aligned} \tag{4.8}$$

Proof. The result follows by taking into consideration that CPs of the third kind $V_r(x)$ are special cases of the polynomials $J_r^{(\gamma,\gamma+1)}(x)$, for $\gamma = -\frac{1}{2}$. □

Corollary 4.3. *$D^q P_{j,M}(x)$, $j \geq q$ can be expanded in terms of the special class of CPs of the fourth kind as follows:*

$$\begin{aligned} D^q P_{j,M}(x) =& 2^q\,(M)_j \left(\sum_{\ell=0}^{\left\lfloor \frac{j-q}{2} \right\rfloor} \frac{1}{\ell!(j-\ell-q)!}\ {}_2F_1\!\left(\begin{matrix} -\ell, -j+\ell+q \\ 1-j-M \end{matrix} \middle| -1 \right) W_{j-q-2\ell}(x) \right. \\ & \left. - \sum_{\ell=0}^{\left\lfloor \frac{1}{2}(j-q-1) \right\rfloor} \frac{1}{\ell!(j-\ell-q)!}\ {}_2F_1\!\left(\begin{matrix} -\ell, -j+\ell+q \\ 1-j-M \end{matrix} \middle| -1 \right) W_{j-q-2p-1}(x) \right). \end{aligned} \tag{4.9}$$

Proof. The result follows from the link between the CPs of the third and fourth kinds: $W_n(x) = (-1)^n\, V_n(-x)$. □

Theorem 4.4. *$D^q P_{j,M}(x)$, $j \geq q$ can be expanded in terms of the shifted JPs $\tilde{J}_j^{(\gamma,\delta)}(x)$ as follows:*

$$D^q P_{j,M}(x) = \gamma_j \left(\sum_{\ell=0}^{\left\lfloor \frac{j-q}{2} \right\rfloor} S_{\ell,j,q}\, \tilde{J}_{j-q-2\ell}^{(\gamma,\delta)}(x) + \sum_{\ell=0}^{\left\lfloor \frac{1}{2}(j-q-1) \right\rfloor} \bar{S}_{\ell,j,q}\, \tilde{J}_{j-q-2\ell-1}^{(\gamma,\delta)}(x) \right), \tag{4.10}$$

where

$$\begin{aligned} \gamma_j =& \frac{2^{2+j}\,\Gamma\left(\frac{5}{2}+j\right)\Gamma(1+j-q+\delta)}{3\,\sqrt{\pi}\,\Gamma(1+\gamma)}, \\ S_{\ell,j,q} =& \frac{(1+2j-4\ell-2q+\gamma+\delta)\,\Gamma(1+j-2\ell-q+\gamma)\,\Gamma(1+j-2\ell-q+\gamma+\delta)}{(2\ell)!\,(j-2\ell-q)!\,\Gamma(1+j-2\ell-q+\delta)\,\Gamma(2+2j-2\ell-2q+\gamma+\delta)} \\ & \times\ {}_4F_3\!\left(\begin{matrix} \frac{1}{2}-\ell, -\ell, -\frac{1}{2}-j+\ell+q-\frac{\gamma}{2}-\frac{\delta}{2}, -j+\ell+q-\frac{\gamma}{2}-\frac{\delta}{2} \\ -\frac{3}{2}-j, -\frac{j}{2}+\frac{q}{2}-\frac{\delta}{2}, \frac{1}{2}-\frac{j}{2}+\frac{q}{2}-\frac{\delta}{2} \end{matrix} \middle| -1 \right), \\ \bar{S}_{\ell,j,q} =& \frac{(-1+2j-4\ell-2q+\gamma+\delta)\,\Gamma(j-2\ell-q+\gamma)\,\Gamma(j-2\ell-q+\gamma+\delta)}{(2\ell+1)!\,(j-2\ell-q-1)!\,\Gamma(j-2\ell-q+\delta)\,\Gamma(1+2j-2\ell-2q+\gamma+\delta)} \\ & \times\ {}_4F_3\!\left(\begin{matrix} -\frac{1}{2}-\ell, -\ell, -j+\ell+q-\frac{\gamma}{2}-\frac{\delta}{2}, \frac{1}{2}-j+\ell+q-\frac{\gamma}{2}-\frac{\delta}{2} \\ -\frac{3}{2}-j, -\frac{j}{2}+\frac{q}{2}-\frac{\delta}{2}, \frac{1}{2}-\frac{j}{2}+\frac{q}{2}-\frac{\delta}{2} \end{matrix} \middle| -1 \right). \end{aligned}$$

Remark 4.1. *There are six special formulas that can be deduced from formula* (4.10) *by taking into consideration the six special classes of the shifted JPs.*

4.2. Derivatives in terms of some other polynomials

Now, we present some derivative formulas of the convolved Pell polynomials as combinations of generalized Fibonacci, generalized Lucas, and Bernoulli polynomials. Similar steps to those followed in the previous subsection can be followed for the derivation of these formulas, so the proofs are omitted.

Theorem 4.5. *$D^q P_{j,M}(x),\ j \geq q$ can be expanded in terms of the $F_j^{A,B}(x)$ which are constructed by* (2.6) *as follows:*

$$
\begin{aligned}
D^q P_{j,M}(x) =& 2^j (M)_j A^{-j+q} \sum_{r=0}^{\left\lfloor \frac{j-q}{2} \right\rfloor} \frac{(-1)^{1+r}(-1-j+2r+q)B^r}{r!(1+j-r-q)!} \\
& \times\ {}_2F_1\left(\begin{matrix} -r, -1-j+r+q \\ 1-j-M \end{matrix} \,\middle|\, \frac{A^2}{4B} \right) F_{j-q-2r}^{A,B}(x).
\end{aligned}
\tag{4.11}
$$

Remark 4.2. *The term ${}_2F_1\left(\frac{A^2}{4\,B}\right)$ that appears in* (4.11) *may be reduced for the choice $A^2 = 4\,B$; thus, a simplified formula can be obtained. The following corollary displays such a result.*

Corollary 4.4. *If $A^2 = 4\,B$, then* (4.11) *reduces to:*

$$
D^q P_{j,M}(x) = (M)_j \sum_{r=0}^{\left\lfloor \frac{j-q}{2} \right\rfloor} \frac{(-1)^{1+r}\, 2^{j-2r}\, A^{-j+q+2r}\, (-1-j+q+2r)\, (-1+M+q)_r}{r!\, (j-q-r+1)!\, (j+M-r)_r}\, F_{j-q-2r}^{A, \frac{A^2}{4}}(x),
$$

and in particular

$$
D^q P_{j,M}(x) = 2^q\, (M)_j \sum_{r=0}^{\left\lfloor \frac{j-q}{2} \right\rfloor} \frac{(-1)^{1+r}\, (-1-j+q+2r)\, (-1+M+q)_r}{r!\, (j-q-r+1)!\, (j+M-r)_r}\, P_{j-q-2r+1}(x). \tag{4.12}
$$

Theorem 4.6. *$D^q P_{j,M}(x),\ j \geq q$ can be expanded in terms of the generalized Lucas polynomials $L_j^{R,S}(x)$ which are constructed by* (2.7) *as follows:*

$$
D^q P_{j,M}(x) = 2^j (M)_j R^{-j+q} \sum_{r=0}^{\left\lfloor \frac{j-q}{2} \right\rfloor} \frac{\xi_{j-2r-2q} (-1)^r S^r}{r!(j-r-q)!}\ {}_2F_1\left(\begin{matrix} -r, -j+r+q \\ 1-j-M \end{matrix} \,\middle|\, \frac{R^2}{4S} \right) L_{j-q-2r}^{R,S}(x), \tag{4.13}
$$

where ξ_m is as defined in (2.8).

Remark 4.3. *The term ${}_2F_1\left(\frac{R^2}{4\,S}\right)$ that appears in* (4.13) *can be summed for the special choice of $R^2 = 4\,S$; hence, a simplified formula can be obtained. The following corollary displays this result.*

Corollary 4.5. *If $R^2 = 4\,S$, then formula* (4.13) *reduces to:*

$$
D^q P_{j,M}(x) = (M)_j \sum_{r=0}^{\left\lfloor \frac{j-q}{2} \right\rfloor} \frac{(-1)^r 2^{j-2r} \xi_{j-q-2r}\, R^{-j+q+2r} (M+q)_r}{r!(j-q-r)!(j+M-r)_r}\, L_{j-q-2r}^{R, \frac{R^2}{4}}(x), \tag{4.14}
$$

and in particular

$$
D^q P_{j,M}(x) = 2^q\, (M)_j \sum_{r=0}^{\left\lfloor \frac{j-q}{2} \right\rfloor} \frac{(-1)^r\, \xi_{j-q-2r}\, (M+q)_r}{(j-q-r)!\, r!\, (j+M-r)_r}\, Q_{j-q-2r}(x). \tag{4.15}
$$

Theorem 4.7. *$D^q P_{j,M}(x)$, $j \geq q$ can be expanded in terms of the Bernoulli polynomials $B_j(x)$ as follows:*

$$D^q P_{j,M}(x) = \frac{2^j \Gamma(j+M)}{\Gamma(M)} \sum_{r=0}^{\lfloor \frac{j-q}{2} \rfloor} \frac{1}{(2r+1)!(j-2r-q)!} \; {}_2F_1\left(\begin{matrix} -r, -\frac{1}{2}-r \\ 1-j-M \end{matrix} \middle| -1 \right) B_{j-q-2r}(x)$$
$$+ \frac{1}{\Gamma(M)} \sum_{r=0}^{\lfloor \frac{1}{2}(j-q-1) \rfloor} \frac{-2^{-2+j-2r}\Gamma(-1+j+M-r)(2r+2)! + 2^j \Gamma(j+M)(r+1)! \; {}_2F_1\left(\begin{matrix} -r, -\frac{1}{2}-r \\ 1-j-M \end{matrix} \middle| -1 \right)}{(r+1)!(2r+2)!(j-2r-q-1)!} \times$$
$$B_{j-q-2r-1}(x). \tag{4.16}$$

5. Moments formulas

This section focuses on establishing the moments formulas for the convolved Pell polynomials. We will show that the moments coefficients involve a certain terminating hypergeometric function of the type ${}_4F_3(1)$. Also, it will be shown that this hypergeometric function can be reduced in some specific cases.

Theorem 5.1. *Consider two non-negative integers j, m. We have*

$$\begin{aligned} x^m P_{j,M}(x) = & \frac{(j+m)!\,\Gamma(j+M)}{2^m\, j!} \sum_{r=0}^{\lfloor \frac{j+m}{2} \rfloor} \frac{(-1)^r (j+m+M-2r)}{r!\,\Gamma(1+j+m+M-r)} \\ & \times {}_4F_3\left(\begin{matrix} -r, \frac{1}{2}-\frac{j}{2}, -\frac{j}{2}, -j-m-M+r \\ -\frac{j}{2}-\frac{m}{2}, \frac{1}{2}-\frac{j}{2}-\frac{m}{2}, 1-j-M \end{matrix} \middle| 1 \right) P_{j+m-2r,M}(x). \end{aligned} \tag{5.1}$$

Proof. The analytic form of $P_{j,M}(x)$ along with the inversion formula (2.3), after some algebraic computations, leads to relation (5.1). □

Remark 5.1. *Although the hypergeometric function that appears in* (5.1) *is balanced and terminating, its sum is not reducible in general. We will give some special cases in which this hypergeometric function can be summed, and accordingly, certain simple moments formulas can be deduced. We will consider the two cases corresponding to $M = 1$ and $M = 2$, respectively.*

Lemma 5.1. *When m and j are positive integers, we get the moments formula for the Pell polynomials $P_j(x)$:*

$$x^m P_j(x) = \frac{1}{2^m} \sum_{r=0}^{m} (-1)^r \binom{m}{r} P_{j+m-2r}(x).$$

Proof. For $M = 1$, (5.1) turns into the following equation:

$$\begin{aligned} x^m P_j(x) = & \frac{(j+m)!}{2^m} \sum_{r=0}^{\lfloor \frac{j+m}{2} \rfloor} \frac{(-1)^r (1+j+m-2r)}{r!\,(j+m-r+1)!} \\ & \times {}_4F_3\left(\begin{matrix} -r, \frac{1}{2}-\frac{j}{2}, -\frac{j}{2}, -1-j-m+r \\ -j, -\frac{j}{2}-\frac{m}{2}, \frac{1}{2}-\frac{j}{2}-\frac{m}{2} \end{matrix} \middle| 1 \right) P_{j+m-2r}(x). \end{aligned} \tag{5.2}$$

Regarding the ${}_4F_3(1)$ term that appears in (5.2), it can be summed by using symbolic algebra. For this purpose, set

$$R_{r,j,m} = {}_4F_3\left(\begin{matrix} -r, \frac{1}{2}-\frac{j}{2}, -\frac{j}{2}, -1-j-m+r \\ -j, -\frac{j}{2}-\frac{m}{2}, \frac{1}{2}-\frac{j}{2}-\frac{m}{2} \end{matrix} \middle| 1\right). \tag{5.3}$$

As a result of Zeilberger's algorithm [47], $R_{r,j,m}$ satisfying the following recurrence relation is obtained:

$$\begin{aligned}&(r-1)(-2-m+r)(j+m-2r+5)R_{r-2,j,m} - \left(jm-2jr-2mr+2r^2+2j+2m-6r+4\right)\\ &\times(3+j+m-2r)R_{r-1,j,m} + (2+j+m-r)(j-r+1)(-2r+1+j+m)\,R_{r,j,m} = 0,\end{aligned}$$

with the initial values $R_{0,j,m} = 1$ and $R_{1,j,m} = \dfrac{m}{-1+j+m}$.
It can be exactly solved to give

$$R_{r,j,m} = \frac{(m-r+1)_r}{(j+m-2r+1)(j+m-r+2)_{r-1}},$$

and this leads to the following formula:

$$x^m\,P_j(x) = \frac{1}{2^m}\sum_{r=0}^{m}(-1)^r\binom{m}{r}P_{j+m-2r}(x).$$

This ends the proof of the lemma. □

Remark 5.2. *Zeilberger's algorithm is a very useful tool for finding a closed form for various sums, especially hypergeometric functions with a negative numerator parameter, such as* (5.3)*. The philosophy of employing such an algorithm depends on the following two fundamental steps:*

1) Obtaining the recurrence relation that corresponds to the sum that we wish to find its closed form.
2) Using suitable computer algebra algorithms such as the Petkovsek and van Hoeij algorithms (see, [47, 48]) to solve the resulting recurrence relation.

Remark 5.3. *There are many contributions that are closely related to Zeilberger's algorithm; see for example [49–51].*

Lemma 5.2. *When m and j are positive integers, we get the moments formula for the Pell polynomials* $P_{j,2}(x)$*:*

$$x^m\,P_{j,2}(x) = \frac{1}{2^m}\sum_{r=0}^{m}\frac{(-1)^r\,m!\,((1+j)(3+j+m)-2(2+j)r)}{r!\,(m-r)!\,(1+j+m-2r)\,(3+j+m-2r)}\,P_{j+m-2r,2}(x).$$

Proof. For $M = 2$, (5.1) gives

$$\begin{aligned}x^m\,P_{j,2}(x) =& \frac{(j+m)!(j+1)}{2^m}\sum_{r=0}^{\left\lfloor\frac{j+m}{2}\right\rfloor}\frac{(-1)^r\,(2+j+m-2r)}{r!\,(j+m-r+2)!}\times\\ &{}_4F_3\left(\begin{matrix} -r, \frac{1}{2}-\frac{j}{2}, -\frac{j}{2}, -2-j-m+r \\ -1-j, -\frac{j}{2}-\frac{m}{2}, \frac{1}{2}-\frac{j}{2}-\frac{m}{2} \end{matrix} \middle| 1\right) P_{j+m-2r,2}(x).\end{aligned}$$

Again, Zeilbereger's algorithm allows one to reduce the sum in the last hypergeometric functions as follows:

$$ {}_4F_3\left(\begin{matrix} -r, \frac{1}{2}-\frac{j}{2}, -\frac{j}{2}, -2-j-m+r \\ -1-j, -\frac{j}{2}-\frac{m}{2}, \frac{1}{2}-\ \frac{j}{2}-\frac{m}{2} \end{matrix} \middle| 1\right) = \frac{(m-r+1)_r\ ((1+j)(3+j+m)-2(2+j)r)}{(1+j)(j+m-2r+1)_3(j+m-r+3)_{r-2}}, $$

and as a result, we get the moments formula below:

$$ x^m\, P_{j,2}(x) = \frac{1}{2^m}\sum_{r=0}^{m} \frac{(-1)^r\, m!\,((1+j)(3+j+m)-2(2+j)r)}{r!\,(m-r)!\,(1+j+m-2r)\,(3+j+m-2r)}\, P_{j+m-2r,2}(x). $$

□

6. Some applications of the introduced formulas

The following formulas will be derived from some of the previous formulas. Particularly, we present the applications as follows:

- In the first application, we give some formulas linking some celebrated numbers.
- In the second application, we evaluate some weighted definite integrals.
- In the third application, we introduce some trigonometric identities.
- In the fourth application, we give a new closed form for the definite integral of $P_{j,M}(x)$.

6.1. New connections between some celebrated numbers

In this section, we give some expressions of the convolved Pell numbers in terms of some famous numbers. These expressions are given as special cases of Theorems 4.5 and 4.6.

Corollary 6.1. *The convolved Pell numbers can be written in terms of the Fibonacci and Pell numbers as in the following expressions*

$$ P_{i,M} = 2^j (M)_j \sum_{p=0}^{\lfloor \frac{j}{2} \rfloor} \frac{(-1)^{1+p}(-1-j+2p)}{p!(j-p+1)!}\ {}_2F_1\left(\begin{matrix} -p, -1-j+p \\ 1-j-M \end{matrix} \middle| \frac{1}{4}\right) F_{j-2p+1}, \tag{6.1} $$

$$ P_{i,M} = (M)_j \sum_{\ell=0}^{\lfloor \frac{j}{2} \rfloor} \frac{(-1)^{1+\ell}(-1-j+2\ell)(-1+M)_\ell}{\ell!(j-\ell+1)!(j+M-\ell)_\ell}\, P_{j-2\ell+1}. \tag{6.2} $$

Proof. Formula (6.1) is a consequence of (4.11), corresponding to the following choices: $q=0, x=1, A=1$ and $B=1$, while Formula (6.2) can be deduced from (4.12) by setting $x=1$ and $q=0$. □

Corollary 6.2. *The convolved Pell numbers can be written in terms of the Lucas and Pell-Lucas numbers, respectively, as in the following expressions:*

$$ P_{j,M} = 2^j (M)_j \sum_{\ell=0}^{\lfloor \frac{j}{2} \rfloor} \frac{(-1)^\ell\, \xi_{j-2\ell}}{\ell!\,(j-\ell)!}\ {}_2F_1\left(\begin{matrix} -\ell, -j+\ell \\ 1-j-M \end{matrix} \middle| \frac{1}{4}\right) L_{j-2\ell}, \tag{6.3} $$

$$ P_{j,M} = (M)_j \sum_{\ell=0}^{\lfloor \frac{j}{2} \rfloor} \frac{(-1)^\ell\, \xi_{j-2\ell}\,(M)_\ell}{(j-\ell)!\,\ell!\,(j+M-\ell)_\ell}\, Q_{j-2\ell}. \tag{6.4} $$

Proof. Formula (6.3) is a consequence of (4.13), corresponding to the following choices: $q = 0, x = 1, R = 1$ and $S = 1$, while (6.4) can be deduced from (4.15) by setting $x = 1$ and $q = 0$. □

6.2. Some weighted integrals involving the convolved Pell polynomials

Here we compute some weighted definite integrals based on the connection formulas given in Section 4.1 (Theorems 4.1 and 4.2).

Corollary 6.3. *For $\lambda > -\frac{1}{2}$, $j \geq m$ and $(j+m)$ even, the follwoing integral formula holds*

$$\int_{-1}^{1}(1-x^2)^{\lambda-\frac{1}{2}}\,P_{j,M}(x)\,C_m^{(\lambda)}(x)\,dx = \frac{\sqrt{\pi}\,\Gamma(j+M)\,\Gamma\left(\frac{1}{2}+\lambda\right)}{\left(\frac{j-m}{2}\right)!\,\Gamma(M)\,\Gamma\left(\frac{1}{2}(2+j+m)+\lambda\right)}\times\ {}_2F_1\left(\left.\begin{array}{c}\frac{1}{2}(-j+m),\frac{1}{2}(-j-m-2\lambda)\\ 1-j-M\end{array}\right|-1\right). \tag{6.5}$$

Proof. Substituting by $q = 0$ in (4.1) produces the following formula

$$\begin{aligned}P_{j,M}(x) = &\frac{2\Gamma(j+M)\Gamma(1+\lambda)}{\Gamma(M)\Gamma(1+2\lambda)}\sum_{\ell=0}^{\lfloor\frac{j}{2}\rfloor}\frac{(j-2\ell+\lambda)\Gamma(j-2\ell+2\lambda)}{\ell!(j-2\ell)!\Gamma(1+j-\ell+\lambda)}\\ &\times\ {}_2F_1\left(\left.\begin{array}{c}-\ell,-j+\ell-\lambda\\ 1-j-M\end{array}\right|-1\right)C_{j-2\ell}^{(\lambda)}(x).\end{aligned} \tag{6.6}$$

If both sides of (6.6) are multiplied by $(1-x^2)^{\lambda-\frac{1}{2}}\,C_m^{(\lambda)}(x)$, integrate over (-1,1), and apply the orthogonality relation of the ultraspherical polynomials; then, (6.5) can be obtained. □

Another integral formula that involves convolved Pell and Hermite polynomials is given in the following corollary.

Corollary 6.4. *For $j \geq m$ and $(j+m)$ even, the following integral formula holds*

$$\int_{-\infty}^{\infty} e^{-x^2}\,P_{j,M}(x)\,H_m(x)\,dx = \frac{2^{1+m}\Gamma\left(\frac{3}{2}+j\right)\,{}_1F_1\left(\frac{1}{2}(-j+m);-\frac{1}{2}-j;1\right)}{\left(\frac{j-m}{2}\right)!}. \tag{6.7}$$

Remark 6.1. *Other integral formulas can be deduced by using the connection formulas between the convolved Pell polynomials and different orthogonal polynomials.*

6.3. Some trigonometric identities

Based on the connection formulas between the convolved Pell polynomials and the four kinds of CPs, some trigonometric expressions for the convolved Pell polynomials can be deduced. The following two corollaries display these expressions.

Corollary 6.5. *The following trigonometric representations for the convolved Pell polynomials hold*

$$P_{j,M}(\cos\vartheta) = \frac{2\,\Gamma(j+M)}{\Gamma(M)}\sum_{\ell=0}^{\lfloor\frac{j}{2}\rfloor}\frac{1}{\ell!(j-\ell)!}\,{}_2F_1\left(\left.\begin{array}{c}-\ell,-j+\ell\\ 1-j-M\end{array}\right|-1\right)\cos((j-2\ell)\vartheta),$$

$$\sin\vartheta\, P_{j,M}(\cos\vartheta) = \frac{\Gamma(j+M)}{\Gamma(M)} \sum_{\ell=0}^{\left\lfloor \frac{j}{2} \right\rfloor} \frac{1+j-2\ell}{\ell!(j-\ell+1)!}\, {}_2F_1\left(\left.\begin{array}{c} -\ell, -j+\ell-1 \\ 1-j-M \end{array}\right| -1\right) \sin((j-2\ell+1)\vartheta).$$

Proof. The results follow from (4.4) and (4.5) by setting $q = 0$ and applying the following two well-known trigonometric definitions:

$$T_j(\cos\vartheta) = \cos(j\,\vartheta), \qquad U_j(\cos\vartheta) = \frac{\sin((j+1)\,\vartheta)}{\sin\vartheta}.$$

□

Corollary 6.6. *The following trigonometric representations for the convolved Pell polynomials hold*

$$\begin{aligned}\cos\left(\frac{\vartheta}{2}\right) P_{j,M}(\cos\vartheta) =& (M)_j \left(\sum_{\ell=0}^{\left\lfloor \frac{j}{2} \right\rfloor} \frac{1}{\ell!(j-\ell)!}\, {}_2F_1\left(\left.\begin{array}{c} -\ell, -j+\ell \\ 1-j-M \end{array}\right| -1\right) \cos\left(\left(j-2\,\ell+\frac{1}{2}\right)\vartheta\right)\right. \\ &\left. + \sum_{\ell=0}^{\left\lfloor \frac{1}{2}(j-1) \right\rfloor} \frac{1}{\ell!(j-\ell)!}\, {}_2F_1\left(\left.\begin{array}{c} -\ell, -j+\ell \\ 1-j-M \end{array}\right| -1\right) \cos\left(\left(j-2\,\ell-\frac{1}{2}\right)\vartheta\right)\right),\end{aligned} \tag{6.8}$$

$$\begin{aligned}\sin\left(\frac{\vartheta}{2}\right) P_{j,M}(\cos\vartheta) =& (M)_j \left(\sum_{\ell=0}^{\left\lfloor \frac{j}{2} \right\rfloor} \frac{1}{\ell!(j-\ell)!}\, {}_2F_1\left(\left.\begin{array}{c} -\ell, -j+\ell \\ 1-j-M \end{array}\right| -1\right) \sin\left(\left(j-2\,\ell+\frac{1}{2}\right)\vartheta\right)\right. \\ &\left. - \sum_{\ell=0}^{\left\lfloor \frac{1}{2}(j-1) \right\rfloor} \frac{1}{\ell!(j-\ell)!}\, {}_2F_1\left(\left.\begin{array}{c} -\ell, -j+\ell \\ 1-j-M \end{array}\right| -1\right) \sin\left(\left(j-2\,\ell-\frac{1}{2}\right)\vartheta\right)\right).\end{aligned} \tag{6.9}$$

Proof. The results can be derived from (4.8) and (4.9) by setting $q = 0$, and applying the following two well-known trigonometric definitions:

$$V_j(\cos\vartheta) = \frac{\cos\left(\left(j+\frac{1}{2}\right)\vartheta\right)}{\cos\left(\frac{\vartheta}{2}\right)}, \qquad W_j(\cos\vartheta) = \frac{\sin\left(\left(j+\frac{1}{2}\right)\vartheta\right)}{\sin\left(\frac{\vartheta}{2}\right)}.$$

□

6.4. *A specific definite integral formula for the convolved Pell polynomials*

In this part, and based on the connection formula between $P_{j,M}(x)$ and the Bernoulli polynomials, the definite integral $\int_0^1 P_{j,M}(x)\,dx$ can be computed in a closed form.

Corollary 6.7. *Consider a non-negative integer j. The following integral formula holds*

$$\int_0^1 P_{j,M}(x)\,dx = \begin{cases} \dfrac{2^j\,\Gamma(j+M)}{(j+1)!\,\Gamma(M)}\, {}_2F_1\left(\left.\begin{array}{c} -\frac{1}{2}-\frac{j}{2}, -\frac{j}{2} \\ 1-j-M \end{array}\right| -1\right), & j \ \textit{even}, \\ \dfrac{-\dfrac{\Gamma\left(\frac{1}{2}(-1+j)+M\right)}{\left(\frac{j+1}{2}\right)!} + \dfrac{2^{1+j}\,\Gamma(j+M)}{(j+1)!}\, {}_2F_1\left(\left.\begin{array}{c} -\frac{1}{2}-\frac{j}{2}, -\frac{j}{2} \\ 1-j-M \end{array}\right| -1\right)}{2\,\Gamma(M)}, & j \ \textit{odd}. \end{cases} \tag{6.10}$$

Proof. This formula will be proved by using the connection formula for $P_{j,M}(x)$ with the Bernoulli polynomials. Setting $q = 0$ in (4.16) leads to the following connection formula:

$$P_{j,M}(x) = \frac{2^j\,\Gamma(j+M)}{\Gamma(M)} \sum_{r=0}^{\lfloor \frac{j}{2} \rfloor} \frac{1}{(2r+1)!(j-2r)!}\ {}_2F_1\left(\left. \begin{matrix} -r, -\frac{1}{2}-r \\ 1-j-M \end{matrix} \right| -1 \right) B_{j-2r}(x)$$

$$+ \frac{1}{\Gamma(M)} \sum_{r=0}^{\lfloor \frac{j-1}{2} \rfloor} \frac{-2^{-2+j-2r}\,\Gamma(-1+j+M-r)\,(2r+2)! + 2^j\,\Gamma(j+M)\,(r+1)!\ {}_2F_1\left(\left. \begin{matrix} -1-r, -\frac{1}{2}-r \\ 1-j-M \end{matrix} \right| -1 \right)}{(r+1)!(2r+2)!(j-2r-1)!} \times B_{j-2r-1}(x). \tag{6.11}$$

Integrating both sides of (6.11) from 0 to 1, and using the well-known identity (see [52])

$$\int_0^1 B_\ell(x)\,dx = \delta_{0,\ell},$$

where $\delta_{\ell,m}$ is the Kronecker delta function, we get the desired result given by (6.10). □

7. A matrix approach for convolved Pell polynomials

In general, the convolved Pell polynomials of degree i can be written as

$$P_{i,M}(x) = \sum_{k=0}^{i} t_{i,k,M}\, x^k, \qquad t_{i,k} \in \mathbb{R}, \quad i = 0, 1, \ldots, \tag{7.1}$$

with

$$t_{i,k,M} = \begin{cases} \frac{2^k (M)_{\lfloor \frac{i+k}{2} \rfloor}}{(\frac{i-k}{2})!\,k!}, & (i-k) \text{ even}, \\ 0, & (i-k) \text{ odd}. \end{cases}$$

Clearly, from (2.2), $t_{i,k,M}$ satisfies the recurrence relation described below:

$$t_{i,k,M} = \frac{2(i+M-1)}{i}\, t_{i-1,k-1,M} + \frac{i+2M-2}{i}\, t_{i-2,k,M}, \qquad i \geq k \geq 2, \tag{7.2}$$

with

$$t_{0,0,M} = 1, \quad t_{i,0,M} = \begin{cases} \frac{(M)_{\frac{i}{2}}}{(\frac{i}{2})!}, & i \text{ even}, \\ 0, & i \text{ odd}, \end{cases} \quad i \geq 1, \quad t_{1,1,M} = 2M.$$

For $P_{i,M}$ we have that $P_{i,M}(-x) = (-1)^i P_{i,M}(x)$, that is convolved Pell polynomials are even functions if i is even, and they are odd functions if i is odd. Moreover, since the coefficients are all positive, then $\forall k \in \mathbb{N}$, $P_{2k,M}(x) \geq 0\ \forall x \in \mathbb{R}$, while $P_{2k+1,M}(x) \geq 0$ for $x \geq 0$ and $P_{2k+1,M}(x) < 0$ for $x < 0$.

If $T_{i,i,M} = [t_{i,0,M}, t_{i,1,M}, \ldots, t_{i,i,M}]$ and $X_i = [1, x, x^2, \ldots, x^i]^T$, then $P_{i,M}(x)$ can be written in the matrix form: $P_{i,M}(x) = T_{i,i,M} X_i$.

7.1. The convolved Pell sequence

We consider the polynomial sequence $\{P_{i,M}\}_{i\in\mathbb{N}}$ whose elements are the convolved Pell polynomials. If $T_M = (t_{i,j,M})_{i,j\geq 0}$, $X = [1, x, x^2, \dots]$ and $P_M^x = [P_{0,M}(x), P_{1,M}(x), \dots, P_{i,M}(x), \dots]^T$, the elements of the polynomial sequence $\{P_{i,M}\}_{i\in\mathbb{N}}$ can be written as

$$P_M^x = T_M X.$$

T_M is a lower, triangular, infinite matrix. It can be factorized [53] as

$$T_M = D_1 B_M D_1^{-1},$$

where $D_1 = diag\{i! \mid i = 0, 1, \dots\}$ and B_M is the lower triangular matrix with the entries

$$b_{i,j,M} = \frac{j! t_{i,j,M}}{i!}, \qquad j \leq i.$$

B_M is a Toeplitz matrix, since it can be written as

$$B_M = \begin{pmatrix} b_{0,M} & 0 & 0 & \cdots & \cdots & 0 \\ b_{1,M} & b_{0,M} & 0 & \cdots & \cdots & 0 \\ \frac{b_{,M}}{2!} & b_{1,M} & b_{0,M} & \ddots & \cdots & 0 \\ \frac{b_{3,M}}{3!} & \frac{b_{2,M}}{2!} & b_{1,M} & b_{0,M} & \ddots & 0 \\ \vdots & & & \ddots & \ddots & \vdots \\ \vdots & & & & \ddots & \ddots \end{pmatrix},$$

with $b_{i-j,M} = \dfrac{b_{i,j,M}}{(i-j)!}$.

Observe that the diagonal elements of T_M are given by: $t_{k,k,M} = \dfrac{2^k (M)_k}{k!}$, $\forall k$, hence, T_M is an invertible matrix. Its inverse $S_M = (s_{i,j,M})_{i,j\geq 0} = T_M^{-1}$ is a lower, triangular, infinite matrix, too. S_M can be calculated as follows:

$$S_M = D_1 B_M^{-1} D_1^{-1},$$

where B_M^{-1} can be easily computed (see for example [54–56]).

Alternatively, the entries of S_M can be directly calculated in the following way (see [57]): let $T_{l,m,M}$, with $l \geq m$, be the following $(l-m+1) \times (l-m+1)$ block of T_M:

$$T_{l,m,M} = \begin{bmatrix} t_{m,m,M} & 0 & \cdots & \cdots & 0 \\ t_{m+1,m,M} & t_{m+1,m+1,M} & 0 & \ddots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & 0 \\ t_{l,m,M} & t_{l,m+1,M} & \cdots & \cdots & t_{l,l,M} \end{bmatrix}.$$

We consider the vectors $[s_{m,m,M}, s_{m+1,m,M}, \ldots, s_{l,m,M}]^T$ and $[e_{m,m}, e_{m+1,m}, \ldots, e_{l,m}]^T = [1, 0, \ldots, 0]^T$, whose elements are $l-m+1$ entries of the m-th columns of the matrices S_M and of the infinite identity matrix I, respectively.

From

$$\sum_{l=m}^{i} t_{i,l,M} s_{l,m,M} = e_{i,m}, \qquad i \geq 0, \quad m = 0, \ldots, i,$$

we get

$$\begin{bmatrix} t_{m,m,M} & 0 & \cdots & \cdots & 0 \\ t_{m+1,m,M} & t_{m+1,m+1,M} & 0 & \ddots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & 0 \\ t_{l,m,M} & t_{l,m+1,M} & \cdots & \cdots & t_{l,l} \end{bmatrix} \begin{bmatrix} s_{m,m,M} \\ \vdots \\ \vdots \\ \vdots \\ s_{l,m,M} \end{bmatrix} = \begin{bmatrix} e_{m,m} \\ \vdots \\ \vdots \\ \vdots \\ e_{l,m} \end{bmatrix}. \tag{7.3}$$

We solve system (7.3) in the unknowns m$s_{i,m,M}$, $m \leq i \leq l$, by using Cramer's rule. For the last element $s_{l,m,M}$ we get

$$s_{l,m,M} = \frac{(-1)^{l-m}}{\left|T_{l,m,M}\right|} \begin{vmatrix} e_{m,m} & t_{m,m,M} & 0 & \cdots & 0 \\ e_{m+1,m} & t_{m+1,m,M} & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & & & \ddots & t_{l-1,l-1,M} \\ e_{l,m} & t_{l,m,M} & \cdots & t_{l,l-2,M} & t_{l,l-1,M} \end{vmatrix}. \tag{7.4}$$

Hence, we obtain that $s_{l,l,M} = \dfrac{1}{t_{l,l,M}} = \dfrac{l!}{2^l (M)_l}$ and, for $l > m$,

$$s_{l,m,M} = \frac{(-1)^{l-m}}{\prod_{i=m}^{l} t_{i,i,M}} \begin{vmatrix} t_{m+1,m,M} & t_{m+1,m+1,M} & 0 & \cdots & \cdots & 0 \\ t_{m+2,m,M} & t_{m+2,m+1,M} & t_{m+2,m+2,M} & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & \ddots & 0 \\ \vdots & & & & \ddots & t_{l-1,l-1,M} \\ t_{k,m,M} & \cdots & \cdots & \cdots & t_{l,l-2,M} & t_{l,l-1,M} \end{vmatrix}.$$

Note that the above determinant is a Hessenberg determinant. It can be calculated recursively (see [58]) or by Gauss elimination. It is known (see, for example, [59]) that Gauss elimination without pivoting for the calculation of the determinant of an Hessenberg matrix is stable.

7.2. The conjugate sequence

Let $\{\widehat{P}_{i,M}\}_{i\in\mathbb{N}}$ be the conjugate sequence of $\{P_{i,M}\}_{i\in\mathbb{N}}$, that is the polynomial sequence related to the matrix S_M with elements

$$\widehat{P}_{i,M}(x) = \sum_{k=0}^{i} s_{i,k,M} x^k, \qquad s_{i,k,M} \in \mathbb{R}, \quad i \geq 0.$$

We know that [53] $P_{i,M} \circ \widehat{P}_{i,M} = id_i$, where " $\circ$ " denotes the umbral composition and $\{id_i\}_{i\in\mathbb{N}}$, with $id_i\,(x) = x^i$, i.e., the polynomial sequence related to the identity matrix I.

Of course, the following relations hold:

$$P^x_M = T_M X = T^2_M \widehat{P}^x_M, \qquad \widehat{P}^x_M = S_M X = S^2_M P^x_M,$$

where $\widehat{P}^x_M = \left[\widehat{P}_{0,M}\,(x)\,, \widehat{P}_{1,M}\,(x)\,, \ldots, \widehat{P}_{i,M}\,(x)\,, \ldots\right]^T$.

for all $i \in \mathbb{N}$, let $T_{M,i}$ and $S_{M,i}$ be the principal submatrices of order i of T_M and S_M, respectively. Moreover, let $P^x_{M,i} = \left[P_{0,M}\,(x)\,, P_{1,M}\,(x)\,, \ldots, P_{i,M}\,(x)\right]^T$, and $\widehat{P}^x_{M,i} = \left[\widehat{P}_{0,M}\,(x)\,, \widehat{P}_{1,M}\,(x)\,, \ldots, \widehat{P}_{i,M}\,(x)\right]^T$. Then,

$$P^x_{M,i} = T_{M,i} X_i = T^2_{M,i}\,\widehat{P}^x_{M,i} \quad \text{and} \quad \widehat{P}^x_{M,i} = S_{M,i} X_i = S^2_{M,i}\,P^x_{M,i}. \tag{7.5}$$

Remark 7.1. *From (7.5), $\forall\ i \in \mathbb{N}$, we get*

$$x^i = \sum_{k=0}^{i} s_{i,k,M}\,P_{k,M}(x). \tag{7.6}$$

Comparing (2.3) and (7.6), we can obtain the following expression of the elements of the matrix S_M:

$$s_{i,k,M} = \begin{cases} \dfrac{i!\,(-1)^{\alpha_{ik}}}{2^i\,\alpha_{ik}!\,(M)_{i-2\alpha_{ik}}\,(M+i-2\alpha_{ik}+1)_{\alpha_{ik}}}, & (i-k) \text{ even}, \\ 0, & (i-k) \text{ odd}, \end{cases}$$

with $\alpha_{ik} = \lfloor \frac{i-k}{2} \rfloor$.

7.3. Basis

Observe that each of the two sets $\{P_{0,M}\,(x)\,, \ldots, P_{i,M}\,(x)\}$ and $\{\widehat{P}_{0,M}\,(x)\,, \ldots, \widehat{P}_{i,M}\,(x)\}$ form a basis for $\mathcal{P}_i$, which is the set of polynomials of degree $\leq i$.

Hence, if $q_i(x) = \sum\limits_{k=0}^{i} q_{i,k} x^k \in \mathcal{P}_i$, then,

$$q_i(x) = \sum_{k=0}^{i} c_{i,k} P_{k,M}(x) = \sum_{k=0}^{i} d_{i,k} \widehat{P}_{k,M}(x),$$

with $c_{i,k} = \sum\limits_{j=0}^{\lfloor \frac{i-k}{2} \rfloor} q_{i,i-k-2j} s_{k+2j,k,M}\ d_{i,k} = \sum\limits_{j=0}^{\lfloor \frac{i-k}{2} \rfloor} q_{i,i-k-2j} t_{k+2j,k,M}, k = 0, 1, \ldots, i.$

It is known (see [60, 61] and the references therein) that the condition of polynomials, that is their sensitivity with respect to small perturbations in the coefficients, depends on the basis. Generally, condition numbers are used for the study of the condition of polynomials (see, [62, 63]).

Particularly, for a polynomial in the convolved Pell basis $q_i(x) = \sum_{k=0}^{i} c_{i,k} P_{k,M}(x)$ the condition number of the root α is

$$C_\alpha = \frac{\sum\limits_{k=0}^{i} |c_{i,k} P_{k,M}\,(\alpha)\,|}{|q'_i\,(\alpha)\,|}.$$

As an example we consider the Wilkinson polynomial [64] of degree 10:

$$p(x) = \prod_{k=1}^{10} (x-k)\,;$$

we calculate the condition number of each root. The outcomes obtained by employing each basis are compared in Table 1.

Table 1. Comparison between condition numbers for different bases.

Root	Power basis	Bessel basis	Fibonacci basis	Conv. Pell basis ($\forall M$)
1	$1.1 \cdot 10^2$	$5.1 \cdot 10^2$	$1.4 \cdot 10^2$	$1.1 \cdot 10^2$
2	$5.9 \cdot 10^3$	$1.5 \cdot 10^4$	$6.3 \cdot 10^3$	$5.9 \cdot 10^3$
3	$1.0 \cdot 10^5$	$1.9 \cdot 10^5$	$1.0 \cdot 10^5$	$1.0 \cdot 10^5$
4	$8.4 \cdot 10^5$	$1.4 \cdot 10^6$	$8.5 \cdot 10^5$	$8.4 \cdot 10^5$
5	$3.8 \cdot 10^6$	$5.6 \cdot 10^6$	$3.8 \cdot 10^6$	$3.8 \cdot 10^6$
6	$1.0 \cdot 10^7$	$1.4 \cdot 10^7$	$1.0 \cdot 10^7$	$1.0 \cdot 10^7$
7	$1.6 \cdot 10^7$	$2.2 \cdot 10^7$	$1.6 \cdot 10^7$	$1.6 \cdot 10^7$
8	$1.5 \cdot 10^7$	$2.0 \cdot 10^7$	$1.6 \cdot 10^7$	$1.5 \cdot 10^7$
9	$8.3 \cdot 10^6$	$1.0 \cdot 10^7$	$8.3 \cdot 10^6$	$8.3 \cdot 10^6$
10	$1.8 \cdot 10^6$	$2.3 \cdot 10^6$	$1.8 \cdot 10^6$	$1.8 \cdot 10^6$

7.4. Recurrence relations and determinant forms

From (7.6), $P_{i,M}(x)$ can be written as:

$$P_{i,M}(x) = x^i - \sum_{k=1}^{i-1} s_{i,k,M} P_{k,M}(x)\,.$$

Furthermore, (7.6), for $i \geq 0$, can be thought of as an infinite linear system in the unknowns $P_{j,M}(x)$, $j \geq 0$. Solving the first $(i+1)$ equations in the unknowns $P_{0,M}(x), \dots, P_{i,M}(x)$ by using Cramer's rule yields the following determinant form

$$P_{i,M}(x) = (-1)^i \prod_{k=0}^{i} \frac{2^k (M)_k}{k!} \begin{vmatrix} 1 & x & x^2 & x^3 & \cdots & x^{i-1} & x^i \\ s_{0,0,M} & s_{1,0,M} & s_{2,0,M} & \cdots & \cdots & s_{i-1,0,M} & s_{i,0,M} \\ 0 & s_{1,1,M} & s_{2,1,M} & \cdots & \cdots & s_{i-1,1,M} & s_{i,1,M} \\ \vdots & \ddots & \ddots & & & \vdots & \vdots \\ \vdots & & \ddots & \ddots & & \vdots & \vdots \\ \vdots & & & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 & s_{i-1,i-1,M} & s_{i,i-1,M} \end{vmatrix}, \quad i \geq 1.$$

Analogously, from (7.5), $\forall\ i \in \mathbb{N}$; we get that $x^i = \sum_{k=0}^{i} t_{i,k,M} \widehat{P}_{k,M}(x)$. Therefore, the conjugate polynomials can be expressed as

$$\widehat{P}_{i,M}(x) = x^i - \sum_{k=1}^{i-1} t_{i,k,M} \widehat{P}_{k,M}(x),$$

and we get

$$\widehat{P}_{i,M}(x) = (-1)^i \prod_{k=0}^{i} \frac{k!}{2^k (M)_k} \begin{vmatrix} 1 & x & x^2 & x^3 & \cdots & x^{i-1} & x^i \\ t_{0,0,M} & 0 & t_{2,0,M} & 0 & \cdots & t_{i-1,0,M} & t_{i,0,M} \\ 0 & t_{1,1,M} & 0 & t_{3,1,M} & \cdots & t_{i-1,1,M} & t_{i,1,M} \\ \vdots & \ddots & \ddots & \ddots & & \vdots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & & & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 & t_{i-1,i-1,M} & 0 \end{vmatrix}, \quad i \geq 1.$$

8. Conclusions

In this paper, we studied the convolved Pell polynomials, which generalize the well-known Pell polynomials. After providing an overview of the convolved Pell polynomials and recalling some important properties of Jacobi, generalized Fibonacci, and generalized Lucas polynomials, we derived many new formulas related to the convolved Pell polynomials. These include new expressions for their high-order derivatives in terms of their original polynomials and also in terms of various classical polynomials as well as connection formulas between two convolved Pell polynomials of different parameters, and between convolved Pell polynomials and some other polynomials. Moreover, we derived the moments formulas of the convolved Pell polynomials and provided some applications of the new formulas. In addition, we have presented a matrix approach for this kind of polynomials. We do believe that most of the formulas presented in this paper are new. In future work, we aim to employ these polynomials from a numerical point of view. In addition, we aim to investigate other convolved polynomials from both theoretical and practical points of view.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interests.

References

1. D. S. Kim, T. Kim, Representing polynomials by degenerate Bernoulli polynomials, *Quaest. Math.*, **46** (2023), 959–980. https://doi.org/10.2989/16073606.2022.2049912
2. T. Kim, D. S. Kim, J. Kwon, Some identities related to degenerate r-Bell and degenerate Fubini polynomials, *Appl. Math. Sci. Eng.*, **31** (2023), 2205642. https://doi.org/10.1080/27690911.2023.2205642
3. T. Kim, D. S. Kim, H. K. Kim, On generalized degenerate Euler-Genocchi polynomials, *Appl. Math. Sci. Eng.*, **31** (2023), 2159958. https://doi.org/10.1080/27690911.2022.2159958
4. H. M. Ahmed, Computing expansions coefficients for Laguerre polynomials, *Integr. Transf. Spec. F.*, **32** (2021), 271–289. https://doi.org/10.1080/10652469.2020.1815727
5. W. M. Abd-Elhameed, S. O. Alkhamisi, New results of the fifth-kind orthogonal Chebyshev polynomials, *Symmetry*, **13** (2021), 2407. https://doi.org/10.3390/sym13122407
6. W. M. Abd-Elhameed, New product and linearization formulae of Jacobi polynomials of certain parameters, *Integr. Transf. Spec. F.*, **26** (2015), 586–599. https://doi.org/10.1080/10652469.2015.1029924
7. F. A. Costabile, M. I. Gualtieri, A. Napoli, Recurrence relations and determinant forms for general polynomial sequences. Application to Genocchi polynomials, *Integr. Transf. Spec. F.*, **30** (2019), 112–127. https://doi.org/10.1080/10652469.2018.1537272
8. A. S. Alali, S. Ali, N. Hassan, A. M. Mahnashi, Y. Shang, A. Assiry, Algebraic structure graphs over the commutative ring Z_m: Exploring topological indices and entropies using M-polynomials, *Mathematics*, **11** (2023), 3833. https://doi.org/10.3390/math11183833
9. Y. Shang, A remark on the solvability of Diophantine matrix equation over $M_2(Q)$, *Southeast Asian Bull. Math.*, **38** (2014), 275–282.
10. Y. Shang, A remark on the chromatic polynomials of incomparability graphs of posets, *Int. J. Pure Appl. Math.*, **67** (2011), 159–164.
11. Z. Fan, W. Chu, Convolutions involving Chebyshev polynomials, *Electron. J. Math.*, **3** (2022), 38–46. https://doi.org/10.47443/ejm.2022.012
12. W. M. Abd-Elhameed, A. Napoli, Some novel formulas of Lucas polynomials via different approaches, *Symmetry*, **15** (2023), 185. https://doi.org/10.3390/sym15010185
13. T. Koshy, *Fibonacci and Lucas numbers with applications*, New York: John Wiley & Sons, 2019. https://doi.org/10.1002/9781118033067
14. W. M. Abd-Elhameed, A. N. Philippou, N. A. Zeyada, Novel results for two generalized classes of Fibonacci and Lucas polynomials and their uses in the reduction of some radicals, *Mathematics*, **10** (2022), 2342. https://doi.org/10.3390/math10132342
15. E. Özkan, M. Taştan, On Gauss Fibonacci polynomials, on Gauss Lucas polynomials and their applications, *Commun. Algebra*, **48** (2020), 952–960. https://doi.org/10.1080/00927872.2019.1670193

16. E. Özkan, I. Altun, Generalized Lucas polynomials and relationships between the Fibonacci polynomials and Lucas polynomials, *Commun. Algebra*, **47** (2019), 4020–4030. https://doi.org/10.1080/00927872.2019.1576186

17. T. Du, Z. Wu, Some identities involving the bi-periodic Fibonacci and Lucas polynomials, *AIMS Mathematics*, **8** (2023), 5838–5846. https://doi.org/10.3934/math.2023294

18. S. Haq, I. Ali, Approximate solution of two-dimensional Sobolev equation using a mixed Lucas and Fibonacci polynomials, *Eng. Comput.*, **38** (2022), 2059–2068. https://doi.org/10.1007/s00366-021-01327-5

19. A. S. Mohamed, Fibonacci collocation pseudo-spectral method of variable-order space-fractional diffusion equations with error analysis, *AIMS Mathematics*, **7** (2022), 14323–14337. https://doi.org/10.3934/math.2022789

20. T. Kim, D. S. Kim, L. C. Jang, D. V. Dolgy, Representing by several orthogonal polynomials for sums of finite products of Chebyshev polynomials of the first kind and Lucas polynomials, *Adv. Differ. Equ.*, **2019** (2019), 162. https://doi.org/10.1186/s13662-019-2092-6

21. D. S. Kim, D. V. Dolgy, D. Kim, T. Kim, Representing by orthogonal polynomials for sums of finite products of Fubini polynomials, *Mathematics*, **7** (2019), 319. https://doi.org/10.3390/math7040319

22. D. V. Dolgy, D. S. Kim, T. Kim, J. Kwon, Connection problem for sums of finite products of Chebyshev polynomials of the third and fourth kinds, *Symmetry*, **10** (2018), 617. https://doi.org/10.3390/sym10110617

23. T. Kim, D. S. Kim, D. V. Dolgy, J. W. Park, Sums of finite products of Legendre and Laguerre polynomials, *Adv. Differ. Equ.*, **2018** (2018), 277. https://doi.org/10.1186/s13662-018-1740-6

24. W. M. Abd-Elhameed, A. M. Alkenedri, Spectral solutions of linear and nonlinear BVPs using certain Jacobi polynomials generalizing third-and fourth-kinds of Chebyshev polynomials, *CMES Comp. Model. Eng.*, **126** (2021), 955–989. https://doi.org/10.32604/cmes.2021.013603

25. W. M. Abd-Elhameed, Novel expressions for the derivatives of sixth-kind Chebyshev polynomials: Spectral solution of the non-linear one-dimensional Burgers' equation, *Fractal Fract.*, **5** (2021), 53. https://doi.org/10.3390/fractalfract5020053

26. A. Napoli, W. M. Abd-Elhameed, An innovative harmonic numbers operational matrix method for solving initial value problems, *Calcolo*, **54** (2017), 57–76. https://doi.org/10.1007/s10092-016-0176-1

27. E. Tohidi, A. H. Bhrawy, K. Erfani, A collocation method based on Bernoulli operational matrix for numerical solution of generalized pantograph equation, *Appl. Math. Model.*, **37** (2013), 4283–4294. https://doi.org/10.1016/j.apm.2012.09.032

28. W. M. Abd-Elhameed, H. M. Ahmed, Tau and Galerkin operational matrices of derivatives for treating singular and Emden-Fowler third-order-type equations, *Int. J. Mod. Phys. C*, **33** (2022), 2250061. https://doi.org/10.1142/S0129183122500619

29. E. H. Doha, W. M. Abd-Elhameed, On the coefficients of integrated expansions and integrals of Chebyshev polynomials of third and fourth kinds, *Bull. Malays. Math. Sci. Soc.*, **37** (2014), 383–398.

30. T. Koshy, *Pell and P-Lucas numbers with applications*, New York: Springer, 2014. https://doi.org/10.1007/978-1-4614-8489-9

31. A. F. Horadam, B. Swita, P. Filipponi, Integration and derivative sequences for Pell and Pell-Lucas polynomials, *Fibonacci Quart.*, **32** (1994) 130–135.

32. A. F. Horadam, Chebyshev and Pell connections, *Fibonacci Quart.*, **43** (2005), 108–121.

33. W. Wang, H. Wang, Some results on convolved (p, q)-Fibonacci polynomials, *Integr. Transf. Spec. F.*, **26** (2015), 340–356. https://doi.org/10.1080/10652469.2015.1007502

34. P. M. C. Catarino, Diagonal functions of the k-Pell and k-Pell-Lucas polynomials and some identities, *Acta Math. Univ. Comen.*, **87** (2018), 147–159.

35. D. Tasci, F. Yalcin, Vieta-Pell and Vieta-Pell-Lucas polynomials, *Adv. Differ. Equ.*, **2013** (2013), 244. https://doi.org/10.1186/1687-1847-2013-224

36. W. Chu, N. N. Li, Power sums of Pell and Pell-Lucas polynomials, *Fibonacci Quart.*, **49** (2011), 139–150.

37. J. P. O. Santos, M. Ivkovic, Polynomial generalizations of the Pell sequences and the Fibonacci sequence, *Fibonacci Quart.*, **43** (2005), 328–338.

38. J. J. Bravo, J. L. Herrera, F. Luca, Common values of generalized Fibonacci and Pell sequences, *J. Number Theory*, **226** (2021), 51–71. https://doi.org/10.1016/j.jnt.2021.03.001

39. S. E. Rihane, Y. Akrour, A. E. Habibi, Fibonacci numbers which are products of three Pell numbers and Pell numbers which are products of three Fibonacci numbers, *Bol. Soc. Mat. Mex.*, **26** (2020), 895–910. https://doi.org/10.1007/s40590-020-00296-x

40. S. Yüzbaşı, G. Yıldırım, A collocation method to solve the parabolic-type partial integro-differential equations via Pell-Lucas polynomials, *Appl. Math. Comput.*, **421** (2022), 126956. https://doi.org/10.1016/j.amc.2022.126956

41. S. Yüzbaşı, G. Yıldırım, Pell-Lucas collocation method for solving a class of second order nonlinear differential equations with variable delays, *Turk. J. Math.*, **47** (2023), 37–55. https://doi.org/10.55730/1300-0098.3344

42. A. S. Mohamed, Pell collocation pseudo spectral scheme for one-dimensional time-fractional convection equation with error analysis, *J. Funct. Space.*, **2022** (2022), 9734604 https://doi.org/10.1155/2022/9734604

43. G. E. Andrews, R. Askey, R. Roy, *Special functions*, Cambridge: Cambridge University Press, 1999.

44. W. M. Abd-Elhameed, Novel formulae of certain generalized Jacobi polynomials, *Mathematics*, **10** (2022), 4237. https://doi.org/10.3390/math10224237

45. J. C. Mason, D. C. Handscomb, *Chebyshev polynomials*, New Youk: CRC Press, 2002. https://doi.org/10.1201/9781420036114

46. W. M. Abd-Elhameed, H. M. Ahmed, A. Napoli, V. Kowalenko, New formulas involving Fibonacci and certain orthogonal polynomials, *Symmetry*, **15** (2023), 736. https://doi.org/10.3390/sym15030736

47. W. Koepf, *Hypergeometric summation: An algorithmic approach to summation and special function identities*, 2 Eds., London: Springer, 2014. https://doi.org/10.1007/978-1-4471-6464-7

48. M. van Hoeij, Finite singularities and hypergeometric solutions of linear recurrence equations, *J. Pure Appl. Algebra*, **139** (1999), 109–131. https://doi.org/10.1016/S0022-4049(99)00008-0

49. J. M. Campbell, A Wilf-Zeilberger-based solution to the Basel problem with applications, *Discrete Math. Lett.*, **10** (2022), 21–27. https://doi.org/10.47443/dml.2022.030

50. J. M. Campbell, Applications of Zeilberger's algorithm to Ramanujan-inspired series involving harmonic-type numbers, *Discrete Math. Lett.*, **11** (2023), 7–13. https://doi.org/10.47443/dml.2022.050

51. D. A. Wolfram, Solving recurrences for Legendre-Bernstein basis transformations, *Examples and Counterexamples*, **4** (2023), 100117. https://doi.org/10.1016/j.exco.2023.100117

52. G. B. Djordjevic, G. V. Milovanovic, *Special classes of polynomials*, Leskovac: University of Nis, Faculty of Technology, 2014.

53. F. A. Costabile, *Modern umbral calculus: An elementary introduction with applications of linear interpolation and operator approximation theory*, Berlin, Boston: De Gruyter, 2019. https://doi.org/10.1515/9783110652925

54. X. G. Lv, T. Z. Huang, A note on inversion of Toeplitz matrices, *Appl. Math. Lett.*, **20** (2007), 1189–1193. https://doi.org/10.1016/j.aml.2006.10.008

55. P. G. Martinsson, V. Rokhlin, M. Tygert, A fast algorithm for the inversion of general Toeplitz matrices, *Comput. Math. Appl.*, **50** (2005), 741–752. https://doi.org/10.1016/j.camwa.2005.03.011

56. M. K. Ng, K. Rost, Y. W. Wen, On inversion of Toeplitz matrices, *Linear Algebra Appl.*, **348** (2002), 145–151. https://doi.org/10.1016/S0024-3795(01)00592-4

57. F. A. Costabile, M. I. Gualtieri, A. Napoli, *Polynomial sequences: Basic methods, special classes, and computational applications*, Berlin: De Gruyter, 2023.

58. N. D. Cahill, J. R. D'Errico, D. A. Narayan, J. Y. Narayan, Fibonacci determinants, *The College Mathematics Journal*, **33** (2002), 221–225. https://doi.org/10.1080/07468342.2002.11921945

59. N. J. Higham, *Accuracy and stability of numerical algorithms*, Philadelphia: SIAM, 2002.

60. R. T. Farouki, The Bernstein polynomial basis: A centennial retrospective, *Comput. Aided Geom. D.*, **29** (2012), 379–419. https://doi.org/10.1016/j.cagd.2012.03.001

61. H. Zhang, Numerical condition of polynomials in different forms, *Electron. T. Numer. Anal.*, **12** (2001), 66–87.

62. R. T. Farouki, V. T. Rajan, On the numerical condition of polynomials in Bernstein form, *Comput. Aided Geom. D.*, **4** (1987), 191–216. https://doi.org/10.1016/0167-8396(87)90012-4

63. W. Gautschi, On the condition of algebraic equations, *Numer. Math.*, **21** (1973), 405–424. https://doi.org/10.1007/BF01436491

64. J. H. Wilkinson, The perfidious polynomial, *Studies in Numerical Analysis*, **24** (1984), 1–28.

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