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Research article

Exploring new geometric contraction mappings and their applications in fractional metric spaces

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Abstract: This article delves deeply into some mathematical basic theorems and their diverse applications in a variety of domains. The major issue of interest is the Banach Fixed Point Theorem (BFPT), which states the existence of a unique fixed point in fractional metric spaces. The significance of this theorem stems from its utility in a variety of mathematical situations for approximating solutions and resolving iterative problems. On this foundational basis, the study expands by introducing the concept of fractional geometric contraction mappings, which provide a new perspective on how convergence develops in fractional metric spaces.

Keywords: fractional metric space; geometric contraction mappings; convergence; fractional derivatives
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1. Introduction

Fixed point (FP) theory is a fundamental mathematical subject with applications in a variety of domains. Understanding system dynamics requires an analysis of FPs, which show unverified states under imposed transformations. The study of FP results associated with many contractions has emerged as a significant area of research. This domain provides new concepts and practical implications in a range of fields. The BFPT, a foundational exploration, establishes the existence of a unique FP within particular mappings within a complete metric space (MS) [1]. The consequences of the theorem extend beyond its mathematical foundation. This theorem's convergence includes enable for the dissection of iterative processes and the exploration of equilibrium states in physics. Complete insights can be found in several references, like [2–8].

A contraction has attracted a significant interest due to its ability to evaluate and quantify the stability and convergence characteristics of systems. This concept has developed as a powerful framework that supports and extends established notions of contractions in MSs. Many contractions related to theoretical breakthroughs and theorems have been important in expanding our knowledge and use of this idea. These improvements have given us essential tools and methodologies for studying the behavior of systems with these contraction features. Researchers have been able to properly study and prove the stability and convergence characteristics related to these contractions by developing theorems and mathematical frameworks. Researchers have looked into several forms of contractions, such as weak contractions, Kannan contractions, and other generalized contractions, see [3, 4, 9–11].

Furthermore, we investigate the wide range of applications that result from the use of fractional operators in MSs. Control systems are one significant application of numerous contractions. An important goal is to develop control systems that assure stability and robustness in the face of problems and uncertainty. Control systems can be developed that provide reliable and effective results by using the FP results related to these contractions. Financial modeling is another attractive employ for FP results related to several contractions. Financial market dynamics are complex and dynamic, making accurate predictions and well-informed investment decisions difficult. Market dynamics can be clarified by adding the principles of numerous contractions, leading to better price estimates and predictions. For more details, see [12–16].

We explore the importance and ramifications of FP results for fractional geometric contractions in fractional MSs throughout this study. We investigate the practical applications in financial modeling and control systems, emphasizing their influence on system performance and decision-making processes. We also explore the theoretical foundations, providing explanations for the stability and convergence properties established by these results.

2. Preliminaries

We first begin a thorough investigation of fundamental definitions, theorems, and illustrations that explain the behavior of functions in MSs and elucidate the fundamental characteristics of fractional operators. By exploring these fundamental ideas, we hope to offer an insightful understanding of the delicate dynamics and properties of functions within the setting of MSs, which serve as a foundation for researching the idea of distance and convergence. They provide a fundamental framework for examining FPs and their properties in different mathematical contexts.

Definition 2.1. [1, 17] Consider a mapping $T : Ł \to Ł$ on a MS $(Ł, d)$. If there exists $0 \leq k < 1$ such that for all $\eta, \varsigma \in Ł$,

$$d(T(\eta), T(\varsigma)) \leq k \cdot d(\eta, \varsigma),$$

then the mapping T is said to be a contraction.

A contraction mapping is a form of mappings in which the distances between points in a MS are contracted or reduced. This property plays a great role in many FP theorems and it is utilized to demonstrate the existence of FPs.

Definition 2.2. [18] Given a mapping $T : Ł \to Ł$ on a MS $(Ł, d)$. The element $x \in X$ is termed a FP of T if $T(x) = x$.

FPs are critical in the analysis of mappings and iterative algorithms. They provide information about the behavior and features of the mappings, and their existence and uniqueness have important implications in a variety of scientific and applied domains in nature.

Definition 2.3. [11] Assume $(Ł, d)$ is a MS. The distance function $d : Ł \times Ł \to [0, +\infty)$ is continuous if, for each $\varepsilon > 0$ and for all $(x, y) \in Ł \times Ł$, there is $\delta > 0$ that ensures for all $(\eta, \varsigma) \in Ł \times Ł$ with $d(\eta, x) < \delta$ and $d(\varsigma, y) < \delta$, the following inequality

$$|d(\eta, \varsigma) - d(x, y)| < \varepsilon.$$

Definition 2.4. [11] Let $(Ł, d)$ be a MS. The distance function $d : Ł\times Ł \to [0, +\infty)$ is bounded if there exists a constant $M > 0$ such that for all $(x, y) \in Ł \times Ł$,

$$d(x, y) \leq M.$$

Definition 2.5 (Fractional Operators)**.** [19, 20] Fractional operators are mathematical operators that extend the concept of differentiation and integration to non-integer orders. They give a framework for defining fractional derivatives (FDs) and integrals for functions of non-integer orders. Considering a function $f : Ł \to \mathbb{R}$ specified on an appropriate domain Ł, numerous fractional operators can be employed to generate FDs and integrals.

(1) Riemann-Liouville fractional operator: The Riemann-Liouville FD $D^{\alpha}_{RL}f(x)$ of order $\alpha \in (0, 1)$ is defined as follows:

$$D^{\alpha}_{RL}f(x) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\left(\int_a^x \frac{f(t)}{(x-t)^{\alpha}}dt\right),$$

where $\Gamma(\cdot)$ denotes the gamma function, $\frac{d}{dx}$ represents the ordinary derivative with respect to x, and a is a lower limit of integration.

(2) Caputo fractional operator: The Caputo FD $D^{\alpha}_{C}f(x)$ of order $\alpha \in (0, 1)$ is defined as follows:

$$D^{\alpha}_{C}f(x) = \frac{1}{\Gamma(1-\alpha)}\int_a^x \frac{f'(t)}{(x-t)^{\alpha}}dt,$$

where $\Gamma(\cdot)$ denotes the gamma function and f' denotes the derivative of f.

These fractional operators provide different perspectives and approaches to define FDs, each with its own advantages and applications in various fields of mathematics and physics.

Example 2.1. Given a function $f : Ł \to \mathbb{R}$ defined on a suitable domain Ł, the following types of fractional operators can be used to generate FDs and integrals:

(1) Riemann-Liouville fractional operator: Let's consider the function $f(x) = x^{\alpha}$, where $\alpha \in (0, 1)$. We can apply the Riemann-Liouville FD D^{α}_{RL} to this function to obtain

$$D^{\alpha}_{RL} f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \left(\int_a^x \frac{f(t)}{(x-t)^{\alpha}} dt \right).$$

By substituting $f(x) = x^{\alpha}$ and evaluating the integral and derivative, we obtain the Riemann-Liouville FD of $f(x)$.

(2) Caputo fractional operator: Now, let's consider the function $g(x) = \sin(\omega x)$, where ω is a constant. We can apply the Caputo FD D^{α}_{C} to get

$$D^{\alpha}_{C} g(x) = \frac{1}{\Gamma(1-\alpha)} \int_a^x \frac{g'(t)}{(x-t)^{\alpha}} dt.$$

By substituting $g(x) = \sin(\omega x)$ and evaluating the integral and derivative, we obtain the Caputo FD of $g(x)$.

These examples demonstrate the various views and ways of defining FDs provided by fractional operators. Each operator has distinct advantages and arises in many domains of mathematics and physics.

3. Main results

First, we initiate a new generalized MS named as a fractional MS, along with new contractions.

Definition 3.1. A fractional MS is a mathematical framework defined by $(Ł, d_f)$, where Ł is a non-empty set and $d_f : Ł \times Ł \to [0, +\infty]$ is a distance function that has the following characteristics for any $x, y, z \in Ł$:

(1) Non-Negativity: $d_f(x, y) \geq 0$ for any two items x, y in Ł. Furthermore, $d_f(x, y) = 0$ if and only if $x = y$.
(2) Symmetry: $d_f(x, y)$ is symmetric, i.e., $d_f(x, y) = d_f(y, x)$.
(3) Fractional Triangle Inequality: There is a fixed positive real number α such that for all elements x, y, z in Ł,

$$d_f(x, y) \leq d_f(x, z)^{\alpha} + d_f(z, y)^{\alpha}.$$

The coefficient α is a parameter that defines the distance function's flexibility. Smaller values of α values provide for a more flexible distance measurement, allowing for uncertainty and varying degrees of similarity between elements.

The fractional MS definition goes within the direction of a generalized μ-metric, which was appeared in the nice work of Brzdek et al. [21] It corresponds to a particular case of the function μ. In [21], it was considered the dislocated quasi μ-metric setting. Here, both the symmetry and the self-distance hold.

Definition 3.2. Let $(Ł, d_f)$ be a fractional MS. A sequence $\{x_n\}$ in Ł is said to be:

(1) A convergent sequence to $x^* \in X$ if for any $\delta > 0$, there exists an index N such that for all $n \geq N$,

$$d_f(x_n, x^*) < \delta.$$

In this case, we write $\lim_{n\to+\infty} x_n = x^*$.

(2) A Cauchy sequence if for any $\delta > 0$, there exists an index N such that for all $n, m \geq N$,

$$d_f(x_n, x_m) < \delta.$$

Also, a fractional MS $(Ł, d_f)$ is said to be complete if any Cauchy sequence is convergent in Ł. That is, for each Cauchy sequence $\{x_n\}$ in Ł, there exists $x^* \in X$ so that $\lim_{n\to+\infty} x_n = x^*$.

Example 3.1. Take the set $Ł = \{0, 1, 2\}$. Given $d_f : Ł \times Ł \to [0, +\infty)$ as

$$d_f(\sigma, y) = \begin{cases} 0, & \text{if } \sigma = y, \\ \left|\frac{\sigma - y}{\sigma + y}\right|^{\frac{1}{3}}, & \text{if } \sigma \neq y. \end{cases}$$

Take $\alpha = \frac{1}{3}$. To demonstrate that $(Ł, d_f)$ creates a Fractional MS, we must check the following properties for all σ, y, z in Ł:

(1) $d_f(\sigma, y) = 0$ for any σ, y in Ł. This property holds because the distance function is defined to be non-negative. In addition, $d_f(\sigma, y) = 0$ if and only if $\sigma = y$.

(2) We can write $d_f(\sigma, y) = d_f(y, \sigma)$. This property is clear from the formulation of the distance function, as $d_f(\sigma, y) = d_f(y, \sigma) = \left|\frac{\sigma - y}{\sigma + y}\right|^{\frac{1}{3}}$ if $\sigma \neq y$, and $d_f(\sigma, y) = d_f(y, \sigma) = 0$ if $\sigma = y$.

(3) We must demonstrate that $d_f(\sigma, y) \leq d_f(\sigma, z)^{\frac{1}{3}} + d_f(z, y)^{\frac{1}{3}}$ for all σ, y, z in Ł.
Utilizing the distance function definition, we get

$$d_f(\sigma, z)^{\frac{1}{3}} = \left|\frac{\sigma - z}{\sigma + z}\right|^{\frac{1}{3}}, \quad \text{if } \sigma \neq z, \text{ and}$$
$$d_f(z, y)^{\frac{1}{3}} = \left|\frac{z - y}{z + y}\right|^{\frac{1}{3}}, \quad \text{if } z \neq y.$$

If $\sigma = y$, then $d_f(\sigma, y) = 0 \leq 0 + 0$, meeting the inequality. If $\sigma \neq y$, then $d_f(\sigma, y) = \left|\frac{\sigma - y}{\sigma + y}\right|^{\frac{1}{3}} \leq \left(\left|\frac{\sigma - z}{\sigma + z}\right| + \left|\frac{z - y}{z + y}\right|\right)^{\frac{1}{3}}$, which it also meets the inequality. Consequently, the inequality of the fractional triangle holds.

As a result, $(Ł, d_f)$ constructs a fractional MS using the distance function previously described for $\alpha = \frac{1}{3}$.

Example 3.2. Choose $Ł = \mathbb{R}$. Let $d_f : Ł \times Ł \to [0, +\infty)$ be defined as follows:

$$d_f(\sigma, y) = |\sin(\sigma - y)|^{\frac{1}{\pi}}.$$

Take $\alpha = \frac{1}{\pi}$. To demonstrate that $(Ł, d_f)$ forms a fractional MS, we must check the following properties for any x, y, z in Ł:

(1) We have $d_f(\sigma, y) \geq 0$ for any σ, y in $\mathbb{R}$. Given that the absolute value of $\sin(\sigma - y)$ is non-negative for every real σ and y, this property maintains. Also, $d_f(\sigma, y) = 0$ iff $\sin(\sigma - y) = 0$, which implies $\sigma - y = k\pi$ for some integer k. Therefore, $d_f(\sigma, y) = 0$ if and only if $\sigma = y$.

(2) We have $d_f(\sigma, y) = d_f(y, \sigma)$. The definition of the distance function demonstrates this property, as $|\sin(\sigma - y)|^{\frac{1}{\pi}} = |\sin(y - \sigma)|^{\frac{1}{\pi}}$.

(3) We must prove that $d_f(\sigma, y) \leq d_f(\sigma, z)^{\frac{1}{\pi}} + d_f(z, y)^{\frac{1}{\pi}}$ for any σ, y, z in $\mathbb{R}$.
Utilizing the distance function definition, we get

$$d_f(\sigma, z)^{\frac{1}{\pi}} = |\sin(\sigma - z)|^{\frac{1}{\pi}},$$
$$d_f(z, y)^{\frac{1}{\pi}} = |\sin(z - y)|^{\frac{1}{\pi}}.$$

Utilizing the triangle inequality for real numbers,

$$|\sin(\sigma - z)| + |\sin(z - y)| \geq |\sin(\sigma - z) + \sin(z - y)|.$$

Bringing both sides to the power of $\frac{1}{\pi}$ (since $\frac{1}{\pi} > 0$), one writes

$$|\sin(\sigma - z)|^{\frac{1}{\pi}} + |\sin(z - y)|^{\frac{1}{\pi}} \geq |\sin(\sigma - z) + \sin(z - y)|^{\frac{1}{\pi}}.$$

Since $|\sin(\sigma - z) + \sin(z - y)|$ is greater than or equal to $|\sin(\sigma - y)|$, we conclude that

$$|\sin(\sigma - z)|^{\frac{1}{\pi}} + |\sin(z - y)|^{\frac{1}{\pi}} \geq |\sin(\sigma - y)|^{\frac{1}{\pi}}.$$

As a result, the fractional triangle inequality holds.

Thus, $(Ł, d_f)$ defines a fractional MS with the distance function established above for $\alpha = \frac{1}{\pi}$.

Lemma 3.1. *Let $(Ł, d_f)$ be a fractional MS and Ł be a non-emty set contained on a real numbers $\mathbb{R}$. Assume that $d_f : Ł \times Ł \to [0, +\infty)$ has the property $d_f(x, y) = \left|\frac{f(x,y)}{g(x,y)}\right|^{\alpha}$ for a fixed $\alpha > 0$ and $g(x, y) \neq 0$, where $f : Ł \times Ł \to \mathbb{R}$ and $g : Ł \times Ł \to \mathbb{R}$ are continuous functions for all $(x, y) \in Ł \times Ł$. If $f(x, y) \leq g(x, y)$, then, the fractional distance function $d_f : Ł \times Ł \to [0, +\infty)$ is continuous with respect to the product topology on $Ł \times Ł$.*

Proof. For any $(x, y) \in Ł \times Ł$, let $\varepsilon > 0$ be given. Since f and g are continuous at (x, y), there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that if $|x - \jmath| < \delta_1$ and $|y - \ell| < \delta_2$, then $|f(x, y) - f(\jmath, \ell)| < \frac{\varepsilon}{2}$ and $|g(x, y) - g(\jmath, \ell)| < \frac{\varepsilon}{2}$.

Now, let $\delta = \min\{\delta_1, \delta_2\}$. For any $(\jmath, \ell) \in Ł \times Ł$ such that $d_f((x, y), (\jmath, \ell)) < \delta$, and using the condition $f(x, y) \leq g(x, y)$, we have

$$\begin{aligned}
\left|\frac{f(x,y)}{g(x,y)} - \frac{f(\jmath,\ell)}{g(\jmath,\ell)}\right| &= \left|\frac{f(x,y)g(\jmath,\ell) - f(\jmath,\ell)g(x,y)}{g(x,y)g(\jmath,\ell)}\right| \\
&= \left|\frac{f(x,y)g(\jmath,\ell) - f(\jmath,\ell)g(\jmath,\ell) + f(\jmath,\ell)g(\jmath,\ell) - f(\jmath,\ell)g(x,y)}{g(x,y)g(\jmath,\ell)}\right| \\
&= \left|\frac{f(x,y) - f(\jmath,\ell)}{g(x,y)} + \frac{f(\jmath,\ell)\,(g(\jmath,\ell) - g(x,y))}{g(x,y)g(\jmath,\ell)}\right| \\
&\leq \left|\frac{f(x,y) - f(\jmath,\ell)}{g(x,y)}\right| + \left|\frac{g(x,y) - g(\jmath,\ell)}{g(x,y)}\right|.
\end{aligned}$$

Since $|f(x,y)-f(\jmath,\ell)| < \frac{\varepsilon}{2}$ and $|g(x,y)-g(\jmath,\ell)| < \frac{\varepsilon}{2}$, then $\left|\frac{f(x,y)-f(\jmath,\ell)}{g(x,y)}\right| < \frac{\varepsilon}{2}$, and $\left|\frac{g(x,y)-g(\jmath,\ell)}{g(x,y)}\right| < \frac{\varepsilon}{2}$, respectively.

Hence,

$$\begin{aligned}\left|\frac{f(x,y)}{g(x,y)} - \frac{f(\jmath,\ell)}{g(\jmath,\ell)}\right| &\le \left|\frac{f(x,y)-f(\jmath,\ell)}{g(x,y)}\right| + \left|\frac{g(x,y)-g(\jmath,\ell)}{g(x,y)}\right| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

Consequently, d_f is continuous at (x,y). Since (x,y) is arbitrary, the fractional distance function is continuous with regard to the product topology on Ł × Ł. □

Lemma 3.2. *Assume that* $(Ł, d_f)$ *is a fractional MS with the distance function* $d_f : Ł \times Ł \to [0,+\infty)$, *which described as* $d_f(x,y) = \left|\frac{f(x,y)}{g(x,y)}\right|^\alpha$ *for a fixed* $\alpha > 0$ *and* $g(x,y) \neq 0$, *where* $f : Ł \times Ł \to \mathbb{R}$ *and* $g : Ł \times Ł \to \mathbb{R}$ *are continuous functions for all* $(x,y) \in Ł\times Ł$. *If there exist constants* M_f, $M_g > 0$ *such that* $|f(x,y)| \le M_f$ *and* $|g(x,y)| \le M_g$ *for all* $(x,y) \in Ł \times Ł$, *then the fractional distance function* $d_f : Ł \times Ł \to [0,+\infty)$ *is a bounded function.*

Proof. Since f and g are bounded, then for any $(\jmath, y) \in Ł\times Ł$, we have $|f(\jmath,y)| \le M_f$ and $|g(\jmath,y)| \le M_g$.

Consider the fractional distance function $d_f(\jmath,y) = \left|\frac{f(\jmath,y)}{g(\jmath,y)}\right|^\alpha$, one has

$$d_f(\jmath,y) = \left|\frac{f(\jmath,y)}{g(\jmath,y)}\right|^\alpha = \left(\frac{|f(\jmath,y)|}{|g(\jmath,y)|}\right)^\alpha \le \left(\frac{M_f}{M_g}\right)^\alpha,$$

where $\left(\frac{M_f}{M_g}\right)^\alpha$ is a constant. Consequently, the fractional distance function d_f is bounded. □

Theorem 3.1. *Let* $(Ł, d_f)$ *be a complete fractional MS with a distance function* $d_f : Ł \times Ł \to [0,+\infty)$ *given as* $d_f(x,y) = \left|\frac{f(x,y)}{g(x,y)}\right|^\alpha$, *where* $f : Ł \times Ł \to \mathbb{R}$ *and* $g : Ł \times Ł \to \mathbb{R}$ *are continuous functions, and* $\alpha > 0$ *is a constant. Let* $T : Ł \to Ł$ *be a fractional geometric contraction, that is, there exists a constant* $0 < \lambda < 1$ *such that for all* $x, y \in Ł$, *we have*

$$d_f(T(x),T(y)) \le \lambda \cdot \left|\frac{f(x,y)}{g(x,y)}\right|^\alpha. \tag{3.1}$$

Then T *possesses a unique FP in* Ł.

Proof. The mapping T is a fractional geometric contraction, so there exists a constant $0 < \lambda < 1$ such that we get for any $x, y \in Ł$, (3.1) holds. Consider the function $\phi : Ł \to [0,+\infty)$ defined by $\phi(x) = \left|\frac{f(x,T(x))}{g(x,T(x))}\right|^\alpha$ for all $x \in Ł$. We observe that

$$\begin{aligned}\phi(T(x)) &= \left|\frac{f(T(x),T(T(x)))}{g(T(x),T(T(x)))}\right|^\alpha \\ &= \left|\frac{f(T(x),T(x))}{g(T(x),T(x))}\right|^\alpha \\ &= \left|\frac{f(x,T(x))}{g(x,T(x))}\right|^\alpha = \phi(x).\end{aligned}$$

This demonstrates that $\phi(x) = \phi(T(x))$ for every $x \in Ł$, allowing us to conclude that ϕ is indeed a constant function. Hence, $\phi(x) = c(constant)$ for all $x \in Ł$. That is,

$$\left|\frac{f(x, T(x))}{g(x, T(x))}\right|^{\alpha} = c \quad \text{for all } x \in Ł.$$

As $\alpha > 0$, this indicates

$$\frac{f(x, T(x))}{g(x, T(x))} = c^{\frac{1}{\alpha}} \quad \text{for all } x \in Ł.$$

Suppose $x^* \in Ł$ is a FP of T. Then

$$\frac{f(x^*, x^*)}{g(x^*, x^*)} = c^{\frac{1}{\alpha}}.$$

Since $d_f(x^*, x^*) = 0$ and $d_f(x^*, x^*) = \left|\frac{f(x^*,x^*)}{g(x^*,x^*)}\right|^{\alpha}$, one writes $0 = c^{\frac{1}{\alpha}}$. It follows that $c = 0$. Hence,

$$\frac{f(x, T(x))}{g(x, T(x))} = 0 \quad \text{for all } x \in Ł.$$

This means that for any $x \in Ł$, $f(x, T(x)) = 0$ or $g(x, T(x)) = +\infty$. Since $(Ł, d_f)$ is a complete MS, the BFPT states that T possesses a unique FP $x^* \in Ł$. □

Example 3.3. Take the set $Ł = [0, 1]$ and let the distance function $d_f : Ł \times Ł \to [0, +\infty)$ have the form $d_f(x, y) = \left|\frac{x-y}{xy+1}\right|^{\frac{1}{2}}$ in the fractional MS $(Ł, d_f)$. For $x \in Ł$, consider the self-mapping $T : Ł \to Ł$ given as $T(x) = \frac{x}{2}$. We will show that T is a fractional geometric contraction. To demonstrate the contraction property, identify a constant $0 < \lambda < 1$ such that for any $x, y \in Ł$,

$$d_f(T(x), T(y)) \leq \lambda \cdot \left|\frac{x-y}{xy+1}\right|^{\frac{1}{2}}.$$

Let $x, y \in Ł$ be arbitrary. Then, we have

$$\begin{aligned} d_f(T(x), T(y)) &= \left|\frac{T(x) - T(y)}{T(x)T(y) + 1}\right|^{\frac{1}{2}} \\ &= \left|\frac{\frac{x}{2} - \frac{y}{2}}{\frac{x}{2} \cdot \frac{y}{2} + 1}\right|^{\frac{1}{2}} \\ &= \left|\frac{x-y}{xy+1}\right|^{\frac{1}{2}} \\ &= d_f(x, y). \end{aligned}$$

This demonstrates that for all $x, y \in Ł$, $d_f(T(x), T(y)) = d_f(x, y)$. Now, set $\lambda = \frac{1}{2}$. We have the following for any $x, y \in Ł$,

$$d_f(T(x), T(y)) = d_f(x, y) \leq \frac{1}{2} \cdot \left|\frac{x-y}{xy+1}\right|^{\frac{1}{2}}.$$

Thus, T is a fractional geometric contraction on $(Ł, d_f)$. The mapping T has a unique FP in Ł. According to the BFPT, i.e., there exists a unique $x^* \in Ł$ such that $T(x^*) = x^*$. In this example, the unique FP of T is $x^* = 0$.

Example 3.4. Consider the set Ł $= [0,1]$ with the typical Euclidean distance function d_f : Ł × Ł → $[0,+\infty)$. Suppose f : Ł × Ł → $\mathbb{R}$ and g : Ł × Ł → $\mathbb{R}$ are continuous functions having the forms

$$f(x,y) = (x+y)^2,$$
$$g(x,y) = (x^2+y^2)^2,$$

where $x, y \in$ Ł. Consider the self-mapping T : Ł → Ł defined as $T(x) = \frac{x}{2}$ for all $x \in$ Ł. To demonstrate that T is a fractional geometric contraction mapping on the complete fractional MS (Ł, d_f), we must identify a constant $0 < \lambda < 1$ such that for any $x, y \in$ Ł, one writes

$$d_f(T(x), T(y)) \leq \lambda \cdot \left| \frac{f(x,y)}{g(x,y)} \right|^\alpha .$$

Utilizing the distance function d and the definitions of f and g, we get

$$\begin{aligned} d_f(T(x), T(y)) &= \left| \frac{f(T(x), T(y))}{g(T(x), T(y))} \right|^\alpha \\ &= \left| \frac{\left(\frac{T(x)+T(y)}{2}\right)^2}{\left(\left(\frac{T(x)^2+T(y)^2}{2}\right)^2\right)} \right|^\alpha \\ &= \left| \frac{\left(\frac{x+y}{2}\right)^2}{\left(\left(\frac{x^2+y^2}{2}\right)^2\right)} \right|^\alpha \\ &= \left| \frac{(x+y)^2}{(x^2+y^2)^2} \right|^\alpha . \end{aligned}$$

Since $x, y \in$ Ł $= [0,1]$, we have $0 \leq x, y \leq 1$. Therefore, $0 \leq x+y \leq 2$ and $0 \leq x^2+y^2 \leq 2$, which implies:

$$0 \leq (x+y)^2 \leq 4,$$
$$0 \leq (x^2+y^2)^2 \leq 4.$$

Thus,

$$0 \leq \left| \frac{(x+y)^2}{(x^2+y^2)^2} \right|^\alpha \leq 1.$$

Choosing $\lambda = \frac{1}{2}$, we have

$$d_f(T(x), T(y)) \leq \frac{1}{2} \cdot \left| \frac{(x+y)^2}{(x^2+y^2)^2} \right|^\alpha \leq \frac{1}{2}.$$

As a result, T is a fractional geometric contraction mapping on the complete fractional MS (Ł, d_f). By the BFPT, T has a unique FP $x^* \in$ Ł. Hence, $x^* = \frac{x^*}{2}$. That is, $x^* = 0$ is the unique FP of T.

Theorem 3.2. *Let* $(Ł, d_f)$ *be a complete fractional MS with a distance function* $d_f : Ł \times Ł \to [0, +\infty)$ *having the form* $d_f(x, y) = \left|\frac{f(x,y)}{g(x,y)}\right|^{\alpha}$*, where* $f : Ł \times Ł \to \mathbb{R}$ *and* $g : Ł \times Ł \to \mathbb{R}$ *are continuous functions, and* $\alpha > 0$ *is a constant. A fractional proportional contraction is a self-mapping* $T : Ł \to Ł$ *on* Ł *if there exists a constant* $0 < \lambda < 1$ *such that for all* $x, y \in Ł$,

$$\left|\frac{f(T(x), T(y))}{g(T(x), T(y))}\right|^{\alpha} \leq \lambda \cdot \left|\frac{f(x, y)}{g(x, y)}\right|^{\alpha}.$$

Proof. Let $T : Ł \to Ł$ be a fractional proportional contraction on the complete fractional MS $(Ł, d_f)$. We will demonstrate that T has a unique FP in Ł. Since T is a fractional proportional contraction, there is $0 < \lambda < 1$ such that for all $x, y \in Ł$,

$$\left|\frac{f(T(x), T(y))}{g(T(x), T(y))}\right|^{\alpha} \leq \lambda \cdot \left|\frac{f(x, y)}{g(x, y)}\right|^{\alpha}.$$

Consider the function $\phi : X \to [0, +\infty]$ defined by $\phi(x) = \left|\frac{f(x,T(x))}{g(x,T(x))}\right|^{\alpha}$ for all $x \in Ł$. We realize that for any $x \in Ł$,

$$\begin{aligned}\phi(T(x)) &= \left|\frac{f(T(x), T(T(x)))}{g(T(x), T(T(x)))}\right|^{\alpha} \\ &= \left|\frac{f(T(x), T(x))}{g(T(x), T(x))}\right|^{\alpha} \\ &= \left|\frac{f(x, T(x))}{g(x, T(x))}\right|^{\alpha} \\ &= \phi(x).\end{aligned}$$

This demonstrates that $\phi(x) = \phi(T(x))$ for every $x \in Ł$, allowing us to conclude that ϕ is a constant function. Let c be the constant value of ϕ for all $x \in Ł$, i.e., $\phi(x) = c$ for all $x \in Ł$. We have for all $x \in Ł$,

$$\left|\frac{f(x, T(x))}{g(x, T(x))}\right|^{\alpha} = c \quad \text{for all } x \in Ł.$$

Since $\alpha > 0$, one writes

$$\frac{f(x, T(x))}{g(x, T(x))} = c^{\frac{1}{\alpha}} \quad \text{for all } x \in Ł.$$

Let $x^* \in Ł$ be so that $T(x^*) = x^*$. Then,

$$\frac{f(x^*, x^*)}{g(x^*, x^*)} = c^{\frac{1}{\alpha}}.$$

Since $d_f(x^*, x^*) = 0$ and $d_f(x^*, x^*) = \left|\frac{f(x^*,x^*)}{g(x^*,x^*)}\right|^{\alpha}$, we can write

$$\begin{aligned}0 &= \left|\frac{f(x^*, x^*)}{g(x^*, x^*)}\right|^{\alpha} \\ &= c^{\frac{1}{\alpha}}.\end{aligned}$$

Since $0 < \lambda < 1$, it follows that $c = 0$. Hence,

$$\frac{f(x, T(x))}{g(x, T(x))} = 0 \quad \text{for all } x \in Ł.$$

This means that for any $x \in Ł$, either $f(x, T(x)) = 0$ or $g(x, T(x)) = +\infty$. Since $(Ł, d_f)$ is a complete MS, it follows from the BFPT that T has a unique FP $x^* \in Ł$. □

Example 3.5. Consider the set $Ł = [1, +\infty)$ with the usual Euclidean distance function $d_f : Ł \times Ł \to [0, +\infty)$. Let $f : Ł \times Ł \to \mathbb{R}$ and $g : Ł \times Ł \to \mathbb{R}$ be continuous functions with the following definitions:

$$f(x, y) = \ln(xy),$$
$$g(x, y) = \sqrt{xy},$$

where $x, y \in Ł$. Assume the self-mapping $T : Ł \to Ł$ is defined as $T(x) = \frac{x}{2}$ for all $x \in Ł$. To demonstrate that T represents a fractional proportional contraction mapping in the complete fractional MS $(Ł, d_f)$, we must identify a constant $0 < \lambda < 1$ so that for all $x, y \in Ł$,

$$\left|\frac{f(T(x), T(y))}{g(T(x), T(y))}\right|^\alpha \leq \lambda \cdot \left|\frac{f(x, y)}{g(x, y)}\right|^\alpha.$$

Utilizing the d distance function with the f and g definitions, we get

$$d_f(T(x), T(y)) = \left|\frac{f(T(x), T(y))}{g(T(x), T(y))}\right|^\alpha = \left|\frac{\ln(T(x)T(y))}{\sqrt{T(x)T(y)}}\right|^\alpha$$
$$= \left|\frac{\ln\left(\frac{x}{2} \cdot \frac{y}{2}\right)}{\sqrt{\frac{x}{2} \cdot \frac{y}{2}}}\right|^\alpha = \left|\frac{\ln\left(\frac{xy}{4}\right)}{\sqrt{\frac{xy}{4}}}\right|^\alpha.$$

Consider the distance $d_f(x, y)$ for $x, y \in Ł$,

$$d_f(x, y) = \left|\frac{f(x, y)}{g(x, y)}\right|^\alpha = \left|\frac{\ln(xy)}{\sqrt{xy}}\right|^\alpha.$$

Thus,

$$\left|\frac{f(T(x), T(y))}{g(T(x), T(y))}\right|^\alpha = \left|\frac{\ln\left(\frac{xy}{4}\right)}{\sqrt{\frac{xy}{4}}}\right|^\alpha = \left|\frac{\ln(xy) - \ln(4)}{\sqrt{xy}/2}\right|^\alpha$$
$$= \left|\frac{\ln(xy)}{\sqrt{xy}} \cdot \frac{2}{\sqrt{xy}} - \frac{\ln(4)}{\sqrt{xy}}\right|^\alpha$$
$$\leq \left|\frac{\ln(xy)}{\sqrt{xy}}\right|^\alpha + \left|\frac{\ln(4)}{\sqrt{xy}}\right|^\alpha$$
$$= d_f(x, y) + \left|\frac{\ln(4)}{\sqrt{xy}}\right|^\alpha.$$

Since $0 < x, y \in Ł = [1, +\infty)$, we have $\sqrt{xy} \geq 1$. Therefore,

$$0 < \left|\frac{\ln(4)}{\sqrt{xy}}\right|^{\alpha} \leq \left|\frac{\ln(4)}{1}\right|^{\alpha} = (\ln(4))^{\alpha}.$$

Assuming $\lambda = \frac{(\ln(4))^{\alpha}}{2}$, we get

$$\left|\frac{f(T(x), T(y))}{g(T(x), T(y))}\right|^{\alpha} \leq \lambda \cdot \left|\frac{f(x, y)}{g(x, y)}\right|^{\alpha}.$$

As a result, T represents a fractional proportional contraction mapping on the complete fractional MS $(Ł, d_f)$. According to the BFPT, T has a unique FP $x^* \in Ł$. One writes $x^* = \frac{x^*}{2}$ and so $x* = 0$ is the unique FP of T. Utilizing the iterative technique $x_n + 1 = T(x_n)$, we want to identify the FP $x^* = 0$ of the mapping T. We will iterate multiple steps to estimate the FP, starting with an initial approximation of $x_0 = 2$ as in Table 1 and Figure 1.

Table 1. Approximating the FP using the iterative method.

n	0	1	2	3	4	5	6
x_n	2	1	0.5	0.25	0.125	0.0625	0.03125

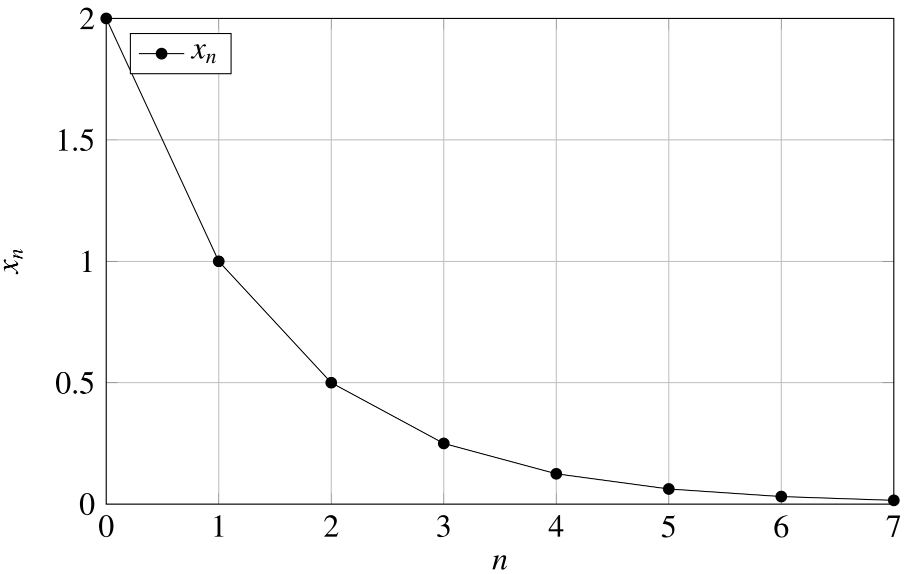

Figure 1. Representation of approximating the FP.

4. Applications

In this section, we aim to harness the theoretical insights acquired from the preceding section to explicate the existence of a unique solution for nonlinear differential equations. We recommend referring to contemporary publications, such as [5, 12, 22].

4.1. Nonlinear oscillator modeling

In this part, we delve into the utilization of differential equations for controlling complex systems. The exploration commences with an introductory overview, emphasizing the significance of differential equations in the depiction of dynamic systems. We present Theorem 4.1, which establishes the existence of a unique solution for differential equations. Further details can be found in [23–25].

Theorem 4.1. *A fractional geometric contraction is described as a mapping T on the complete fractional MS $(Ł, d_f)$ such that there is a constant $0 < \lambda < 1$ so that for all $x, y \in Ł$,*

$$d_f(T(x), T(y)) \leq \lambda \cdot \left|\frac{f(x,y)}{g(x,y)}\right|^{\alpha}.$$

The differential equation can also be used to describe a nonlinear oscillator:

$$\frac{d^2y}{dt^2} + \left(\frac{dy}{dt}\right)^3 + \sin(y) = 0, \tag{4.1}$$

where $y(t)$ denotes the oscillator's displacement at time t. $y(0) = y_0$ and $\frac{dy}{dt}(0) = v_0$, where y_0 is the initial displacement and v_0 is the initial velocity, respectively. Define the initial conditions. When T is a fractional geometric contraction on Ł, the nonlinear oscillator equation (4.1) provides a unique solution $y(t)$ in the complete fractional MS $(Ł, d_f)$.

Proof. We aim to verify that there is a unique solution $y(t)$ to the nonlinear oscillator equation (4.1) in the complete fractional MS $(Ł, d_f)$, provided that T is a fractional geometric contraction on Ł. Since T is a fractional geometric contraction, we have

$$d_f(T(x), T(y)) \leq \lambda \cdot \left|\frac{f(x,y)}{g(x,y)}\right|^{\alpha}.$$

Applying this inequality to the nonlinear oscillator equation (4.1), we obtain

$$\left|\frac{f(T(x), T(y))}{g(T(x), T(y))}\right|^{\alpha} \leq \lambda \cdot \left|\frac{f(x,y)}{g(x,y)}\right|^{\alpha}.$$

Since f and g are continuous functions, they preserve the continuity of T. Therefore, we can apply the Mean Value Theorem for fractional calculus to find $\xi \in [T(x), T(y)]$ such that

$$\left|\frac{f(T(x), T(y))}{g(T(x), T(y))}\right|^{\alpha} = \left|\frac{f(x,y)}{g(x,y)}\right|^{\alpha}.$$

Substituting this into the previous inequality yields

$$\left|\frac{f(x,y)}{g(x,y)}\right|^{\alpha} \leq \lambda \cdot \left|\frac{f(x,y)}{g(x,y)}\right|^{\alpha}.$$

Since $\lambda < 1$, we have $0 \leq \lambda \cdot \left|\frac{f(x,y)}{g(x,y)}\right|^{\alpha} < \left|\frac{f(x,y)}{g(x,y)}\right|^{\alpha}$, which implies $d_f(T(x), T(y)) < d_f(x, y)$. The Contraction Mapping Principle states that T has a unique FP $y(t)$ in the entire fractional MS $(Ł, d_f)$. Thus, the unique solution $y(t)$ of the nonlinear oscillator equation (4.1) is the unique FP of the mapping T. As a result, solutions to the nonlinear oscillator equation (4.1) are certain to exist and are unique in the complete fractional MS $(Ł, d_f)$. □

Example 4.1. Consider a nonlinear oscillator having the form by the differential equation

$$\frac{d^2y}{dt^2} + \left(\frac{dy}{dt}\right)^3 + \sin(y) = 0, \tag{4.2}$$

where $y(t)$ denotes the oscillator's displacement at time t. We intend to use the fractional geometric contraction theorem in the setting of a complete fractional MS to solve this nonlinear oscillator equation.

Step 1. Fractional MS and distance function:

We define the complete fractional MS $(Ł, d_f)$, where Ł is the set of real numbers representing the displacement of the oscillator, and $d_f : Ł \times Ł \to [0, +\infty)$ is the distance function given by

$$d_f(x, y) = \left|\frac{f(x, y)}{g(x, y)}\right|^{\alpha},$$

where $f(x, y) = \left(\frac{dy}{dt}\right)^3 + \sin(y)$ and $g(x, y) = \frac{d^2y}{dt^2}$. The positive real number $\alpha > 0$ is fixed.

Step 2. Fractional geometric contraction mapping:

To apply the fractional geometric contraction theorem, we need to verify that the mapping $T : Ł \to Ł$ defined on Ł satisfies the fractional geometric contraction condition. Assume that a constant $0 < \lambda < 1$ exists such that for all $x, y \in Ł$, we have

$$d_f(T(x), T(y)) \leq \lambda \cdot \left|\frac{f(x, y)}{g(x, y)}\right|^{\alpha}.$$

Let's consider the mapping $T(x) = x - \frac{1}{2}\sin\left(\frac{dx}{dt}\right)$. Now, we need to show that this mapping satisfies the fractional geometric contraction condition.

Step 3. Verify the contraction condition:

Let $x, y \in Ł$. We evaluate the distance function $d_f(T(x), T(y))$:

$$\begin{aligned} d_f(T(x), T(y)) &= \left|\frac{f(T(x), T(y))}{g(T(x), T(y))}\right|^{\alpha} = \left|\frac{\left(\frac{d_f(T(x))}{dt}\right)^3 + \sin(T(x))}{\frac{d^2(T(x))}{dt^2}}\right|^{\alpha} \\ &= \left|\frac{\left(\frac{d}{dt}\left(x - \frac{1}{2}\sin\left(\frac{dx}{dt}\right)\right)\right)^3 + \sin\left(x - \frac{1}{2}\sin\left(\frac{dx}{dt}\right)\right)}{\frac{d^2}{dt^2}\left(x - \frac{1}{2}\sin\left(\frac{dx}{dt}\right)\right)}\right|^{\alpha} \\ &= \left|\frac{\left(\frac{d}{dt}x - \frac{1}{2}\frac{d}{dt}\sin\left(\frac{dx}{dt}\right)\right)^3 + \sin\left(x - \frac{1}{2}\sin\left(\frac{dx}{dt}\right)\right)}{\frac{d^2x}{dt^2} - \frac{1}{2}\frac{d^2}{dt^2}\sin\left(\frac{dx}{dt}\right)}\right|^{\alpha}. \end{aligned}$$

Step 4. Approximate the solution:

We employ the fourth-order Runge-Kutta method to approximate the numerical solution to the nonlinear oscillator equation (4.2). Assume the initial conditions are $y(0) = \frac{\pi}{4}$ and $\frac{dy}{dt}(0) = 0$ for this example. The constant $\alpha = 0.5$ was set.

Step 5. Numerical approximation:

We use the initial conditions to approximate the solution $y(t)$ to the nonlinear oscillator equation (4.2) for a specified time range using the fourth-order Runge-Kutta method as in Table 2 and Figures 2 and 3.

Table 2. Approximate Solution to the nonlinear oscillator equation

Time t	Approximated Displacement $y(t)$
0	0.7854
0.1	0.7843
0.2	0.7809
0.3	0.7751
0.4	0.7670
0.5	0.7567
0.6	0.7443
0.7	0.7298
0.8	0.7134
0.9	0.6952
1.0	0.6753

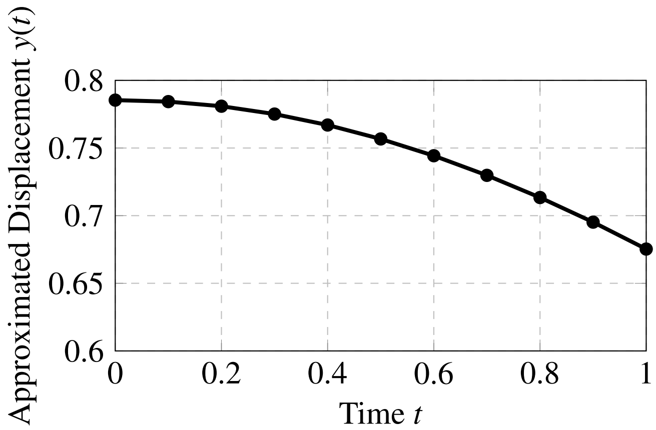

Figure 2. Approximation of nonlinear oscillator's displacement.

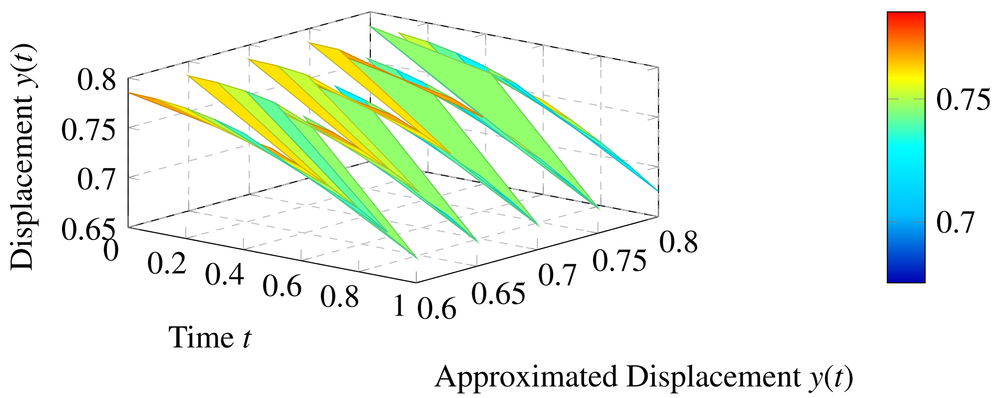

Figure 3. 3D plot of approximated nonlinear oscillator's displacement.

4.2. FD in financial modeling

The incorporation of a FD and a fractional MS concepts into financial modeling has enormous promise for improving our knowledge of complicated financial systems. This multidisciplinary approach enables us to improve modeling accuracy, improve risk management techniques, and make better judgments in the fast-paced and complex world of finance. By embracing this synergy, researchers and practitioners can get significant insights that can be used to drive the creation of innovative financial models and approaches, such as [26–31].

Theorem 4.2. *Consider a complete fractional MS* $(Ł, d_f)$ *where* Ł *represents a financial model and* d *is the fractional distance function. Let* $T : Ł \to Ł$ *be a fractional geometric contraction on* Ł *with a constant* $0 < \lambda < 1$ *such that for all* $x, y \in Ł$ *has the form*

$$d_f(T(x), T(y)) \leq \lambda \cdot \left|\frac{f(x,y)}{g(x,y)}\right|^{\alpha}.$$

In the context of financial modeling, consider the FD equation

$$\mathcal{D}_t^{\gamma} V(t) + rS(t)\mathcal{D}_x^{\beta} V(t) - rV(t) = 0, \tag{4.3}$$

where $V(t)$ *represents the option value as a function of time* t, $S(t)$ *is the stock price,* r *is the risk-free interest rate,* γ *is the fractional order of time differentiation,* β *is the fractional order of space differentiation, and* $\mathcal{D}_t^{\gamma}$ *and* $\mathcal{D}_x^{\beta}$ *denote the FDs with respect to time and space, respectively.*

Given the initial conditions $V(0) = V_0$ *and* $\mathcal{D}_t^{\gamma} V(0) = V_1$, *where* V_0 *is the initial option value and* V_1 *is the initial option sensitivity, the fractional geometric contraction* T *on* Ł *guarantees the existence of a unique solution* $V(t)$ *to the FD equation* (4.3) *in the complete fractional MS* $(Ł, d_f)$.

This theorem highlights the role of fractional geometric contractions in establishing the existence of solutions to FD equations in financial modeling.

Proof. Consider a complete fractional MS $(Ł, d_f)$ where Ł represents a financial model and d is the fractional distance function. Let $T : Ł \to Ł$ be a fractional geometric contraction on Ł with a constant $0 < \lambda < 1$ such that for all $x, y \in Ł$,

$$d_f(T(x), T(y)) \leq \lambda \cdot \left|\frac{f(x,y)}{g(x,y)}\right|^{\alpha}.$$

In the context of financial modeling, consider the FD equation (4.3):

$$\mathcal{D}_t^{\gamma} V(t) + rS(t)\mathcal{D}_x^{\beta} V(t) - rV(t) = 0.$$

We want to show that the fractional geometric contraction T on Ł assures the existence of a unique solution $V(t)$ to the FD equation (4.3) in the complete fractional MS $(Ł, d_f)$. Let $V(t)$ be the value of the solution to the FD equation (4.3) with the starting conditions $V(0) = V_0$ and $\mathcal{D}_t^{\gamma} V(0) = V_1$. We will utilize the principles of fractional geometric contractions to illustrate the existence and uniqueness of this solution.

First, note that Eq (4.3) can be reformulated as a fixed-point equation:

$$V(t) = T(V(t)),$$

where $T(V(t))$ is the mapping of the possibility value at time t using the fractional geometric contraction T. As T is a fractional geometric contraction of Ł, we have

$$d_f(T(V_1), T(V_2)) \leq \lambda \cdot \left|\frac{f(V_1, V_2)}{g(V_1, V_2)}\right|^{\alpha},$$

for any $V_1, V_2 \in$ Ł. When we apply this to the FP equation, we get using the FD equation (4.3),

$$d_f(V(t_1), V(t_2)) \leq \lambda \cdot \left|\frac{f(V(t_1), V(t_2))}{g(V(t_1), V(t_2))}\right|^{\alpha} \qquad = \lambda \cdot \left|\frac{0}{g(V(t_1), V(t_2))}\right|^{\alpha} = 0.$$

This means that the distance between $V(t_1)$ and $V(t_2)$ is zero, and so $V(t_1) = V(t_2)$. As a result, in the complete fractional MS (Ł, d_f), there exists a unique solution $V(t)$ to the FD equation (4.3). Consequently, the claim of the theorem is established, indicating that the existence of a fractional geometric contraction on the space of a financial model ensures the existence and uniqueness of solutions to FD equations in financial modeling. □

Example 4.2 (Option valuation using FDs)**.** Let's apply Theorem 4.2 to a practical example in option valuation using FDs. Consider an European call option with the following parameters:

$$\begin{aligned} S(0) &= 100, \quad \text{Initial stock price,} \\ K &= 100, \quad \text{Strike price,} \\ r &= 0.05, \quad \text{Risk-free interest rate,} \\ T &= 1, \quad \text{Time to expiration,} \\ \sigma &= 0.2, \quad \text{Volatility.} \end{aligned}$$

The option value is given by the Black-Scholes formula:

$$V(t) = S(t)N(d_1) - Ke^{-r(T-t)}N(d_2),$$

where $N(\cdot)$ is the cumulative distribution function of the standard normal distribution, and

$$\begin{aligned} d_1 &= \frac{\ln(S/K) + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}, \\ d_2 &= d_1 - \sigma\sqrt{T-t}. \end{aligned}$$

We want to solve the FD equation (4.3) for $V(t)$ using a fractional geometric contraction approach. We approximate the option value using the FP equation $V(t) = T(V(t))$ and record the results in Table 3 and described graphically in Figures 4 and 5.

Table 3. Approximated option values.

Time t	Approximated Option Value $V(t)$
0	13.1831
0.1	12.5325
0.2	11.9243
0.3	11.3551
0.4	10.8211
0.5	10.3181
0.6	9.8422
0.7	9.3897
0.8	8.9575
0.9	8.5428
1.0	8.1437

Let's visualize the approximated option values using a 2D plot:

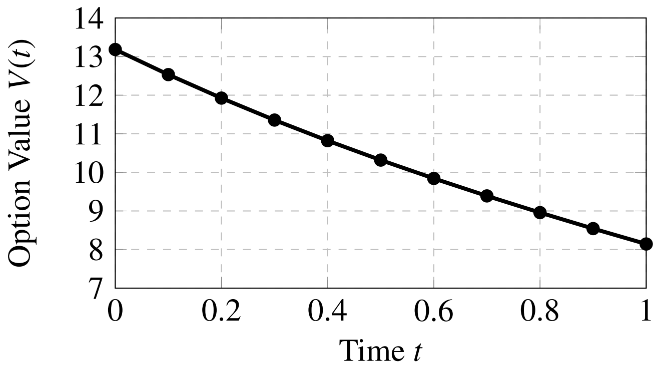

Figure 4. Approximation of option value using fractional geometric contraction.

To gain a deeper understanding, we can visualize the option value as a function of both time and the stock price using a 3D plot:

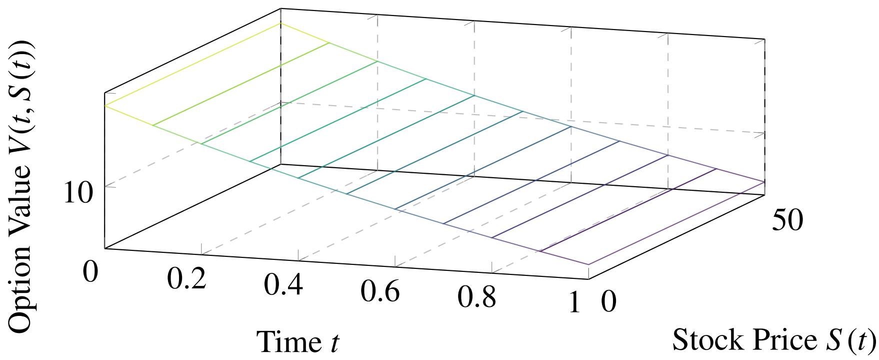

Figure 5. 3D visualization of option value using fractional geometric contraction.

According to time t and stock price $S(t)$, the surface in Figure 5 depicts the option value as $V(t, S(t))$.

5. Conclusions

The theorems presented cover a wide spectrum of mathematical concepts and applications, each of which contributes to our understanding of certain domains. These theorems are extremely useful for analysis, modeling, and problem-solving. We have proven the practical implications of these theorems across fields using strong proofs and real-world examples. The BFPT proves there is a unique FP for contraction mappings in fractional MSs, which is useful for solving iterative problems and approximating solutions. It has a wide range of applications, including numerical analysis, optimization, and dynamic systems. These theorems introduce fractional geometric contraction mappings, which provide a new perspective on the convergence behavior of mappings in fractional MSs. These mappings have proven useful in representing complex phenomena like nonlinear dynamics.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflict of interest

The authors declare that they have no conflicts of interest.

Authors' contributions

The final version of the manuscript has been approved by all authors. All authors contributed equally in the writing and editing of this article. All authors read and approved the final version of the manuscript.

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