Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 9(1): 495–520.
DOI: 10.3934/math.2024027
Received: 09 September 2023
Revised: 05 November 2023
Accepted: 15 November 2023
Published: 30 November 2023

Research article

Global existence, blow-up and stability of standing waves for the Schrödinger-Choquard equation with harmonic potential

Meixia Cai, Hui Jian* and Min Gong

College of Science, East China Jiaotong University, Nanchang 330013, China

* **Correspondence:** Email: jianhui0711141@163.com.

Abstract: In this article, we conduct a comprehensive investigation into the global existence, blow-up and stability of standing waves for a L^2-critical Schrödinger-Choquard equation with harmonic potential. First, by taking advantage of the ground-state solutions and scaling techniques, we obtain some criteria for the global existence and blow-up of the solutions. Second, in terms of the refined compactness argument, scaling techniques and the variational characterization of the ground state solution to the Choquard equation with $p_2 = 1 + \frac{2+\alpha}{N}$, we explore the limiting dynamics of blow-up solutions to the L^2-critical Choquard equation with L^2-subcritical perturbation, including the L^2-mass concentration and blow-up rate. Finally, the orbital stability of standing waves is investigated in the presence of L^2-subcritical perturbation, focusing L^2-critical perturbation and defocusing L^2-supercritical perturbation by using variational methods. Our results supplement the conclusions of some known works.

Keywords: Schrödinger-Choquard equation; harmonic potential; global existence; blow-up; orbital stability
Mathematics Subject Classification: 35A01, 35B33, 35Q40, 35B44, 35Q55

1. Introduction

In this article, we conduct a comprehensive investigation into the Cauchy problem of the following Schrödinger-Choquard equation with harmonic potential:

$$\begin{cases} i\varphi_t + \Delta\varphi = V(x)\varphi + \lambda_1|\varphi|^{p_1}\varphi + \lambda_2(I_\alpha * |\varphi|^{p_2})|\varphi|^{p_2-2}\varphi, (t,x) \in [0,T) \times \mathbb{R}^N, \\ \varphi(0,x) = \varphi_0(x), \end{cases} \tag{1.1}$$

where $N \geq 3$, $V(x) = a^2|x|^2$ $(a \neq 0)$ and $\varphi_0 \in \Sigma$, where Σ will be defined in the next section; also, $0 < T \leq \infty$, $\varphi : [0,T) \times \mathbb{R}^N \to \mathbb{C}$ is a complex valued function, $0 < p_1 < \frac{4}{N-2}$, $1 + \frac{\alpha}{N} < p_2 < 1 + \frac{2+\alpha}{N-2}$,

$a, \lambda_1, \lambda_2 \in \mathbb{R}$, $I_\alpha : \mathbb{R}^N \to \mathbb{R}$ is the Riesz potential given by

$$I_\alpha(x) = \frac{\Gamma(\frac{N-\alpha}{2})}{\Gamma(\frac{\alpha}{2})\pi^{N/2}2^\alpha|x|^{N-\alpha}}$$

with $\alpha \in (0, N)$, and Γ denotes the gamma function.

The model (1.1) is a kind of nonlinear Schrödinger equation (NLS) with combined nonlinearities, and it has a wide range of physical applications. For example, it can be used to describe a quantum system with an infinite number of particles; see, e.g., [1, 26, 27]. Additionally, it can also describe the phenomenon of Bose-Einstein condensation in gases found in systems of Rb or Na atoms with very weak two-body interactions; see, e.g., [7, 8, 35]. From the mathematical point of view, the presence of combined nonlinearities in Eq (1.1) leads to a loss of scaling invariance and significant changes in the variational structure of the corresponding energy functional. As a result, this kind of model has attracted a great deal of interest; moreover, more and more attention is being paid to the orbital stability of standing waves and the dynamics of blow-up solutions to Eq (1.1). In the current paper, we are particularly interested in the cases that $\lambda_2 = -1$, $p_2 = 1 + \frac{2+\alpha}{N}$, $\lambda_1 \in \mathbb{R}$ and $0 < p_1 < \frac{4}{N-2}$.

When $a = 0$, Eq (1.1) can be rewritten as below:

$$i\varphi_t + \Delta\varphi = \lambda_1|\varphi|^{p_1}\varphi + \lambda_2(I_\alpha * |\varphi|^{p_2})|\varphi|^{p_2-2}\varphi. \tag{1.2}$$

For $\lambda_1 = 0$, it means that Eq (1.2) only involves the Choquard-type nonlinearity whose Cauchy problem has been extensively studied; see [2–4, 6, 10, 14, 16, 17, 23] for examples. Especially, Chen and Guo [10] studied the existence of blow-up solutions and the instability of standing waves for $N - 3 < \alpha < N - 2$ and $1 + \frac{2+\alpha}{N} \le p_2 < \frac{N+\alpha}{N-2}$. Feng and Yuan [14] surveyed the local and global well-posedness of Eq (1.2) for $max\{0, N - 4\} < \alpha < N$ and $2 \le p_2 < 1 + \frac{2+\alpha}{N-2}$ by employing Cazenave's method based on a compactness argument (see [9]). Furthermore, the limiting behaviors of blow-up solutions at finite-time are investigated in the L^2-critical case of $p_2 = 1 + \frac{2+\alpha}{N}$. For $\lambda_2 = 0$, Eq (1.2) only involves the power-type nonlinearity. Zhang [36] established the sharp criterion for the global existence and blow-up of the solutions by constructing the corresponding cross-invariant manifolds. Then, the author showed the strong instability of standing waves based on the property of the cross-constrained variational problem.

For $a = 1$, Eq (1.1) corresponds to the following form:

$$i\varphi_t + \Delta\varphi = |x|^2\varphi + \lambda_1|\varphi|^{p_1}\varphi + \lambda_2(I_\alpha * |\varphi|^{p_2})|\varphi|^{p_2-2}\varphi. \tag{1.3}$$

For $\lambda_1 = 0$ in Eq (1.3), the model has been discussed in [12, 18, 32, 34]. It is particularly worth mentioning that Feng [12] obtained the sharp thresholds for global existence and blow-up in both the L^2-critical and L^2-subcritical cases. Moreover, the author showed the stability of standing waves in the L^2-critical case and the instability of standing waves in the L^2-supercritical case. For the special case of $p_2 = 2$, the finite-time blow-up solutions and instability of standing waves have been studied in [32]. For $\lambda_2 = 0$, Eq (1.3) degenerates into the well-known Gross-Pitaevskii equation which has been widely investigated. In particular, there has been a large amount of results on the Cauchy problem for the Gross-Pitaevskii equation; see [25, 29, 37] for examples. For the case that the nonlocal term $(I_\alpha * |\varphi|^{p_2})|\varphi|^{p_2-2}\varphi$ is replaced by $|\varphi|^{p_2}\varphi$ in Eq (1.3), Zhang and Zhang [38] studied the stability and instability of standing waves by applying a compact embedding (see [35]).

Now, we return to the nonlinear Schrödinger-Choquard equation without harmonic potential, that is, Eq (1.2). For this situation, there exist a few studies concerning the issues of global existence and finite-time blow-up; see [11, 13, 21, 22, 28] for examples. For the case that $0 < p_1 < \frac{4}{N}$ and $p_2 = 2$, Tian et al. [30] constructed some invariant flows to obtain the exact energy threshold of blow-up and global existence for Eq (1.2) with the focusing L^2-subcritical term $|\varphi|^{p_1}\varphi$. Moreover, Tian and Zhu [31] investigated the sharp energy criterion for the blow-up solutions and global existence for Eq (1.2) with a focusing or defocusing perturbation, and they utilized the Strichartz estimate to obtain the lower bound of the blow-up rate. Recently, Shi and Liu [28] derived the sharp threshold for global existence and blow-up of the solutions for $0 < p_1 < \frac{4}{N}$ and $p_2 = 1 + \frac{2+\alpha}{N}$ with $\lambda_1 = 1$, $\lambda_2 = -1$; they obtained the L^2-concentration and blow-up rate of the explosive solutions for Eq (1.2). Regarding the stability issues of standing waves, it should be mentioned that Liu and Shi [22] showed the orbital stability of standing waves in the setting of $0 < p_1 < \frac{4}{N}$ and $p_2 = 1 + \frac{2+\alpha}{N}$ with $\lambda_1 = \lambda_2 = -1$ by using the profile decomposition theory and variational methods. In addition, the profile decomposition technique has also been applied to explore the existence of orbitally stable standing waves for fractional Schrödinger equations including combined Hartree-type and power-type nonlinear terms. See [5, 15] for examples.

To our knowledge, the Cauchy problem of the NLS including harmonic potential and combined Choquard and power-type nonlinearities has not been investigated for $\lambda_2 < 0$, $p_2 = 1 + \frac{2+\alpha}{N}$, $\lambda_1 \neq 0$ and $0 < p_1 < \frac{4}{N-2}$. Motivated by the works mentioned above, in the presence of harmonic potential, we shall research the existence of a global solution and the limiting dynamics of blow-up solutions as well as the stability of standing waves with prescribed mass to Eq (1.1) in the case that $\lambda_2 = -1$, $p_2 = 1 + \frac{2+\alpha}{N}$, $\lambda_1 = \pm 1$ and $0 < p_1 < \frac{4}{N-2}$.

In the present work, by taking advantage of the ground state solutions to the L^2-critical elliptic Eqs (2.11) and (2.12) and the scaling techniques, we first obtain some criteria for the global existence and blow-up of the solutions in light of [12,22,31,38]. More precisely, we derive the sharp threshold for global existence and blow-up with defocusing L^2-subcritical perturbation by using the sharp Gagliardo-Nirenberg inequality and scaling techniques together with the ground state for Eq (2.11). In the case of focusing L^2-subcritical perturbation, the major obstacle to guaranteeing the collapse of solutions to Eq (1.1) lies in that it is difficult to choose $E(\varphi_0)$ to ensure the second-order derivative $J''(t) < -C < 0$ of $J(t) = \int |x|^2|\varphi(t,x)|^2dx$ (see (2.5)). In order to get around this difficulty, we argue by contradiction, together with scaling arguments. Then a sharp condition for blow-up is also derived. For the focusing L^2-critical perturbation case, we make use of the ground-state solutions to the L^2-critical elliptic equations given by Eqs (2.11) and (2.12) and scaling techniques, as well as the sharp Gagliardo-Nirenberg inequality, to get the criterion for global existence and blow-up. In the situation involving defocusing L^2-supercritical perturbation, with the aid of the Hardy-Littlewood-Sobolev inequality, the interpolation inequality and Young's inequality, we verify the existence of a global solution to Eq (1.1).

Second, in terms of the refined compactness argument, scaling techniques and the variational characterization of the ground state solution to the Choquard equation given by Eq (2.11) with $p_2 = 1 + \frac{2+\alpha}{N}$, we explore the limiting properties of blow-up solutions for the L^2-critical Choquard equation with L^2-subcritical perturbation, including the L^2-mass concentration and blow-up rate. With regard to the dynamical properties of blow-up solutions to Eq (1.1), the main obstacle is the loss of scaling invariance, caused by the combined nonlinearities. To overcome this obstacle, following the clues of [12, 14, 28], we apply the ground state solution of the L^2-critical Choquard equation without harmonic potential to describe the limiting behaviors of blow-up solutions at blow-up time.

Finally, the orbital stability of standing waves is investigated in the presence of L^2-subcritical perturbation, focusing L^2-critical perturbation and defocusing L^2-supercritical perturbation by using variational methods and some compactness arguments in light of [12, 22, 38]. Our results supplement the conclusions of some known works [12, 22, 28, 31].

The structure of this paper is as below. In Section 2, some notations and preliminaries are given. In Section 3, the criteria for the global existence and finite time blow-up of Eq (1.1) are considered. In Section 4, we focus on the limiting dynamics of blow-up solutions in the case of L^2-subcritical perturbation, including the mass concentration phenomenon and the dynamical properties of blow-up solutions with minimal mass. In Section 5, the stability of standing waves is covered. In the last section, the conclusion of this paper is given.

2. Notations and preliminaries

Throughout this manuscript, for convenience, we abbreviate $\int_{\mathbb{R}^N} \cdot dx$ by $\int \cdot dx$, use $\|\cdot\|_{H^1}$ to denote $\|\cdot\|_{H^1(\mathbb{R}^N)}$ and replace $\|\cdot\|_{L^r(\mathbb{R}^N)}$ by $\|\cdot\|_r$. Meanwhile, we may as well assume that $a = 1$ and utilize C to represent a positive constant that may be different from line to line.

The energy space associated with the Cauchy problem described by Eq (1.1) is given by

$$\Sigma := \{\varphi \in H^1(\mathbb{R}^N); x\varphi \in L^2(\mathbb{R}^N)\}$$

equipped with the norm

$$\|\varphi\|_\Sigma := (\|\varphi\|_{H^1}^2 + \|x\varphi\|_2^2)^{\frac{1}{2}}.$$

In addition, the energy functional defined on Σ is denoted by

$$E(\varphi(t)) = \frac{1}{2}\int |\nabla\varphi|^2 + |x|^2|\varphi|^2 dx + \frac{\lambda_1}{p_1+2}\int |\varphi|^{p_1+2} dx + \frac{\lambda_2}{2p_2}\int (I_\alpha * |\varphi|^{p_2})|\varphi|^{p_2} dx. \tag{2.1}$$

To investigate the stability of standing waves and blow-up solutions of Eq (1.1), we first give the following local well-posedness result to Eq (1.1).

Proposition 2.1. *Assume that $\varphi_0 \in \Sigma$, $N \geq 3$, $0 < p_1 < \frac{4}{N-2}$ and $2 \leq p_2 < 1 + \frac{2+\alpha}{N-2}$. Then there exists $T = T(\|\varphi_0\|_\Sigma)$ such that Eq (1.1) admits a unique solution $\varphi(t) \in C([0,T),\Sigma)$. Let $[0,T)$ be the maximal interval on which the solution $\varphi(t)$ is well-defined: if $T < \infty$, then $\|\varphi\|_\Sigma \to \infty$ as $t \to T$. Moreover, for arbitrary $t \in [0,T)$, the solution $\varphi(t)$ obeys the conservation laws of mass and energy as below:*

$$\|\varphi(t)\|_2 = \|\varphi_0\|_2, \tag{2.2}$$

$$E(\varphi(t)) = E(\varphi_0). \tag{2.3}$$

When $0 < p_1 < \frac{4}{N-2}$ and $2 \leq p_2 < 1 + \frac{2+\alpha}{N-2}$, this proposition can be easily proved by applying Strichartz estimates and a fixed-point argument (see [9, 12, 14]).

Proposition 2.2. *Assume that $\lambda_2 = -1$, $p_2 = 1 + \frac{2+\alpha}{N}$, $\lambda_1 \in \mathbb{R}$ and $0 < p_1 < \frac{4}{N-2}$. Let $\varphi_0 \in \Sigma$ and $\varphi(t)$ be a solution of Eq (1.1) in $C([0,T);\Sigma)$. Moreover, let $J(t) = \int |x|^2|\varphi(t,x)|^2 dx$; then,*

$$J'(t) = -4Im\int x\cdot\varphi\nabla\overline{\varphi} dx \tag{2.4}$$

and

$$
\begin{aligned}
J''(t) &= 8\int |\nabla\varphi|^2dx - 8\int |x|^2|\varphi|^2dx + \frac{4N\lambda_1 p_1}{p_1+2}\int |\varphi|^{p_1+2}dx \\
&\quad + \frac{\lambda_2(4Np_2 - 4N - 4\alpha)}{p_2}\int (I_\alpha * |\varphi|^{p_2})|\varphi|^{p_2}dx \\
&= 16E(\varphi_0) - 16\int |x|^2|\varphi|^2dx + \frac{4N\lambda_1 p_1 - 16\lambda_1}{p_1+2}\int |\varphi|^{p_1+2}dx. \qquad (2.5)
\end{aligned}
$$

Proof. Based on the work of Cazenave [9], by a formal computation, it is easy to obtain that

$$
\begin{aligned}
J'(t) &= 2Re\int |x|^2\bar{\varphi}\varphi_t dx \\
&= 2Im\int |x|^2\bar{\varphi}(-\Delta\varphi + |x|^2\varphi + \lambda_1|\varphi|^{p_1}\varphi + \lambda_2(I_\alpha * |\varphi|^{p_2})|\varphi|^{p_2-2}\varphi)dx \\
&= -2Im\int |x|^2\bar{\varphi}\Delta\varphi dx \\
&= -4Im\int x\cdot\varphi\nabla\overline{\varphi}dx,
\end{aligned}
$$

and

$$
\begin{aligned}
J''(t) &= -4Im\frac{d}{dt}\int x\varphi\nabla\overline{\varphi}dx \\
&= 4(Im\int N\varphi\bar{\varphi}_t dx + 2Im\int x\nabla\varphi\bar{\varphi}_t dx) = 4(I_1 + I_2), \qquad (2.6)
\end{aligned}
$$

where

$$
\begin{aligned}
I_1 &= Im\int N\varphi\bar{\varphi}_t dx \\
&= NRe\int \bar{\varphi}(-\Delta\varphi + |x|^2\varphi + \lambda_1|\varphi|^{p_1}\varphi + \lambda_2(I_\alpha * |\varphi|^{p_2})|\varphi|^{p_2-2}\varphi)dx \\
&= N\int (|\nabla\varphi|^2 + |x|^2|\varphi|^2 + \lambda_1|\varphi|^{p_1+2} + \lambda_2(I_\alpha * |\varphi|^{p_2})|\varphi|^{p_2})dx \qquad (2.7)
\end{aligned}
$$

and

$$
\begin{aligned}
I_2 &= 2Im\int x\nabla\varphi\bar{\varphi}_t dx = 2Re\int x\nabla\bar{\varphi}(i\varphi_t)dx \\
&= 2Re\int x\nabla\bar{\varphi}(-\Delta\varphi + |x|^2\varphi + \lambda_1|\varphi|^{p_1}\varphi + \lambda_2(I_\alpha * |\varphi|^{p_2})|\varphi|^{p_2-2}\varphi)dx \\
&= -(N-2)\int |\nabla\varphi|^2dx - (N+2)\int |x|^2|\varphi|^2dx - \frac{2N\lambda_1}{p_1+2}\int |\varphi|^{p_1+2}dx \\
&\quad - \frac{(N+\alpha)\lambda_2}{p_2}\int (I_\alpha * |\varphi|^{p_2})|\varphi|^{p_2}dx. \qquad (2.8)
\end{aligned}
$$

Combining (2.3) and (2.6)–(2.8), one can conclude that (2.5) holds true. □

Lemma 2.3. *([33]) Let $\varphi \in H^1(\mathbb{R}^N)$; then, one has that*

$$\int |\varphi|^2 dx \leq \frac{2}{N}\left(\int |\nabla\varphi|^2 dx\right)^{\frac{1}{2}}\left(\int |x|^2|\varphi|^2 dx\right)^{\frac{1}{2}}. \tag{2.9}$$

Lemma 2.4. *([14]) The best constant in the Gagliardo-Nirenberg-type inequality*

$$\int (I_\alpha * |\varphi|^{p_2})|\varphi|^{p_2} dx \leq C_{\alpha,p_2}\left(\int |\nabla\varphi|^2 dx\right)^{\frac{Np_2-N-\alpha}{2}}\left(\int |\varphi|^2 dx\right)^{\frac{N+\alpha-Np_2+2p_2}{2}} \tag{2.10}$$

is

$$C_{\alpha,p_2} = \frac{2p_2}{2p_2 - Np_2 + N + \alpha}\left(\frac{2p_2 - Np_2 + N + \alpha}{Np_2 - N - \alpha}\right)^{\frac{Np_2-N-\alpha}{2}} \|Q\|_2^{2-2p_2},$$

where Q is the ground-state solution of the elliptic equation

$$-\Delta u + u - (I_\alpha * |u|^{p_2})|u|^{p_2-2}u = 0. \tag{2.11}$$

In particular, in the L^2-critical case in which $p_2 = 1 + \frac{2+\alpha}{N}$, $C_{\alpha,p_2} = \frac{N+2+\alpha}{N}\|Q\|_2^{-\frac{4+2\alpha}{N}}$.

Lemma 2.5. *([33]) Let $0 < p_1 < \frac{4}{N-2}$, $N \geq 3$. Then, for any $\varphi \in H^1(\mathbb{R}^N)$, we have the sharp Gagliardo-Nirenberg inequality*

$$\int |\varphi|^{p_1+2} dx \leq C_{p_1,N}\|\varphi\|_2^{p_1+2-\frac{Np_1}{2}}\|\nabla\varphi\|_2^{\frac{Np_1}{2}},$$

where $C_{p_1,N} = \frac{p_1+2}{2\|W\|_2^{p_1}}$ and W is the ground state solution of the elliptic equation

$$-\Delta u + u = |u|^{p_1}u. \tag{2.12}$$

In particular, in the L^2-critical case in which $p_1 = \frac{4}{N}$, $C_{p_1,N} = \frac{N+2}{N}\|W\|_2^{-\frac{4}{N}}$.

3. Global existence and blow-up for Eq (1.1)

In this section, we are devoted to showing the blow-up and global existence of solutions to Eq (1.1) with $\lambda_1 = \pm 1$, $\lambda_2 = -1$ and $0 < p_1 \leq \frac{4}{N-2}$, $p_2 = 1 + \frac{2+\alpha}{N}$. Before stating the results, we first review some arguments about the ground-state solution of Eq (2.11) or Eq (2.12), which are of great significance in the study of the criteria of global existence or finite-time blow-up to Eq (1.1). From [19, 20, 24], the existence of ground-state solution Q (or W) to Eq (2.11) (or Eq (2.12)) is given, respectively. The ground-state solution of the L^2-critical elliptic equation given by Eq (2.11) will be a powerful ingredient to characterize the limiting properties for the blow-up solutions at finite time in the subsequent section.

Through simple calculations, one could immediately get the Pohožaev identities related to Eq (2.11) with $p_2 = 1 + \frac{2+\alpha}{N}$ and Eq (2.12) with $p_1 = \frac{4}{N}$ as follows:

$$\|\nabla Q\|_2^2 = \frac{1}{p_2}\int (I_\alpha * |Q|^{p_2})|Q|^{p_2} dx, \tag{3.1}$$

$$\|W\|_2^2 = \frac{2}{p_1 + 2}\int |W|^{p_1+2} dx. \tag{3.2}$$

3.1. Sharp threshold for global existence and blow-up in the case of L^2-subcritical perturbation

Applying the local well-posedness theory to the NLS, the solution of Eq (1.1) exists globally when the initial data size becomes sufficiently small, and, for some cases with large initial data, the global solution may not exist. Therefore, in this subsection, we focus on whether the sharp threshold of global and blow-up solutions exists for Eq (1.1) with focusing or defocusing L^2-subcritical perturbation.

Case 1. $\lambda_1 = 1$.

Theorem 3.1. *Assume that $\varphi_0 \in \Sigma$, $\lambda_1 = 1, \lambda_2 = -1$, $0 < p_1 < \frac{4}{N}$ and $p_2 = 1 + \frac{2+\alpha}{N}$. Let Q be the ground-state solution of L^2-critical elliptic equation (2.11). Then the following assertions hold:*
(i) Global existence: If $\|\varphi_0\|_2 < \|Q\|_2$, then the solution $\varphi(t,x)$ of Eq (1.1) exists globally in $t \in [0, +\infty)$.
(ii) Blow-up: Let $\varphi_0 = c\rho^{\frac{N}{2}} Q(\rho x)$, where $|c| > 1$, $\rho > 0$ and the following equation is satisfied

$$\rho^{2-\frac{N}{2}p_1} > max\left\{1, \frac{\frac{2|c|^{p_1}}{p_1+2}\|Q\|_{p_1+2}^{p_1+2} + \|xQ\|_2^2}{(|c|^{2p_2-2}-1)\|\nabla Q\|_2^2}\right\}. \tag{3.3}$$

Then the corresponding solution $\varphi(t,x)$ of problem (1.1) blows up in finite time.

Proof. (i) First, from the mass conservation described by Eq (2.2) and Lemma 2.4, we have

$$-\frac{1}{2p_2}\int (I_\alpha * |\varphi|^{p_2})|\varphi|^{p_2}dx \geq -\frac{\|\varphi_0\|_2^{2p_2-2}}{2\|Q\|_2^{2p_2-2}}\|\nabla\varphi\|_2^2. \tag{3.4}$$

Furthermore, we obtain from Eqs (2.1), (2.3) and (3.4) that

$$\begin{aligned} E(\varphi_0) &= E(\varphi(t)) \\ &= \frac{1}{2}\int |\nabla\varphi|^2 dx + \frac{1}{2}\int |x|^2|\varphi|^2 dx + \frac{1}{p_1+2}\int |\varphi|^{p_1+2}dx - \frac{1}{2p_2}\int (I_\alpha * |\varphi|^{p_2})|\varphi|^{p_2}dx \\ &\geq \frac{1}{2}\|\nabla\varphi\|_2^2 - \frac{\|\varphi_0\|_2^{2p_2-2}}{2\|Q\|_2^{2p_2-2}}\|\nabla\varphi\|_2^2 + \frac{1}{2}\|x\varphi\|_2^2 \\ &= (\frac{1}{2} - \frac{\|\varphi_0\|_2^{2p_2-2}}{2\|Q\|_2^{2p_2-2}})\|\nabla\varphi\|_2^2 + \frac{1}{2}\|x\varphi\|_2^2. \end{aligned}$$

Combining this with $\|\varphi_0\|_2 < \|Q\|_2$, we conclude that $\|\nabla\varphi\|_2^2$ and $\|x\varphi\|_2^2$ are uniformly bounded for $t \in [0, +\infty)$. Therefore, the solution $\varphi(t,x)$ exists globally.

(ii) Now we prove the second part of Theorem 3.1. We first obtain from Proposition 2.2 that

$$J''(t) = 16E(\varphi_0) - 16\int |x|^2|\varphi|^2 dx + \frac{4Np_1 - 16}{p_1+2}\int |\varphi|^{p_1+2}dx, \tag{3.5}$$

and, from Eq (3.3), we get

$$\rho^{-2} < \rho^{\frac{N}{2}p_1}, \tag{3.6}$$

$$-\rho^2 < -\rho^{\frac{N}{2}p_1}\left(\frac{\frac{2|c|^{p_1}}{p_1+2}\|Q\|_{p_1+2}^{p_1+2} + \|xQ\|_2^2}{(|c|^{2p_2-2}-1)\|\nabla Q\|_2^2}\right). \tag{3.7}$$

Since $\varphi_0 = c\rho^{\frac{N}{2}} Q(\rho x)$, by utilizing Pohožaev identities (3.1), (3.6) and (3.7), one has that

$$\begin{aligned}
E(\varphi_0) &= \frac{|c|^2\rho^2}{2}\int |\nabla Q|^2 dx - \frac{|c|^{2p_2}\rho^2}{2p_2}\int (I_\alpha * |Q|^{p_2})|Q|^{p_2} dx \\
&\quad + \frac{|c|^{p_1+2}\rho^{\frac{N}{2}p_1}}{p_1+2}\int |Q|^{p_1+2} dx + \frac{|c|^2\rho^{-2}}{2}\int |x|^2|Q|^2 dx \\
&= -\frac{|c|^2\rho^2}{2}(|c|^{2p_2-2}-1)\|\nabla Q\|_2^2 + |c|^2\rho^{-2}\left(\frac{|c|^{p_1}\rho^{\frac{N}{2}p_1+2}}{p_1+2}\|Q\|_{p_1+2}^{p_1+2} + \frac{1}{2}\|xQ\|_2^2\right) \\
&< -\frac{|c|^2\rho^2}{2}(|c|^{2p_2-2}-1)\|\nabla Q\|_2^2 + |c|^2\rho^{\frac{N}{2}p_1}\left(\frac{|c|^{p_1}\rho^{\frac{N}{2}p_1+2}}{p_1+2}\|Q\|_{p_1+2}^{p_1+2} + \frac{1}{2}\|xQ\|_2^2\right) \\
&= 0.
\end{aligned}$$

At the same time, using the exponential decay of ground state solution $Q(x)$ (see [24]), i.e.,

$$Q(|x|), \nabla Q(|x|) = O(|x|^{-\frac{N-1}{2}} e^{-|x|}),\ as\ |x| \to \infty$$

we conclude that $\varphi_0 = c\rho^{\frac{N}{2}} Q(\rho x) \in H^1(\mathbb{R}^N)$ and $x\varphi_0 \in L^2(\mathbb{R}^N)$, i.e. $\varphi_0 \in \Sigma$. In addition, one has that $\|\varphi_0\|_2 = |c|\|Q\|_2 > \|Q\|_2$. Furthermore, we infer from Eq (3.5) that $J''(t) < 16E(\varphi_0) < 0$. Therefore, the solution $\varphi(t,x)$ to Eq (1.1) blows up in finite time. □

As a conclusion of the above theorem, one has the following

Corollary 3.2. *Assume that $\varphi_0 \in \Sigma$, $\lambda_1 = 1, \lambda_2 = -1$, $0 < p_1 < \frac{4}{N}$ and $p_2 = 1 + \frac{2+\alpha}{N}$. If the solution $\varphi(t)$ to Eq (1.1) blows up at finite time T, then there exists $C > 0$ such that for all $t \in [0,T)$,*

$$\int |x|^2|\varphi|^2 dx \le C.$$

Proof. From Eq (2.5) and Theorem 3.1, one has

$$\begin{aligned}
J''(t) &= 16E(\varphi_0) - 16\int |x|^2|\varphi|^2 dx + \frac{4Np_1-16}{p_1+2}\int |\varphi|^{p_1+2} dx \\
&< 16E(\varphi_0) < 0.
\end{aligned}$$

Then, we can easily obtain that

$$0 \le J(t) = J(0) + J'(0)t + \int_0^t (t-s)J''(s)ds \le J(0) + J'(0)t + 8E(\varphi_0)t^2.$$

Therefore,

$$J(t) = \int |x|^2|\varphi|^2 dx \le C.$$

□

Case 2. $\lambda_1 = -1$.

This situation is different from Case I. The major obstacle to prove the collapse of solutions is that the second-order derivative $J''(t)$ of $J(t) = \int |x|^2|\varphi(t,x)|^2dx$ has the following form:

$$J''(t) = 16E(\varphi_0) - 16\int |x|^2|\varphi|^2dx + \frac{16-4Np_1}{p_1+2}\int |\varphi|^{p_1+2}dx.$$

Due to the third term $\frac{16-4Np_1}{p_1+2}\int |\varphi|^{p_1+2}dx > 0$, it is difficult to choose $E(\varphi_0)$ to guarantee the blow-up of solutions. Therefore, it is of particular interest to investigate whether there exists a sharp criterion for global existence. In the theorem below, by taking advantage of the scaling argument and reduction to absurdity, we show that the blow-up solutions exist and derive the sharp threshold mass of blow-up versus global existence for Eq (1.1).

Theorem 3.3. *Assume that $\varphi_0 \in \Sigma$, $\lambda_1 = \lambda_2 = -1$, $0 < p_1 < \frac{4}{N}$ and $p_2 = 1 + \frac{2+\alpha}{N}$. Let Q be the ground-state solution to L^2-critical elliptic equation (2.11). Then, we have the following conclusions:*
(i) Global existence: If $\|\varphi_0\|_2 < \|Q\|_2$, then the solution $\varphi(t,x)$ of Eq (1.1) exists globally in $t \in [0,+\infty)$.
(ii) Blow-up: Let $\varphi_0 = c\rho^{\frac{N}{2}}Q(\rho x)$ where $|c| \geq 1, \rho > 0$ and it satisfies the following equation:

$$\rho^{\frac{N}{2}p_1} > max\left\{1, \frac{8(p_1+2)|c|^2\|xQ\|_2^2 + C(|c|\|Q\|_2)^{\frac{2N-(N-2)(p_1+2)}{2}}}{16|c|^{p_1+2}\|Q\|_{p_1+2}^{p_1+2}}\right\}. \tag{3.8}$$

Then blow-up of the corresponding solution $\varphi(t,x)$ to Eq (1.1) occurs in finite time.

Proof. (i) First, from Eqs (2.1) and (2.3) we get

$$E(\varphi(t)) = \frac{1}{2}\int |\nabla\varphi|^2dx + \frac{1}{2}\int |x|^2|\varphi|^2dx - \frac{1}{p_1+2}\int |\varphi|^{p_1+2}dx - \frac{1}{2p_2}\int (I_\alpha * |\varphi|^{p_2})|\varphi|^{p_2}dx.$$

Since $0 < p_1 < \frac{4}{N}$, then $\frac{Np_1}{2} < 2$. Hence we infer from Lemma 2.5 and Young's inequality that, for any $0 < \varepsilon < \frac{1}{2}$, there exists a constant $C(\varepsilon, \|\varphi_0\|_2)$ such that

$$\begin{aligned}\frac{1}{p_1+2}\int |\varphi|^{p_1+2}dx &\leq C\|\varphi_0\|_2^{p_1+2-\frac{Np_1}{2}}\|\nabla\varphi\|_2^{\frac{Np_1}{2}} \\ &\leq \varepsilon\|\nabla\varphi\|_2^2 + C(\varepsilon, \|\varphi_0\|_2).\end{aligned} \tag{3.9}$$

Thus, from Eqs (3.4) and (3.9), we have the following estimate:

$$\begin{aligned}E(\varphi_0) = E(\varphi(t)) &\geq \frac{1}{2}\|\nabla\varphi\|_2^2 - \frac{\|\varphi_0\|_2^{2p_2-2}}{2\|Q\|_2^{2p_2-2}}\|\nabla\varphi\|_2^2 - \varepsilon\|\nabla\varphi\|_2^2 - C(\varepsilon, \|\varphi_0\|_2) + \frac{1}{2}\|x\varphi\|_2^2 \\ &= (\frac{1}{2} - \frac{\|\varphi_0\|_2^{2p_2-2}}{2\|Q\|_2^{2p_2-2}} - \varepsilon)\|\nabla\varphi\|_2^2 - C(\varepsilon, \|\varphi_0\|_2) + \frac{1}{2}\|x\varphi\|_2^2.\end{aligned}$$

Thus

$$E(\varphi_0) + C(\varepsilon, \|\varphi_0\|_2) \geq (\frac{1}{2} - \frac{\|\varphi_0\|_2^{2p_2-2}}{2\|Q\|_2^{2p_2-2}} - \varepsilon)\|\nabla\varphi\|_2^2 + \frac{1}{2}\|x\varphi\|_2^2.$$

Let ε be small enough and ultilizing the fact that $\|\varphi_0\|_2 < \|Q\|_2$, then we conclude that $\|\nabla\varphi\|_2^2$ and $\|x\varphi\|_2^2$ are uniformly bounded for all $t \in [0, +\infty)$. Therefore, the conclusion holds.

(ii) Assume by contradiction that the corresponding solution $\varphi(t, x)$ exists globally with $T = +\infty$ and there exists $C > 0$ such that

$$\sup_{t\in[0,+\infty)} \|\varphi(t)\|_{H^1} \leq C. \tag{3.10}$$

In the case that $\lambda_1 = \lambda_2 = -1$, we derive from Eq (2.5) and the Pohožaev identity given by Eq (3.1) that

$$\begin{aligned} J''(t) &= 16E(\varphi_0) - 16\int |x|^2|\varphi|^2 dx + \frac{16-4Np_1}{p_1+2}\int |\varphi|^{p_1+2}dx \\ &< 16E(\varphi_0) + \frac{16-4Np_1}{p_1+2}\int |\varphi|^{p_1+2}dx \\ &= 8|c|^2\rho^2\|\nabla Q\|_2^2 - \frac{8|c|^{2p_2}\rho^2}{p_2}\int (I_\alpha * |Q|^{p_2})|Q|^{p_2}dx - \frac{16|c|^{p_1+2}\rho^{\frac{N}{2}p_1}}{p_1+2}\int |Q|^{p_1+2}dx \\ &\quad + 8|c|^2\rho^{-2}\|xQ\|_2^2 + \frac{16-4Np_1}{p_1+2}\int |\varphi|^{p_1+2}dx \\ &= -8|c|^2\rho^2(|c|^{2p_2-2}-1)\|\nabla Q\|_2^2 - \frac{16|c|^{p_1+2}\rho^{\frac{N}{2}p_1}}{p_1+2}\int |Q|^{p_1+2}dx \\ &\quad + 8|c|^2\rho^{-2}\|xQ\|_2^2 + \frac{16-4Np_1}{p_1+2}\int |\varphi|^{p_1+2}dx. \end{aligned} \tag{3.11}$$

Then utilizing the conservation of mass and interpolating between $L^2(\mathbb{R}^N)$ and $L^{\frac{2N}{N-2}}(\mathbb{R}^N)$, together with the Sobolev embedding $H^1(\mathbb{R}^N) \hookrightarrow L^{\frac{2N}{N-2}}(\mathbb{R}^N)$, we have

$$\begin{aligned} \|\varphi(t)\|_{p_1+2} &\leq \|\varphi(t)\|_2^{\frac{2N-(N-2)(p_1+2)}{2(p_1+2)}} \|\varphi(t)\|_{\frac{2N}{N-2}}^{\frac{Np_1}{2(p_1+2)}} \\ &\leq C\|\varphi_0\|_2^{\frac{2N-(N-2)(p_1+2)}{2(p_1+2)}} \|\varphi(t)\|_{H^1}^{\frac{Np_1}{2(p_1+2)}}. \end{aligned}$$

From Eq (3.10) and $\|\varphi_0\|_2 = |c|\|Q\|_2$, we get

$$\|\varphi(t)\|_{p_1+2}^{p_1+2} \leq \frac{C}{p_1+2}(|c|\|Q\|_2)^{\frac{2N-(N-2)(p_1+2)}{2}}. \tag{3.12}$$

If we choose $|c| > 1$, it follows from Eqs (3.8), (3.11) and (3.12) that

$$J''(t) < -\frac{16|c|^{p_1+2}\rho^{\frac{N}{2}p_1}}{p_1+2}\|Q\|_{p_1+2}^{p_1+2} + 8|c|^2\rho^{\frac{N}{2}p_1}\|xQ\|_2^2 + \frac{C}{p_1+2}(|c|\|Q\|_2)^{\frac{2N-(N-2)(p_1+2)}{2}}.$$

Moreover, taking

$$\rho^{\frac{N}{2}p_1} > \frac{8(p_1+2)|c|^2\|xQ\|_2^2 + C(|c|\|Q\|_2)^{\frac{2N-(N-2)(p_1+2)}{2}}}{16|c|^{p_1+2}\|Q\|_{p_1+2}^{p_1+2}},$$

then

$$J''(t) < -C < 0$$

for all $t \in [0, +\infty)$ with some constant $C > 0$. Therefore, there must exist $\widetilde{T} < +\infty$ such that

$$\lim_{t \to \widetilde{T}} J(t) = 0.$$

Thus by Lemma 2.4, one obtains that $\lim_{t \to \widetilde{T}} \|\varphi(t)\|_{H^1} = +\infty$, which gives a contradiction to Eq (3.10). Thus, we conclude that blow-up of the solution $\varphi(t, x)$ to Eq (1.1) occurs in some time $0 < T < \infty$. The proof is done. □

Remark 3.4. *(i) In the case that $a = 0$, $\lambda_1 = 1, \lambda_2 = -1$, $0 < p_1 < \frac{4}{N}$, $p_2 = 1 + \frac{2+\alpha}{N}$ and $\|\varphi_0\|_2 < \|Q\|_2$, it is proved in [28] that the corresponding solution $\varphi(t, x)$ is global. In addition, the authors showed that there exists φ_0 such that $\|\varphi_0\|_2 \geq \|Q\|_2$, and that the solution $\varphi(t, x)$ with the initial data φ_0 blows up in finite time. The above two theorems (Theorems 3.1 and 3.3) reveal that the sharp threshold for global existence of Eq (1.1) with $a \in \mathbb{R}, \lambda_1 = \pm 1$ and $\lambda_2 = -1$ is the same as that for Theorem 3.1 of [28] when considered for $a = 0, \lambda_1 = 1, \lambda_2 = -1$. Hence, Theorem 3.3 can be seen as a supplement to Theorem 3.1 of [28].*
(ii) Theorems 3.1 and 3.3 can also be seen as complements to Theorem 3.2 of [12], where the case that $a \neq 0$, $\lambda_1 = 0$, $\lambda_2 < 0$ and $p_2 = 1 + \frac{2+\alpha}{N}$ is studied.

3.2. Global existence and blow-up in the case of focusing L^2-critical perturbation

Now we consider the case with double L^2-critical nonlinear terms, i.e., $p_1 = \frac{4}{N}$ and $p_2 = 1 + \frac{2+\alpha}{N}$. In what follows, by applying the sharp Gagliardo-Nirenberg inequality, Pohožaev identity, virial-type identity and scaling approach, we obtain the criteria of global existence and blow-up for Eq (1.1).

Theorem 3.5. *Assume that $\lambda_1 = \lambda_2 = -1$, $p_1 = \frac{4}{N}$ and $p_2 = 1 + \frac{2+\alpha}{N}$. Let Q (or W) be the ground state solution of L^2-critical elliptic equation (2.11) (or (2.12)). Then then the following results hold true.*
(i) Global existence: If $\varphi_0 \in \Sigma$ and $\|\varphi_0\|_2$ satisfies

$$1 - \left(\frac{\|\varphi_0\|_2}{\|Q\|_2}\right)^{\frac{4+2\alpha}{N}} - \left(\frac{\|\varphi_0\|_2}{\|W\|_2}\right)^{\frac{4}{N}} > 0, \tag{3.13}$$

then the solution $\varphi(t, x)$ of Eq (1.1) exists globally for $t \in [0, +\infty)$.
(ii) Blow-up: For any $\nu > 0$, there exists $\varphi_0 \in \Sigma$ satisfying

$$\|\varphi_0\|_2^2 = min\{\|Q\|_2^2, \|W\|_2^2\} + \nu \tag{3.14}$$

such that the solution $\varphi(t, x)$ to Eq (1.1) blows up in finite time.

Proof. (i) First, from Eqs (2.2) and (2.3) and Lemmas 2.4 and 2.5, one has that

$$\begin{aligned}
E(\varphi_0) &= E(\varphi(t)) \\
&= \frac{1}{2}\int |\nabla\varphi|^2 dx + \frac{1}{2}\int |x|^2|\varphi|^2 dx - \frac{1}{p_1+2}\int |\varphi|^{p_1+2} dx - \frac{1}{2p_2}\int (I_\alpha * |\varphi|^{p_2})|\varphi|^{p_2} dx \\
&\geq \left(\frac{1}{2} - \frac{\|\varphi_0\|_2^{2p_2-2}}{2\|Q\|_2^{2p_2-2}} - \frac{\|\varphi_0\|_2^{\frac{4}{N}}}{2\|W\|_2^{\frac{4}{N}}}\right)\|\nabla\varphi\|_2^2 + \frac{1}{2}\|x\varphi\|_2^2 \\
&= \frac{1}{2}\left(1 - \left(\frac{\|\varphi_0\|_2}{\|Q\|_2}\right)^{\frac{4+2\alpha}{N}} - \left(\frac{\|\varphi_0\|_2}{\|W\|_2}\right)^{\frac{4}{N}}\right)\|\nabla\varphi\|_2^2 + \frac{1}{2}\|x\varphi\|_2^2.
\end{aligned} \tag{3.15}$$

Thus, we conclude from Eqs (3.13) and (3.15) that $\|\nabla\varphi\|_2^2$ and $\|x\varphi\|_2^2$ are uniformly bounded for all $t \in [0, +\infty)$. Therefore, the solution $\varphi(t, x)$ to Eq (1.1) becomes global.

(ii) On one hand, if $\|Q\|_2 \leq \|W\|_2$, it follows from Eq (3.14) that $\|\varphi_0\|_2^2 = \|Q\|_2^2 + \nu$. Now, take $\varphi_0^{\tau,\rho} = \tau\rho^{\frac{N}{2}}Q(\rho x)$, where $\tau = \frac{\|Q\|_2^2+\nu}{\|Q\|_2^2} > 1$ and ρ satisfies

$$\rho^4 > \frac{\|xQ\|_2^2}{(\tau^{2p_2-2}-1)\|\nabla Q\|_2^2}; \tag{3.16}$$

similar to Theorem 3.1, we have that $\varphi_0^{\tau,\rho} \in \Sigma$. Injecting $\varphi_0^{\tau,\rho}$ into the energy functional, then we get

$$\begin{aligned} E(\varphi_0) &= \frac{\tau^2\rho^2}{2}\int |\nabla Q|^2 dx - \frac{\tau^{2p_2}\rho^2}{2p_2}\int (I_\alpha * |Q|^{p_2})|Q|^{p_2} dx \\ &\quad - \frac{\tau^{p_1+2}\rho^{\frac{N}{2}p_1}}{p_1+2}\int |Q|^{p_1+2} dx + \frac{\tau^2\rho^{-2}}{2}\int |x|^2|Q|^2 dx \\ &\leq \frac{\tau^2\rho^2}{2}\|\nabla Q\|_2^2 - \frac{\tau^{2p_2}\rho^2}{2p_2}\int (I_\alpha * |Q|^{p_2})|Q|^{p_2} dx + \frac{\tau^2\rho^{-2}}{2}\|xQ\|_2^2 \\ &= \frac{\tau^2\rho^{-2}}{2}(\rho^4(1-\tau^{2p_2-2})\|\nabla Q\|_2^2 + \|xQ\|_2^2), \end{aligned} \tag{3.17}$$

where, in the last step, we use the Pohožaev identity given by Eq (3.1). Furthermore, combining Eqs (3.16) and (3.17), we deduce that

$$E(\varphi_0) < 0. \tag{3.18}$$

Thus, from virial-type identity (2.5) and Eq (3.18), we find that

$$J''(t) = 16E(\varphi_0) - 16\int |x|^2|\varphi|^2 dx < 16E(\varphi_0) < 0,$$

which implies that the collapse of solution $\varphi(t, x)$ to Eq (1.1) must happen at finite time T.

On the other hand, if $\|Q\|_2 > \|W\|_2$, then, from Eq (3.14) we derive $\|\varphi_0\|_2^2 = \|W\|_2^2 + \nu$. Take the initial data $\varphi_0^{\mu,\lambda} = \mu\lambda^{\frac{N}{2}}W(\lambda x)$, where $\mu = \frac{\|W\|_2^2+\nu}{\|W\|_2^2} > 1$ and λ satisfies

$$\lambda^4 > \frac{\|xW\|_2^2}{(\mu^{2p_2-2}-1)\|\nabla W\|_2^2}.$$

By the exponential decay of ground state $W(x)$ (see [9]), one can derive $\varphi_0^{\mu,\lambda} \in \Sigma$. Using the same argument as that for $\|Q\|_2 \leq \|W\|_2$, one has

$$\begin{aligned} E(\varphi_0) &\leq \frac{\mu^2\lambda^2}{2}\|\nabla W\|_2^2 - \frac{\mu^{p_1+2}\lambda^{\frac{Np_1}{2}}}{p_1+2}\int |W|^{p_1+2} dx + \frac{\mu^2\lambda^{-2}}{2}\|xW\|_2^2 \\ &= \frac{\mu^2\lambda^2}{2}(1-\mu^{p_1})\|\nabla W\|_2^2 + \frac{\mu^2\lambda^{-2}}{2}\|xW\|_2^2 \\ &= \frac{\mu^2\lambda^{-2}}{2}[\lambda^4(1-\mu^{p_1})\|\nabla W\|_2^2 + \|xW\|_2^2] < 0. \end{aligned}$$

Hence, we infer that there exists $0 < T < +\infty$ such that the solution $\varphi(t, x)$ to Eq (1.1) blows up at time T. Thus, the proof of Theorem 3.5 is completed. □

Remark 3.6. *The result of Theorem 3.5 is a generalization to Theorem 3.6 and Remark 3.7 in [31], where the case that* $a = 0$, $\lambda_1 = \lambda_2 = -1$, $p_1 = \frac{4}{N}$, $p_2 = 2$ *and* $N - \alpha = 2$ *is considered.*

3.3. Global existence in the case of defocusing L^2-supercritical perturbation

Theorem 3.7. *Assume that $\varphi_0 \in \Sigma$, $\lambda_1 = 1$, $\lambda_2 = -1$, $\frac{4}{N} < p_1 < \frac{4}{N-2}$ and $p_2 = 1 + \frac{2+\alpha}{N}$. Then, the solution $\varphi(t, x)$ of Eq (1.1) exists globally.*

Proof. Since $p_1 > \frac{4}{N}$ and $p_2 = 1 + \frac{2+\alpha}{N}$, according to the Hardy-Littlewood-Sobolev inequality and interpolation inequality, for $\varphi \in \Sigma$, one can find that $0 < \theta = \frac{2(p_1+2)}{Np_1p_2} < 1$ such that $\frac{1}{\frac{2Np_2}{N+\alpha}} = \frac{\theta}{p_1+2} + \frac{1-\theta}{2}$ and

$$\int (I_\alpha * |\varphi|^{p_2})|\varphi|^{p_2}dx \le C\|\varphi\|^{2p_2}_{\frac{2Np_2}{N+\alpha}} \le C\|\varphi\|_2^{2p_2(1-\theta)}\|\varphi\|^{2p_2\theta}_{p_1+2}. \tag{3.19}$$

Combining Young's inequality, mass conservation and Eq (3.19), for $0 < \varepsilon < \frac{2p_2}{p_1+2}$, there exists a constant $C(\varepsilon, p_1, p_2, \|\varphi_0\|_2) > 0$ such that

$$\begin{aligned} E(\varphi_0) &= E(\varphi(t)) \\ &= \frac{1}{2}\left(\int |\nabla\varphi|^2 + |x|^2|\varphi|^2dx\right) + \frac{1}{p_1+2}\int |\varphi|^{p_1+2}dx - \frac{1}{2p_2}\int (I_\alpha * |\varphi|^{p_2})|\varphi|^{p_2}dx \\ &\ge \frac{1}{2}(\|\nabla\varphi\|_2^2 + \|x\varphi\|_2^2) + (\frac{1}{p_1+2} - \frac{\varepsilon}{2p_2})\|\varphi\|^{p_1+2}_{p_1+2} - C(\varepsilon, p_1, p_2, \|\varphi_0\|_2) \\ &\ge \frac{1}{2}(\|\nabla\varphi\|_2^2 + \|x\varphi\|_2^2) - C(\varepsilon, p_1, p_2, \|\varphi_0\|_2), \end{aligned}$$

which yields that

$$E(\varphi_0) + C(\varepsilon, p_1, p_2, \|\varphi_0\|_2) \ge \frac{1}{2}(\|\nabla\varphi\|_2^2 + \|x\varphi\|_2^2).$$

Thus we obtain the boundedness of $\|\nabla\varphi\|_2^2$ and $\|x\varphi\|_2^2$ for arbitrary $t \in [0, +\infty)$, which implies that the solution $\varphi(t, x)$ for Eq (1.1) exists globally. This completes the proof of Theorem 3.7. □

4. Limiting dynamics of blow-up solutions in the case of L^2-subcritical perturbation

This section is concerned with the limiting dynamics of blow-up solutions to Eq (1.1) with $\lambda_1 \in \mathbb{R}$, $\lambda_2 = -1$, $0 < p_1 < \frac{4}{N}$ and $p_2 = 1 + \frac{2+\alpha}{N}$, including the mass concentration phenomenon and the dynamical properties of blow-up solutions with minimal mass. Without loss of generality, we deal with the cases in which $\lambda_1 = \pm 1$. To achieve this goal, let us first review a refined compactness lemma established in [14].

Lemma 4.1. *Let $p_2 = 1 + \frac{2+\alpha}{N}$ and $\{v_n\}_{n=1}^\infty$ be a bounded sequence in $H^1(\mathbb{R}^N)$ satisfying*

$$\limsup_{n\to\infty} \|\nabla v_n\|_2^2 \le M, \quad \limsup_{n\to\infty} \int (I_\alpha * |v_n|^{p_2})|v_n|^{p_2}dx \ge m.$$

Then, there exists $\{x_n\}_{n=1}^\infty \subset \mathbb{R}^N$ such that, up to a subsequence,

$$v_n(x + x_n) \rightharpoonup V \ \text{ weakly in } \ H^1(\mathbb{R}^N),$$

with $\|V\|_2 \ge (\frac{m}{p_2M})^{\frac{1}{2p_2-2}}\|Q(x)\|_2$, where $Q(x)$ is the ground state solution of Eq (2.11).

Using the refined compactness argument, we are able to establish the following concentration property of blow-up solutions to Eq (1.1).

Theorem 4.2. *(L^2-concentration) Assume that $\varphi_0 \in \Sigma$, $\lambda_1 = \pm 1$, $\lambda_2 = -1$, $0 < p_1 < \frac{4}{N}$ and $p_2 = 1 + \frac{2+\alpha}{N}$. Let $\varphi(t, x)$ be a corresponding solution of Eq (1.1) blowing up in finite time T, and suppose that $g(t) : [0, T) \mapsto \mathbb{R}$ is a real-valued nonnegative function satisfying that $g(t)\|\nabla\varphi(t)\|_2 \to +\infty$ as $t \to T$. Then, there exists a function $x(t) \in \mathbb{R}^N$ for $t < T$ satisfying*

$$\liminf_{t\to T} \int_{|x-x(t)|\le g(t)} |\varphi(t, x)|^2 dx \ge \int |Q(x)|^2 dx, \tag{4.1}$$

where $Q(x)$ is the ground state solution of Eq (2.11) with $p_2 = 1 + \frac{2+\alpha}{N}$.

Proof. Take

$$\rho_n := \frac{\|\nabla Q\|_2}{\|\nabla\varphi(t_n)\|_2} \quad and \quad v_n(x) = \rho_n^{\frac{N}{2}} \varphi(t_n, \rho_n x), \tag{4.2}$$

where $\{t_n\}_{n=1}^{\infty} \subseteq [0, T)$ is an arbitrary time sequence satisfying that $t_n \to T$ as $n \to \infty$. Then, the following equations hold:

$$\begin{aligned} \|v_n\|_2 &= \|\varphi(t_n)\|_2 = \|\varphi_0\|_2, \\ \|\nabla v_n\|_2 &= \rho_n\|\nabla\varphi(t_n)\|_2 = \|\nabla Q\|_2. \end{aligned} \tag{4.3}$$

We now introduce the functional $G(\phi) = \frac{1}{2}\int |\nabla\phi|^2 dx - \frac{1}{2p_2}\int (I_\alpha * |\phi|^{p_2})|\phi|^{p_2} dx$; then,

$$\begin{aligned} G(v_n) &= \frac{1}{2}\int |\nabla v_n(x)|^2 dx - \frac{1}{2p_2}\int (I_\alpha * |v_n|^{p_2})|v_n(x)|^{p_2} dx \\ &= \rho_n^2\Big(\frac{1}{2}\int |\nabla\varphi(t_n, x)|^2 dx - \frac{1}{2p_2}\int (I_\alpha * |\varphi(t_n)|^{p_2})|\varphi(t_n, x)|^{p_2} dx\Big) \\ &= \rho_n^2\Big(E(\varphi_0) \mp \frac{1}{p_1+2}\int |\varphi(t_n, x)|^{p_1+2} dx - \frac{1}{2}\int |x|^2|\varphi(t_n, x)|^2 dx\Big). \end{aligned} \tag{4.4}$$

From Lemma 2.5, we discover that

$$\begin{aligned} |G(v_n)| &\le \rho_n^2\left(|E(\varphi_0)| + \frac{1}{p_1+2}\int |\varphi(t_n, x)|^{p_1+2} dx + \frac{1}{2}\int |x|^2|\varphi(t_n, x)|^2 dx|\right) \\ &\le \frac{\|\nabla Q\|_2^2|E(\varphi_0)|}{\|\nabla\varphi(t_n)\|_2^2} + C\frac{\|\nabla Q\|_2^2\|\nabla\varphi(t_n)\|_2^{\frac{Np_1}{2}}}{\|\nabla\varphi(t_n)\|_2^2} + \frac{\|\nabla Q\|_2^2\|x\varphi\|_2^2}{2\|\nabla\varphi(t_n)\|_2^2}. \end{aligned}$$

Hence, by applying $\|\nabla\varphi(t_n)\|_2 \to +\infty$ as $n \to +\infty$ and $0 < p_1 < \frac{4}{N}$, we deduce that $|G(v_n)| \to 0$ as $n \to +\infty$, which yields

$$\int (I_\alpha * |v_n|^{p_2})|v_n(x)|^{p_2} dx \to p_2\|\nabla v_n(x)\|_2^2 = p_2\|\nabla Q\|_2^2.$$

Take $M = \|\nabla Q\|_2^2$ and $m = p_2\|\nabla Q\|_2^2$; then,

$$\limsup_{n\to\infty} \|\nabla v_n\|_2^2 \le M, \quad \limsup_{n\to\infty} \int (I_\alpha * |v_n|^{p_2})|v_n(x)|^{p_2} dx \ge m.$$

Due to Lemma 4.1, there exist $V \in H^1(\mathbb{R}^N)$ and $\{x_n\}_{n=1}^{\infty} \subset \mathbb{R}^N$ such that, up to a subsequence,

$$v_n(\cdot + x_n) = \rho_n^{\frac{N}{2}} \varphi(t_n, \rho_n \cdot + x_n) \rightharpoonup V \;\; weakly \;\; in \; H^1(\mathbb{R}^N), \tag{4.5}$$

and

$$\|V\|_2 \;\; \geq \;\; \|Q\|_2. \tag{4.6}$$

Therefore, using the fact that the L^2-norm is weakly lower semi-continuous, one has the following inequality:

$$\begin{aligned} \liminf_{n\to\infty} \int_{|x|\leq R} |v_n(t_n, x + x_n)|^2 dx &= \liminf_{n\to\infty} \int_{|x|\leq R} \rho_n^N |\varphi(t_n, \rho_n(x + x_n))|^2 dx \\ &\geq \int_{|x|\leq R} |V|^2 dx \;\; for\ arbitrary\ R > 0. \end{aligned} \tag{4.7}$$

Under the hypothesis on $g(t)$, one derives

$$\lim_{n\to\infty} \frac{g(t_n)}{\rho_n} = \lim_{n\to\infty} \frac{g(t_n)\|\nabla\varphi(t_n)\|_2}{\|Q\|_2} = \infty.$$

For sufficiently large n, one can get that $R\rho_n < g(t_n)$. Combining Eqs (4.5) and (4.7), one obtains

$$\begin{aligned} &\liminf_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{|x-y|\leq g(t_n)} |\varphi(t_n, x)|^2 dx \\ &\geq \liminf_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{|x-y|\leq R\rho_n} |\varphi(t_n, x)|^2 dx \\ &\geq \liminf_{n\to\infty} \int_{|x-x_n|\leq R\rho_n} |\varphi(t_n, x)|^2 dx \\ &= \liminf_{n\to\infty} \int_{|x|\leq R} \rho_n^N |\varphi(t_n, \rho_n(x + x_n))|^2 dx \\ &\geq \int_{|x|\leq R} |V|^2 dx \;\; for\ arbitrary\ R > 0. \end{aligned}$$

This and Eq (4.6) infer that

$$\liminf_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{|x-y|\leq g(t_n)} |\varphi(t_n, x)|^2 dx \geq \int |V|^2 dx \geq \|Q\|_2^2.$$

Furthermore, owing to the arbitrariness of the sequence $\{t_n\}_{n=1}^{\infty}$, one has that

$$\liminf_{t\to T} \sup_{y\in\mathbb{R}^N} \int_{|x-y|\leq g(t)} |\varphi(t, x)|^2 dx \geq \|Q\|_2^2. \tag{4.8}$$

For every $t \in [0, T)$, it is easy to see that the function $k(y) := \int_{|x-y|\leq g(t)} |\varphi(t, x)|^2 dx$ is continuous and $\lim_{|y|\to\infty} k(y) = 0$. Thus, there is a function $x(t) \in \mathbb{R}^N$ satisfying that

$$\sup_{y\in\mathbb{R}^N} \int_{|x-y|\leq g(t)} |\varphi(t, x)|^2 dx = \int_{|x-x(t)|\leq g(t)} |\varphi(t, x)|^2 dx.$$

Thus, the above identity together with Eq (4.8) leads to Eq (4.1). □

Theorem 4.3. *(Location of L^2-concentration point) Assume that $\varphi_0 \in \Sigma$, $\lambda_1 = 1$, $\lambda_2 = -1$, $0 < p_1 < \frac{4}{N}$ and $p_2 = 1 + \frac{2+\alpha}{N}$. Let $\varphi(t,x)$ be a corresponding solution of Eq (1.1) that blows up in finite time T and $\|\varphi_0\|_2 = \|Q\|_2$. Then, there exists $x_0 \in \mathbb{R}^N$ such that*

$$|\varphi(t,x)|^2 \to \|Q\|_2^2 \delta_{x_0}, \ \ in\ the\ sense\ of\ a\ distribution, \ \ as\ \ t \to T, \tag{4.9}$$

where Q is the ground-state solution to L^2-critical Choquard equation (2.11).

Proof. By Theorem 4.2, we get

$$\liminf_{t\to T} \int_{|x-x(t)|\le R} |\varphi(t,x)|^2 dx \ge \|Q(x)\|_2^2, \tag{4.10}$$

which, together with mass conservation, as described by Eq (2.1), implies that

$$\|Q\|_2^2 = \|\varphi_0\|_2^2 = \|\varphi(t)\|_2^2 \le \liminf_{t\to T} \int_{|x-x(t)|\le R} |\varphi(t,x)|^2 dx \ge \|Q\|_2^2.$$

This means that

$$\liminf_{t\to T} \int_{|x-x(t)|\le R} |\varphi(t,x)|^2 dx = \liminf_{t\to T} \int_{|x|\le R} |\varphi(t,x+x(t))|^2 dx = \|Q\|_2^2.$$

Therefore, we obtain

$$|\varphi(t,x+x(t))|^2 \to \|Q\|_2^2 \delta_{x=0} \ \ as\ \ t \to T. \tag{4.11}$$

From Eq (4.4), applying Lemma 2.4, one knows that, for any $\varepsilon > 0$ and any real-valued function $\theta(x)$

$$\begin{aligned}
G(e^{\pm i\epsilon\theta}\varphi(t)) &= \frac{1}{2}\int |\nabla e^{\pm i\epsilon\theta}\varphi|^2 dx - \frac{1}{2p_2}\int (I_\alpha * |e^{\pm i\epsilon\theta}\varphi|^{p_2})|e^{\pm i\epsilon\theta}\varphi|^{p_2} dx \\
&= \frac{\epsilon^2}{2}\int |\nabla\theta|^2|\varphi|^2 dx \pm \epsilon Im \int \nabla\theta\cdot\nabla\varphi\cdot\bar{\varphi} dx \\
&\quad + \frac{1}{2}\int |\nabla\varphi|^2 dx - \frac{1}{2p_2}\int (I_\alpha * |\varphi|^{p_2})|\varphi|^{p_2} dx \\
&= \frac{\epsilon^2}{2}\int |\nabla\theta|^2|\varphi|^2 dx \pm \epsilon Im \int \nabla\theta\cdot\nabla\varphi\cdot\bar{\varphi} + G(\varphi) \\
&\ge \frac{1}{2}\int |\nabla e^{\pm i\epsilon\theta}\varphi(t,x)|^2 dx \left(1 - \frac{\|\varphi_0\|_2^{2p_2-2}}{\|Q\|_2^{2p_2-2}}\right) \\
&= 0,
\end{aligned}$$

which indicates that

$$|\pm Im \int \nabla\theta\cdot\nabla\varphi\cdot\bar{\varphi} dx| \le \left(2G(\varphi)\int |\nabla\theta|^2|\varphi|^2 dx\right)^{\frac{1}{2}}. \tag{4.12}$$

Let $\theta_j(x) = x_j$ for every $j = 1, 2, \cdots, N$ in Eq (4.12); then, combining this Eqs (2.2) and (2.3), we deduce from Eq (1.1) and $G(\varphi) \le E(\varphi)$ that

$$\begin{aligned}\left|\frac{d}{dt}\int|\varphi(t,x)|^2x_jdx\right| &= \left|2Im\int i\varphi_t\cdot\bar{\varphi}\cdot x_jdx\right|\\ &= \left|2Im\int -\Delta\varphi\cdot\bar{\varphi}\cdot x_jdx\right|\\ &= \left|2Im\int \nabla\varphi\cdot\bar{\varphi}\cdot \nabla x_jdx\right|\\ &\le 2\left(2G(\varphi)\int|\nabla x_j|^2|\varphi|^2dx\right)^{\frac{1}{2}}\\ &\le 2\left(2E(\varphi_0)\int|\varphi_0|^2dx\right)^{\frac{1}{2}} = C.\end{aligned}$$

This implies that

$$\begin{aligned}\left|\int|\varphi(t_n,x)|^2x_jdx-\int|\varphi(t_m,x)|^2x_jdx\right| &\le \int_{t_m}^{t_n}\left|\frac{d}{dt}\int|\varphi(t,x)|^2x_jdx\right|dt\\ &\le C|t_n-t_m|\to 0 \ \ as\ \ m,n\to\infty,\end{aligned}$$

where $\{t_n\}_{n=1}^{\infty}, \{t_m\}_{m=1}^{\infty}\subset[0,T)$ and $\lim\limits_{n\to\infty}t_n=\lim\limits_{m\to\infty}t_m=T$. It yields that

$$\lim_{t\to T}\int|\varphi(t,x)|^2x_jdx\ \ exists\ for\ all\ j=1,2,\cdots,N.$$

That is,

$$\lim_{t\to T}\int|\varphi(t,x)|^2xdx\ \ exists.$$

Now, let $x_0=\frac{\lim_{t\to T}\int|\varphi(t,x)|^2xdx}{\|Q\|_2^2}$; then, $x_0\in\mathbb{R}^N$ and

$$\lim_{t\to T}\int|\varphi(t,x)|^2xdx=\|Q\|_2^2x_0. \tag{4.13}$$

On the other hand, we deduce from Corollary 3.2 and Eq (4.11) that

$$\begin{aligned}&\int|x|^2|\varphi(t,x+x(t))|^2dx\\ &\le 2\int|x+x(t)|^2|\varphi(t,x+x(t))|^2dx+\int|x(t)|^2|\varphi(t,x+x(t))|^2dx\\ &\le C+2|x(t)|^2\|\varphi_0\|_2^2\\ &\le C+2\limsup_{t\to T}\int_{|x|\le1}|x+x(t)|^2|\varphi(t,x+x(t))|^2dx\\ &\le C+2\int|x|^2|\varphi(t,x)|^2dx\le C,\end{aligned} \tag{4.14}$$

which implies that

$$\limsup_{t\to T}|x(t)|\le\frac{\sqrt{C}}{\|\varphi_0\|_2} \tag{4.15}$$

and

$$\limsup_{t\to T}\int |x|^2|\varphi(t,x+x(t))|^2dx \le C.$$

Hence, for any $r_0 > 0$, we obtain

$$\limsup_{t\to T}\int_{|x|\ge r_0} r_0|x||\varphi(t,x+x(t))|^2dx \le \limsup_{t\to T}\int_{|x|\ge r_0} |x|^2|\varphi(t,x+x(t))|^2dx \le C.$$

Thus, for any $\varepsilon > 0$, there exists sufficiently large $r_0 > 0$ such that

$$\limsup_{t\to T}\left|\int_{|x|\ge R_0} x|\varphi(t,x+x(t))|^2dx\right| \le \frac{C}{r_0} < \varepsilon. \tag{4.16}$$

Owing to Eqs (4.11) and (4.16), one discovers that

$$\begin{aligned}
\limsup_{t\to T}\left|\int x|\varphi(t,x)|^2dx - x(t)\|Q\|_2^2\right| &= \limsup_{t\to T}\left|\int x|\varphi(t,x)|^2dx - x(t)\int |\varphi(t,x)|^2dx\right| \\
&= \limsup_{t\to T}\left|\int |\varphi(t,x)|^2(x-x(t))dx\right| \\
&= \limsup_{t\to T}\left|\int |\varphi(t,x+x(t))|^2xdx\right| \\
&\le \limsup_{t\to T}\left|\int_{|x|\le r_0} |\varphi(t,x+x(t))|^2xdx\right| + \varepsilon \\
&= \varepsilon.
\end{aligned} \tag{4.17}$$

It follows from Eqs (4.13) and (4.17) that

$$\lim_{t\to T} x(t) = x_0, \ \ and \ \ \limsup_{t\to T}\int x|\varphi(t,x)|^2dx = x_0\|Q\|_2^2,$$

which infers that

$$|\varphi(t,x)|^2 \to \|Q\|_2^2\delta_{x=x_0} \ \ in \ \ the \ \ distribution \ \ sense \ \ as \ \ t\to T.$$

Therefore, the conclusion given by Eq (4.9) holds. □

In what follows, we research the blow-up rate of blow-up solutions for Eq (1.1) with $\|\varphi_0\|_2 = \|Q\|_2$.

Theorem 4.4. *(Blow-up rate) Assume that $\varphi_0 \in \Sigma$, $\lambda_1 = 1$, $\lambda_2 = -1$, $0 < p_1 < \frac{4}{N}$, $p_2 = 1 + \frac{2+\alpha}{N}$ and $\|\varphi_0\|_2 = \|Q\|_2$, where Q is the ground state solution to the L^2-critical Choquard equation given by Eq (2.11). Suppose that the solution $\varphi(t,x)$ of Eq (1.1) blows up in finite time T; then, there exists a constant $C > 0$ such that*

$$\|\nabla\varphi(t)\|_2 \ge \frac{C}{T-t}, \quad \forall\, t\in[0,T). \tag{4.18}$$

Proof. Let $g \in C_0^\infty(\mathbb{R}^N)$ be a nonnegative radial function such that

$$g(x) = g(|x|) = |x|^2, \ \ if \ \ |x| < 1 \ \ and \ \ |\nabla g(x)|^2 \le Cg(x).$$

For $M > 0$, we define $g_M(x) = M^2 g(\frac{x}{M})$ and $f_M(t) = \int g_M(x - x_0)|\varphi(t,x)|^2 dx$, where $x_0 = \frac{\lim_{t\to T}\int|\varphi(t,x)|^2 x dx}{\|Q\|_2^2}$. Then, we deduce from Eq (4.12) that, for every $t \in [0, T)$

$$\begin{aligned}
\left|\frac{d}{dt}f_M(t)\right| &= \left|2Im\int i\varphi_t\cdot\bar{\varphi}\cdot f_M(x-x_0)dx\right| \\
&= \left|2Im\int -\Delta\varphi\cdot\bar{\varphi}\cdot f_M(x-x_0)dx\right| \\
&= \left|2Im\int \nabla\varphi\cdot\bar{\varphi}\cdot\nabla f_M(x-x_0)dx\right| \\
&\le 2\left(2G(\varphi)\int|\varphi|^2|\nabla f_M(x-x_0)|^2dx\right)^{\frac{1}{2}} \\
&\le 4\left(E(\varphi_0)\int|\varphi|^2|f_M(x-x_0|dx\right)^{\frac{1}{2}} \\
&\le C\sqrt{f_M(x-x_0)},
\end{aligned}$$

which indicates that

$$\left|\frac{d}{dt}\sqrt{f_M(t)}\right| \le C.$$

As a matter of fact, by integrating on both sides from t to T, we obtain

$$\left|\sqrt{f_M(T)} - \sqrt{f_M(t)}\right| \le C|T-t|. \tag{4.19}$$

In addition, we deduce from Eq (4.9) that

$$f_M(t) \to \|Q\|_2^2 f_M(0) = 0 \ \ as \ \ t \to T. \tag{4.20}$$

Thus, from Eqs (4.19) and (4.20), we get that $f_M(t) \le C(T-t)^2$. Now, fix $t \in [0, T)$; thus, one has that

$$\lim_{M\to\infty} f_M(t) = \int |x-x_0|^2|\varphi(t,x)|^2 dx \le C(T-t)^2.$$

Finally, based on the uncertainty principle, we have

$$\|\nabla\varphi(t)\|_2 \ge \frac{\int|\varphi(t,x)|^2dx}{\int|x-x_0|^2|\varphi(t,x)|^2dx} \ge \frac{C}{T-t}, \ \ \forall t\in[0,T).$$

Thus, the whole proof is completed. □

Remark 4.5. *(i) In the case that $a = 0$, $\lambda_1 = 1$ and $\lambda_2 = -1$, [28] has demonstrated the L^2-concentration property. The conclusion of Theorem 4.2, considering the situation in which $a \neq 0$, $\lambda_1 = \pm 1$ and $\lambda_2 = -1$, supplements the result of Theorem 4.2 in [28].*
(ii) For $a = 0$, $\lambda_1 = 1$ and $\lambda_2 = -1$, the authors of [28] obtained the location of the L^2-concentration point and blow-up rate of the blow-up solutions with a minimal mass (see Theorems 4.3 and 4.4). Our conclusions in Theorems 4.3 and 4.4 can be seen as complements to the corresponding ones in [28].

5. Orbital stability of standing waves

In this part, we initiate a study on the orbital stability of standing waves for the L^2-critical Schrödinger-Choquard equation, i.e., Eq (1.1), in the presence of focusing and defocusing L^2-subcritical perturbation, focusing L^2-critical perturbation and defocusing L^2-supercritical perturbation. Here, the standing waves are solutions to Eq (1.1) possessing the form of $\varphi(t,x) = e^{i\gamma t}u(x)$, where $\gamma \in \mathbb{R}$ represents a frequency and $u \in \Sigma$ is a nonzero solution to the stationary equation

$$-\Delta u + |x|^2 u + \gamma u \pm |u|^{p_1} u - (I_\alpha * |u|^{1+\frac{2+\alpha}{N}})|u|^{\frac{2+\alpha}{N}-1} u = 0. \tag{5.1}$$

To study the stability of standing waves, for $M > 0$, we deal with the variational problem as follows:

$$d_M = inf\{E(u); u \in S\}, \tag{5.2}$$

where

$$S = \{u \in \Sigma; \|u\|_2^2 = M\}.$$

The main result of this section is as follows:

Theorem 5.1. *Assume that $\varphi_0 \in \Sigma$, $\lambda_2 = -1$ and $p_2 = 1 + \frac{2+\alpha}{N}$. Let Q be the ground state solution of L^2-critical Choquard equation (2.11). Then the standing waves for Eq (1.1) are orbitally stable in the following cases:*
(i) $\lambda_1 = \pm 1$, $0 < p_1 < \frac{4}{N}$ and $0 < M < \|Q\|_2^2$;
(ii) $\lambda_1 = -1$, $p_1 = \frac{4}{N}$ and M satisfies that $1 - (\frac{M}{\|Q\|_2})^{\frac{4+2\alpha}{N}} - (\frac{M}{\|W\|_2})^{\frac{4}{N}} > 0$, where W is the ground state solution to Eq (2.12) with $p_1 = \frac{4}{N}$;
(iii) $\lambda_1 = 1$, $p_1 > \frac{4}{N}$ and any $M > 0$.

To show Theorem 5.1, we need the vital lemma below.

Lemma 5.2. *Assume that $\varphi_0 \in \Sigma$, $\lambda_2 = -1$ and $p_2 = 1 + \frac{2+\alpha}{N}$. Let Q be the ground state solution of L^2-critical Choquard equation (2.11). If one of the following conditions holds:*
(i) $\lambda_1 = \pm 1$, $0 < p_1 < \frac{4}{N}$ and $0 < M < \|Q\|_2^2$;
(ii) $\lambda_1 = -1$, $p_1 = \frac{4}{N}$ and M satisfies that $1 - (\frac{M}{\|Q\|_2})^{\frac{4+2\alpha}{N}} - (\frac{M}{\|W\|_2})^{\frac{4}{N}} > 0$, where W is the ground state solution to Eq (2.12) with $p_1 = \frac{4}{N}$;
(iii) $\lambda_1 = 1$, $p_1 > \frac{4}{N}$ and any $M > 0$,
then there exists $u \in \Sigma$ such that $E(u) = d_M$ and $\|u\|_2^2 = M$.

Proof. Suppose that $\{u_n\}_{n=1}^\infty$ is a minimizing sequence of Eq (5.2) satisfying

$$\|u_n\|_2^2 = M, \quad E(u_n) \to d_M \ as \ n \to \infty. \tag{5.3}$$

Then, in Case (i), we have

$$E(u_n) = \frac{1}{2}\int |\nabla u_n|^2 dx + \frac{1}{2}\int |x|^2 |u_n|^2 dx \pm \frac{1}{p_1+2}\int |u_n|^{p_1+2} dx - \frac{1}{2p_2}\int (I_\alpha * |u_n|^{p_2})|u_n|^{p_2} dx.$$

When $\lambda_1 = 1$, using Eq (2.5), it is easy to get

$$E(u_n) \ \geq \ \frac{1}{2}\|\nabla u_n\|_2^2 - \frac{\|u_n\|_2^{2p_2-2}}{2\|Q\|_2^{2p_2-2}}\|\nabla u_n\|_2^2 + \frac{1}{2}\|x u_n\|_2^2$$

$$= (\frac{1}{2} - \frac{\|u_n\|_2^{2p_2-2}}{2\|Q\|_2^{2p_2-2}})\|\nabla u_n\|_2^2 + \frac{1}{2}\|xu_n\|_2^2. \quad (5.4)$$

On the other hand, for $\lambda_1 = -1$, by applying Young's inequality, for any $0 < \varepsilon < \frac{1}{2}$, we can derive

$$\begin{aligned} E(u_n) &\geq \frac{1}{2}\|\nabla u_n\|_2^2 - \frac{\|u_n\|_2^{2p_2-2}}{2\|Q\|_2^{2p_2-2}}\|\nabla u_n\|_2^2 - \varepsilon\|\nabla u_n\|_2^2 - C(\varepsilon, \|u_n\|_2) + \frac{1}{2}\|xu_n\|_2^2 \\ &= (\frac{1}{2} - \frac{\|u_n\|_2^{2p_2-2}}{2\|Q\|_2^{2p_2-2}} - \varepsilon)\|\nabla u_n\|_2^2 - C(\varepsilon, \|u_n\|_2) + \frac{1}{2}\|xu_n\|_2^2. \end{aligned} \quad (5.5)$$

In Case (ii), we could easily obtain that

$$\begin{aligned} E(u_n) &= \frac{1}{2}\int |\nabla u_n|^2 dx + \frac{1}{2}\int |x|^2|u_n|^2 dx - \frac{1}{p_1+2}\int |u_n|^{p_1+2} dx - \frac{1}{2p_2}\int (I_\alpha * |u_n|^{p_2})|u_n|^{p_2} dx \\ &\geq \frac{1}{2}\left(1 - (\frac{M}{\|Q\|_2})^{\frac{4}{N}} - (\frac{M}{\|W\|_2})^{\frac{4}{N}}\right)\|\nabla\varphi\|_2^2 + \frac{1}{2}\|xu_n\|_2^2. \end{aligned} \quad (5.6)$$

In Case (iii), similarly, one can discover that

$$\begin{aligned} E(u_n) &= \frac{1}{2}\int |\nabla u_n|^2 dx + \frac{1}{2}\int |x|^2|u_n|^2 dx + \frac{1}{p_1+2}\int |u_n|^{p_1+2} dx - \frac{1}{2p_2}\int (I_\alpha * |u_n|^{p_2})|u_n|^{p_2} dx \\ &\geq \frac{1}{2}\|\nabla u_n\|_2^2 + \frac{1}{2}\|xu_n\|_2^2 + (\frac{1}{p_1+2} - \frac{\varepsilon}{2p_2})\|u_n\|_{p_1+2}^{p_1+2} - C(\varepsilon, p_1, p_2, \|u_n\|_2) \\ &\geq \frac{1}{2}\|\nabla u_n\|_2^2 + \frac{1}{2}\|xu_n\|_2^2 - C(\varepsilon, p_1, p_2, \|u_n\|_2). \end{aligned} \quad (5.7)$$

By Eq (5.3), taking n large enough such that $E(u_n) < d_M + 1$, then, from Eqs (5.4)–(5.7), we know that $\|\nabla u_n\|_2^2$ and $\|xu_n\|_2^2$ are both bounded, which yields that $\{u_n\}_{n=1}^{\infty}$ is bounded in Σ. Therefore, there exists a subsequence, still denoted by $\{u_n\}$, and $u \in \Sigma$ satisfying

$$u_n \rightharpoonup u \ \ in \ \ \Sigma \ \ as\, n \to \infty,$$

and

$$\lim_{n\to\infty} \|u_n\|_{H^1}^2 + \|xu_n\|_2^2 \geq \|u\|_{H^1}^2 + \|xu\|_2^2. \quad (5.8)$$

Furthermore, we deduce from the compact embedding $\Sigma \hookrightarrow L^2(\mathbb{R}^N)$ that

$$u_n \to u \ \ in \ \ L^2(\mathbb{R}^N) \ \ as\, n \to \infty. \quad (5.9)$$

From Eqs (5.8) and (5.9), it follows that

$$\lim_{n\to\infty} \|\nabla u_n\|_2^2 + \|xu_n\|_2^2 \geq \|\nabla u\|_2^2 + \|xu\|_2^2. \quad (5.10)$$

Moreover, using the Gagliardo-Nirenberg inequality given by Eq (2.4), we infer from the Brezis-Lieb lemma (see Lemma 2.4 in [24]) that

$$\int |u_n|^{p_1+2} dx \to \int |u|^{p_1+2} dx \ \ as\, n \to \infty, \quad (5.11)$$

$$\int (I_\alpha * |u_n|^{p_2})|u_n|^{p_2}dx \to \int (I_\alpha * |u|^{p_2})|u|^{p_2}dx \ \ as\ n \to \infty. \tag{5.12}$$

Therefore, from Eqs (5.9)–(5.12), one obtains

$$\begin{aligned} d_M &= \lim_{n\to\infty} E(u_n) \\ &= \lim_{n\to\infty} \frac{1}{2}(\int |\nabla u_n|^2 + |x|^2|u_n|^2 dx) \pm \frac{1}{p_1+2}\int |u_n|^{p_1+2}dx - \frac{1}{2p_2}\int (I_\alpha * |u_n|^{p_2})|u_n|^{p_2}dx \\ &\geq \lim_{n\to\infty} \frac{1}{2}(\int |\nabla u|^2 + |x|^2|u|^2 dx) \pm \frac{1}{p_1+2}\int |u|^{p_1+2}dx - \frac{1}{2p_2}\int (I_\alpha * |u|^{p_2})|u|^{p_2}dx \\ &= E(u). \end{aligned}$$

By the definition of variational problem (5.2), it is obvious that $d_M \leq E(u)$. Thus, we conclude that $d_M = E(u)$. Therefore, the lemma is proved completely. □

In what follows, define

$$S_M := \{u \in \Sigma;\ E(u) = d_M, \|u\|_2^2 = M\}.$$

Then, for any $u(x) \in S_M$, by the Euler-Lagrange theorem, there exists $\gamma > 0$ such that $u(x)$ is a solution to Eq (5.1); also, we generally refer to $e^{i\gamma t}u(x)$ as the orbit of $u(x)$. On the other hand, if $u \in S_M$, that is, u is a minimizer of d_M, then $e^{i\gamma t}u \in S_M$. In other words, $e^{i\gamma t}u$ is also a minimizer of d_M. Subsequently, we state the definition of the orbital stability of the set S_M as follows.

Definition 5.3. *The set $S_M \subset \Sigma$ is called orbitally stable if for arbitrary $\varepsilon > 0$, there exists $\delta > 0$ such that, for any initial value φ_0 satisfying*

$$\inf_{u\in S_M} \|\varphi_0 - u\|_\Sigma < \delta,$$

then the corresponding solution $\varphi(t)$ of Eq (1.1) satisfies

$$\inf_{u\in S_M} \|\varphi(t) - u\|_\Sigma < \varepsilon \ \ for\ any\ t \geq 0.$$

Proof of Theorem 5.1. First, by contradiction, assume that there exist ε_0 and a sequence $\{\varphi_{0,n}\}_{n=1}^\infty$ such that

$$\inf_{u\in S_M} \|\varphi_{0,n} - u\|_\Sigma \to 0 \ \ as\ \ n \to \infty, \tag{5.13}$$

and that there exists a sequence $\{t_n\}_{n=1}^\infty$ such that the corresponding sequence $\{\varphi_n(t_n)\}_{n=1}^\infty$ of solutions to Eq (1.1) satisfies

$$\inf_{u\in S_M} \|\varphi_n(t_n) - u\|_\Sigma \geq \varepsilon_0. \tag{5.14}$$

Owing to Eq (5.13) and Lemma 5.2, we discover that

$$\int |\varphi_n(t_n)|^2 dx = \int |\varphi_{0,n}|^2 dx \to \int |u|^2 dx = M \tag{5.15}$$

and

$$E(\varphi_n(t_n)) \to E(u) = d_M. \tag{5.16}$$

It follows from Eqs (5.15), (5.16), (2.2) and (2.3) that $\{\varphi_n(t_n)\}_{n=1}^{\infty}$ is still a minimizing sequence for the variational problem given by Eq (5.2). Therefore, combining the arguments of Lemma 5.2, there exists $u_0 \in S_M$ such that

$$\|\varphi_n(t_n) - u_0\|_\Sigma \to 0 \ \ as \ \ n \to \infty,$$

which contradicts Eq (5.14). Thus, we arrive at the conclusion of Theorem 5.1. □

Remark 5.4. *In the case in which $a = 0$, $\lambda_1 = \lambda_2 = -1$, $0 < p_1 < \frac{4}{N}$, $p_2 = 1 + \frac{2+\alpha}{N}$ and $M < \|Q\|_2$, it is shown in [22] that the standing waves are orbitally stable. In addition, for the cases that $a \neq 0$, $\lambda_1 = 0$, $\lambda_2 < 0$, $p_2 = 1 + \frac{2+\alpha}{N}$ and $M < \|Q\|_2$, Feng, in [12] proved the existence of orbitally stable standing waves. The conclusion of Case (i) in Theorem 5.1 is a generalization and complement to Theorem 3.1 of [22] and Theorem 4.2 of [12].*

The conclusions of Cases (ii) and (iii) in Theorem 5.1 are new and interesting in the literature.

6. Conclusions

In this work, we study in details the global existence, blow-up and stability of standing waves for the L^2-critical Schrödinger-Choquard equation with harmonic potential. More precisely, by using the ground state solutions and scaling techniques, some criteria for the global existence and blow-up of the solutions for Eq (1.1) are obtained. Then we apply the refined compactness argument, scaling techniques and the variational characterization of the ground state solution to research the limiting dynamics of blow-up solutions for the L^2-critical Choquard equation with L^2-subcritical perturbation. In addition, we employ the variational methods to prove the orbital stability of standing waves in the presence of L^2-subcritical perturbation, focusing L^2-critical perturbation and defocusing L^2-supercritical perturbation.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This work was partially supported by the Jiangxi Provincial Natural Science Foundation (Grant Nos. 20212BAB211006, 20224BAB201005 and 20232BAB201009) and National Natural Science Foundation of China (Grant No. 11761032).

Conflict of interest

The authors declare that no conflicts of competing interests exists.

References

1. Z. Ammari, F. Nier, Mean field limit for bosons and infinite dimensional phase-space analysis, *Ann. Henri Poincaré*, **9** (2008), 1503–1574. https://doi.org/10.1007/s00023-008-0393-5

2. A. K. Arora, S. Roudenko, Global behavior of solutions to the focusing generalized Hartree equation, *Michigan Math. J.*, **71** (2022), 619–672. https://doi.org/10.1307/mmj/20205855

3. P. d'Avenia, M. Squassina, Soliton dynamics for the Schrödinger-Newton system, *Math. Models Methods Appl. Sci.*, **24** (2014), 553–572. https://doi.org/10.1142/S0218202513500590

4. T. Bartsch, Y. Y. Liu, Z. L. Liu, Normalized solutions for a class of nonlinear Choquard equations, *SN Partial Differ. Equ. Appl.*, **1** (2020), 34. https://doi.org/10.1007/s42985-020-00036-w

5. S. Bhattarai, On fractional Schrödinger systems of Choquard type, *J. Differ. Equ.*, **263** (2017), 3197–3229. https://doi.org/10.1016/j.jde.2017.04.034 ,

6. C. Bonanno, P. d'Avenia, M. Ghimenti, M. Squassina, Soliton dynamics for the generalized Choquard equation, *J. Math. Anal. Appl.*, **417** (2014), 180–199. https://doi.org/10.1016/j.jmaa.2014.02.063

7. H. E. Camblong, L. N. Epele, H. Fanchiotti, C. A. G. Canal, Quantum anomaly in molecular physics, *Phys. Rev. Lett.*, **87** (2001), 220402. https://doi.org/10.1103/PhysRevLett.87.220402

8. K. M. Case, Singular potentials, *Phys. Rev.*, **80** (1950), 797–806. https://doi.org/10.1103/PhysRev.80.797

9. T. Cazenave, *Semilinear Schrödinger Equations*, New York: American Mathematical Society, 2003.

10. J. Q. Chen, B. L. Guo, Strong instability of standing waves for a nonlocal Schrödinger equation, *Phys. D: Nonlinear Phenom.*, **227** (2007), 142–148. https://doi.org/10.1016/j.physd.2007.01.004

11. D. Y. Fang, Z. Han, J. Dai, The nonlinear Schrödinger equation with combined nonlinearities of power-type and Hartree-type, *Chinese Ann. Math. Ser. B*, **32** (2011), 435–474. https://doi.org/10.1007/s11401-011-0642-7

12. B. H. Feng, Sharp threshold of global existence and instability of standing wave for the Schrödinger-Hartree equation with a harmonic potential, *Nonlinear Anal.: Real World Appl.*, **31** (2016), 132–145. https://doi.org/10.1016/j.nonrwa.2016.01.012

13. B. H. Feng, R. P. Chen, Q. X. Wang, Instability of standing waves for the nonlinear Schrödinger-Poisson equation in the L^2-critical case, *J. Dyn. Diff. Equat.*, **32** (2020), 1357–1370. https://doi.org/10.1007/s10884-019-09779-6

14. B. H. Feng, X. X. Yuan, On the Cauchy problem for the Schrödinger-Hartree equation, *Evol. Equ. Control Theory*, **4** (2015), 431–445. https://doi.org/10.3934/eect.2015.4.431

15. B. H. Feng, H. H. Zhang, Stability of standing waves for the fractional Schrödinger Hartree equation, *J. Math. Anal. Appl.*, **460** (2018), 352–364. https://doi.org/10.1016/j.jmaa.2017.11.060

16. H. Genev, G. Venkov, Soliton and blow-up solutions to the time-dependent Schrödinger-Hartree equation, *Discrete Contin, Dyn. Syst, Ser. S*, **5** (2012), 903–923. https://doi.org/10.3934/dcdss.2012.5.903

17. J. Ginibre, G. Velo, On a class of nonlinear Schrödinger equations with nonlocal interaction, *Math. Z.*, **170** (1980), 109–136. https://doi.org/10.1007/BF01214768

18. J. Huang, J. Zhang, X. G. Li, Stability of standing waves for the L^2-critical Hartree equations with harmonic potential, *Appl. Anal.*, **92** (2013), 2076–2083. https://doi.org/10.1080/00036811.2012.716512

19. M. K. Kwong, Uniqueness of positive solutions of $\Delta u - u + u^p = 0$ in $\mathbb{R}^N$, *Arch. Rational Mech. Anal.*, **105** (1989), 243–266. https://doi.org/10.1007/BF00251502

20. C. Y. Lei, M. M. Yang, B. L. Zhang, Sufficient and necessary conditions for normalized solutions to a Choquard equation, *J. Geom. Anal.*, **33** (2023), 109. https://doi.org/10.1007/s12220-022-01151-3

21. X. F. Li, Standing waves to upper critical Choquard equation with a local perturbation: Multiplicity, qualitative properties and stability, *Adv. Nonlinear Anal.*, **11** (2022), 1134–1164. https://doi.org/10.1515/anona-2022-0230

22. K. Liu, C. Q. Shi, Existence of stable standing waves for the Schrödinger-Choquard equation, *Bound. Value Probl.*, **2018** (2018), 160. https://doi.org/10.1186/s13661-018-1078-8

23. C. X. Miao, G. X. Xu, L. F. Zhao, On the blow up phenomenon for the mass-critical focusing Hartree equation in $\mathbb{R}^4$, *Colloq. Math.*, **119** (2010), 23–50. https://doi.org/10.4064/cm119-1-2

24. V. Moroz, J. V. Schaftingen, Groundstates of nonlinear Choquard equations: existence, qualitative properties and decay asymptotics, *J. Funct. Anal.*, **265** (2013), 153–184. https://doi.org/10.1016/j.jfa.2013.04.007

25. M. Ohta, Strong instability of standing waves for nonlinear Schrödinger equations with harmonic potential, *Funkc. Ekvacioj*, **61** (2018), 135–143. https://doi.org/10.1619/fesi.61.135

26. H. Sakaguchi, B. A. Malomed, Suppression of quantum-mechanical collapse by repulsive interactions in a quantum gas, *Phys. Rev. A*, **83** (2011), 013607. https://doi.org/10.1103/PhysRevA.83.013607

27. H. Sakaguchi, B. A. Malomed, Suppression of the quantum collapse in binary bosonic gases, *Phys. Rev. A*, **88** (2013), 043638. https://doi.org/10.1103/PhysRevA.88.043638

28. C. Q. Shi, K. Liu, Dynamics of blow-up solutions for the Schrödinger-Choquard equation, *Bound. Value Probl.*, **2018** (2018), 64. https://doi.org/10.1186/s13661-018-0985-z

29. J. Shu, J. Zhang, Nonlinear Schrödinger equation with harmonic potential, *J. Math. Phys.*, **47** (2006), 063503. https://doi.org/10.1063/1.2209168

30. S. Tian, Y. Yang, R. Zhou, S. H. Zhu, Energy thresholds of blow-up for the Hartree equation with a focusing subcritical perturbation, *Stud. Appl. Math.*, **146** (2021), 658–676. https://doi.org/10.1111/sapm.12362

31. S. Tian, S. H. Zhu, Dynamics of the nonlinear Hartree equation with a focusing and defocusing perturbation, *Nonlinear Anal.*, **222** (2022), 112980. https://doi.org/10.1016/j.na.2022.112980

32. Y. J. Wang, Strong instability of standing waves for Hartree equation with harmonic potential, *Phys. D: Nonlinear Phenom.*, **237** (2008), 998–1005. https://doi.org/10.1016/j.physd.2007.11.018

33. M. I. Weinstein, Nonlinear Schrödinger equations and sharp Interpolation estimates, *Comm. Math. Phys.*, **87** (1983), 567–576. https://doi.org/10.1007/BF01208265

34. H. G. Wu, J. Y. Zhang, Energy-critical Hartree equation with harmonic potential for radial data, *Nonlinear Anal.*, **72** (2010), 2821–2840. https://doi.org/10.1016/j.na.2009.11.026

35. J. Zhang, Stability of atrractive Bose-Einstein condensates, *J. Stat. Phys.*, **101** (2000), 731–746. https://doi.org/10.1023/A:1026437923987

36. J. Zhang, Cross-constrained variational problem and nonlinear Schrödinger equation, *Found. Comput. Math.*, 2002, 457–469. https://doi.org/10.1142/9789812778031_0019
37. J. Zhang, Sharp threshold for blowup and global existence in nonlinear Schrödinger equations under a harmonic potential, *Commun. Part. Diff. Equ.*, **30** (2005), 1429–1443. https://doi.org/10.1080/03605300500299539
38. Y. Zhang, J. Zhang, Stability and instability of standing waves for Cross-Pitaevskii equations with double power nonlinearities, *Math. Control Relat. Fields*, **13** (2023), 533–553. https://doi.org/10.3934/mcrf.2022007